Archives

Ref.TH.1637-CERN

PHASE SHIFT AMBIGUITIES FOR ANALYTIC AMPLITUDES

C. Itzykson *) and A. Martin

CERN - Geneva

A B S T R A C T

We investigate ambiguities in reconstructing a unitary elastic amplitude at fixed energy given the differential cross-section when one assumes analyticity in the $\cos\theta$ plane. Strong hints are given that not more than a twofold non-trivial ambiguity is present. This is demonstrated for genuine entire functions of finite order. Moreover it is found that within a class of amplitudes which includes polynomials as well as entire functions of order zero, i) the difference between two amplitudes with the same cross-section must be a polynomial, ii) if the cross-section is smaller than 1.38 $(4\pi/k^2)$ there is no ambiguity. Indications are given on directions for future work.

---

*) On leave of absence from DPhT-CEN Saclay, Gif-sur-Yvette.

Ref.TH.1637-CERN
20 March 1973

1. INTRODUCTION

Interest in reconstructing a scattering amplitude from a knowledge of experimental data on cross-sections, polarizations and spin correlations is of course as old (and even older) as particle physics. The problem has many aspects, a small fraction of which only will be discussed in the present article. Experimental uncertainties, incomplete data, complications due to spin and opening of inelastic channels all present their own challenges. A large amount of work has been devoted to the elucidation of these various problems. On the other hand, a priori knowledge of some theoretical principles like unitarity and analyticity in both scattering angle and energy, allows one to a large extent to overcome in practice a number of these basic difficulties. Here, we shall disregard continuity in energy but assume on the other hand perfect knowledge of the angular distribution at a given energy.

It was noted some time ago, however, that even if one is given at a fixed energy the cross-section as a function of angle with infinite accuracy in the simplest of all possible cases, i.e., corresponding to the scattering of two spinless particles below inelastic threshold - it can happen that two completely different sets of phase shifts provide us with two unitary amplitudes corresponding to one and the same cross-section. The original example of Chrichton [1] constructed with sets of S, P and D waves, has been recently extended to S, P, D and F waves by Berends and Ruijsenaars [2] and will be elaborated as we proceed.

Furthermore, it has also been shown by Martin [3] and Newton [4] independently and confirmed by Atkinson, Johnson and Warnock [5] that if a certain criterion is met, no such ambiguity can arise and of course the Crichton example as well as the new ones involving up to $L=3$ waves violate this criterion.

In the sequel we shall restrict our discussion to the rather academic case of the scattering of spinless particles below the inelastic threshold. It is hoped, however, that what can be learned with these restrictions can be useful to tackle more realistic problems.

Beyond unitarity , the information which we want to make use of is a knowledge of a domain of analyticity in the $\cos\Theta$ plane, where Θ is the scattering angle, containing the physical region $-1 \leq \cos\Theta \leq 1$. This allows one to exploit the convergent expansion of the amplitude in a Legendre series and the problem will be then to discuss the possible existence of several amplitudes leading to the same differential cross-sections under the constraints of analyticity and unitarity.

Such an additional constraint has been used by Atkinson, Mahoux and Yndurain [6]. However, their aim is slightly different from ours. They prove at the same time, under certain sufficient conditions, the existence and the local uniqueness (i.e., inside a certain ball in a Banach space) of a solution, while we shall not worry at all about the existence of a solution but try to study the global uniqueness of solutions.

Barring the trivial ambiguity $G(\cos\Theta) = -F^*(\cos\Theta^*)$ the hope is to obtain the following result i) there exist at most two unitary analytic amplitudes F and G corresponding to the same cross-section and ii) these ambiguities occur along closed one-dimensional curves in the space of partial wave amplitudes. Such curves do not intersect.

No counter-example to this conjecture is at present known to the authors and we shall present a large body of evidence in its favour in the present paper.

In spite of its apparent simplicity the problem reveals itself rather difficult. In particular it requires a number of technical tricks which tend to obscure the basic line of argument. Hence, it was thought advisable to present first a series of general results dealing with the (rather unrealistic) case of amplitudes analytic in the whole complex $\cos\Theta$ plane. Even in this framework we can only give partial confirmation of the above conjecture even though highly suggestive.

We hope to present in a future publication further results pertaining both to polynomial amplitudes as well as amplitudes having singularities in the $\cos\Theta$ plane some of which will be briefly stated in the conclusion.

The paper is organized as follows. Section 2 is a summary of the state of the art as well as a list of notations, conventions and elementary remarks. In Section 3 we investigate the properties of entire unitary amplitudes of finite but non-zero order. The latter case is discussed in Section 4 where elaborate reasoning is used to pin down the ambiguity to a polynomial one. In Section 5 we show that within the class of polynomials as well as entire functions of order less than one, if the total (i.e., in the present case elastic) cross-section is less than 1.3 no ambiguity can arise.

The final Section 6 gives an over-all view of what has been achieved and what remains to be proved.

As was alluded above we had to deal with a number of technical fine points. The necessary but dull work of proving inequalities involving Legendre functions and the unitarity kernel is summarized in Appendix A. A second Appendix B collects useful facts on entire functions which might not be familiar to the inexpert reader.

2. PRELIMINARIES

We consider here unitary, analytic, elastic scattering amplitudes normalized in such a way that the partial wave expansion reads

$$F(z) = \sum_0^\infty (2\ell+1) f_\ell P_\ell(z) \tag{1}$$

z is the cosine of the scattering angle, $P_\ell(z)$ is the ℓ^{th} Legendre polynomial and with a convenient normalization of the amplitude, unitarity implies

$$f_\ell = e^{i\delta_\ell} \sin\delta_\ell \qquad \text{i.e. } \operatorname{Im} f_\ell = |f_\ell|^2 \tag{2}$$

where the real phase shift δ_ℓ is defined modulo π. The dimensionless elastic cross-section $\sigma(\cos\theta) = |F(\cos\theta)|^2$ yields upon integration the total cross-section

$$\sigma = \int_{-1}^{+1} \frac{d\cos\theta}{2} \sigma(\cos\theta) = \sum_0^\infty (2\ell+1)|f_\ell|^2 = \sum_0^\infty (2\ell+1)\operatorname{Im} f_\ell = \operatorname{Im} F(1) \tag{3}$$

The assumption of analyticity implies that the expansion (1) converges uniformly and absolutely in a certain ellipse with foci at ± 1. For the time being we assume even more by requiring that (1) converges in the whole complex z plane thus defining a polynomial or a genuine entire function. The problem under investigation is thus to find all possible amplitudes, F, G,... such that

$$\sigma_F(\cos\theta) = \sigma_G(\cos\theta) = \dots \qquad -1 \leq \cos\theta \leq 1 \tag{4}$$

Since with F analytic $\sigma_F(\cos\theta)$ can be extended to an analytic function $\sigma_F(z) = F(z)F^*(z^*)$, the equality (4) can be trivially extended to

$$\sigma_F(z) = \sigma_G(z) = \dots \tag{5}$$

It is convenient to split F in dispersive and absorptive parts $D(z)$ and $A(z)$ defined as

$$D(z) = \frac{1}{2}\left(F(z) + F^*(z^*) \right) = \sum_0^\infty (2\ell+1)\, \mathrm{Re} f_\ell \, P_\ell(z)$$
$$A(z) = \frac{1}{2i}\left(F(z) - F^*(z^*) \right) = \sum_0^\infty (2\ell+1)\, \mathrm{Im} f_\ell \, P_\ell(z) \qquad (6)$$

which coincide with the real and imaginary parts of F on the real axis. Clearly D and A are "real" analytic functions and $\sigma(z)$ is expressed as

$$\sigma(z) = D^2(z) + A^2(z)$$

It is easy to see that the amplitude $G = -F^*(z^*)$ obtained from F by changing all phase shifts in their opposite: $\delta_\ell \to -\delta_\ell$ or $\{D \to -D,\ A \to A\}$ is such that $\sigma_F(z) = \sigma_G(z)$ and introduces a trivial ambiguity. For our purposes we shall always mean by ambiguity a non-trivial one. Also it should be noted that we shall not discuss the question of characterizing a "real" analytic function $\sigma(z)$ such that it can be written $\sigma(z) = F(z)F^*(z^*)$ with F a unitary amplitude of type (1).

A quick survey of the problem immediately reveals that in order for non-trivial ambiguities to arise the amplitude F is severely restricted. In this direction a very general result [3] established with the use of non-linear analysis states that if $[F(ij) \equiv F(\cos\theta\, ij)$; i, j points on the unit sphere$]$

$$\sin\mu = \sup_{(1,2)} \frac{\int d\Omega_{3/4\pi}\, |F(13)|\,|F(23)|}{|F(12)|} < 0.79 \qquad (7)$$

then there exists a unique unitary amplitude $F(\cos\theta)$ with given modulus $|F(\cos\theta)|$ on the physical region $-1 \leq \cos\theta \leq 1$ up to the trivial ambiguity. This remarkable result does not give any hint upon the structure of possible ambiguities and severely restricts in fact the class of amplitudes. The aim of all subsequent work is to go beyond it by using completely different techniques. Also, condition (7) is too restrictive from a physical point of view. It will be obviously violated if the differential cross-section has a sharp minimum at a given angle. This will be the case if a $\ell \neq 0$ resonance dominates the cross-section.

If the amplitude is a polynomial of degree L the knowledge of $\sigma(z)$ and a choice for the trivial ambiguity (taking for instance $\mathrm{Re}\, f_L > 0$) shows that there are at most 2^L amplitudes. The role of unitarity is crucial in reducing drastically this number. A first step was made by Crichton [1] in studying the case of amplitudes limited to $\ell = 0$, 1, and 2 waves. He was able to show that there exists a one-dimensional closed curve in the space δ_0, δ_1, δ_2 along which pairwise points yield totally different amplitudes F and G with $\sigma_F(z) \equiv \sigma_G(z)$. This construction has recently been re-analyzed by Atkinson and collaborators [1] and extended to amplitudes with $\ell = 0$, 1, 2 and 3 waves by Berends and Ruijsenaars [2]. (In fact Cornille and the authors of the present article had for the largest part obtained the same results independently.) The interest in constructing explicitly the ambiguity curves for $\ell_{max} = 3$ is in part to test a number of conjectures that could be drawn by examining the Crichton result, most of which actually are not valid with $\ell_{max} = 3$. Furthermore it can also be shown in this case that it is impossible to have three amplitudes F, G, H such that no pair is related by the trivial ambiguity and with the same cross-section. More results are also known concerning higher degree polynomials and will be included in a further publication. Except in Section 5 we shall not consider polynomials but genuine entire functions.

In one disguise or another it will be seen that the crucial fact on which the proofs are based is the following. Consider the expansion (1) of F which converges for all values of z. If F is entire then the series is infinite and for ℓ large enough $|f_\ell| \to 0$. Now f_ℓ lies on the unitarity circle $\mathrm{Re}\, f_\ell^2 + \mathrm{Im}\, f_\ell^2 = \mathrm{Im}\, f_\ell$; if $|f_\ell| \to 0$, $\mathrm{Im}\, f_\ell$ is of order $\mathrm{Re}\, f_\ell^2$. For very large z the large partial waves will play the essential role and F will be dominated by its dispersive part. If $F = D + iA$ and $G = D' + iA'$ are two amplitudes with the same cross-section

$$D^2 + A^2 = D'^2 + A'^2$$

or

$$(D+D')(D-D') = -(A+A')(A-A')$$

The right-hand side will be vanishingly small for most z as compared to D or D' hence either $D = \pm D'$, or else $D-D'$ or $D+D'$ are very small as compared to D and D'. In many cases one can in fact control a possible cancellation in the growth of the product $(D+D')(D-D')$ as compared to the growth of D and D' and consequently obtain that no ambiguity is present. If not, the growth of $F-G$ or $F+\bar{G}$ [where $\bar{G}(z) \equiv G^*(z^*)$] will be much smaller than the one of F and G. This basic mechanism is very much linked with the fact that F and G are analytic throughout the whole z plane and do not reduce to polynomials.

