Phase retrieval, error reduction algorithm, and Fienup variants: A view from convex optimization*

Heinz H. Bauschke
Department of Mathematics and Statistics, University of Guelph
Guelph, Ontario N1G 2W1, Canada.

Patrick L. Combettes
Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie – Paris 6
75005 Paris, France.

D. Russell Luke
Institut für Numerische und Angewandte Mathematik, Universität Göttingen
37083 Göttingen, Germany.

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Abstract

The phase retrieval problem is of paramount importance in various areas of applied physics and engineering. The state of the art for solving this problem in two dimensions relies heavily on the pioneering work of Gerchberg, Saxton, and Fienup. Despite the widespread use of the algorithms proposed by these three researchers, current mathematical theory cannot explain their remarkable success. Nevertheless, great insight can be gained into the behavior, the shortcomings, and the performance of these algorithms from their possible counterparts in convex optimization theory. An important step in this direction was made two decades ago when the error reduction algorithm was identified as a nonconvex alternating projection algorithm. The purpose of this paper is to formulate the phase retrieval problem with mathematical care and to establish new connections between well established numerical phase retrieval schemes and classical convex optimization methods. Specifically, it is shown that Fienup's basic input-output algorithm corresponds to Dykstra's algorithm, and that Fienup's hybrid input-output algorithm can be viewed as an instance of the Douglas-Rachford algorithm. This work provides a theoretical framework to better understand and, potentially, improve existing phase recovery algorithms.

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1 Introduction

The *phase retrieval problem* consists of estimating the phase of a complex-valued function from measurements of its modulus and additional *a priori* information. It is of fundamental importance in numerous areas of applied physics and engineering[1–8] and has been studied for over forty years (see[9–11] and the references therein). Historically, the roots of the problem can be traced back to 1892: in a letter to A. Michelson, Lord Rayleigh stated that the continuous phase retrieval problem in interferometry was in general impossible to solve without *a priori* information on the symmetry of the data.[12]

As in many inverse problems, a common formulation of the phase retrieval problem is to seek as a solution any function that is consistent with the measurements as well as with *a priori* constraints. The Gerchberg-Saxton algorithm[13] and its descendent in the form of the error reduction algorithm[14] was the first widely used numerical scheme to solve this type of problem. While its intrinsic mechanism is clear physically — it consists of alternating back-substitutions of known information in the spatial and Fourier domains — it was not initially understood mathematically. In particular, failure of convergence and stagnation of the iterates away from solution points were observed from the outset but lacked a sound mathematical explanation. In the early 1980s, with the work of Youla[15] and others,[16–18] the application of Bregman's method of successive projections[19] to the recovery a signal described by convex constraints generated considerable interest in the signal recovery community; see[20–23] and the extensive lists of references therein. It was natural to seek to embed other iterative methods in this powerful projection framework. Thus, the informal use of cyclic projections in the presence of nonconvex sets appears in several places in the literature.[24–27] In,[28] the error reduction algorithm (loosely called the Gerchberg-Saxton algorithm there) was revealed as such an algorithm, featuring a nonconvex magnitude constraint in the underlying signal space. This study (see also the follow-up paper[29]) gave insightful geometrical interpretations of the stagnation problem as well as of other aspects of the error reduction iterative procedure. A local convergence statement for the general nonconvex projection method was then proposed in[30] and followed in[31] by a more formal analysis based on the theory of multi-valued projections (see also[32] for a tutorial review of these two papers, and[33] for further developments). Another approach to the convergence question was proposed in[34,35] based on the projection theory for convex sets. To extend this theory to the nonconvex setting, the authors require the projection operators to be single-valued.[36] Unfortunately, there is no known example of a nonconvex set for which the projection operator is single-valued (see Remark 3.10). Indeed, the projections in the phase retrieval problem are inherently multi-valued: in[38] the projection operator is precisely identified with the multi-valued subdifferential of a related nonsmooth error metric. A smooth approximation to the projection operator is presented in[39] together with results on the local convergence of iterative methods for minimizing a corresponding smooth error metric.

In a series of papers[40–42] which were unified and reviewed in his seminal 1982 paper,[14] Fienup introduced a broad framework for iterative algorithms. Three main classes of algorithms were presented: error reduction, Basic Input-Output (BIO), and Hybrid Input-Output (HIO). Of the three classes, the last is the most widely used by practitioners. This work resulted in new applications in a wide range of imaging modalities. Furthermore, the error reduction and Fienup algorithms continue to constitute the central conceptual framework for phase retrieval algorithms.

While well-known in the field, the BIO and HIO algorithms lack a proper mathematical framework. The aim of this paper is to show that, just like the error reduction algorithm, the BIO and HIO algorithms also have powerful counterparts in the world of convex projection methods. Our discussion is somewhat more formal than what is usually found in the optics literature, as the level of sophistication of the algorithms requires great attention to details. As much as possible, we provide an intuitive and non-technical discussion without sacrificing the mathematical rigor and precision necessary for a meaningful and constructive analysis of Fienup's algorithms. All technical proofs have been relegated to an appendix.

The paper is organized as follows. In Section 2, the phase retrieval problem is posed as a feasibility problem. Section 3 supplies the necessary review of projection theory, convex analysis, and fixed point theory. The classical algorithms for solving the phase retrieval problem are presented in Section 4. Section 5 describes the correspondence between these algorithms and classical algorithms for solving the convex optimization problems: error reduction and alternating projections (Section 5.2); Fienup's BIO and Dykstra's algorithm (Section 5.3); Fienup's HIO and Douglas-Rachford (Section 5.4). Concluding remarks are formulated in Section 6.

2 Phase retrieval and feasibility

In its general form, the signal recovery problem is to estimate the original form of a signal x in a functional space $\mathcal{L}$ from the measurements of physically related signals and *a priori* information.[21,22] In phase retrieval problems, the measurements consist of the modulus m of the Fourier transform $\widehat{x}$ of x. In other words, the imaging model is described by the relationship

$$|\widehat{x}| = m, \tag{1}$$

and x is commonly referred to as the *object* or *input* of the imaging model. For instance, in optical interferometry astronomy, x is the scattering amplitude of some medium. The measurement, m, is a nonnegative function which is proportional to the modulus of the spatial coherence function (Section 7.4 in[43]), that is, m is proportional to the modulus of the Fourier transform of the scattering amplitude.

A general signal space that appropriately models the underlying physics is the complex Hilbert space

$$\mathcal{L} = L^2[\mathbb{R}^N, \mathbb{C}]. \tag{2}$$

Hence, a signal x in $\mathcal{L}$ is a square-integrable function mapping a continuous variable $t \in \mathbb{R}^N$ to a complex number $x(t) \in \mathbb{C}$. The set of signals that satisfy the *Fourier domain constraint* (1) is[44]

$$M = \big\{y \in \mathcal{L} \colon |\widehat{y}| = m \ \text{ a.e.}\big\}. \tag{3}$$

In addition to the imaging model, an important piece of information that is typically available in phase retrieval problems is that the support of x is contained in some set $D \subset \mathbb{R}^N$. If we let

1_E denote the characteristic function of a set $E \subset \mathbb{R}^N$ and $\complement E$ its complement, this *object domain constraint* confines x to the set

$$S = \{y \in \mathcal{L} \colon y \cdot 1_{\complement D} = 0\}. \tag{4}$$

The *phase retrieval problem* can be posed as that of finding a function $x \in \mathcal{L}$ that satisfies these two constraints, namely,

$$\text{find some } \ x \in S \cap M. \tag{5}$$

This formulation exhibits the phase retrieval problem as a problem of finding a point in the intersection of constraint sets, i.e., a *set theoretic estimation problem* in the sense of.[20] In mathematics (especially in optimization) problems of this kind are called *feasibility problems*. In this paper we shall restrict our attention to the case when (5) is consistent, i.e., $S \cap M \neq \emptyset$. It should be noted, however, that occurrences of inconsistent set theoretic formulations in phase retrieval or other signal recovery problems are far from being academic due to noisy data, measurement errors, or inaccurate *a priori* information.[21,46–49] Several investigations have been devoted to analyzing and coping with this situation in convex problems.[50–55]

While the infinite-dimensional space $\mathcal{L}$ appears to be the most appropriate signal space to model the physics of the problem and to describe the subtle properties of the algorithms in their full generality, we shall also call attention to finite-dimensional versions of the results whenever these happen to differ from their infinite-dimensional counterparts. The reason for this is that in most numerical applications, the signals are sampled on a finite grid and the algorithms are implemented on a digital computer.[56] In this context, the underlying Hilbert space is a Euclidean space whose dimension is determined by the number of samples.

3 Fundamentals of numerical theory

Before discussing the most common and successful algorithms for solving the phase retrieval problem, we establish the mathematical definitions, properties, and results that constitute the theoretical foundation of projection algorithms. We begin with a few basic definitions.

3.1 Distances, projections, and projectors

3.1.1 Distances

As we shall deal with different Hilbert spaces, we assume in this section that

$$\boxed{\mathcal{H} \text{ is a general Hilbert space, with inner product } \langle\cdot,\cdot\rangle \text{ and norm } \|\cdot\| \colon x \mapsto \sqrt{\langle x, x\rangle}.}$$

For instance, if $\mathcal{H} = \mathcal{L}$, then $\langle x, y\rangle = \int x\overline{y}$, for $x, y \in \mathcal{H}$. Or $\langle x, y\rangle = x^T y$ in $\mathbb{R}^N$. The quantity $\|x\|^2$ is simply the *energy* of a signal $x \in \mathcal{H}$.

