Phase-separation dynamics of circular domain walls in the degenerate optical parametric oscillator

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We study the dynamics of the formation of circular domain walls, which are large-intensity structures, in a degenerate optical parametric oscillator. We show that the mean-field and the propagation models predict the same increase in the domain size proportional to $t^{1/3}$.

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The existence and stability of circular domain walls in a single-longitudinal-mode degenerate optical parametric oscillator were recently reported for three different limits above the oscillation threshold: close to the threshold, where the field is described by a Swift–Hohenberg equation,[1] and away from the threshold, either in the mean-field limit,[1,2] which averages the propagation effects inside the cell, or out of the mean-field limit.[3] The circular domain walls result from the existence of two solutions, A and $A_\pi = A\exp(i\pi)$, with the same intensity. Robust structures are generated in the form of circular or striped[4] walls separating domains in which either of the two solutions exists. The Swift–Hohenberg equation describes small-amplitude solutions of the Maxwell–Bloch equations. Large-amplitude solutions, which cannot be described by a Swift–Hohenberg equation, exist in a wide range of parameters.[5] In this case a circular domain wall of small diameter is formed, with a large peak amplitude in its center after a long transient.[2,3] The predictions of the mean-field and the propagation models for the intensity and the stability of these peaks were recently compared.[5] Only qualitative agreement between the models was found, in a limited range of the input pump amplitude in which the signal intensities were not too large. However, the dynamics of the peak formation obeys simple power laws, and there are good reasons to suspect that these power laws are generic. The conjecture is therefore that the power laws should be verified by the mean-field and the propagation models, even when these models are not compatible. The purpose of this Letter is to report on the confirmation of this conjecture.

The kinetics of localized pattern formation in optics was recently shown to be slow[6,7] and governed by the same power laws as phase separation in solid[8,9] and fluid[10] systems, in which, in the late time dynamics of grain formation (or spinodal decomposition), the grain size varies according to the Lifshitz–Slyozov–Wagner power law, $\sim t^{1/3}$; this regime is often followed by a linear power-law regime.

The mean-field equations for the pump and the signal field amplitudes, A_0 and A_1, respectively, are[11]

$$\partial_t A_0 = E - (\gamma + i\Delta_0)A_0 + \frac{i}{2}\nabla_\perp{}^2 A_0 - A_1{}^2,$$

$$\partial_t A_1 = -(1 + i\Delta_1)A_1 + i\nabla_\perp{}^2 A_1 + A_0 A_1{}^*,$$

where E is the driving field amplitude, $\gamma = \gamma_0/\gamma_1$ is the ratio of cavity-decay rates, $\Delta_0 = (\omega_0 - \omega_{\mathrm{ext}})/\gamma_\perp$, $\Delta_1 = (\omega_1 - \omega_{\mathrm{ext}}/2)/\gamma_\perp$, and $\nabla_\perp{}^2$ is the transverse Laplace operator. The propagation model[3,5] consists of the propagation equations for the pump and the signal field amplitudes inside the medium along the longitudinal coordinate z:

$$\partial_z \alpha_0 = \frac{i}{2}\nabla_\perp{}^2\alpha_0 + i\alpha_1{}^2\exp(iz\Delta k),$$

$$\partial_z \alpha_1 = i\nabla_\perp{}^2\alpha_1 + i\alpha_0\alpha_1{}^*\exp(-iz\Delta k),$$

where $l\Delta k = 2\theta_1 - \theta_0$ is the phase mismatch between the pump and the signal, with l as the cavity length, and the boundary conditions at the cell incoupling mirror are

$$\alpha_0(0,x,y,t) = \alpha_0{}^{\mathrm{in}} + R_0\exp(i\theta_0)\alpha_0(l,x,y,t-\tau),$$

$$\alpha_1(0,x,y,t) = R_1\exp(i\theta_1)\alpha_1(l,x,y,t-\tau).$$

The variables in the two models are connected by the relations $(A_0, A_1, E, \Delta_{0,1}) = \beta(i\alpha_0, \alpha_1, i\alpha_0{}^{\mathrm{in}}\beta, -\theta_{0,1})$ and $(t, x\sqrt{\beta})_{MF} = \beta(t, x)_{\mathrm{propag}}$, with $1 - R_1 = 1/\beta$. τ is the delay time.