We shall now investigate this question of dominance of the dispersive part for F a unitary entire amplitude.

First, observe that

$$f_\ell = \frac{1}{2}\int_{-1}^{+1} dx \;\; F(x)\, P_\ell(x) \tag{8}$$

Using Cauchy's theorem

$$F(x) = \frac{1}{2\pi i}\int_{E_\theta} dz \;\; \frac{F(z)}{z-x}$$

with E_Θ the ellipse $z = \mathrm{ch}(\Theta + i\varphi)$, Θ fixed, $0 \leq \varphi \leq 2\pi$. Then

$$f_\ell = \frac{1}{2\pi i}\int_{E_\theta} dz \; F(z) \int_{-1}^{+1}\frac{dx}{2} \;\; \frac{P_\ell(x)}{z-x} = \int_{E_\theta}\frac{dz}{2\pi i} \; F(z) Q_\ell(z) \tag{9}$$

where Q_ℓ is the associated Legendre function. The prominent role of the ellipses E_Θ stems from the structure of Legendre functions. For instance it is recalled in Appendix A that for $z \in E_\Theta$, $|P_\ell(z)|$ and $|Q_\ell(z)|$ are maximum at $z = \mathrm{ch}\Theta = x$. Denoting by $M_F(x)$ the maximum modulus of $F(z)$ inside the closed domain bounded by E_Θ we have then

$$|f_\ell| \leq M_F(x)\, Q_\ell(x)\, x \quad , \; x = \mathrm{ch}\theta = \text{ semi-major} \tag{9'}$$

axis of the ellipse

E_Θ (with foci ± 1).

We shall have to make frequent use of the spherical convolution. We write

$$f(z) = g * h \; (z) \tag{10}$$

with for z real between -1 and $+1$

$$f(12) = \int d\Omega_3/_{4\pi} \; g(13)\, h(23) \tag{11}$$

In terms of partial waves if g_ℓ and h_ℓ denote the partial wave coefficients of g and h defined as in (1), then clearly:

$$f_\ell = g_\ell\, h_\ell$$

showing that the convolution is commutative, associative and preserves the property that if g and h are entire so is $g*h$.

Let F denote a unitary entire amplitude (not a polynomial) we shall now prove a fundamental inequality. Write $F = D + iA$ then there exists a constant k and $x_0 > 1$ such that for $x \geq x_0$

$$M_A(2x^2-1) \leq k\left(M_D(x)\right)^2 \tag{12}$$

Since F is a genuine entire function there exists an infinite number of non-vanishing f_ℓ tending to zero for $\ell \to \infty$. Hence there exists L such that, for $\ell > L$, $\mathrm{Im}\, f_\ell < \frac{1}{2}$.

Notice that an upper bound of L can be obtained from the knowledge of $|F|$ only since

$$\mathrm{Im}\, f_\ell < \frac{1}{2\ell+1}\ \frac{1}{2}\int_{-1}^{+1} dx\ |F(x)|^2$$

This upper bound is therefore common to all possible solutions corresponding to a given cross-section. For $\mathrm{Im}\, f_\ell < \frac{1}{2}$ $|\mathrm{Re}\, f_\ell| \geq \mathrm{Im}\, f_\ell$, then

$$\mathrm{Im}\, f_\ell = \mathrm{Re}\, f_\ell^{\,2} + \mathrm{Im}\, f_\ell^{\,2} \leq 2\ \mathrm{Re}\, f_\ell^{\,2} \tag{13}$$

The equal sign takes into account the possibility that for certain ℓ, f_ℓ vanishes. Let us break F into two parts $F = F_< + F_>$, where $F_<$ is the polynomial including all partial waves with $\ell \leq L$. Correspondingly we have $A = A_< + A_>$, $D = D_< + D_>$.

Consider now the maximum modulus of $A_>(z)$ inside the ellipse $E_{2\theta}$. Now

$$|A_>(z)| \leq \sum_{\ell > L} (2\ell+1)\,\mathrm{Im}\, f_\ell\ |P_\ell(z)|$$

It is shown in Appendix A that $|P_\ell(z)|$ assumes its maximum value inside $E_{2\theta}$ at the point $z = \mathrm{ch}\, 2\theta = 2x^2-1$ $(x = \mathrm{ch}\, \theta)$ at which point $P_\ell(z)$ is real and positive. Hence, $|A_>(z)|$ assumes its maximum value at this point and

$$M_{A_>}(2x^2-1) = \sum_{l>L} (2l+1)\,\mathrm{Im} f_l\, P_l(2x^2-1) \leq 2 \sum_{l>L} (2l+1)\,\mathrm{Re} f_l^2\, P_l(2x^2-1) \leq 2\, D * D(2x^2-1) \tag{14}$$

In the same Appendix one recalls the construction of the Mandelstam kernel which allows one to write

$$2\, D * D(2x^2-1) = 2 \int_{E_\theta} \frac{dz_1}{2\pi i} \int_{E_\theta} \frac{dz_2}{2\pi i}\; D(z_1)\, D(z_2)\; M(z_1, z_2, 2x^2-1) \tag{15}$$

with $z_{1,2} = \mathrm{ch}(\theta + i\varphi_{1,2})$, $z_+ = \mathrm{ch}(2\theta + i(\varphi_1 + \varphi_2))$, $z_- = \cos(\varphi_1 - \varphi_2)$ and

$$|M(z_1, z_2, x)| \leq \frac{\text{constant}}{\left|\sqrt{(z_+ - x)(x - z_-)}\right|}$$

for θ larger than some fixed θ_0, the constant above being independent from θ. Then

$$2\,|D * D(2x^2-1)| \leq \frac{2}{(2\pi)^2} \iint_{E_\theta} |dz_1 dz_2|\; M_D(x)^2\, \frac{\text{constant}}{\left|\sqrt{(z_+ - x)(x - z_-)}\right|}$$

The last integral can be performed by going over to the variables $\varphi = \varphi_1 + \varphi_2$ $\psi = \varphi_1 - \varphi_2$. Then the initial domain of integration $0 \leq \varphi_{1,2} < 2\pi$ can be brought to $0 \leq \psi \leq 2\pi$, $0 \leq \varphi \leq 2\pi$ integrated over twice. Furthermore, $|dz_1 dz_2| = \frac{1}{4}|z_+ - z_-|\, d\varphi\, d\psi$

$$2\,|D * D(2x^2-1)| \leq 4\,\text{constant}\; M_D^2(x) \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_0^{2\pi} \frac{d\psi}{2\pi}\, \frac{|\mathrm{ch}(2\theta + i\varphi) - \cos\psi|}{\sqrt{|\mathrm{ch}(2\theta + i\varphi) - \mathrm{ch}\, 2\theta|\,|\mathrm{ch}\, 2\theta - \cos\psi|}}$$

After rather crude majorizations one finds

$$2\, D * D(2x^2-1) \leq 4\,\text{constant}\; M_D^2(x) \left\{ \frac{\mathrm{ch}\, 2\theta + 1}{\sqrt{(\mathrm{ch}\, 2\theta - 1)\,\mathrm{sh}\, 2\theta}} \right\} \int_0^{\pi} \frac{d\varphi}{\pi} \frac{1}{\sqrt{\sin\varphi}}$$

The last integral is convergent being equal to $(1/\sqrt{2}\,\pi)\,\Gamma(\frac{1}{4})^2/\Gamma(\frac{1}{2})$. The bracket equals $[\mathrm{cth}\,\theta]^{\frac{3}{2}}$ and tends to 1 for $\theta \to \infty$. Hence we conclude that there exists a constant independent of D such that for $\theta > \theta_0$, or $x > x_0$

$$M_{A_>}(2x^2-1) \leq \text{constant} \;\; M_D^2(x) \tag{16}$$

Returning to our original functions A and D it is clear that for x large enough

$$M_A(2x^2-1) \leq 2\, M_{A_>}(2x^2-1) \tag{17}$$

the factor 2 being a typical choice, How big x_0 should be now depends on the amplitude F under consideration. Choosing the biggest of the values x_0 where (17) and (16) become both valid their combination enables one to obtain the desired inequality (12) with a constant k say twice as large as the one appearing in (16), i.e., independent of F.

Before attacking the problem of possible ambiguities let us draw from (12) a conclusion that will prove useful in the sequel. As $M_f(\mathrm{ch}\,\theta)$ was the maximum modulus of a function $f(z)$ in the ellipse E_θ we define similarly $\mathcal{M}_f(x)$ as the maximum modulus of $f(z)$ inside a circle of radius x:

$$\mathcal{M}_f(x) = \sup_{|z| \leq x} |f(z)| \tag{18}$$

Then from (12) it follows that for any entire unitary amplitude $F(z)$

$$\lim_{x \to \infty} \frac{\mathcal{M}_A(x)}{\mathcal{M}_D(x)} = 0 \tag{19}$$

Indeed, observe first that the ellipse E_Θ is entirely contained in the circle of radius $\mathrm{ch}\,\Theta$ and contains entirely the circle of radius $\mathrm{sh}\,\Theta$. Then, for any analytic function $\mathfrak{M}_f(\mathrm{sh}\,\Theta) \leq M_f(\mathrm{ch}\,\Theta) \leq \mathfrak{M}_f(\mathrm{ch}\,\Theta)$. For Θ large enough Eq. (12) tells us that

$$\mathfrak{M}_A(\mathrm{sh}\,2\theta) \leq M_A(\mathrm{ch}\,2\theta) \leq k\, M_D^2(\mathrm{ch}\,\theta) \leq k\, \mathfrak{M}_D^2(\mathrm{ch}\,\theta)$$

Now, $\mathrm{sh}2\Theta = 2\mathrm{ch}\Theta\sqrt{\mathrm{ch}\Theta^2 - 1}$; again for Θ large enough $\mathrm{sh}2\Theta$ is clearly bigger than $\mathrm{ch}^2\Theta$ and since $\mathfrak{M}(x)$ is a monotonically increasing function

$$\mathfrak{M}_A(x^2) \leq k\, \mathfrak{M}_D^2(x) \qquad x > x_0 \tag{20}$$

Now assume that (19) does not hold. Then there would exist an ϵ and an infinite increasing sequence of points $x_p \to \infty$ such that

$$\mathfrak{M}_A(x_p^2)/\mathfrak{M}_D(x_p^2) \geq \epsilon$$

and thus

$$\mathfrak{M}_D(x_p^2) \leq \frac{1}{\epsilon}\, \mathfrak{M}_A(x_p^2) \leq \frac{k}{\epsilon}\, \mathfrak{M}_D^2(x_p)$$

Set $d(z) = (k/\epsilon)D(z)$. The above inequality translates as

$$\mathfrak{M}_d(x_p^2) \leq \mathfrak{M}_d^2(x_p) \tag{21}$$

According to Hadamard's three circles theorem $\log \mathfrak{M}_d(x)$ is an increasing convex function of $t = \log x$; writing

$$\varphi(t) = \log \mathfrak{M}_d(e^t) \tag{22}$$

this means

$$\varphi(\alpha t_1 + (1-\alpha) t_2) \leq \alpha\varphi(t_1) + (1-\alpha)\varphi(t_2) \qquad 0 \leq \alpha \leq 1$$