Definition 3.1 (distances) Suppose $x \in \mathcal{H}$.

(i) If y is a point in $\mathcal{H}$, then the distance from x to y is $d(x,y) = \|x-y\|$.

(ii) If Y is a set in $\mathcal{H}$, then the distance from x to Y is $d(x,Y) = \inf_{y \in Y} d(x,y)$.[58]

As we show in the following example, the distance from a point to a set may not be attained.

Example 3.2 Let $\mathcal{H} = \mathbb{R}^2$ be the Euclidean plane.

(i) If $x = (2,0)$ and $Y = \{y \in \mathcal{H} \colon \|y\| \leq 1\}$ is the unit ball, then $d(x,Y) = 1$ and the distance is attained at $y = (1,0) \in Y \colon d(x,Y) = d(x,y)$. Moreover, y is the only point in Y with this property — every other point in Y is further away from x than y is.

(ii) If $x = (2,0)$ and $Y = \{y \in \mathcal{H} \colon \|y\| < 1\}$ is the open unit ball, then still $d(x,Y) = 1$ yet there is no point $y \in Y$ with $d(x,y) = d(x,Y)$; the distance from x to Y is not attained.

(iii) If $x = (0,0)$ and $Y = \{y \in \mathcal{H} \colon \|y\| = 1\}$ is the unit circle, then $d(x,Y) = 1 = d(x,y)$, for all $y \in Y$; the distance is attained at every point in Y.

The points at which the distance to a set is attained are of great importance and the subject of the following subsection.

3.1.2 Projection operators (projectors)

Definition 3.3 (projection operator) Suppose Y is a set in $\mathcal{H}$. If $x \in \mathcal{H}$, then the set of points in Y nearest to x, namely

$$\big\{y \in Y \colon d(x,y) = d(x,Y)\big\}, \tag{6}$$

is denoted $\Pi_Y(x)$, and called the *projection* of x onto Y. The induced operator Π_Y is called the *projection operator* or *projector* onto Y.

An interpretation from signal processing helps at this point: if Y contains the signals satisfying a certain property, then the signals in $\Pi_Y(x)$ are the closest signals to x satisfying this property. It is crucial to realize that the output of a projector Π_Y are *subsets* of Y. These may be empty, reduced to a single element, or have more than one element: revisiting Example 3.2 and borrowing its notation, we see that (i) $\Pi_Y(x) = \{(1,0)\}$, (ii) $\Pi_Y(x) = \emptyset$, (iii) $\Pi_Y(x) = Y$, respectively. To bring out this behavior clearly, we say that the projector is a *multifunction* or a *multi-valued map*.[59]

Remark 3.4 (single-valued selections of projectors) Let Π_Y be the projector onto a *proximinal set* Y, i.e., $\Pi_Y(x) \neq \emptyset$ for all $x \in \mathcal{H}$. Then we shall denote by P_Y a *selection* of Π_Y, i.e., $P_Y(x) \in \Pi_Y(x)$, for all $x \in \mathcal{H}$. P_Y is therefore a single-valued operator. When Y is a *Chebyshev set*, i.e., $\Pi_Y(x)$ is a singleton for all $x \in \mathcal{H}$, then Π_Y has a *unique* selection P_Y, which is itself called the projector onto Y. This always occurs when the set Y is closed and convex, see Fact 3.9 below.

3.1.3 A preview: Projections for the phase retrieval problem

In the setting of the phase retrieval problem, the abstract Hilbert space $\mathcal{H}$ is simply the function space $\mathcal{L}$ introduced in Section 2. The most common approach for solving the phase retrieval problem is to enforce the known object domain and Fourier domain constraints in some alternating fashion. Thus, given a signal x, the support constraint is naturally enforced by setting x equal to zero outside the given domain D, via the transformation $x \mapsto x \cdot 1_D$. As we shall now see, this simple operation is actually a projection.

Example 3.5 (support constraint) Suppose D is a measurable[60] set in $\mathbb{R}^N$ and fix $x \in \mathcal{L}$. Then the projection (recall the notation of Remark 3.4) of x onto the set S of (4) is

$$P_S(x) = x \cdot 1_D. \tag{7}$$

The same observation is true for the image modulus constraint. Approaches to enforce it are described below; again, these operations turn out to be projections.

Example 3.6 (image modulus constraint) Let m be a nonnegative function in $\mathcal{L}$ and fix $x \in \mathcal{L}$. Then $y \in \mathcal{L}$ belongs to the projection $\Pi_M(x)$ of x onto the set M of (3) if and only if it satisfies a.e.

$$\widehat{y}(\omega) = \begin{cases} m(\omega)\frac{\widehat{x}(\omega)}{|\widehat{x}(\omega)|}, & \text{if } \widehat{x}(\omega) \neq 0; \\ m(\omega)\exp[i\varphi(\omega)], & \text{otherwise,} \end{cases} \tag{8}$$

for some measurable function $\varphi : \mathbb{R}^N \to \mathbb{R}$.

Example 3.6 shows that every function $y \in \Pi_M(x)$ satisfies

$$d\big(\widehat{x}(\omega), m(\omega)\mathbb{S}\big) = d\big(\widehat{x}(\omega), \widehat{y}(\omega)\big) \quad \text{a.e. on } \mathbb{R}^N, \tag{9}$$

where $m(\omega)\mathbb{S} = \{u \in \mathbb{C} : |u| = m(\omega)\}$ denotes a circle in the complex plane, with radius $m(\omega)$ and centered at the origin. The *multi-valuedness* of the projection is now evident: whenever $\widehat{x}(\omega) = 0$, any phase φ will work. Consequently, if the set $\{\omega \in \mathbb{R}^N : m(\omega) \neq 0 \text{ and } \widehat{x}(\omega) = 0\}$ is sufficiently large,[61] then $\Pi_M(x)$ contains *infinitely many* elements (see[39] and Example 3.15 below).

In practice, one picks the *particular selection* $y_0 \in \Pi_M(x)$ corresponding to zero phase $\varphi \equiv 0$:

$$\widehat{y_0}(\omega) = \begin{cases} m(\omega)\frac{\widehat{x}(\omega)}{|\widehat{x}(\omega)|}, & \text{if } \widehat{x}(\omega) \neq 0; \\ m(\omega), & \text{otherwise.} \end{cases} \tag{10}$$

Analogous formulae hold if one considers a modulus constraint in the object domain (as in the original set-up of the Gerchberg-Saxton algorithm for reconstructing phase from two intensity measurements; see[13]).

3.2 Convexity and closedness

In what follows we assume that[62]

$$\boxed{\mathcal{H} \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle \text{ and induced norm } \|\cdot\|.} \tag{11}$$

Definition 3.7 (vector subspace and convex set) Suppose C is a nonempty set in $\mathcal{H}$. Then C is a

(i) *vector subspace* if it contains the zero vector and if the line joining any two points in C lies entirely in C (algebraically: $\lambda c_1 + (1-\lambda)c_2 \in C$ whenever c_1, $c_2 \in C$ and $\lambda \in \mathbb{R}$);

(ii) *convex* set if the line segment joining any two points in C lies entirely in C (algebraically: $\lambda c_1 + (1-\lambda)c_2 \in C$ whenever c_1, $c_2 \in C$ and $\lambda \in [0,1]$).

Definition 3.8 (closed set) Suppose C is a set in $\mathcal{H}$. Then C is *closed* if, whenever (c_n) is a sequence in C that converges to some point $c \in \mathcal{H}$, the limit point c belongs necessarily to C.

Closedness is important for algorithmic purposes: one often wants the limit of a sequence to inherit good properties from the terms of the sequence. Closedness is certainly a necessary condition for the existence of projections. Indeed, a point x in the closure (smallest closed superset) of C but not in C has no projection onto C. In finite-dimensional spaces, closedness is also sufficient to guarantee the existence of projections, e.g.[31] ; however, this is no longer true in infinite-dimensional spaces (see Example III.4.3.2.b in[63] for a counterexample).

In tandem with convexity, closedness guarantees that projections are extremely well-behaved.

Fact 3.9 (projection onto a closed convex set) Suppose C is a nonempty closed convex set in $\mathcal{H}$. Then for every $x \in \mathcal{H}$, the projection of x onto C is a singleton; moreover, the point[64] $P_C(x)$ is characterized by

$$P_C(x) \in C, \quad \text{and } \langle c - P_C(x), x - P_C(x)\rangle \leq 0, \ \text{ for all } c \in C. \tag{12}$$

In addition, the projector P_C satisfies

$$\|P_C(x) - P_C(y)\|^2 + \|(I - P_C)(x) - (I - P_C)(y)\|^2 \leq \|x - y\|^2, \quad \text{for all } x, y \text{ in } \mathcal{H}. \tag{13}$$

Remark 3.10 (Chebyshev problem) Suppose C is a closed nonempty set in $\mathcal{H}$. If C is convex, then Fact 3.9 states that the projector $\Pi_C = \{P_C\}$ is a single-valued map. The converse implication is the famous *Chebyshev problem*: if the projector Π_C is a single-valued map, must the set C be convex? The answer is affirmative in finite-dimensional spaces, but remains open to date for the general Hilbert space case. If it turns out to be affirmative in general, then the results of[34,35] discussed in the Introduction are essentially void. The reader is referred to Chapter 12 in[65] for further information.

For the remainder of this section, we use the notation of Section 2. The following result is quite useful.