To characterize the pattern kinetics we measure the average domain size, defined as $R(t) = 2N_xN_y/[n_x(t) + n_y(t)]$, where $n_x(t)$ and $n_y(t)$ are the total number of

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sign changes of the real part of the signal field in the transverse x and y directions, respectively. The power laws that we seek are $R(t) \sim t^{\alpha}$. Two different methods were used to solve the mean-field model, with similar conclusions for the power laws: the explicit Euler method with periodic transverse boundary conditions and the implicit Crank–Nicolson method, in both time and transverse space with Neuman boundary conditions. For the propagation equations a split-step method was used, with $N_x = N_y = 192$. The noise amplitude was 10^{-2} to 10^{-1}. The calculations were done with the homogeneous input-pump field amplitude plus a small-amplitude inhomogeneous random signal.

The mean-field model was investigated for $\Delta_0 = \Delta_1 = 0$. With Neuman boundary conditions, there is a single power-law regime with a slope of $\sim 1/3$, followed by an abrupt increase of the domain size. For periodic boundary conditions one can clearly distinguish after a short transient two successive power-law regimes, with a slope of $\sim 1/2$ for $20 < t < 90$, followed by a slope of $\sim 1/3$ for $90 < t < 1310$, and finally a regime showing a rapid variation of $R(t)$, corresponding to the shrinking of the large negative domain toward a single small negative circular domain. The propagation model with $\Delta_0 = \Delta_1 = 0$ displays mainly the $t^{1/3}$ regime, which covers the range $200 < t_{\mathrm{prop}} = 10t < 8100$ (see Fig. 1). Similar results were obtained for other mistunings. Figure 2 displays the phase separation of the real part of the signal amplitude for $10 < t < 1200$. A stable profile is reached for $t = 1400$. The same results were also obtained by use of two simpler propagation models defined previously.[3]

A striking result of these simulations is the convergence of the results of the three propagation models and the mean-field model toward the $t^{1/3}$ power law, despite the fact that the peak formation is ten times slower in the mean-field model than in the propagation models.

The $t^{1/3}$ power law seems to be a fairly general law found in many systems and generated by different mechanisms. Other power laws have been derived in similar contexts.[12] For instance, Siggia[10] predicted an increase of the grain size with $R(t) \sim t$ after the coalescence stage, $R(t) \sim t^{1/3}$, in concentrated fluid systems. This result is limited: It holds only in three dimensions, as shown by San Miguel *et al.*,[13] who instead found a power law $R(t) \sim t^{1/2}$ that is not continued in two dimensions by the $R(t) \sim t$ regime reported by Siggia. However, this result is clearly model dependent, as shown, for example, by the results obtained in Refs. 14 and 15.

In gradient systems with potential $F(A)$ of the form $\partial_t A = -\delta F/\delta A$, one can show that a $t^{1/2}$ power law is associated with the stage of contraction (or expansion) of a large circular domain of radius r_0, provided that the order parameter $A(x, y)$ is a function of the radial variable r only. Indeed, in this case[1] the two- and one-dimensional potentials obey the relation $F_{2D} = 2\pi r_0 F_{1D}$. Then the radial velocity of the large ring is $\mathrm{d}r_0/\mathrm{d}t = -(\partial_t F/\partial r_0)/[2\pi \int(\partial_t A/\partial r)^2 r \mathrm{d}r] \sim 1/r_0$, and therefore $r_0 \sim t^{1/2}$. In our study the formation of peaks occurs well above threshold, i.e., outside the validity range of gradient-form amplitude equations (of the Ginzburg–Landau type for positive detuning or the Swift–Hohenberg type for negative detuning). Moreover, the signal amplitude $A_1(x, y)$ is not a function of the radial variable, because several peaks are located inside the large circular domain. Coalescence of these peaks is observed during contraction of the large circular domain. Actually the contraction of the large circular domains containing several peaks is ruled by the $\sim t^{1/3}$ power law for all the models analyzed in this Letter.