In terms of φ , (21) reads for two consecutive values t_p, t_{p+1}

$$\begin{aligned}\varphi(2t_p) &\leq 2\varphi(t_p)\\ \varphi(2t_{p+1}) &\leq 2\varphi(t_{p+1})\end{aligned} \tag{23}$$

Two cases can occur

i) $t_{p+1} \leq 2t_p$

Then λ exists with $0 \leq \lambda < 1$ such that $t_{p+1} = \lambda t_p + (1-\lambda)2t_p$.
Thus, according to (22) and (23)

$$\varphi(t_{p+1}) \leq \lambda\varphi(t_p) + (1-\lambda)\varphi(2t_p) \leq [\lambda + 2(1-\lambda)]\varphi(t_p) = \frac{t_{p+1}}{t_p}\varphi(t_p)$$

ii) $2t_p \leq t_{p+1}$

Then t_{p+1} is between $2t_p$ and $2t_{p+1}$ and $0 < \lambda \leq 1$ exists with $t_{p+1} = \lambda 2t_p + (1-\lambda)2t_{p+1}$, in which case

$$\varphi(t_{p+1}) \leq \lambda\varphi(2t_p) + (1-\lambda)\varphi(2t_{p+1}) \leq 2\left(\lambda\varphi(t_p) + (1-\lambda)\varphi(t_{p+1})\right)$$

This means

$$\varphi(t_{p+1})\left[1 - 2(1-\lambda)\right] \leq 2\lambda\varphi(t_p)$$

But $1-2(1-\lambda) = 2\lambda(t_p/t_{p+1})$ and since λ, t_p, t_{p+1} are all positive (recall that $t_p \to \infty$)

$$\varphi(t_{p+1}) \leq \frac{t_{p+1}}{t_p}\varphi(t_p)$$

In both cases we reach the conclusion that

$$\frac{\varphi(t_{p+1})}{t_{p+1}} \leq \frac{\varphi(t_p)}{t_p}$$

Consequently as $t_p \to \infty$, $(\varphi(t_p)/t_p) < A$ or $\varphi(t_p) < At_p$.

But the convexity of φ as a function of t shows that $\varphi(t) < At$ for all large t. Returning to the function d, for all x there would exist constants α and β such that

$$\mathfrak{M}_d(x) \leq \alpha + \beta x^A$$

This can only hold if d and thus D is a polynomial. If D is a polynomial F would also be a polynomial and we have a contradiction if F is a genuine entire function. For such amplitudes (19) must then hold.

3. AMPLITUDES WITH NON-VANISHING FINITE ORDER

For a large class of entire functions a standard indicator of growth at infinity is the order ρ_F defined in the following way. Let $\mathfrak{M}(x)$ be the maximum modulus of F inside the circular domain $|z| \leq x$ as defined at the end of the previous section. Then

$$\rho_F = \overline{\lim} \quad \frac{\log(\log \mathcal{M}(x))}{\log x} \tag{1}$$

The order ρ can be zero, finite, or infinite. Clearly the order of a sum or of a product is smaller or equal to the largest order of the terms in the sum or the product. From the previous section we know that

$$\mathcal{M}_A(x^2) \leq k\, \mathcal{M}_D^2(x) \tag{2}$$

Consequently, applying the definition of the order

$$\rho_A \leq \frac{1}{2} \rho_D \tag{3}$$

When this is combined with $\rho_D \leq \mathrm{Sup}(\rho_F, \rho_A)$ it immediately yields

$$\rho_D = \rho_F \tag{4}$$

Assume now that the two amplitudes F and G have the same differential cross-section. We write

$$F = D + iA \qquad G = D' + iA'$$
$$\overline{F} = F^*(z^*) = D - iA \qquad \overline{G} = G^*(z^*) = D' - iA'$$

The assumption of equal cross-sections reads

$$F\overline{F} = D^2 + A^2 = D'^2 + A'^2 = G\overline{G}$$

It is clear that $\mathfrak{M}_F(x) = \mathfrak{M}_{\overline{F}}(x)$ and hence $\rho_F = \rho_{\overline{F}}$. Furthermore $\rho_{F\overline{F}} \leq \rho_F$ and from $D^2 = F\overline{F} - A^2$ since the order of any power of a function is equal to the order of this function $\rho_D \leq \mathrm{Sup}(\rho_{F\overline{F}}, \rho_A)$. Hence either

i) $\operatorname{Sup}(\rho_{F\bar{F}}, \rho_A) = \rho_A$ then $\rho_D \leq \rho_A \leq \frac{1}{2}\rho_D$ and $\rho_A = \rho_D = \rho_F = \rho_{F\bar{F}} = 0$ or ∞.

ii) $\operatorname{Sup}(\rho_{F\bar{F}}, \rho_A) = \rho_{F\bar{F}}$ then $\rho_F = \rho_D \leq \rho_{F\bar{F}} \leq \rho_F$ again $\rho_{F\bar{F}} = \rho_F$.

If an ambiguity exists we have $F\bar{F} = G\bar{G}$ and from the above equality follows that the two amplitudes have necessarily the same order

$$\rho_F = \rho_G \tag{5}$$

We shall denote this common order by ρ. According to (4) this is also the order of the absorptive parts D and D'. From now on we assume this order to be finite. Let us state as a theorem the results to be proved in this section.

Theorem 1 If F and G are two entire, unitary amplitudes of finite order with the same differential cross-section, then $G = F$, or $G = -\bar{F}$ (trivial ambiguity) or else

i) the common order ρ is an integer;

ii) if this integer order is larger or equal to one, again apart from the trivial ambiguity, there exists at most two distinct amplitudes with the same cross-section, i.e., the pair F G is unique;

iii) when an ambiguity is present and the order is greater or equal to one neither $F-G$ nor $F+\bar{G}$ is a polynomial, i.e., infinitely many partial waves are genuinely different.

In fact for amplitudes with $\rho = 0$ then ii) is also valid and only iii) is changed as we shall see. We defer this discussion to the next section. Let us turn to the proof of the theorem. From the equality of differential cross-sections we learn that

$$-(D+D')(D-D') = (A+A')(A-A') = Q \tag{6}$$

This quantity can only be zero if $F = G$ or $F = -\bar{G}$. Assume this is not the case. Than Q is a priori an entire function of order smaller or equal to $\rho/2$. As recalled in Appendix B it has a standard representation

$$Q(z) = z^m e^{P_n(z)} \prod_{i \in N} \left\{ \left(1 - \frac{z}{z_i}\right) e^{\left(\frac{z}{z_i}\right) + \frac{1}{2}\left(\frac{z}{z_i}\right)^2 + \cdots + \frac{1}{p}\left(\frac{z}{z_i}\right)^p} \right\} \tag{7}$$

where m denotes the order of its zero at the origin (possibly $m=0$), P_n is a polynomial of degree n and z_i are the non-vanishing zeros of Q. The index i runs over a denumerable set N. The order of Q is $\rho_Q \leq \rho/2$ and $n \leq \rho_Q, \rho_Q - 1 \leq p \leq \rho_Q$. In any case $(Q(z)/z^m)e^{-P_n(z)}$ i.e., the quantity built from the zeros, is of order smaller or equal to the order of Q. Now (D+D') and (D-D') vanish only where Q vanishes, hence they both have a representation of the type (7) (with m and P_n different) but the function constructed upon the zeros is obtained by retaining - in (D+D') a subset of the factors - and in (D-D') the complementary subset. Since by deleting a certain number of terms in an absolutely convergent product can only give a convergent product again we have

$$D \pm D' = z^{m_\pm} e^{P_{n_\pm}} \prod_{i \in N_\pm} \left\{ \left(1 - \frac{z}{z_i}\right) e^{\left(\left(\frac{z}{z_i}\right) + \cdots + \frac{1}{p}\left(\frac{z}{z_i}\right)^p\right)} \right\}$$

with $N_+ \cup N_- = N$ and the same p as before. It is shown in Appendix B that if we delete in an infinite product, such as the ones occurring above, a number of factors, the order can only decrease. Hence the orders of the products $\prod_{i \in N_\pm}$ are smaller or equal to $\rho/2$, and we can write

$$D \pm D' = 2\, e^{\lambda_\pm z^{n_\pm}} \psi_\pm(z)$$

where

$$\rho_{\psi_\pm} \leq \operatorname{Sup}\left(\rho/2 \,,\, n_\pm - 1 \right)$$

But computing $Q = (D+D')(D-D')$ must produce a function of order smaller or equal to $\rho/2$. Hence either $n_\pm \leq \rho/2$ or $n_+ = n_-$, $\lambda_+ = -\lambda_-$. The first case is clearly excluded since this would lead to D and D' having order less than ρ. But then $n_+ = n_-$ is the order ρ and is an integer. Point i) is thus established. Furthermore, with ρ an integer

$$\begin{aligned} D &= e^{\lambda z^\rho} \psi_+(z) + e^{-\lambda z^\rho} \psi_-(z) \\ D' &= e^{\lambda z^\rho} \psi_+(z) - e^{-\lambda z^\rho} \psi_-(z) \end{aligned} \qquad (8)$$

Assume now that D has a decomposition as indicated in (8) and ρ is a non-vanishing integer. Then we shall obtain the fact that at most two amplitudes lead to the same cross-section by establishing that the decomposition given in (8) is unique.

First observe that D satisfies the condition $D(z) = D^*(z^*)$ so does D' and hence D±D' also. Their zeros occur by complex conjugate pairs or are real, in any case the entire functions constructed upon the zeros are "real" and hence λ is real. Then $|\lambda|$ is the type of the function D, and $\lambda = \pm|\lambda|$. Since the type is uniquely determined by the function we see that if there exists more than a two-fold ambiguity there would exist two non-identically vanishing functions f, g of order $\leq \sup(\rho/2, \rho - 1)$ such that:

$$0 = e^{|\lambda| z^\rho} f(z) + e^{-|\lambda| z^\rho} g(z)$$

or

$$g(z) = - e^{2|\lambda| z^\rho} f(z)$$

Take z along the real positive axis, call it then x

$$\log|g(x)| = 2|\lambda| x^\rho + \log|f(x)|$$

In Appendix B (Lemma 1) we show that there exists a constant $\lambda_0 > 1$ such that

$$\overline{\lim_{x\to\infty}} \; |f(x)| \, M_f(\lambda_0 x) = \infty$$

Consequently, no matter how large K is there exists an infinite sequence of points $\{x_p\}$ such that $x_p \to \infty$ and

$$\log|f(x_p)| + \log M_f(\lambda_0 x_p) > K$$

This would imply

$$\log|g(x_p)| > 2|\lambda| x_p^\rho + K - \log M_f(\lambda_0 x_p)$$

or

$$\log|g(x_p)| M_f(\lambda_0 x_p) > 2|\lambda| x_p^\rho + K$$

However,

$$\log|g(x)| \leq A x^{\rho_1} + B$$

$$\log \mathfrak{M}_f(x) \leq C x^{\rho_1} + D$$

with

$$\rho_1 = \operatorname{Sup}\left(\frac{\rho}{2}, \rho - 1\right) + \epsilon < \rho$$

Then for infinitely many points $x_p \to \infty$ one would have an inequality of the form

$$2|\lambda| x_p^{\rho} + K < \left(A + C\lambda_0^{\rho_1}\right) x_p^{\rho_1} + B + D$$

This is clearly impossible with $\rho_1 < \rho$ since $A+C\lambda_0^{\rho_1} > 0$. Hence f and g have to be zero, and we find that the decomposition is unique. Clearly then apart from the trivial ambiguity D' is uniquely determined in terms of D and there are at most two amplitudes with the same cross-section.