Proposition 3.11 (separable constraints and projections) Suppose $(A(t))_{t\in\mathbb{R}^N}$ is a family of sets in $\mathbb{C}$. Let $\mathcal{A} = \{a \in \mathcal{L} : a(t) \in A(t) \text{ a.e.}\}$.

(i) If each $A(t)$ is convex, then so is $\mathcal{A}$.

(ii) If each $A(t)$ is closed, then so is $\mathcal{A}$.

Theorem 3.12 (projection onto a separably closed convex constraint) Let $(A(t))_{t\in\mathbb{R}^N}$ be a family of closed convex sets in $\mathbb{C}$ such that $\mathcal{A} = \{a \in \mathcal{L} : a(t) \in A(t) \text{ a.e.}\}$ is nonempty. Assume that $t \mapsto A(t)$ is a measurable multifunction[66] from $\mathbb{R}^N$ to $\mathbb{C}$. Fix $x \in \mathcal{L}$. Then $y = P_{\mathcal{A}}(x)$ is given by $y(t) = P_{A(t)}(x(t))$ a.e.

Remark 3.13 In Theorem 3.12, the condition that $t \mapsto A(t)$ be measurable may be difficult to verify in practice. For our purpose, however, it is sufficient to work with the following criterion, taken from Section 14.A in:[67]

> if μ is a measurable function from $\mathbb{R}^N$ to $\mathbb{C}$ and $Z \subset \mathbb{C}$,
> then $A(t) = \mu(t) \cdot Z$ defines a measurable multifunction.

Note that there is no restriction whatsoever on the set Z.

Example 3.14 (object domain constraint) Let $x \in \mathcal{L}$. Then

(i) (*support*) $\mathcal{A} = \{y \in \mathcal{L} : y \cdot 1_{\complement D} = 0\}$ is closed and convex, with $P_{\mathcal{A}}(x) = x \cdot 1_D$;

(ii) (*real-valuedness*) $\mathcal{A} = \{y \in \mathcal{L} : y(t) \in \mathbb{R} \text{ a.e.}\}$ is closed and convex, with $P_{\mathcal{A}}(x) = \mathrm{Re}(x)$;

(iii) (*nonnegativity*) $\mathcal{A} = \{y \in \mathcal{L} : y(t) \text{ is real and nonnegative a.e.}\}$ is closed and convex (but not a vector subspace), with[68] $P_{\mathcal{A}}(x) = (\mathrm{Re}(x))^+$.

Theorem 3.12 provides a convenient expression for the projection in the infinite-dimensional space $\mathcal{L}$ in terms of the finite-dimensional pointwise projections when the constraint is convex. Unfortunately, it is well-known that the phase retrieval problem involves nonconvex constraints, as the next example illustrates.

Example 3.15 (Fourier domain constraint is closed but not convex) For closedness, see the proof in Appendix A. Unless $m = 0$ (in which case the Fourier modulus constraint encompasses only the zero function), the Fourier modulus constraint is never a convex set. To see this, pick $x \in M$. Then $-x \in M$; however, the convex combination $\frac{1}{2}x + \frac{1}{2}(-x) = 0$ does not belong to M.

The next theorem states that certain nonconvex constraints can also be dealt with pointwise. This justifies the expression of the projection in Example 3.6.

Theorem 3.16 (projection onto a separably compact constraint) Let $(A(t))_{t\in\mathbb{R}^N}$ be a family of compact (i.e., closed and bounded) sets in $\mathbb{C}$ such that $\mathcal{A} = \{a \in \mathcal{L} : a(t) \in A(t) \text{ a.e. } \}$ is nonempty. Assume that $t \mapsto A(t)$ is a measurable multifunction from $\mathbb{R}^N$ to $\mathbb{C}$. Then, for every $x \in \mathcal{L}$, $\Pi_{\mathcal{A}}(x) \neq \emptyset$; in fact, $y \in \Pi_{\mathcal{A}}(x)$ if and only if y is measurable and $y(t) \in \Pi_{A(t)}\big(x(t)\big)$ a.e.

While the phase retrieval problem (5) is not convex, some related problems which are convex can be found in the literature.

(i) In the problem considered by Gerchberg in,[69] the object domain constraint is again $x \cdot 1_{\complement D} = 0$ and the Fourier domain constraint is $\widehat{x} \cdot 1_{\Omega} = f$, i.e., the Fourier transform of x (not just its modulus) on a domain Ω is a known function f. This constraint forms a convex set (actually an affine subspace, i.e., the translation of a vector subspace). Hence, the resulting feasibility problem is convex (affine), which explains the good convergence properties of the alternating projection algorithm proposed by Gerchberg to solve it. This observation was made by Youla[70] in the case of the Papoulis extrapolation algorithm for band-limited signals[71] (this algorithm is identical to Gerchberg's, except that the roles played by the object and Fourier domains are interchanged).

(ii) In some problems, e.g., in holography, the Fourier domain constraint arises from the knowledge of the phase φ of the Fourier transform of x rather than from its modulus. In,[15] Youla observed that the phase constraint $\angle\widehat{x} = \varphi$ leads to a convex set (actually a convex cone). This fact was fully exploited in[17] (see also[29] and references therein).

(iii) In,[15] Youla pointed out that the submodulus constraint $|\widehat{x}| \leq m$ is convex. In most phase retrieval problems, this convexification of the exact constraint is too coarse and it will typically produce poor results.

(iv) Convexity is an algebraic notion which, by definition, depends on the choice of the underlying vector space structure. In,[72] Çetin exhibited an alternative (discrete) signal space in which the constraint $|\widehat{x}| = m$ is convex. Unfortunately, this approach is not suitable for the phase retrieval problem since the constraint $x \cdot 1_{\complement D} = 0$ is no longer convex in this space.

Ultimately, the difficulty of the phase retrieval problem is caused by the lack of convexity of the Fourier domain constraint and the lack of good convex approximations to it.

3.3 Some fixed point theory

Throughout this section, we continue to make assumption (11).

Definition 3.17 (firm nonexpansivity and nonexpansivity) Suppose T is a map from $\mathcal{H}$ to $\mathcal{H}$. Then T is *firmly nonexpansive*, if

$$\|T(x) - T(y)\|^2 + \|(I - T)(x) - (I - T)(y)\|^2 \leq \|x - y\|^2, \quad \text{for all } x, y \in \mathcal{H}; \tag{14}$$

and T is *nonexpansive*, if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \text{for all } x, y \text{ in } \mathcal{H}. \tag{15}$$

Fact 3.9 states that the projector onto a nonempty closed convex set is firmly nonexpansive. From the definition, it is immediate that every firmly nonexpansive map is nonexpansive. Firmly nonexpansive and nonexpansive maps are actually very closely related.

Fact 3.18 (Theorem 12.1 in[73]) Suppose T is a map from $\mathcal{H}$ to $\mathcal{H}$. Then the following are equivalent:

(i) T is firmly nonexpansive;

(ii) $2T - I$ is nonexpansive;

(iii) $T = \frac{1}{2}\widetilde{T} + \frac{1}{2}I$, for some nonexpansive map $\widetilde{T}$.

Many problems can be reduced to finding a fixed point of a nonexpansive mapping: the *fixed point set* of a mapping T from $\mathcal{H}$ to $\mathcal{H}$ is

$$\operatorname{Fix} T = \big\{x \in \mathcal{H} \colon T(x) = x\big\}. \tag{16}$$

For example, the set of fixed points of a projector onto a closed convex set C is just $\operatorname{Fix} P_C = C$.

Fixed points are usually found as limit points of sequences. Discussing convergence in infinite-dimensional spaces requires care, because there exist distinct notions of convergence. The following concepts are appropriate in our present setting.

Definition 3.19 (norm and weak convergence) Suppose (x_n) is a sequence in $\mathcal{H}$ and $x \in \mathcal{H}$. Then

(i) (x_n) *converges (in norm, or strongly)* to x, if $\|x_n - x\| \to 0$ (in symbols $x_n \to x$);

(ii) (x_n) *converges weakly* to x, if $\langle x_n - x, y\rangle \to 0$, for all $y \in \mathcal{H}$ (in symbols $x_n \overset{\mathrm{w}}{\rightharpoonup} x$).

Physically, $x_n \to x$ in $\mathcal{H}$ means that the energy of the residual signal $\|x_n - x\|^2$ becomes arbitrarily small as n increases; on the other hand, $x_n \overset{\mathrm{w}}{\rightharpoonup} x$ means only that any measurement of the residual signal that can be modeled by a linear operation from $\mathcal{H}$ to $\mathbb{R}$ becomes arbitrarily small.

It is easy to see that if a sequence converges in norm, then it does so weakly (indeed, if $x_n \to x$ and $y \in \mathcal{H}$ then, by the Cauchy-Schwarz inequality: $|\langle x_n - x, y\rangle| \leq \|x_n - x\| \cdot \|y\| \to 0$); in finite-dimensional spaces, the converse is true as well. However, in every infinite-dimensional space, there exist sequences that converge weakly but not in norm (for instance, every orthonormal sequence converges weakly to zero, but not in norm).

We shall see in the next section that it is very desirable to have algorithms that find fixed points of nonexpansive mappings. The strong interest in firmly nonexpansive mappings stems from the ease of finding their fixed points by simple iteration.

Fact 3.20 (Opial[74]) Suppose T is a firmly nonexpansive mapping from $\mathcal{H}$ to $\mathcal{H}$ with $\operatorname{Fix} T \neq \emptyset$. Then for every $x \in \mathcal{H}$, the sequence $(T^n x)$ converges weakly to some point in $\operatorname{Fix} T$.