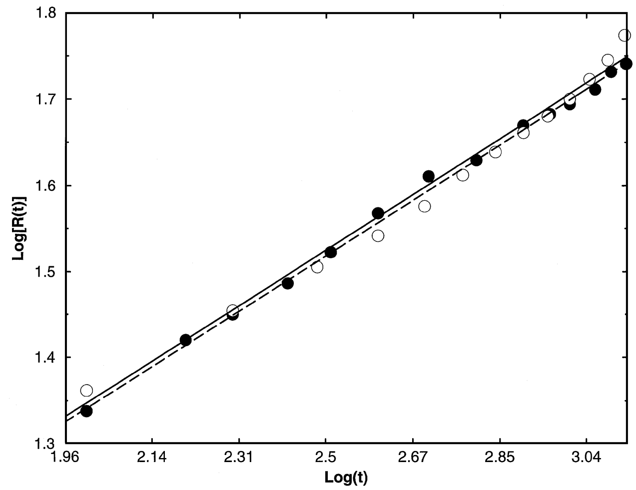

Fig. 1. Plot of $\log_{10} R$ versus $\log_{10} t$. The solid line and the filled circles are obtained from the mean-field model. The dotted line and the open circles correspond to the propagation model. The propagation parameters are $\theta_0 = \theta_1 = 0$, $\alpha_0^{\mathrm{in}} = 0.03$, $R_{0,1} = 0.9$, and $\beta = 10$. The corresponding mean-field parameters are $\Delta_0 = \Delta_1 = 0$ and $E = 3$. The grid step size is 0.06.

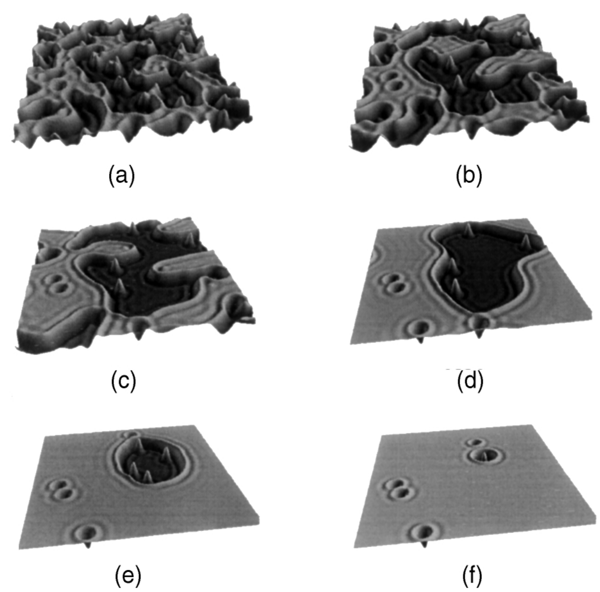

Fig. 2. Time evolution of the real part of the signal amplitude with the propagation model. (a) $t = 10$, (b) $t = 30$, (c) $t = 100$, (d) $t = 250$, (e) $t = 800$, (f) $t = 1200$. The parameters are the same as in Fig. 1.

The formation of localized structures or circular phase domains was first reported in a parameter range in which homogeneous steady-state solution $\bar{A}_1$ is modulationally stable.[1–3] Nevertheless, it was proved in the propagation models that such structures also occur beyond the modulational instability threshold of the nontrivial branch in a large range of parameter values. In this case a circular domain with large negative amplitude is formed inside a small-amplitude square or a hexagonal solution. The dynamics of such structures also displays the $t^{1/3}$ regime.

The $t^{1/3}$ regime was first observed in optics during the formation of a striped localized structure in the presence of long-wavelength instability, with quadratic Laplacian operators playing the role of surface tension, with both the mean-field and the Swift–Hohenberg models.[6] The present study shows that the $t^{1/3}$ power law also appears during the formation of intense peaks, in a large range of parameters above the threshold at which no small-amplitude equation is valid.

Finally, we emphasize that the law $R(t) \sim t^{1/3}$ is a scaling law for the peak formation averaged over many different initial conditions; i.e., this law corresponds to a domain of self-similarity. That this is so has been shown with the help of the correlation function for the formation of the peaks.

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