It is amusing to see that the trivial ambiguity manifests itself again here in the sense that given the type $|\lambda|$ of D we do not know if $\lambda = +|\lambda|$ or $\lambda = -|\lambda|$ and hence we can permute the meanings of ψ_+ and ψ_-, with the result that there are two possible D' differing by the sign. This does not, however, affect A' the sign of which cannot be changed since the absorptive part is positive on the real axis beyond $x = 1$. The absorptive part A' is given by $A'^2 = D^2 + A^2 - D'^2$ of which in virtue of the above remark only one root is relevant. Point ii) is thus established. Note that our considerations relied heavily on the fact that the order was a positive integer.

Finally let us show that $F-G$ or $F+\bar{G}$ cannot be a polynomial if ρ is a positive integer. Assume the contrary, then by changing eventually G in $-\bar{G}$ we can arrange that it is $F-G$ which is a polynomial.

Set then $D' = D + \Delta D$, $A' = A + \Delta A$ where ΔD and ΔA are polynomials. Then from (6) we see that

$$2D = -\Delta D - \frac{\Delta A}{\Delta D}(2A + \Delta A)$$

This says then that $\rho_D \leq \rho_A \leq \rho_D/2$ and can only hold if $\rho_D = \rho = 0$ (or infinity in fact but we deal here with functions of finite order).

Theorem 1 is thus completely proved.

4. AMPLITUDES OF ORDER ZERO

We now concentrate on the case where F and hence G are genuine entire functions of order zero, that is not polynomials.

Theorem For such F and G with the same differential cross-section:

i) $F-G$ or $F+\tilde{G}$ is a polynomial;

ii) assuming the sign of D' has been chosen in such a way that $F-G$ is a polynomial, then the degree of $D-D'$ is smaller than the degree of $A-A'$;

iii) there exists at most a twofold non-trivial ambiguity.

Functions of zero order are in a sense very close to polynomials. That is up to a constant factor they are entirely defined in terms of their zeros as inspection of formula (7) reveals. In particular their asymptotic behaviour is dictated by the distribution of these zeros. This will be the main tool for the subsequent proofs.

Consider

$$A(z)-A'(z) \equiv \Delta A(z) = \sum_{0}^{\infty} (2\ell+1)\left(\mathrm{Im} f_\ell - \mathrm{Im}\, g_\ell\right) P_\ell(z) \qquad (1)$$

We shall prove i) by contradiction. Hence assume for the time being that ΔA is not a polynomial. For ℓ large enough $\mathrm{Im}\, f_\ell$ and $\mathrm{Im}\, g_\ell$ tend to zero. Hence there exists L such that $\ell > L$ entails $\mathrm{Im}\, f_\ell < \frac{1}{4}$, $\mathrm{Im}\, g_\ell < \frac{1}{4}$ and thus $\mathrm{Im}\, f_\ell + \mathrm{Im}\, g_\ell < \frac{1}{2}$. From unitarity we have

$$\mathrm{Im}\, f_\ell - \mathrm{Im}\, f_\ell^2 = \mathrm{Re}\, f_\ell^2$$
$$\mathrm{Im}\, g_\ell - \mathrm{Im}\, g_\ell^2 = \mathrm{Re}\, g_\ell^2$$

Subtracting

$$\left(\mathrm{Im}\, f_\ell - \mathrm{Im}\, g_\ell\right)\left(1 - \mathrm{Im}\, f_\ell - \mathrm{Im}\, g_\ell\right) = \mathrm{Re}\, f_\ell^2 - \mathrm{Re}\, g_\ell^2$$

but for $\ell > L$

$$\frac{1}{2} < 1 - \mathrm{Im}\, f_\ell - \mathrm{Im}\, g_\ell \leq 1$$

and, therefore

$$\left| \mathrm{Im} f_\ell - \mathrm{Im} g_\ell \right| \leq 2 \left| \mathrm{Re} f_\ell^2 - \mathrm{Re} g_\ell^2 \right| \qquad \mathrm{Im} f_\ell, \mathrm{Im} g_\ell < \frac{1}{4} \tag{2}$$

Correspondingly, we split F and G in $F_< + F_>$, $G_< + G_>$ with the first part containing all the partial waves up to L. In view of (2)

$$\left| \Delta A_> (z) \right| \leq 2 \sum_{\ell > L} (2\ell+1) \left| \mathrm{Re} f_\ell^2 - \mathrm{Re} g_\ell^2 \right| \left| P_\ell(z) \right|$$

Now

$$\left| \mathrm{Re} f_\ell^2 - \mathrm{Re} g_\ell^2 \right| = \left| \mathrm{Re} f_\ell - \mathrm{Re} g_\ell \right| \left| \mathrm{Re} f_\ell + \mathrm{Re} g_\ell \right| \leq \mathrm{ch}\, 2\theta \, Q_\ell(\mathrm{ch}\, 2\theta) M_{\Delta D * \Sigma D}(\mathrm{ch}\, 2\theta)$$

where we have used formula (9') of Section 2, applied to the function

$$\Delta D * \Sigma D(z) = \sum_0^\infty (2\ell+1)(\mathrm{Re} f_\ell - \mathrm{Re} g_\ell)(\mathrm{Re} f_\ell + \mathrm{Re} g_\ell) P_\ell(z)$$

on an ellipse $E_{2\theta}$. Using now the properties of the convolution integral and a reasoning completely similar to the one leading to the inequality (12), Section 2, we obtain

$$M_{\Delta D * \Sigma D}(\mathrm{ch}\, 2\theta) \leq k \, M_{\Delta D}(\mathrm{ch}\, \theta) \, M_{\Sigma D}(\mathrm{ch}\, \theta)$$

for $\theta > \theta_0$. Choosing now $\theta' < \theta$ (see below for a more precise choice for θ').

$$M_{\Delta A_>}(\mathrm{ch}\, 2\theta') \leq 2k \, M_{\Delta D}(\mathrm{ch}\, \theta) M_{\Sigma D}(\mathrm{ch}\, \theta) \, \mathrm{ch}\, 2\theta \sum_{\ell > L} (2\ell+1) P_\ell(\mathrm{ch}\, 2\theta') Q_\ell(\mathrm{ch}\, 2\theta)$$

The last sum can be extended down to $\ell = 0$, by increasing the right-hand side and sums to $1/\mathrm{ch}2\theta - \mathrm{ch}2\theta'$:

$$M_{\Delta A_>}(\mathrm{ch}\, 2\theta') \leq 2k \, M_{\Delta D}(\mathrm{ch}\, \theta) M_{\Sigma D}(\mathrm{ch}\, \theta) \frac{\mathrm{ch}\, 2\theta}{\mathrm{ch}\, 2\theta - \mathrm{ch}\, 2\theta'}$$

It is always possible to pick $\theta' < \theta$ in such a way that $\mathrm{ch}2\theta' < \rho \, \mathrm{ch}2\theta$ with $\frac{1}{2} < \rho < 1$ and $\mathrm{ch}^2\theta < \mathrm{sh}2\theta'$. Indeed, compatibility requires

$$\mathrm{ch}^2\theta < \mathrm{sh}\, 2\theta' < \mathrm{ch}\, 2\theta' < \rho \, \mathrm{ch}\, 2\theta$$

or

$$\frac{ch^2\theta}{ch\,2\theta} < \rho < 1$$

since $ch^2\theta/ch2\theta$ tends to $\frac{1}{2}$ for $\theta \to \infty$ it is seen that the choice will always be possible with $1 > \rho > \frac{1}{2}$ for θ large enough. But then with $k' = 2k/1-\rho$ and θ large enough

$$\mathcal{M}_{\Delta A_>}(ch^2\theta) < \mathcal{M}_{\Delta A_>}(sh\,2\theta') \leq M_{\Delta A_>}(ch\,2\theta') \leq k'\mathcal{M}_{\Delta D}(ch\theta)\,\mathcal{M}_{\Sigma D}(ch\theta)$$

By increasing perhaps k' by a finite factor we can replace $\mathcal{M}_{\Delta A_>}$ by $\mathcal{M}_{\Delta A}$ and conclude that there exists a constant q such that for x large enough

$$\mathcal{M}_{\Delta A}(x^2) \leq q\,\mathcal{M}_{\Delta D}(x)\,\mathcal{M}_{\Sigma D}(x) \tag{3}$$

For

$$A(z) + A'(z) \equiv \Sigma A(z) = \sum_0^\infty (2\ell+1)\left(\mathrm{Im} f_\ell + \mathrm{Im} g_\ell\right) P_\ell(z)$$

we split the sum in a similar manner in $\Sigma A_<$ and $\Sigma A_>$ where in $\Sigma A_>$, $\mathrm{Im} f_\ell$ and $\mathrm{Im} g_\ell$ are smaller than $\frac{1}{2}$ and hence

$$\mathrm{Im} f_\ell + \mathrm{Im} g_\ell \leq 2\left(\mathrm{Re} f_\ell^2 + \mathrm{Re} g_\ell^2\right) = \left(\mathrm{Re} f_\ell + \mathrm{Re} g_\ell\right)^2 + \left(\mathrm{Re} f_\ell - \mathrm{Re} g_\ell\right)^2$$

The technique is now very similar as above

$$\begin{aligned} M_{\Sigma A_>}(ch2\theta) &\leq \Sigma D * \Sigma D(ch2\theta) + \Delta D * \Delta D(ch2\theta) \\ &\leq k\left(M^2_{\Sigma D}(ch\theta) + M^2_{\Delta D}(ch\theta)\right) \end{aligned}$$

$$\mathcal{M}_{\Sigma A_>}(x^2) \leq k\left(\mathcal{M}^2_{\Sigma D}(ch\theta) + \mathcal{M}^2_{\Delta D}(ch\theta)\right) \qquad x = ch\theta$$

And again for x large we can replace $\Sigma A_>$ by ΣA, obtaining

$$\mathcal{M}_{\Sigma A}(x^2) \leq k\left(\mathcal{M}^2_{\Sigma D}(x) + \mathcal{M}^2_{\Delta D}(x)\right) \tag{4}$$

Multiplying together inequalities (3) and (4) yields:

$$\mathcal{M}_{\Delta A}(x^2)\,\mathcal{M}_{\Sigma A}(x^2) \leq kq\,\mathcal{M}_{\Delta D}(x)\,\mathcal{M}_{\Sigma D}(x)\left(\mathcal{M}^2_{\Delta D}(x)+\mathcal{M}^2_{\Sigma D}(x)\right) \tag{5}$$

For x going to infinity $\mathcal{M}_{\Delta D}(x)$ and $\mathcal{M}_{\Sigma D}(x)$ go to infinity. Now, if two real quantities a and b are such that $a \geq 2$, $b \geq 2$ then $a+b \leq ab$. Hence, there exists a constant K and $x_0 > 1$, such that for $x > x_0$

$$\mathcal{M}_{\Delta A}(x^2)\,\mathcal{M}_{\Sigma A}(x^2) \leq K\left(\mathcal{M}_{\Delta D}(x)\,\mathcal{M}_{\Sigma D}(x)\right)^3 \tag{6}$$

Note that up to now use has not yet been made of the fact that F and G have the same differential cross-section. This we take into account by noticing that

$$\mathcal{M}_{\Delta D.\Sigma D}(x^2) = \mathcal{M}_{\Delta A.\Sigma A}(x^2) \leq \mathcal{M}_{\Delta A}(x^2)\,\mathcal{M}_{\Sigma A}(x^2)$$

and thus for $x > x_0$

$$\mathcal{M}_{\Delta D.\Sigma D}(x^2) \leq K\left(\mathcal{M}_{\Delta D}(x)\,\mathcal{M}_{\Sigma D}(x)\right)^3 \tag{7}$$