Remark 3.21 An ingenious example by Genel and Lindenstrauss[75] shows that it is not possible to strengthen the conclusion of Fact 3.20 to norm convergence. Also, if T is nonexpansive (but not firmly), then $(T^n x)$ need not converge to a fixed point: consider $T = -I$. Then $\operatorname{Fix} T = \{0\}$ and $T^n x = (-1)^n x$ is not convergent for all $x \neq 0$.

4 Classical algorithms

We now discuss three popular algorithms designed for solving the phase retrieval problem (5): given a starting point x_0, each of these algorithms constructs a sequence (x_n) of functions that in practice often converges to a solution of (5). The features common to all three algorithms are these: the construction of a function x_{n+1} depends only on the predecessor x_n, and x_{n+1} is found by applying the projection operators P_S and P_M in some fashion to x_n. Because of this, each algorithm is entirely characterized by its updating rule.[76]

We follow Fienup's framework.[14] To bring out the results as clearly as possible, we assume that the object domain constraint is only a support constraint. In addition, for the sake of definiteness, P_M designates the selection of the Fourier domain projector Π_M defined through (10) (see Remark 3.4).

4.1 Error reduction algorithm

The error reduction algorithm updates a current iterate x_n via[78]

$$x_{n+1}(t) = \begin{cases} \big(P_M(x_n)\big)(t), & \text{if } t \in D; \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Hence $x_{n+1} = 1_D \cdot P_M(x_n)$; equivalently, by Example 3.5,

$$x_{n+1} = (P_S P_M)(x_n). \tag{18}$$

Local convergence results for this and other types of nonconvex successive projection methods can be found in.[31]

4.2 Fienup's basic input-output (BIO) algorithm

The update x_{n+1} in the BIO algorithm is obtained from x_n by setting

$$x_{n+1}(t) = \begin{cases} x_n(t), & \text{if } t \in D; \\ x_n(t) - \big(P_M(x_n)\big)(t), & \text{otherwise.} \end{cases} \tag{19}$$

Note that $x_{n+1} = 1_D \cdot x_n + 1_{\complement D} \cdot \big(x_n - P_M(x_n)\big) = x_n - (1 - 1_D) \cdot P_M(x_n)$, which we rewrite as

$$x_{n+1} = (P_S P_M + I - P_M)(x_n). \tag{20}$$

4.3 Fienup's hybrid input-output (HIO) algorithm

Given a parameter $\beta > 0$, the HIO algorithm constructs the successor of x_n via

$$x_{n+1}(t) = \begin{cases} \big(P_M(x_n)\big)(t), & \text{if } t \in D; \\ x_n(t) - \beta\big(P_M(x_n)\big)(t), & \text{otherwise.} \end{cases} \tag{21}$$

For clarity, we henceforth set $\beta = 1$; in practice, values of β different from 1 are important as they may result in better performance.[3,14] The recursion (21) now becomes

$$\begin{aligned} x_{n+1} &= 1_D \cdot P_M(x_n) + 1_{\complement D} \cdot (x_n - P_M(x_n)) \\ &= 1_D \cdot P_M(x_n) + (1 - 1_D) \cdot (x_n - P_M(x_n)) \\ &= 1_D \cdot \big(2P_M(x_n) - x_n\big) + x_n - P_M(x_n) \\ &= \big(P_S(2P_M - I) + (I - P_M)\big)(x_n). \end{aligned} \tag{22}$$

Note that this can also be written as

$$x_{n+1} = \big(P_S P_M + (I - P_S)(I - P_M)\big)(x_n), \tag{23}$$

because the projector onto a closed vector space is linear.

Remark 4.1 The description of the HIO algorithm in,[14] specialized to our setting, actually reads[79]

$$x_{n+1}(t) = \begin{cases} \big(P_M(x_n)\big)(t), & \text{if } t \in D \text{ or } \big(P_M(x_n)\big)(t) = 0; \\ x_n(t) - \big(P_M(x_n)\big)(t), & \text{otherwise.} \end{cases} \tag{24}$$

The updates differ precisely at points t that belong to $\complement D$ and that satisfy $\big(P_M(x_n)\big)(t) = 0$. However, it is not easy to determine which formulation is used in the community, as the papers we are aware of are not specific on this question. Notable exceptions are,[80–82] which use the formulation we shall employ,[83] and,[84] which uses the literal definition. We should note that the main results of this paper would remain essentially unchanged if we followed the literal definition. Interestingly, the same issue formally arises for the error reduction algorithm and for the BIO algorithm; however, a careful inspection reveals that this does not lead to different algorithms!

5 Main connections with convex optimization algorithms

We are now ready to establish the correspondence between classical algorithms for solving (5) and their counterparts for solving a two-set convex feasibility problem. Throughout this section, the standing assumption is that

$$\boxed{A \text{ and } B \text{ are two nonempty closed convex sets in a real Hilbert space } \mathcal{H}.}$$

5.1 The convex feasibility problem

The *convex feasibility problem* associated with A and B is to

$$\text{find some } \ x \in A \cap B. \tag{25}$$

Note the similarity between (25) and the formulation (5) of the phase retrieval problem as a feasibility problem. However, (5) is *not* a convex feasibility problem as the image modulus constraint is not convex (Example 3.15).

We now revisit the three classical algorithms for solving the phase retrieval problem described above. It will turn out that each algorithm corresponds to a classical algorithm for solving (25).[85]

5.2 Error reduction algorithm and POCS

The method of alternating *projections onto convex sets (POCS)* generates, for the present setting of two constraints, sequences (a_n) and (b_n) as follows: pick an arbitrary starting point $a_0 \in \mathcal{H}$. Then update for $n \geq 0$ via

$$b_n = P_B(a_n) \quad \text{and} \quad a_{n+1} = P_A(b_n). \tag{26}$$

This process is depicted in Fig. 6.

The following basic result shows that POCS does find a solution of (25).

Fact 5.1 (Theorem 1 in[19]) Suppose $A \cap B \neq \emptyset$. Then both sequences (a_n) and (b_n) in (26) converge weakly to a point in $A \cap B$.

Remark 5.2 Although we do not go into details, we mention in passing two possible paths to proving Fact 5.1 The first approach is fixed point theoretic and consists of showing that (i) $\mathrm{Fix}(P_A P_B) = A \cap B$ and (ii) iterating the composition $P_A P_B$ produces — analogously to Fact 3.20 — fixed points.[89] The second approach is more elementary and builds on the notion of *Fejér monotonicity*; see.[91–93]

Observation 5.3 (Error reduction algorithm as a nonconvex POCS algorithm) Replace the set A with the (convex) object domain constraint set S and the set B with the (nonconvex) Fourier domain constraint set M. Then the sequence (a_n) generated by (26) corresponds to the sequence (x_n) generated by the error reduction algorithm (18). This connection was established by Levi and Stark[28] in 1984.

5.3 Fienup's BIO algorithm and Dykstra's algorithm

Dykstra's algorithm was first developed for closed convex cones in,[94] and subsequently generalized to closed convex sets in.[95] For two closed convex sets A and B, it produces four sequences (a_n), (b_n), (p_n), and (q_n) as follows (see Fig. 6). Fix a starting point a_0, set $q_{-1} = 0 = p_0$, and update for $n \geq 0$ via

$$\begin{aligned} b_n &= P_B(a_n + q_{n-1}), \quad q_n = (I - P_B)(a_n + q_{n-1}) = a_n + q_{n-1} - b_n; \\ a_{n+1} &= P_A(b_n + p_n), \quad p_{n+1} = (I - P_A)(b_n + p_n) = b_n + p_n - a_{n+1}. \end{aligned} \tag{27}$$

Clearly, Dykstra's algorithm is more involved than POCS and is more demanding in terms of storage. However, its convergence properties are superior in the sense that (i) it converges in norm and, (ii) it provides a well-defined limit point, namely the feasible signal that lies closest to the starting point.[96]

Fact 5.4 (Boyle-Dykstra[95]) Suppose $A \cap B \neq \emptyset$. Then both sequences (a_n) and (b_n) in (27) converge in norm to $P_{A\cap B}(a_0)$, the point in $A \cap B$ closest to a_0.

Fact 5.4 is quite remarkable because the sequences converge in norm, and their limit is explicitly identified as the nearest feasible point to the starting point. This explains the popularity of Dykstra's algorithm in approximation theory, where this method is well understood and many extensions have been found; see, for instance,.[98–102] For applications of Dykstra's algorithm to signal recovery, see.[103]

For the rest of this subsection, we assume additionally that A is a closed vector space. Then (p_n) lies entirely in $A^\perp$, the orthogonal complement of A, and the computation of a_{n+1} becomes $a_{n+1} = P_A b_n + P_A p_n = P_A b_n$. Thus, the sequence (p_n) is not needed, and Dykstra's algorithm simplifies to

$$b_n = P_B(a_n + q_{n-1}), \quad q_n = (I - P_B)(a_n + q_{n-1}), \quad a_{n+1} = P_A(b_n). \tag{28}$$

Hence

$$a_{n+1} + q_n = (P_A P_B + I - P_B)(a_n + q_{n-1}) = (P_A P_B + I - P_B)^{n+1}(a_0). \tag{29}$$

The next observation appears to be new; it establishes the correspondence between BIO and Dykstra's algorithm.

Observation 5.5 (BIO algorithm as a nonconvex Dykstra algorithm) Replace the set A with the (convex) object domain constraint set S and the set B with the (nonconvex) Fourier domain constraint set M. Then the sequence $(a_n + q_{n-1})$ generated by (29) corresponds to the sequence (x_n) generated by Fienup's BIO algorithm (20).