This is the key inequality. Functions of order zero are such that there is a close control on the relation between the product of maxima and the maximum of the product. It is indeed shown in Appendix B (lemma 2) that condition (7) has the consequence that

$$\frac{\mathcal{M}_{\Sigma D}(x)\,\mathcal{M}_{\Delta D}(x)}{\mathcal{M}_{\Delta D.\Sigma D}(x)} < A_\epsilon\, e^{B_\epsilon (\log x)^{\left[\frac{\log 3}{\log 2}+\epsilon-1\right]}} \qquad x > x_0 \tag{8}$$

where ϵ can be chosen as small as one wants. Then $(\log 3/\log 2)+\epsilon-1$ is a quantity between 0 and 1 and as a result the right-hand side grows slower than any power of x, no matter how small this power is. The idea now is to find a lower bound for the ratio in (8) with a growth at least as fast as a power of x in such a way as to arrive at a contradiction. This will be achieved by looking again at ΔA. We have

$$M_{\Delta A}(x) \leq \sum_0^\infty (2\ell+1)\,|\mathrm{Im} f_\ell - \mathrm{Im} g_\ell|\, P_\ell(x) \tag{9}$$

For a fixed $\lambda > 1$ let $(2L+1)|\mathrm{Im}\, f_L - \mathrm{Im}\, g_L|P_L(\lambda x)$ be the largest of all terms among those appearing on the right-hand side of (19) when written for $M_{\Delta A}(\lambda x)$. L depends on $x\lambda$. If the maximum is reached for several values we choose the largest possible L. Also $L(x\lambda) \to \infty$ as $x \to \infty$ for fixed λ otherwise $\Delta A(z)$ would be bounded by a polynomial and hence would be a polynomial. Since according to (A-2) $P_\ell(x) \leq (P_\ell(\lambda x)/\lambda^\ell)$

$$M_{\Delta A}(x) \leq \sum_0^\infty (2\ell+1)\,|\mathrm{Im} f_\ell - \mathrm{Im} g_\ell|\, P_\ell(\lambda x)\lambda^{-\ell} \leq \frac{(2L+1)}{1-1/\lambda}\,|\mathrm{Im} f_L - \mathrm{Im} g_L|\, P_L(\lambda x) \tag{10}$$

We can always assume x big enough and hence L big enough in such a way that $\mathrm{Im}\, f_L < \frac{1}{4}$ $\mathrm{Im}\, g_L < \frac{1}{4}$ and thus

$$|\mathrm{Im} f_L - \mathrm{Im} g_L| \leq 2\,|\mathrm{Re} f_L^2 - \mathrm{Re} g_L^2| = 2\,|\mathrm{Re} f_L - \mathrm{Re} g_L|\,|\mathrm{Re} f_L + \mathrm{Re} g_L|$$

$$M_{\Delta A}(x) \leq \frac{2\lambda}{\lambda-1}\,(2L+1)\,|\mathrm{Re} f_L - \mathrm{Re} g_L|\,|\mathrm{Re} f_L + \mathrm{Re} g_L|\; P_L(\lambda x) \tag{11}$$

Similarly, if $(2M+1)(\mathrm{Im}\, f_M + \mathrm{Im}\, g_M)P_M(\lambda x)$ denotes the largest of all quantities $(2\ell+1)(\mathrm{Im}\, f_\ell + \mathrm{Im}\, g_\ell)P_\ell(\lambda x)$ we have

$$M_{\Sigma A}(x) \leq \frac{\lambda}{\lambda-1}\,(2M+1)\left(\mathrm{Im} f_M + \mathrm{Im} g_M\right) P_M(\lambda x) \tag{12}$$

Again for x large enough $\mathrm{Im}\, f_M < \frac{1}{2}$ $\mathrm{Im}\, g_M < \frac{1}{2}$ and

$$\mathrm{Im} f_M + \mathrm{Im} g_M \leq 2\left(\mathrm{Re} f_M^2 + \mathrm{Re} g_M^2\right) \leq 2\left(|\mathrm{Re} f_M| + |\mathrm{Re} g_M|\right)^2$$

The quantity $|\mathrm{Re}\, f_M| + |\mathrm{Re}\, g_M|$ is equal either to $|\mathrm{Re}\, f_M + \mathrm{Re}\, g_M|$ or to $|\mathrm{Re}\, f_M - \mathrm{Re}\, g_M|$. Assume for definiteness that the first case is valid, then

$$M_{\Sigma A}(x) \leq \frac{2\lambda}{\lambda-1}\left(\mathrm{Re}\, f_M + \mathrm{Re}\, g_M\right)^2 P_M(\lambda x) \tag{12'}$$

Assuming $x > \lambda$ in (11) we then use

$$\left|\mathrm{Re}\, f_L - \mathrm{Re}\, g_L\right| \leq \frac{x}{\lambda} Q_L\left(\frac{x}{\lambda}\right) M_{\Delta D}\left(\frac{x}{\lambda}\right)$$

and in (12)

$$\left|\mathrm{Re}\, f_M + \mathrm{Re}\, g_M\right| \leq \frac{x}{\lambda} Q_M\left(\frac{x}{\lambda}\right) M_{\Sigma D}\left(\frac{x}{\lambda}\right)$$

and obtain using the product of the two

$$M_{\Delta A}(x) M_{\Sigma A}(x) \leq \left(\frac{2\lambda}{\lambda-1}\right)^2 \left\{(2M+1)\frac{x}{\lambda} P_M(\lambda x) Q_M\left(\frac{x}{\lambda}\right)\right\}\left\{(2L+1)\frac{x}{\lambda} P_L(\lambda x) Q_L\left(\frac{x}{\lambda}\right)\right\} \times$$
$$\left(|\mathrm{Re}\, f_L| + |\mathrm{Re}\, g_L|\right)\left(|\mathrm{Re}\, f_M| + |\mathrm{Re}\, g_M|\right) M_{\Delta D}\left(\frac{x}{\lambda}\right) M_{\Sigma D}\left(\frac{x}{\lambda}\right) \tag{13}$$

(The reason for introducing $M_{\Delta D}$ and $M_{\Sigma D}$ at x/λ rather than x will appear subsequently.) It is readily seen that had $|\mathrm{Re}\, f_M| + |\mathrm{Re}\, g_M|$ been equal to $|\mathrm{Re}\, f_M - \mathrm{Re}\, g_M|$ by inverting the roles of ΔD and ΣD we would have come to the same conclusion as (13).

Now observe that by virtue of (A-2) $P_M(\lambda x) \leq P_M(\lambda^2) P_M(x/\lambda)$ and [cf.(A-3)] for $x/\lambda > 1$

$$(2M+1)\frac{x}{\lambda} P_M(\lambda x) Q_M\left(\frac{x}{\lambda}\right) \leq P_M(\lambda^2)\left[(2M+1)\frac{x}{\lambda} P_M\left(\frac{x}{\lambda}\right) Q_M\left(\frac{x}{\lambda}\right)\right] \leq \frac{x}{x-\lambda} P_M(\lambda^2)$$

with a similar inequality for the second bracket on the right-hand side of (13). For x large enough $x/x-\lambda$ is say smaller than 2. Then

$$\frac{M_{\Delta A}(x) M_{\Sigma A}(x)}{M_{\Delta D}\left(\frac{x}{\lambda}\right) M_{\Sigma D}\left(\frac{x}{\lambda}\right)} \leq \left(\frac{4\lambda}{\lambda-1}\right)^2 P_M(\lambda^2) P_L(\lambda^2)\left(|\mathrm{Re}\, f_L| + |\mathrm{Re}\, g_L|\right)\left(|\mathrm{Re}\, f_M| + |\mathrm{Re}\, g_M|\right) \tag{14}$$

Consider the term $(|\mathrm{Re}\, f_M| + |\mathrm{Re}\, g_M|)$ then from unitarity again we have

$$2\,|\mathrm{Re} f_M|\,|\mathrm{Re} g_M| \leq |\mathrm{Re} f_M|^2 + |\mathrm{Re} g_M|^2 \leq \mathrm{Im} f_M + \mathrm{Im} g_M$$

and hence

$$|\mathrm{Re} f_M| + |\mathrm{Re} g_M| \leq \sqrt{2(\mathrm{Im} f_M + \mathrm{Im} g_M)}$$

Now recall that M was defined as the largest M such that the inequality

$$(2\ell+1)\left(\mathrm{Im} f_\ell + \mathrm{Im} g_\ell\right) P_\ell(\lambda x) \leq (2M+1)\left(\mathrm{Im} f_M + \mathrm{Im} g_M\right) P_M(\lambda x)$$

holds. For λ fixed as x increases M eventually will become as large as one wants. Since M takes integer values, there will exist an increasing sequence of points x_p at which M has jumps. In other words at these points

$$\left(2(M-k_p)+1\right)\left(\mathrm{Im} f_{M-k_p} + \mathrm{Im} g_{M-k_p}\right) P_{M-k_p}(\lambda x_p) = (2M+1)\left(\mathrm{Im} f_M + \mathrm{Im} g_M\right) P_M(\lambda x_p)$$

where k_p is a positive integer which could increase in size. Of course, $M-k_p$ as well as M must go to infinity as $x_p \to \infty$. Consequently recalling that M is a function of x, if we restrict ourselves to the sequence x_p

$$\left(|\mathrm{Re} f_M| + |\mathrm{Re} g_M|\right)^2 \leq 2\left(\mathrm{Im} f_M + \mathrm{Im} g_M\right) = 2\,\frac{2(M-k_p)+1}{2M+1}\left(\mathrm{Im} f_{M-k_p} + \mathrm{Im} g_{M-k_p}\right)\frac{P_{M-k_p}(\lambda x_p)}{P_M(\lambda x_p)}$$

But for p large enough we have also the inequality

$$\mathrm{Im} f_{M-k_p} + \mathrm{Im} g_{M-k_p} \leq 2\left(|\mathrm{Re} f_{M-k_p}| + |\mathrm{Re} g_{M-k_p}|\right)^2$$

and hence

$$\left(|\mathrm{Re} f_M|+|\mathrm{Re} g_M|\right)^2 \leq 4\ \frac{P_{M-k_p}(\lambda x_p)}{P_M(\lambda x_p)}\left(|\mathrm{Re} f_{M-k_p}|+|\mathrm{Re} g_{M-k_p}|\right)^2 \qquad (15)$$

Inserting (15) in (14) yields

$$\sqrt{x_p}\ \frac{M_{\Delta A}(x_p)M_{\Sigma A}(x_p)}{M_{\Delta D}\left(\frac{x_p}{\lambda}\right)M_{\Sigma D}\left(\frac{x_p}{\lambda}\right)} \leq 2\left(\frac{4\lambda}{\lambda-1}\right)^2\left\{P_L(\lambda^2)\left(|\mathrm{Re} f_L|+|\mathrm{Re} g_L|\right)\right\}\times$$

$$\left\{P_M(\lambda^2)\sqrt{x_p\ \frac{P_{M-k_p}(\lambda x_p)}{P_M(\lambda x_p)}}\left(|\mathrm{Re} f_{M-k_p}|+|\mathrm{Re} g_{M-k_p}|\right)\right\} \qquad (16)$$

It should be noted that a factor x_p has been added to both sides of this inequality. The claim now is as $p\to\infty$ and $x_p\to\infty$ the right-hand side goes to zero.

Indeed λ is a fixed quantity but $L\to\infty$, since

$$\sum_0^\infty (2\ell+1)\,P_\ell(\lambda^2)\,|\mathrm{Re} f_\ell|$$

and

$$\sum_0^\infty (2\ell+1)\,P_\ell(\lambda^2)\,|\mathrm{Re} g_\ell|$$

are convergent series one has certainly

$$P_L(\lambda^2)\left(|\mathrm{Re} f_L|+|\mathrm{Re} g_L|\right)\longrightarrow 0$$

For the second bracket we get rough estimates by using (A-2)

$$x_p\ \frac{P_{M-k_p}(\lambda x_p)}{P_M(\lambda x_p)} \leq \frac{(2\lambda x_p)^{M-k_p}}{(\lambda x_p)^M}\,x_p \leq \frac{4^{M-k_p}}{\lambda}\ \frac{1}{(\lambda x_p)^{k_p-1}}$$

Note that $k_p-1\geq 0$. Then as $p\to\infty$ x_p will eventually get larger than $4\lambda^3$ and hence for p large enough

$$\sqrt{\frac{P_{M-k_p}(\lambda x_p)\, x_p}{P_M(\lambda x_p)}} \leq \frac{2^{M-k_p}}{\sqrt{\lambda}} \frac{1}{(2\lambda^2)^{k_p-1}}$$

and

$$P_M(\lambda^2)\sqrt{x_p \frac{P_{M-k_p}(\lambda x_p)}{P_M(\lambda x_p)}} \leq \frac{(2\lambda^2)^M}{\sqrt{\lambda}} \cdot \frac{2^{M-k_p}}{(2\lambda^2)^{k_p-1}} = (2\lambda^2)^{M-k_p} (\lambda)^{3/2}\, 2^{M-k_p+1}$$

$$\leq 2\,(\lambda)^{3/2}\,(4\lambda^2)^{M-k_p}$$

Consequently, using again (A-2)

$$P_M(\lambda^2)\sqrt{\frac{x_p\, P_{M-k_p}(\lambda x_p)}{P_M(\lambda x_p)}} \leq 2\,(\lambda)^{3/2}\, P_{M-k_p}(4\lambda^2)$$

Returning to (16) we see that the second bracket is smaller than

$$2\,(\lambda)^{3/2}\, P_{M-k_p}(4\lambda^2)\left(\left|\mathrm{Re}\, f_{M-k_p}\right| + \left|\mathrm{Re}\, g_{M-k_p}\right|\right)$$

which as $M-k_p \to \infty$ goes to zero. Our contention that

$$\sqrt{x_p}\quad \frac{M_{\Delta A}(x_p)\, M_{\Sigma A}(x_p)}{M_{\Delta D}\left(\frac{x_p}{\lambda}\right) M_{\Sigma D}\left(\frac{x_p}{\lambda}\right)} \to 0 \quad \text{as} \quad x_p \to 0 \tag{17}$$

is thus proved. Since again $M_{\Delta D.\Sigma D}(x) = M_{\Delta A.\Sigma A}(x) \leq M_{\Delta A}(x) M_{\Sigma A}(x)$ we have obtained the following result: there exists an infinite sequence of points $\{x_p\}$ with $x_p \to \infty$ such that

$$\sqrt{x_p}\quad \frac{M_{\Delta D.\Sigma D}(x_p)}{M_{\Delta D}\left(\frac{x_p}{\lambda}\right) M_{\Sigma D}\left(\frac{x_p}{\lambda}\right)} \to 0 \tag{18}$$

Now compare this result with the previous one given by the inequality (8). We have to add a last touch of delicacy since in (18) it is the maxima over ellipses which occur while in (8) we had the maxima over circles. At any rate (18) tells us that no matter how small a is for x_p large enough we have

$$\sqrt{x_p}\; M_{\Delta D.\Sigma D}(x_p) \leq a\, M_{\Delta D}\left(\frac{x_p}{\lambda}\right) M_{\Sigma D}\left(\frac{x_p}{\lambda}\right)$$

hence combining this with (8)

$$\mathfrak{M}_{\Delta D.\Sigma D}\left(\sqrt{x_p^2-1}\right) \leq a\, A_\epsilon\, e^{\left\{ B_\epsilon \left(\log \frac{x_p}{\lambda}\right)^{\left[\frac{\log 3}{\log 2}+\epsilon-1\right]} - \frac{1}{2}\log x_p \right\}}\, \mathfrak{M}_{\Delta D.\Sigma D}\left(\frac{x_p}{\lambda}\right)$$

As $x_p \to \infty$ the exponent of e becomes more and more negative, consequently

$$\frac{\mathfrak{M}_{\Delta D.\Sigma D}\left(\sqrt{x_p^2-1}\right)}{\mathfrak{M}_{\Delta D \Sigma D}\left(\frac{x_p}{\lambda}\right)} \longrightarrow 0 \qquad (19)$$

But, and this was the motivation for introducing this extra factor λ, for x_p large enough $\sqrt{x_p^2-1} > x_p/\lambda$ and the ratio in (19) is certainly bigger than one, thus leading to a contradiction and proving point i) of the theorem.

Admittedly the proof is rather cumbersome, we were unable to find a short cut. The remainder will now follow more easily. We know now that if an ambiguity exists of non-trivial nature then $F-G$ or $F+\bar{G}$ is a polynomial. By a judicious choice of the sign of D' it can be assumed that $F-G$ is a polynomial. Now consider F say. Recall that for any entire amplitude no matter its order in fact provided it is not a polynomial:

$$\lim_{x\to\infty} \frac{\mathfrak{M}_A(x)}{\mathfrak{M}_D(x)} = 0 \qquad (20)$$

as was proved at the end of Section 2, formula (19).

Let ℓ and m be the degrees of $A-A'$ and $D-D'$, in other words

$$A(z) - A'(z) = \alpha z^{\ell} + \cdots \qquad \alpha \neq 0$$
$$D(z) - D'(z) = \beta z^{m} + \cdots \qquad \beta \neq 0$$

Noticing that for $x \to +\infty$ A increases faster than any power we easily find

$$\mathcal{M}_D(x) \sim -\left|\frac{\alpha}{\beta}\right| x^{\ell-m} \mathcal{M}_A(x)$$

From (20) we would then have a contradiction if m were larger or equal to ℓ. Hence the degree of ΔD is smaller than the degree of ΔA and point ii) is thus proved.

The last item in the theorem deals with the impossibility of a three-fold non-trivial ambiguity in the case of amplitudes of order zero. Again let us prove this by showing that it would lead to an absurdity. Assume F, $G = F + \Delta_1 F$, $H = F + \Delta_2 F$ to be three unitary amplitudes of order zero with the same differential cross-section. By choosing properly the signs of the real parts $\Delta_1 F$ and $\Delta_2 F$ are polynomials and by hypothesis none of $\Delta_1 D$, $\Delta_2 D$, $\Delta_1 A$, $\Delta_2 A$ vanishes identically.

Observe that the equality of differential cross-section can be written

$$\begin{aligned} \frac{D}{A} &= -\frac{\Delta_i A}{\Delta_i D} + R_i \\ R_i &= -\frac{(\Delta_i A)^2}{2A\,\Delta_i D} - \frac{\Delta_i D}{2A} \end{aligned} \tag{22}$$

Use this equality along the real positive axis for x larger than 1 and the largest possible real positive zero of $\Delta_i D$. Then $\Delta_i D$ and A do not vanish.

If F is a genuine entire function $R_i = O(1/x^\infty)$ i.e., vanishes faster than any inverse power of x as $x \to \infty$. Consequently

$$\frac{\Delta_1 A}{\Delta_1 D} - \frac{\Delta_2 A}{\Delta_2 D} = O\left(\frac{1}{x^\infty}\right) \qquad x \text{ real} \to \infty$$

But the left-hand side is a rational function so that it has to vanish and

$$\frac{\Delta_1 A}{\Delta_1 D} = \frac{\Delta_2 A}{\Delta_2 D} \tag{23}$$

This equation means that in the Argand diagram F, G and H lie on a straight line. On the other hand, they are on the same circle since $|F| = |G| = |H|$. Therefore, two of them coincide and by continuity and analyticity it is always the same pair.

This concludes the proof of Theorem 2.

However, the preceding proof suggests to extend somehow the previous result to the case where F, G and H are instead polynomials of degree L (they clearly have to have the same degree if $F\bar{F} = G\bar{G} = H\bar{H}$) in which case

$$R_i(z) = O\left(\frac{1}{|z|^{L-L_i}}\right) \tag{24}$$

where L_i denotes the degree of $\Delta_i D$ and we shall see immediately that the degree of $\Delta_i A$ is smaller or equal to the one of $\Delta_i D$. Now, the equality of differential cross-sections has two trivial consequences. The first is that the modulus of the highest partial wave is common to F, G and H. We choose the signs of the real parts so that $f_L = g_L = h_L$ and hence $L_i \leq L-1$. The second fact which readily follows from $\sigma_F(z) = \sigma_G(z) = \sigma_H(z)$ is that

$$\mathrm{Re}\left[f_L^*\left(\Delta_i f_{L_i}\right)\right] = 0 \tag{25}$$

In other words $\Delta_i f_{L_i}$ is in quadrature with f_L (see Fig. 2). With $f_L \neq 0$ $0 < \mathrm{Arg}\, f_L < \pi$ and $\Delta_i \mathrm{Re}\, f_{L_i} \neq 0$. So that degree $\Delta_i D =$ degree $\Delta_i F \geq$ degree $\Delta_i A$. The reader will remark that what Theorem 2 says can be considered as a limiting case of the preceding situation when $L \to \infty$, $f_L \to 0$ along the unitary circle and $\Delta_i f_{L_i}$ tends to a pure imaginary quantity.

Denote by M the maximum of L_i then

$$\frac{\Delta_1 A}{\Delta_1 D} - \frac{\Delta_2 A}{\Delta_2 D} = O\left(\frac{1}{|z|^{L-M}}\right)$$

This means

$$\Delta_1 A\ \Delta_2 D - \Delta_2 A\ \Delta_1 D = O\left(\frac{1}{|z|^{L-M-L_1-L_2}}\right)$$

If L_1, say, is equal to the maximum M of L_1, L_2 then $L_2 \leq L_1 - 1$. Indeed f_{L_1} occupies one of the positions (a) or (b) on Fig. 1. g_{L_1} occupies the other position (b) or (a); since $\Delta_2 f_{L_2}$ is in quadrature with f_L, either $L_2 \leq M-1$ or $L_2 = L_1 = M$. In the last case h_M has to be at the same position as g_M: $g_M = h_M$ all higher waves for F, G, H being equal. But then relabelling the amplitudes we would consider G as playing the role assigned to F previously in which case one would have again $L_2 \leq L_1 - 1$. Hence, with this understanding

$$\Delta_1 A \; \Delta_2 D - \Delta_1 D \, \Delta_2 A = O\left(\frac{1}{|z|^{L+1-3M}}\right)$$

Furthermore, $\Delta_1 A$ and $\Delta_2 A$ have to vanish in the forward direction $z = 1$. Dividing both sides by $(z-1)$ we are still left on the left-hand side with a polynomial and

$$\frac{\Delta_1 A \; \Delta_2 D - \Delta_1 D \, \Delta_2 A}{z-1} = O\left(\frac{1}{|z|^{L+2-3M}}\right)$$

From this we conclude that if

$$L \geqslant 3M-1 \tag{26}$$

the left-hand side polynomial vanishes and as a result

$$\frac{\Delta_1 A}{\Delta_1 D} = \frac{\Delta_2 A}{\Delta_2 D}$$

which by the same argument as above means that two of the three amplitudes coincide.