Remark 5.6 Even when $A \cap B \neq \emptyset$, it is possible that the sequences (p_n) and (q_n) generated by Dykstra's algorithm (in its general form) are both *unbounded*; see.[101] This suggests the pertinent sequence to monitor in Fienup's BIO algorithm is $\big(P_M(x_n)\big)$, rather than (x_n).

5.4 Fienup's HIO algorithm and the Douglas-Rachford algorithm

When specialized to the convex feasibility problem (25), the *Douglas-Rachford algorithm*[104] generates a sequence (x_n), from an arbitrary starting point x_0, by

$$x_{n+1} = \big(P_A(2P_B - I) + (I - P_B)\big)(x_n). \tag{30}$$

For brevity, we set

$$T = P_A(2P_B - I) + (I - P_B). \tag{31}$$

If A is a closed vector space, then T can be written more symmetrically as $T = P_A P_B + (I - P_A)(I - P_B)$. Now let $R_A = 2P_A - I$ be the *reflector* with respect to A and define R_B likewise. The following proposition gives an alternative description of the Douglas-Rachford algorithm that lends itself to a simple geometrical interpretation (see Fig. 6).

Proposition 5.7 The mapping T in (31) can be written as $T = \big(R_A R_B + I\big)/2$. Hence, (30) is equivalent to

$$x_{n+1} = \tfrac{1}{2}\big(R_A R_B + I\big)(x_n). \tag{32}$$

The next two basic results on the Douglas-Rachford iteration are due to Lions and Mercier;[106] see also.[87,88] We include some proofs in Appendix A, as they appear to be simpler than those found in the literature.

Fact 5.8 The mapping T in (31) is firmly nonexpansive.

Fact 5.9 (Lions-Mercier[106]) Suppose $A \cap B \neq \emptyset$. Then the sequence (x_n) generated by (32) converges weakly to some point $x \in \operatorname{Fix} T$ and $P_B(x) \in A \cap B$. Moreover, the sequence $\big(P_B(x_n)\big)$ is bounded, and every weak cluster point[107] of $\big(P_B(x_n)\big)$ lies in $A \cap B$. If $\mathcal{H}$ is finite-dimensional, then $x_n \to x$ and $P_B(x_n) \to P_B(x) \in A \cap B$.

The following connection, which identifies HIO with the Douglas-Rachford algorithm, does not seem to have been drawn elsewhere.

Observation 5.10 (HIO algorithm as a nonconvex Douglas-Rachford algorithm) Replace the set A with the (convex) object domain constraint set S and the set B with the (nonconvex) Fourier domain constraint set M. Then the sequence generated by the Douglas-Rachford algorithm (30) corresponds to the sequence generated by the HIO algorithm (22).[108]

6 Concluding remarks

The contribution of this paper is two-fold. First, an analysis of the phase retrieval problem has been carried out in the mathematical context of multi-valued projection operators. This analysis provides rigorous and easily verifiable criteria for calculating projections. Second, new connections have been established between some classical phase retrieval methods and some standard convex optimization algorithms.

While the mathematical theory remains unable to completely analyze the convergence behavior of these algorithms in nonconvex settings, the analogies drawn here open the door for experimentation with variations that are well understood in convex settings. We believe that the convex-analytical viewpoint adopted in this paper can be exploited further in order to develop alternative phase retrieval schemes.

Appendix A – Proofs

Proof of Example 3.5: See Example 3.14 for a rigorous proof. ☐

Proof of Example 3.6: This is a sketch of the proof; see Theorem 4.2 in[39] for full details. Fix $x \in \mathcal{L}$. Because the Fourier transform is unitary, it follows that $\widehat{\Pi_M(x)} = \Pi_{\mathcal{A}}(\widehat{x})$. In turn, by Theorem 3.16, the projection onto $\mathcal{A}$ can be found separably, provided that the selection is measurable. But the projection onto the circle in $\mathbb{C}$ is easy: radially scale the point, and observe the multi-valuedness at the origin. The step from the measurable selection to the measurable phase φ requires a measure-theoretical argument; see Proof of Theorem 4.2 in.[39] ☐

Proof of Fact 3.9: This is part of the folklore. See, for instance, Lemma 1.1 in.[109] ☐

Proof of Proposition 3.11: (i) is easily verified. (ii) Let (a_n) be a sequence in $\mathcal{A}$ converging to some $z \in \mathcal{L}$. A result by Riesz (see Theorem 12.6 in[45] or Theorem 2.8.2 in[110]) implies that there exists a subsequence (a_{k_n}) of (a_n) that converges to z a.e. Since $(a_{k_n}(t))_n$ lies in the closed set $A(t)$ a.e., it follows that $z(t) \in A(t)$ a.e. Consequently, $\mathcal{A}$ is closed. □

Proof of Theorem 3.12: Since each set $A(t)$ is closed convex and nonempty, the function $y \colon t \mapsto P_{A(t)}(x(t))$, is well-defined. By Corollary 8.2.13(3) in[59] (see also Exercise 14.17.(b) in[67]), y is measurable. Pick an arbitrary $a \in \mathcal{A}$. Then $|x(t) - y(t)| \leq |x(t) - a(t)|$ a.e. Squaring and integrating yields $y \in \mathcal{L}$ and hence $y \in \Pi_{\mathcal{A}}(x)$. Since $\mathcal{A}$ is closed and convex by Proposition 3.11, $\Pi_{\mathcal{A}}(x)$ is a singleton (Fact 3.9), which we write as $y = P_{\mathcal{A}}(x)$. □

Proof of Example 3.14: We first define $A(t)$, depending on the case considered: (i) $A(t) = \mathbb{C}$, if $t \in D$; $A(t) = \{0\}$, otherwise, (ii) $A(t) \equiv \mathbb{R}$, (iii) $A(t) \equiv \mathbb{R}_+$, respectively. Now let $\mathcal{A}$ be as in Theorem 3.12. Note that $\mathcal{A}$ is nonempty, as it contains the zero function. Next, observe that in each case, we can write $A(t) = \mu(t) \cdot Z$, where $\mu \colon \mathbb{R}^N \to \mathbb{C}$ is measurable and $Z \subset \mathbb{C}$: indeed, (i) $Z = \mathbb{C}$ and $\mu = 1_D$, (ii) $Z = \mathbb{R}$ and $\mu \equiv 1$, (iii) $Z = \mathbb{R}_+$ and $\mu \equiv 1$. In view of Remark 3.13, $t \mapsto A(t)$ is a measurable multifunction in each case. The result now follows from Theorem 3.12. □

Proof of Theorem 3.16: Since A is a measurable multifunction, the function $\mathbb{R}^N \to \mathbb{R}_+ : t \mapsto d(\rho, A(t))$ is measurable, for every $\rho \in \mathbb{C}$ (Corollary 8.2.13(2) in,[59] Theorem 14.3 in[67]). On the other hand, the function $\mathbb{C} \to \mathbb{R}_+ : \rho \mapsto d(\rho, A(t))$ is continuous (even nonexpansive), for every $t \in \mathbb{R}^N$. Altogether, the function $\mathbb{R}^N \times \mathbb{C} \to \mathbb{R}_+ : (t, \rho) \mapsto d(\rho, A(t))$ is a *Carathéodory function*; see Definition 4.49 in.[45] Now $\mathcal{A} \neq \emptyset$, hence A is nonempty-valued. By Theorem 17.5 in,[45] the multifunction A is *weakly measurable*. Now fix $x \in \mathcal{L}$, and let $f \colon \mathbb{R}^N \times \mathbb{C} \to \mathbb{R} : (t, \rho) \mapsto -|x(t) - \rho|$. Then $f(t, \rho)$ is measurable in t, and continuous in ρ; thus, f is also a Carathéodory function. The *Measurable Maximum Theorem* (Theorem 17.18 in[45]) yields (i) $t \mapsto d(x(t), A(t))$ is a measurable function, (ii) $t \mapsto \Pi_{A(t)}(x(t))$ is a measurable multifunction, with nonempty compact values, and (iii) there exists a measurable selection $z(t) \in \Pi_{A(t)}(x(t))$. Clearly, $z \in \Pi_{\mathcal{A}}(x)$. (This first part of the proof can also be contemplated from a higher perspective, see Section 8.2 in[59]). It remains to prove the equivalence concerning $\Pi_{\mathcal{A}}(x)$. The "if" part is clear. We now suppose to the contrary that the "only if" part is false. Then there exists a measurable function $y \in \Pi_{\mathcal{A}}(x)$ such that the set $E = \{t \in \mathbb{R}^N \colon |x(t) - y(t)| > d(x(t), A(t))\}$ has strictly positive measure. But then

$$\begin{aligned} d^2(x, \mathcal{A}) &= \|x - y\|^2 \\ &= \int_E |x(t) - y(t)|^2 dt + \int_{\complement E} |x(t) - y(t)|^2 dt \\ &> \int_E |x(t) - z(t)|^2 dt + \int_{\complement E} |x(t) - z(t)|^2 dt \\ &= \|x - z\|^2 \\ &= d^2(x, \mathcal{A}), \end{aligned} \tag{A1}$$

which is absurd. □

Proof of Example 3.15: Recall that $m \in \mathcal{L}$ is the prescribed nonnegative modulus function. The set of all functions satisfying the image modulus constraint is $M = \{z \in \mathcal{L} : |\widehat{z}| = m\}$. Note that

$M \neq \emptyset$, since $m \in M$. Let $\mathbb{S}$ be the closed unit circle in $\mathbb{C}$, and set $A(\omega) = m(\omega) \cdot \mathbb{S}$, for all $\omega \in \mathbb{R}^N$. The multifunction $\omega \mapsto A(\omega)$ is compact-valued, nonempty-valued, and measurable (Remark 3.13). Now let $\mathcal{A} = \{z \in \mathcal{L} : z(\omega) \in A(\omega) \text{ a.e.}\}$. Then $\mathcal{A}$ is closed by Proposition 3.11.(ii). Consequently, as the pre-image of $\mathcal{A}$ under a continuous operator, namely the the Fourier transform, the set M is closed.[111] □

Proof of Proposition 5.7: By simple expansion

$$R_A R_B = (2P_A - I)(2P_B - I) = 2P_A(2P_B - I) + 2(I - P_B) - I. \tag{A2}$$

Hence, (31) yields $T = \frac{1}{2}(R_A R_B + I)$. □

Proof of Fact 5.8: (See also Proposition 2 in,[106] Corollary 4.1 in.[88]) Because P_A and P_B are projectors onto convex sets, they are firmly nonexpansive (Fact 3.9). Thus, by Fact 3.18, R_A and R_B are nonexpansive. Hence the composition $R_A R_B$ is nonexpansive which, in turn, implies (Fact 3.18 again) that $\frac{1}{2}(R_A R_B + I)$ is firmly nonexpansive. In view of Proposition 5.7, the proof is complete. □

Fact A1 $P_B(\operatorname{Fix} T) = A \cap B \subset \operatorname{Fix} T$.

Proof. Fix an arbitrary $x \in \mathcal{H}$. Write $x = b + q$, where $b = P_B(x)$ and $q = x - b$. Then $b = P_B(x) = P_B(b+q)$. The result follows from the equivalences $x = T(x) \Leftrightarrow x = P_A(2P_B - I)(x) + (I - P_B)(x)$ $\Leftrightarrow b + q = P_A\big(2b - (b+q)\big) + b + q - b \Leftrightarrow b = P_A(b - q)$. □

Proof of Fact 5.9: (See also Theorem 1.(iii) and Remark 7 in[106] and Corollary 6.1 in[88].[112]) By Fact A1, $\operatorname{Fix} T$ contains $A \cap B \neq \emptyset$. In view of Fact 3.20, the sequence $(x_n) = \big(T^n(x_0)\big)$ converges weakly to some fixed point x of T. Since P_B is nonexpansive and (x_n) is bounded, it follows that $\big(P_B(x_n)\big)$ is bounded. Since T is firmly nonexpansive, $\|x_n - x\|^2 \geq \|x_{n+1} - x\|^2 + \|x_n - x_{n+1}\|^2$, which implies (after telescoping)

$$0 \leftarrow x_n - x_{n+1} = P_B(x_n) - P_A\big(2P_B(x_n) - x_n\big). \tag{A3}$$

Hence weak cluster points of $\big(P_B(x_n)\big)$ must lie in $A \cap B$. Finally, if $\mathcal{H}$ is finite-dimensional, then $P_B(x_n) \to P_B(x) = P_A\big(2P_B(x) - x\big) \in A$. □

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E-mails: bauschke@cecm.sfu.ca, plc@math.jussieu.fr, luke@math.uni-goettingen.de

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[42] J. R. Fienup, Iterative method applied to image reconstruction and to computer-generated holograms, *Opt. Eng.*, **19**:297–305 (1980).

[43] J. W. Goodman, *Statistical Optics,* (Wiley Interscience, New York, 1985).

[44] "a.e." stands for "almost everywhere" in the sense of measure theory since, strictly speaking, the elements of $\mathcal{L}$ are classes of equivalence of signals that may differ on a set of zero measure. For technical details on $\mathcal{L}$ see for instance.[45]

[45] C. D. Aliprantis and K. C. Border, *Infinite-Dimensional Analysis*, (Springer-Verlag, Berlin, second edition, 1999).

[46] J. N. Cederquist, J. R. Fienup, C. C. Wackerman, S. R. Robinson, and D. Kryskowski, Wave-front phase estimation from Fourier intensity measurements, *J. Opt. Soc. Amer. A*, **6**:1020–1026 (1989).

[47] P. L. Combettes, Inconsistent signal feasibility problems: Least-squares solutions in a product space, *IEEE Trans. Signal Processing*, **42**:2955–2966 (1994).

[48] G. T. Herman, *Image Reconstruction from Projections – The Fundamentals of Computerized Tomography*, (Academic Press, New York, 1980).

[49] J. H. Seldin and J. R. Fienup, Numerical investigation of the uniqueness of phase retrieval, *J. Opt. Soc. Amer. A*, **7**:412–27 (1990).

[50] H. H. Bauschke, The composition of finitely many projections onto closed convex sets in Hilbert space is asymptotically regular, *Proc. Amer. Math. Soc.* (to be published).

[51] H. H. Bauschke and J. M. Borwein, On the convergence of von Neumann's alternating projection algorithm for two sets, *Set-Valued Anal.*, **1**:185–212 (1993).

[52] H. H. Bauschke, J. M. Borwein, and A. S. Lewis, The method of cyclic projections for closed convex sets in Hilbert space, In *Recent Developments in Optimization Theory and Nonlinear Analysis (Jerusalem, 1995)* (Amer. Math. Soc., Providence, RI, 1997), pp. 1–38.

[53] P. L. Combettes and P. Bondon, Hard-constrained inconsistent signal feasibility problems, *IEEE Trans. Signal Processing*, **47**:2460–2468 (1999).

[54] L. G. Gubin, B. T. Polyak, and E. V. Raik, The method of projections for finding the common point of convex sets, *USSR Comput. Math. Math. Phys.*, **7**:1–24 (1967).

[55] D. C. Youla and V. Velasco, Extensions of a result on the synthesis of signals in the presence of inconsistent constraints, *IEEE Trans. Circuits Syst.*, **33**:465–468 (1986).

[56] Iterative signal recovery projection algorithms have also been implemented optically without sampling the continuous waveforms, e.g.,.[57] In such instances, the underlying signal space is $\mathcal{L}$ itself.

[57] R. J. Marks II, Coherent optical extrapolation of 2-D band-limited signals: Processor theory, *Appl. Opt.*, **19**:1670–1672 (1980).

[58] Let R be a set of real numbers. If $R \neq \emptyset$, then "$\inf(R)$" stands for the *infimum* of R, i.e., the largest number in $[-\infty, +\infty[$ that is smaller than or equal to all elements of R. By convention, $\inf \emptyset = +\infty$.

[59] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis* (Birkhäuser Boston Inc., Boston, 1990).

[60] For theoretical reasons, the sets (and functions) we deal with must be "measurable" — this is not the same "physically measurable" or "observable"! For our purposes, measurable sets and functions constitute a sufficiently large class to work with; thus, all closed and open subsets (and all continuous functions) are measurable as well as various combinations of those.

[61] Mathematically, this set is assumed to have nonzero measure.

[62] The complex Hilbert space $\mathcal{L}$ from the phase retrieval problem is also a real Hilbert space provided that we use the real part of the inner product as the new inner product.

[63] C. Tisseron, *Notions de Topologie – Introduction aux Espaces Fonctionnels* (Hermann, Paris, 1985).

[64] Recall the notation from Remark 3.4.

[65] F. Deutsch, *Best Approximation in Inner Product Spaces* (Springer-Verlag, New York, 2001).

[66] In our setting, this means that the set $\big\{t \in \mathbb{R}^N : A(t) \cap Z \neq \emptyset\big\}$ is measurable, for every closed (or, equivalently, open) set Z in $\mathbb{C}$; see Section 8.1 in[59] and Section 14.A in.[67]

[67] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis* (Springer-Verlag, Berlin, 1998).

[68] If $x \in \mathcal{L}$, then $\mathrm{Re}(x)$ denotes the function defined by $\mathrm{Re}(x)\colon t \mapsto \mathrm{Re}(x(t))$, and $(\mathrm{Re}(x))^+$ subsequently denotes the positive part: $(\mathrm{Re}(x))^+ \colon t \mapsto \max\{0, \mathrm{Re}(x(t))\}$.

[69] R. W. Gerchberg, Super-resolution through error energy reduction, *Optica Acta*, **21**:709–720 (1974).

[70] D. C. Youla, Generalized image restoration by the method of alternating orthogonal projections, *IEEE Trans. Circuits Syst.*, **25**:694–702 (1978).

[71] A. Papoulis, A new algorithm in spectral analysis and band-limited extrapolation, *IEEE Trans. Circuits Syst.*, **22**:735–742 (1975).

[72] A. E. Çetin and R. Ansari, Convolution-based framework for signal recovery and applications, *J. Opt. Soc. Amer. A*, **5**:1193–1200 (1988).

[73] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory* (Cambridge University Press, Cambridge, 1990).

[74] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73**:591–597 (1967).

[75] A. Genel and J. Lindenstrauss, An example concerning fixed points, *Israel J. Math.*, **22**:81–86 (1975).

[76] Alternative descriptions of these algorithms have been proposed, see for instance.[77]

[77] G. T. Herman and D.-W. Ro, Image recovery using iterative data refinement with relaxation, *Opt. Eng.*, **29**:513-523 (1990).