We have thus shown that if there are at most two amplitudes of degree L with the same differential cross-section if one allows to change partial waves up to the M^{th} with $3M \leq L+1$. For instance if $L = 4$ and one changes only S and P waves one cannot have the same differential cross-section for three amplitudes. Furthermore this shows the instability of ambiguities. Assume there would exist three unitary polynomial amplitudes F, G, H with the same differential cross-section and not two of them related by the trivial ambiguity. Then add to all of them an arbitrary equal amount of a very high partial wave, then no matter how small this amount is the three new amplitudes cannot have equal cross-sections.

5. THE CASE OF SMALL TOTAL CROSS-SECTIONS

We have recalled that if the parameter $\sin \mu$ defined as in Section 2, (7) is smaller than 0.79 then no ambiguity is present when reconstructing a unitary amplitude from the differential cross-section. Here we shall show that when the integrated differential cross-section is small enough again no ambiguity can be present. This is of interest since for instance it includes cases where the previous bound is not satisfied with $\sin \mu$ very large, infinite for instance if the differential cross-section vanishes in the physical region.

We shall obtain the required result by showing that if two amplitudes lead to the same cross-section and differ by a polynomial then $\sigma \geq 111/80 = 1.3875$. This will cover the cases where the amplitudes F and G are either polynomials or entire functions of order 0. At the moment we are not in a position to find out if this result is absolutely general though we suspect it is. We shall treat the two cases in turn.

i) Polynomial case

Recall that one can always choose the highest partial wave of F and G to be identical and let L = degree F = degree G, M = degree (F-G).

We have

$$f_L = g_L$$
$$\mathrm{Re}\left[f_L^*(f_M - g_M)\right] = 0 \tag{1}$$

This means that $f_M + g_M/2$ lies on a straight line through the center of the unitary circle and parallel to f_L. Of course it is inside the unitary circle. If the highest phase shift δ_L is chosen, say, between 0 and π we have then (see Fig. 2)

$$\frac{1}{2}\,\mathrm{Im}(f_M + g_M) \geq \frac{1 - \sin\delta_L}{2} \tag{2}$$

Unitarity says that

$$\sigma = \operatorname{Im} F(1) = \operatorname{Im} G(1) = \frac{1}{2} \operatorname{Im}\left(F(1) + G(1)\right) \tag{3}$$

We have, combining (2) and (3)

$$\sigma \geqslant (2L+1)\sin^2\delta_L + (2M+1)\ \frac{1-\sin\delta_L}{2} \tag{4}$$

Expressed as a function of $u = \sin\delta_L$ the right-hand side is a concave parabola with a minimum value

$$\sigma_{L,M} = \frac{(2M+1)\left[8(2L+1) - (2M+1)\right]}{16\,(2L+1)} \tag{5}$$

Now F and G cannot differ only by their s wave $(M=0)$. Indeed from (3) $\operatorname{Im} f_0 = \operatorname{Im} g_0$, $\operatorname{Re} f_0 = -\operatorname{Re} g_0$. This means $\operatorname{Im} F = \operatorname{Im} G$, hence $\operatorname{Re} F = -\operatorname{Re} G$ then by changing $\operatorname{Re} G$ into $-\operatorname{Re} G$ gives $F = G$. So that $M \geq 1$ and as M varies from 1 to $L-1$, $\sigma_{L,M} \geq \sigma_{L,1}$. Furthermore $\sigma_{L,1} \geq \sigma_{2,1}$. A lower bound on σ is then

$$\sigma_{2,1} = \frac{3}{16}\left(8 - \frac{3}{5}\right) = \frac{111}{80} = 1.3875$$

This is not the best possible result (one has to add the other waves). However, it is not of crucial importance to improve this bound by something like ten percent.

ii) <u>Entire functions of order zero</u>

Theorem 2 of the previous section has shown that $F-G$ is also a polynomial in this case and that if M denotes the degree of this polynomial $\operatorname{Re} f_M - \operatorname{Re} g_M = 0$. This appears as a limit of (1) when $L \to \infty$ and $f_L \to 0$ along the unitary circle. Again this gives $\operatorname{Im} f_M + \operatorname{Im} g_M = \frac{1}{2}$ and since $M \geq 1$ as the previous reasoning showed $\sigma > 3 \times \frac{1}{2} = 1.5$ which is nothing but $\sigma_{\infty,1}$.

We conclude that if $\sigma > \sigma_0$ with $\sigma_0 \geq 111/80$ there does not exist any ambiguity.

6. CONCLUDING REMARKS

We are aware of the fact that the problem is not completely solved. In fact we have not touched upon the physically important case of an amplitude analytic inside an ellipse but not entire. However, we know now that for polynomial amplitudes with $L_{max} \leq 3$ and for amplitudes which are genuine entire functions of finite order there are at most two solutions, excluding the trivial ambiguities. We still have to fill the gap between $L = 3$ and a transcendental zero order entire function. In this gap, a heuristic counting argument, which was proposed independently by Berends and Ruijsenaars [2] and by Cornille and the authors, leads to believe that there are at most two solutions. This argument is that a cross-section corresponding to L partial waves depends on $2L+1$ parameters which are the L^{th} phase shift and the real and imaginary parts of the zeros of one of the acceptable amplitudes. The zeros of all other amplitudes with the same cross-section being either equal or complex conjugate of those of the chosen one. If one has n solutions one imposes nL unitarity conditions on partial waves and if one wants to have less constraints than parameters one must take $n = 2$. Then there remains one parameter free. However, there is no proof that the unitarity constraints are really independent. A partial result for $L = 4$ has been obtained by using only the unitarity of $L = 3$ partial wave and conditions on the forward amplitude. One finds that the number of solutions has to be less than five.

To summarize we believe that if the amplitude is an entire function of finite order (including now polynomials) there are at most two solutions. However, we see a drastic change between functions of order less than one and functions of order larger or equal to one. The difference between two amplitudes of order less than one has to be a polynomial (in a future paper we hope to exhibit an explicit example of polynomial ambiguity for the case of zero order amplitudes). On the contrary the difference between two entire functions of order ≥ 1 cannot be a polynomial. Now notice that as the order increases we approach more and more the realistic case since the partial wave amplitudes decrease more and more slowly with ℓ. In the case of an amplitude analytic in a cut plane Burkhart [7] has shown that the knowledge of the discontinuity across part of the cut fixes the amplitude completely. This means that two amplitudes cannot differ by a polynomial without being identical, since they have the same discontinuity. We have generalized this result to any amplitude analytic in the neighbourhood of the physical region but not entire. Such an amplitude has a maximum ellipse of convergence of semi-major axis $\operatorname{ch}\theta_0$. On this ellipse there is at least one singularity. However, the absorptive part is analytic inside an ellipse with semi-major axis $\operatorname{ch}2\theta_0$. Then if ΔD and ΔA are the differences between two acceptable amplitudes we have

$$2D\,\Delta D = -(2A+\Delta A)\,\Delta A - (\Delta D)^2$$

If ΔD and ΔA are polynomials the right-hand side is regular at the singularity while the left-hand side is not, which is a contradiction. So the realistic case appears in that respect like a continuation of the case of entire functions of finite order ≥ 1.

We have also shown, for arbitrary polynomials and for entire functions of order zero that ambiguities disappear if the cross-section is small enough irrespective of whether or not the amplitude has zeros or sharp minima in the physical region. We believe that this feature will persist in general for arbitrary amplitudes with some analyticity, but we have no explicit proof. This will be the object of future investigations. A summary of our findings is to be found in the Table.

Perhaps we should make one more remark. The reason why the problem is so hard is that we gave ourselves, as a rule of the game not to look at the energy dependence of cross-sections. If fact if one knows the modulus of the amplitude at all energies, then it can be shown, as this has been done by Bessis and Martin [8] in a special case and by Alvarez-Estrada [9] in general that the amplitude is unique. However, this is a typically unstable result because any small uncertainty will spoil it since one appeals to the knowledge of the modulus of a function along the border of its domain of analyticity. This is the reason why we tried to solve the fixed energy problem.

7. ACKNOWLEDGMENTS

We are deeply indebted to Dr. H. Cornille who participated in some parts of this work. We acknowledge stimulating discussions with L. Van Hove, R. Stora and I. Sakmar.

APPENDIX A

Recall the standard integral representations

$$P_\ell(z) = \int_0^{2\pi} \frac{d\psi}{2\pi} \left(z + \sqrt{z^2-1}\cos\psi \right)^\ell$$
$$Q_\ell(z) = \int_0^{\infty} d\psi \left(z + \sqrt{z^2-1}\, \mathrm{ch}\psi \right)^{-\ell-1} \qquad (1)$$

Let x and λ denote real quantities larger than one then $P_\ell(x)$ is real positive and

$$\lambda^\ell \leq \frac{P_\ell(\lambda x)}{P_\ell(x)} \leq P_\ell(\lambda) \leq \left(\lambda + \sqrt{\lambda^2-1}\right)^\ell \qquad \lambda, x \geq 1 \qquad (2)$$

Indeed $xP'_\ell(x)$ is an ℓ^{th} degree polynomial with parity $(-1)^\ell$. Since $x(d/dx)\ x^\ell = \ell x^\ell$ it can be written

$$x P'_\ell(x) = \ell P_\ell(x) + a_2\,(2\ell-3)\,P_{\ell-2}(x) + \dots + a_{2k}\left[2(\ell-2k)+1\right] P_{\ell-2k}(x) + \dots$$

where

$$a_{2k} = \int_{-1}^{+1} \frac{dx}{2}\ P_{\ell-2k}\ x P'_\ell(x) = \frac{x}{2} P_\ell(x) P_{\ell-2k}(x)\Big|_{-1}^{+1} - \int_{-1}^{+1} \frac{dx}{2} P_\ell(x) \frac{d}{dx}\left(x P_{\ell-2k}(x)\right)$$

The second integral vanishes since $[xP_{\ell-2k}(x)]'$ is a polynomial of degree $\ell-2k$ and hence orthogonal to $P_\ell(x)$, while the integrated term is 1. Hence $a_{2k} = 1$. Since all Legendre polynomials are positive for $x > 1$ we have $xP'_\ell(x) > \ell P_\ell(x)$ or $(P'_\ell(x)/P_\ell(x)) > \ell/x$. Integrating this inequality from x to λx yields the first inequality in (2). Moreover the addition theorem yields

$$P_\ell(x)P_\ell(\lambda) = \int_0^\pi \frac{d\psi}{\pi} P_\ell\left(\lambda x + \sqrt{\lambda^2-1}\sqrt{x^2-1}\cos\psi\right) = \int_0^\pi \frac{d\psi}{\pi}\frac{1}{2}\left\{ P_\ell\left(\lambda x + \sqrt{\lambda^2-1}\sqrt{x^2-1}\cos\psi\right) + P_\ell\left(\lambda x - \sqrt{\lambda^2-1}\sqrt{x^2-1}\cos\psi\right)\right\}$$

Now, for an argument larger than one P_ℓ is not only positive but has all its derivatives positive. Consequently it is a convex function and hence the bracket in the last integral is larger than the Legendre polynomial evaluated at the half sum of its arguments: λx. Then the second inequality in (2) follows

$$P_\ell(x)P_\ell(\lambda) \geqslant P_\ell(\lambda x)$$

Finally the last inequality is a trivial consequence of (1).