[78] A more general formulation of the algorithm also includes a nonnegativity constraint.[14]

[79] The reason for this difference is that Fienup defines on page 2763 of[14] γ as the set of all points where (in our notation) $P_M(x_n)$ violates the object domain constraints. Hence $\gamma = \{t \in \complement D\colon (P_M(x_n))(t) \neq 0\}$, or: $t \in \gamma$ if and only if $t \notin D$ and $P_M(x_n)(t) \neq 0$. It follows that t belongs to the complement of γ if and only if $t \in D$ or $P_M(x_n)(t) = 0$. The latter condition then leads to this different interpretation of the HIO algorithm. Sticking with this interpretation for another moment, we could set $D(n) = D \cup \big\{t \in \complement D\colon P_M(x_n)(t) = 0\big\}$, $S(n) = \big\{z \in \mathcal{L}\colon z \cdot 1_{\complement D(n)} = 0\big\}$, and obtain analogously

$$x_{n+1} = \big(P_{S(n)}(2P_M - I) + (I - P_M)\big)(x_n).$$

In practical experiments for problem (5), however, this ambiguity has hardly an impact, as the sets γ and $\complement D$ almost always coincide.

[80] H. Takajo, T. Shizuma, T. Takahashi, and S. Takahata, Reconstruction of an object from its noisy Fourier modulus: Ideal estimate of the object to be constructed and a method that attempts to find that object, *Appl. Opt.*, **38**:5568–5576 (1999).

[81] H. Takajo, T. Takahashi, and T. Shizuma, Further study on the convergence property of the hybrid input-output algorithm used for phase retrieval, *J. Opt. Soc. Amer. A*, **16**:2163–2168 (1999).

[82] H. Takajo, T. Takahashi, R. Ueda, and M. Taninaka, Study on the convergence property of the hybrid input-output algorithm used for phase retrieval, *J. Opt. Soc. Amer. A*, **15**:2849–2861 (1998).

[83] The corresponding mask is certainly much easier to implement.

[84] J. R. Fienup and C. C. Wackerman, Phase retrieval stagnation problems and solutions, *J. Opt. Soc. Amer. A*, **3**:1897–1907 (1986).

[85] The algorithms discussed here for solving (25) can be viewed in the broader context of finding a zero of the sum of two maximal monotone operators. Good starting points are[86–88]

[86] P. L. Combettes, Fejér-monotonicity in convex optimization, In C. A. Floudas and P. M. Pardalos, editors, *Encyclopedia of Optimization*, volume 2 (Kluwer, New York, 2001), pp. 106–114.

[87] J. Eckstein, *Splitting Methods for Monotone Operators with Applications to Parallel Optimization*, PhD thesis (Department of Civil Engineering, Massachusetts Institute of Technology, 1989). Available as Report LIDS-TH-1877, Laboratory for Information and Decision Sciences, MIT.

[88] J. Eckstein and D. P. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, *Math. Programming (Ser. A)*, **55**:293–318 (1992).

[89] Unfortunately, $P_A P_B$ is generally *not* firmly nonexpansive. However, it is *strongly nonexpansive* (see[90] for a precise definition and further information) and, for this class of mappings, a result corresponding to Fact 3.20 does exist.[90]

[90] R. E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston J. Math.*, **3**:459–470 (1977).

[91] H. H. Bauschke and J. M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Rev.*, **38**:367–426 (1996).

[92] H. H. Bauschke, Projection algorithms: Results and open problems, In *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, *Studies in Computational Mathematics Volume 8*, D. Butnariu, Y. Censor, and S. Reich, eds. (Elsevier, Amsterdam, 2001), pp. 11–22.

[93] P. L. Combettes, Quasi-fejérian analysis of some optimization algorithms, In *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, *Studies in Computational Mathematics Volume 8*, D. Butnariu, Y. Censor, and S. Reich, eds. (Elsevier, Amsterdam, 2001), pp. 115–152.

[94] R. L. Dykstra, An algorithm for restricted least squares regression, *J. Amer. Stat. Assoc.*, **78**(384):837–842 (1983).

[95] J. P. Boyle and R. L. Dykstra, A method for finding projections onto the intersection of convex sets in Hilbert spaces, In *Advances in Order Restricted Statistical Inference (Iowa City, Iowa, 1985)* (Springer, Berlin, 1986) pp. 28–47.

[96] In the aforementioned context of maximal monotone operators,[85] Dykstra's algorithm can be interpreted as a tight version of the Peaceman-Rachford algorithm. See page 77 in[87] for further information. Let us also note that in the standard linear case, the Peaceman-Rachford and Douglas-Rachford algorithms can be viewed from a unifying standpoint (See Section 7.4 in[97]).

[97] R. S. Varga, *Matrix Iterative Analysis*, second edition (Springer-Verlag, New York, 2000).

[98] H. H. Bauschke and J. M. Borwein, Dykstra's alternating projection algorithm for two sets, *J. Approx. Theory*, **79**:418–443 (1994).

[99] H. H. Bauschke and A. S. Lewis, Dykstra's algorithm with Bregman projections: A convergence proof, *Optimization*, **48**:409–427 (2000).

[100] N. Gaffke and R. Mathar, A cyclic projection algorithm via duality, *Metrika*, **36**:29–54 (1989).

[101] S.-P. Han, A successive projection method, *Math. Programming (Ser. A)*, **40**:1–14 (1988).

[102] H. Hundal and F. Deutsch, Two generalizations of Dykstra's cyclic projections algorithm, *Math. Programming A*, **77**:335–355 (1997).

[103] P. L. Combettes, Signal recovery by best feasible approximation, *IEEE Trans. Image Processing*, **2**:269–271 (1993).

[104] The Douglas-Rachford algorithm was originally developed as a linear implicit iterative method to solve partial differential equations in[105] (see also Chap. 7 in[97]). It was extended to an operator splitting method for finding a zero of the sum of two maximal monotone operators by Lions and Mercier in.[106] Applied to the normal cone maps of the constraints sets, one obtains a method for solving (25). See[86–88, 106] for further information.

[105] J. Douglas and H. H. Rachford, On the numerical solution of heat conduction problems in two or three space variables, *Trans. Amer. Math. Soc.*, **82**:421–439 (1956).

[106] P.-L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.*, **16**:964–979 (1979).

[107] u is a weak cluster point of a sequence (u_n) if there exists a subsequence (u_{k_n}) such that $u_{k_n} \overset{\mathrm{w}}{\rightharpoonup} u$.

[108] If we had used the literal update rule for the HIO algorithm, the present observation would change only in one aspect: the set A would be replaced with $S(n)$ (see Remark 4.1 and[79]), and hence vary with n.

[109] E. H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory, In E. H. Zarantonello, editor, *Contributions to Nonlinear Functional Analysis* (Academic Press, New York, 1971), pp. 237–424.

[110] L. Debnath and P. Mikusiński, *Introduction to Hilbert Spaces with Applications*, second edition (Academic Press Inc., San Diego, CA, 1999).

[111] However, as shown in Property 4.1 in,[39] the set M is not *weakly closed*, i.e., if a sequence (x_n) of points in M converges weakly to a point x, then x may not be in M.

[112] While P_B is nonexpansive and therefore Lipschitz continuous, this property is not sufficient to draw the conclusion advertised in Corollary 6.1 in,[88] namely (in our context), that $(P_B x_n)$ converges weakly to a point in $A \cap B$. Such a conclusion requires additional assumptions, e.g., that P_B be weakly continuous (if so, then $P_B x_n \overset{\mathrm{w}}{\rightharpoonup} P_B x$), as is the case when $\dim \mathcal{H} < +\infty$ (or when B is a closed affine subspace). Note, however, that the projector onto a closed convex set may fail to be weakly continuous Example on page 245 in.[109]

LIST OF CAPTIONS

Figure 1: POCS algorithm. The initial point a_0 is projected onto B and then onto A. The point a_1 thus obtained belongs to both sets and the algorithm therefore converges in two steps. Note that the solution a_1 is not the projection of a_0 onto $A \cap B$.

Figure 2: Dykstra's algorithm. The first two steps of this algorithm are identical to those of POCS (Fig. 6). Here, however, although $a_1 \in A \cap B$, the algorithm does not reach convergence at this point since the outward pointing normal q_0 pulls the vector $a_1 + q_0$ out of B before it is projected onto B. Through this process, two infinite sequences (a_n) and (b_n) are generated that converge to $P_{A\cap B}(a_0)$. Note that since A is an affine subspace in this example, $p_n \perp A$ and, therefore, $a_{n+1} = P_A(b_n)$.

Figure 3: Douglas-Rachford algorithm. The update equation (32) is executed as follows: one first computes the reflection $r_{n+\frac{1}{2}}$ of x_n with respect to B and then the reflection r_{n+1} of $r_{n+\frac{1}{2}}$ with respect to A. The update x_{n+1} is the midpoint of the segment between x_n and r_{n+1}. In this example the algorithm converges in 4 iterations since $x_4 \in A \cap B$.

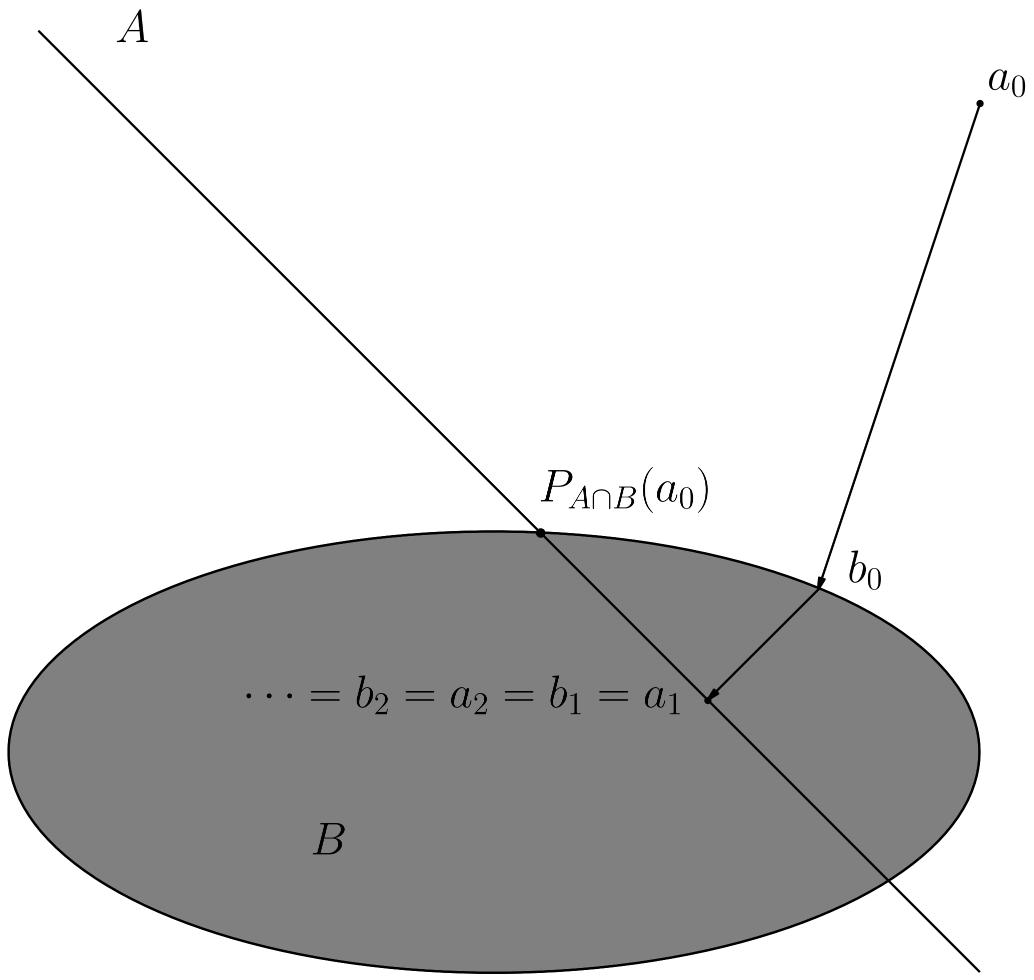

Figure 1

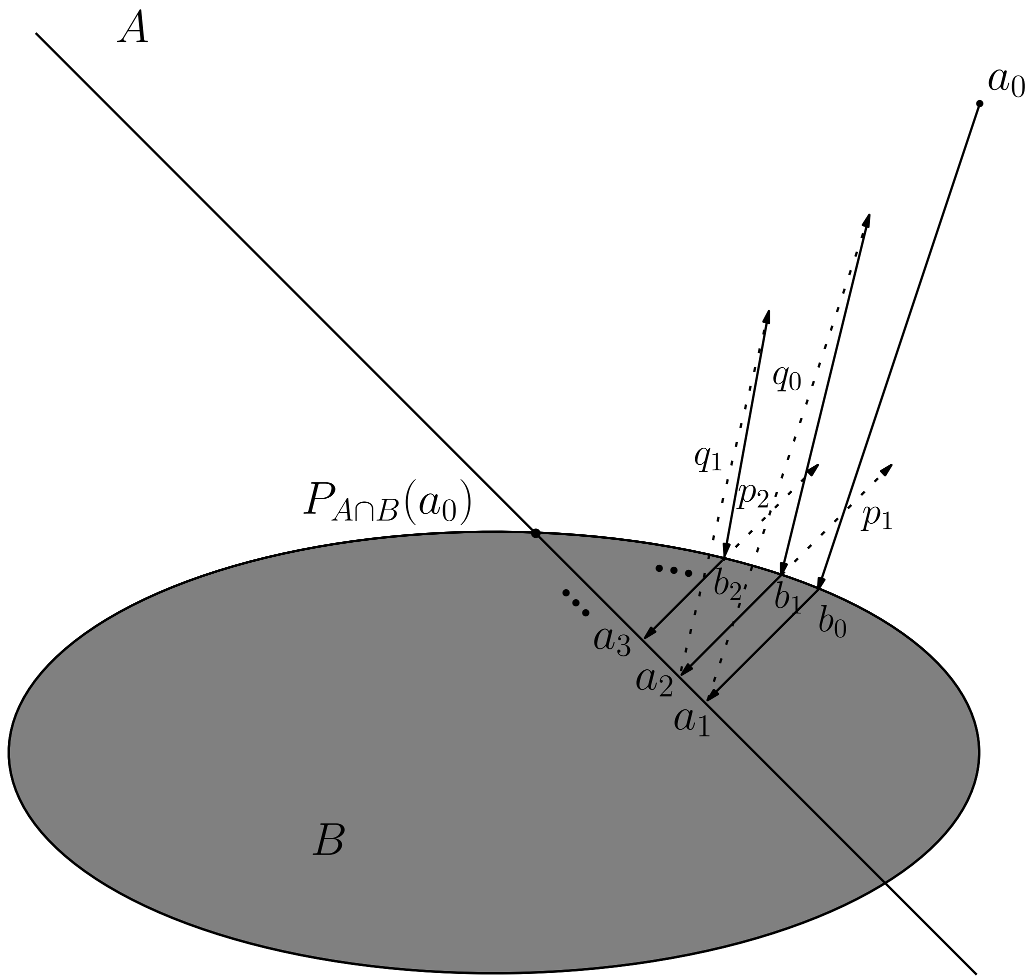

Figure 2

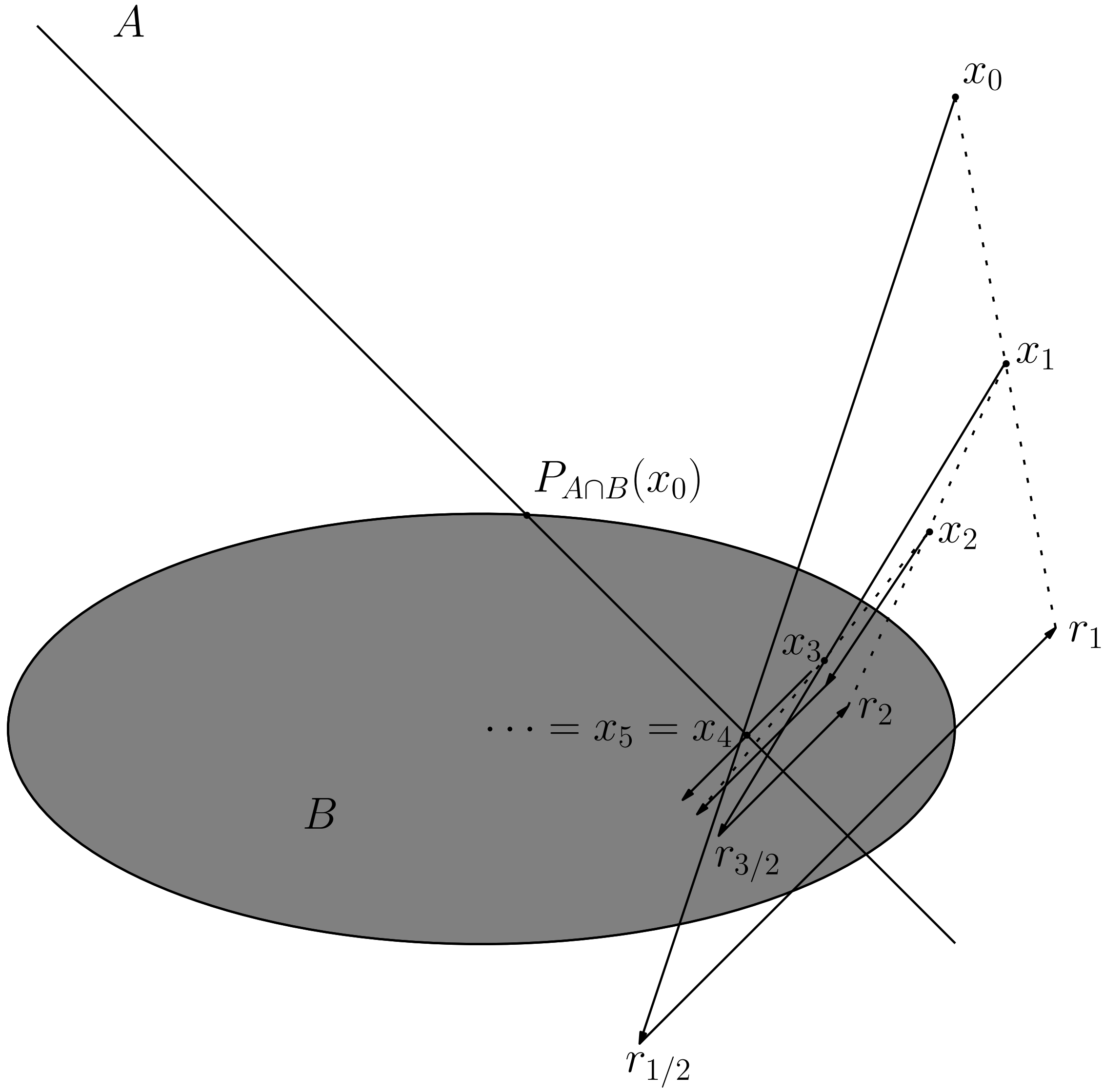

Figure 3