Let us study the maximum modulus of $P_\ell(z)$ and $Q_\ell(z)$ on an ellipse E_Θ, i.e., $z = \mathrm{ch}(\Theta + i\varphi)$, $\sqrt{z^2-1} = \mathrm{sh}(\Theta + i\varphi)$, Θ fixed. Then

$$|P_\ell(z)| \leqslant \int_0^{2\pi} \frac{d\psi}{2\pi} \left|z+\sqrt{z^2-1}\cos\psi\right|^\ell = \int_0^{2\pi} \frac{d\psi}{2\pi} \left|(\mathrm{ch}\Theta + \mathrm{sh}\Theta\cos\psi)\cos\varphi + i(\mathrm{sh}\Theta + \mathrm{ch}\Theta\cos\psi)\sin\varphi\right|^\ell$$

One has $|\mathrm{sh}\Theta + \mathrm{ch}\Theta\cos\psi| \leq \mathrm{ch}\Theta + \mathrm{sh}\Theta\cos\psi$ since the difference of the squares of the positive right-hand side and of the left-hand side is $(1-\cos^2\psi)$. Hence

$$|P_\ell(\mathrm{ch}(\Theta+i\varphi))| \leqslant \int_0^{2\pi} \frac{d\psi}{2\pi} (\mathrm{ch}\Theta + \mathrm{sh}\Theta\cos\psi)^\ell = P_\ell(\mathrm{ch}\Theta)$$

proving that $|P_\ell(z)|$ attains its maximum value at the extremity of the semi-major axis of the ellipse. Similarly on the same ellipse

$$|Q_\ell(z)| \leqslant \int_0^\infty d\psi \frac{1}{\left|z+\sqrt{z^2-1}\,\mathrm{ch}\psi\right|^{\ell+1}}$$

$$\left|z+\sqrt{z^2-1}\,\mathrm{ch}\psi\right| = \left|(\mathrm{ch}\Theta+\mathrm{sh}\Theta\,\mathrm{ch}\psi)\cos\varphi + i(\mathrm{sh}\Theta + \mathrm{ch}\Theta\,\mathrm{ch}\psi)\sin\varphi\right|$$

with the inequality (recall that by definition Θ is positive)

$$0 < \mathrm{ch}\Theta + \mathrm{sh}\Theta\,\mathrm{ch}\psi < \mathrm{sh}\Theta + \mathrm{ch}\Theta\,\mathrm{ch}\psi$$

which follows by taking the difference $(\mathrm{ch}\Theta - \mathrm{sh}\Theta)(\mathrm{ch}\psi - 1) > 0$. Then

$$|Q_\ell(z)| \leqslant \int_0^\infty d\psi \frac{1}{(\mathrm{ch}\Theta + \mathrm{sh}\Theta\,\mathrm{ch}\psi)^{\ell+1}} = Q_\ell(\mathrm{ch}\Theta)$$

Therefore, the same conclusion holds for $|Q_\ell(z)|$ on E_Θ. It reaches its maximum value at the extremity of the semi-major axis.

Finally we need an estimate of $P_\ell(x)Q_\ell(x)$ for x real and larger than one. In general $P_\ell(z)Q_\ell(z)$ is analytic in the z plane cut from -1 to $+1$. From the known discontinuity of Q_ℓ along the cut

$i\pi P_\ell$ we readily derive from the Cauchy formula

$$P_\ell(z)Q_\ell(z) = \int_{-1}^{+1}\frac{dt}{2}\;\frac{P_\ell(t)^2}{z-t}$$

For x larger than 1 we deduce

$$(2\ell+1)P_\ell(x)Q_\ell(x) = (2\ell+1)\int_{-1}^{+1}dt\;\frac{P_\ell(t)^2}{x-t} \leq \frac{1}{x-1} \tag{3}$$

We close this Appendix by an estimate of the size of the Mandelstam unitarity kernel. Formulas (10) and (11) of Section 2 give the definition of the spherical convolution $f = g*h$. Assume g and h to be analytic functions in the neighbourhood of the physical region. If C denotes a closed curve enclosing the segment $[-1,+1]$ we can write with n_1, n_2, n_3 unit vectors

$$\begin{aligned} f(n_1.n_2) &= \int_C \frac{dz_1}{2\pi i}\,g(z_1)\int_C\frac{dz_2}{2\pi i}\,h(z_2)\int\frac{d^2n_3}{4\pi}\;\frac{1}{(z-n_1n_3)(z-n_2.n_3)} \\ &= \int_C\int_C \frac{dz_1}{2\pi i}\;\frac{dz_2}{2\pi i}\;g(z_1)\,h(z_2)\;\;M(z_1,z_2\,;\,n_1.n_2) \end{aligned} \tag{4}$$

with

$$M(z_1,z_2;n_1.n_2) = \int\frac{d^2n_3}{4\pi}\;\frac{1}{(z_1-n_1.n_3)(z_2-n_2.n_3)} = \int_0^1 d\alpha\int\frac{d^2n_3}{4\pi}\;\frac{1}{\left[\alpha z_1+(1-\alpha)z_2-(\alpha n_1+(1-\alpha)n_2).n_3\right]^2}$$

by using the standard Feynman trick. The integral over n_3 is then trivial and

$$M(z_1,z_2;n_1.n_2) = \int_0^1 d\alpha\;\frac{1}{(\alpha p_1+(1-\alpha)p_2)^2}$$

where p_1 and p_2 are complex four vectors $p_1=(z_1,n_1)$ $p_2=(z_2,n_2)$ so that $p_1^2=z_1^2-1$, $p_2^2=z_2^2-1$, $p_1\cdot p_2 = z_1z_2-n_1\cdot n_2$. Denoting now $n_1\cdot n_2 \equiv z$, the last integral is readily performed

$$M = \frac{1}{\sqrt{p_1^2 p_2^2 - (p_1 \cdot p_2)^2}} \operatorname{arctg} \frac{\sqrt{p_1^2 p_2^2 - (p_1 \cdot p_2)^2}}{p_1 \cdot p_2}$$

$$p_1^2 p_2^2 - (p_1 \cdot p_2)^2 = \left(\sqrt{z_1^2-1}\sqrt{z_2^2-1} - z_1 z_2 + z\right)\left(\sqrt{z_1^2-1}\sqrt{z_2^2-1} + z_1 z_2 - z\right)$$

Define now $z_\pm = z_1 z_2 \pm \sqrt{z_1^2-1}\sqrt{z_2^2-1}$

$$M(z_1, z_2; z) = \frac{1}{2i\sqrt{(z_+-z)(z-z_-)}} \log \frac{z_1 z_2 - z + i\sqrt{(z_+-z)(z-z_-)}}{z_1 z_2 - z - i\sqrt{(z_+-z)(z-z_-)}} \tag{5}$$

$$= \frac{1}{i\sqrt{(z_+-z)(z-z_-)}} \log \frac{\sqrt{z_+-z} + i\sqrt{z-z_-}}{\sqrt{z_+-z} - i\sqrt{z-z_-}}$$

Notice now that if the curve C is chosen to be an ellipse E_θ, we have

$$z_{1,2} = \operatorname{ch}(\theta + i\varphi_{1,2}) \qquad z_+ = \operatorname{ch}(2\theta + i\varphi_1 + i\varphi_2) \tag{6}$$
$$z_- = \cos(\varphi_1 - \varphi_2)$$

The apparent singularity at $z = z_-$ is spurious (the argument of the logarithm is such that on the sheet of interest for $z_1 = z_2$ real greater than 1 and z real and smaller than $\sqrt{2z_1^2-1}$ the kernel M is real). Then (4) and (5) show that if g and h are analytic inside an ellipse $E_{2\theta}$, $f = g*h$ is analytic inside a domain bounded by the ellipse $E_{2\theta}$. The phenomenon is of course well known in the context of the analytic S matrix where the ellipses correspond to the saml1 and large Lehman ellipse.

For our purposes it is necessary to have a bound on $|M(z_1, z_2; z)|$ when z reaches the point $x = \operatorname{ch}2\theta$ and z_1 and z_2 have the parametrization given by Eq. (6), i.e., run along the ellipse E_θ.

The choice of the phases of the square roots is readily seen to be immaterial. What is relevant however is to follow the imaginary part of the logarithm and make sure one does not change it by an amount 2π.

We see that z_- is confined to the segment $-1, +1$ while z_+ runs along the ellipse $E_{2\theta}$. With $z = x = ch2\theta$, $\sqrt{z_+ - x}$ is on a curve (L) with equation given by

$$\left(\frac{X^2-Y^2}{ch2\theta}+1\right)^2 + \frac{4X^2Y^2}{sh^2 2\theta} = 1$$

As z_+ circles around one of the arcs of this curve the argument inside the logarithm in M has a phase which varies in an interval $-\pi$ to $+\pi$ [at both ends the factor $1/i\sqrt{(z_+-z)(z-z_-)}$ however also changes sign hence the two extreme values of M are equal] and one can also check that just before z reaches $E_{2\theta}$ this range is $-\pi+\epsilon$, $\pi-\epsilon$. The modulus of the argument $|\sqrt{z_+-z}+i\sqrt{z-z_-}/\sqrt{z_+-z}-i\sqrt{z-z_-}|$ is certainly bounded by the ratio of the largest to the smallest distance of the point $M(x_o = 0,\ y_o = \sqrt{ch2\theta - z_-})$ to the curve L. Now as θ becomes very large the quantity z_- will become negligible as compared to $ch2\theta$ and the curve L will become very much alike a true lemniscate (Fig. 3)

$$\left(\frac{X^2-Y^2}{x}+1\right)^2 + 4\frac{X^2Y^2}{x^2} = 1 \qquad (\tilde{L})$$

the ratio of the largest to smallest distance of the point $\tilde{M}(\tilde{x}_o = 0,\ \tilde{y}_o = \sqrt{x})$ being x independent since x and y as well as $\tilde{x}_o$ $\tilde{y}_o$ scale as $\sqrt{x}$. Geometrically $(\tilde{L})$ has the foci $\pm\sqrt{x}$ along the y axis i.e., the points $\tilde{M}$ and its symmetric with the product of the distances of a point on the curve to $\tilde{M}$ and $-\tilde{M}$ being constant and equal to x. Then a point at minimal distance from $\tilde{M}$ will necessarily be at maximal distance from $-\tilde{M}$. Hence the lines joining it to $\tilde{M}$ and $-\tilde{M}$ will both be orthogonal to the tangent at the curve at this point. This is achieved for the points $\tilde{M}_+$ and $\tilde{M}_-$ on the symmetry axis through the foci the ratio being then $\sqrt{2}+1/\sqrt{2}-1$.

To be on the safe side there exists then some value θ_o such that for $\theta > \theta_o$ the ratio will be smaller than twice this value, say. We have therefore with z_1 and z_2 on E_θ

$$|M(z_1, z_2; ch2\theta)| \leq \frac{1}{|\sqrt{(z_+ - ch2\theta)(ch2\theta - z_-)}|}\sqrt{\pi^2 + \left(\log 2\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^2} \qquad \theta > \theta_o$$

We have given above a complete majorization of $|M(z_1 z_2, x)|$ when x is the extremity of the semi-major axis of the ellipse $E_{2\Theta}$. Since when Θ gets very large the ellipse gets closer and closer to a circle a very similar bound will apply when x is replaced by any point on E_Θ. Indeed if E_Θ were replaced by a circle then the problem would become rotationally invariant.

APPENDIX B

This Appendix is a collection of facts on entire functions. It is partly based on the book "Entire Functions" by Boas [10], and partly presents a number of extra results established for our own needs.

Let $f(z)$ be entire and

$$\mathcal{M}_f(x) = \sup_{|z|<x} |f(z)| \tag{1}$$

The order of f is defined as

$$\rho_f = \overline{\lim_{x\to\infty}} \frac{\log[\log \mathcal{M}_f(x)]}{\log x} \tag{2}$$

and if ρ is finite and positive the type τ is

$$\tau_f = \overline{\lim_{x\to\infty}} \; x^{-\rho} \log \mathcal{M}_f(x) \tag{3}$$

In the sequel we assume $f(z)$ such that $f(0) = 1$ unless otherwise stated and drop the index f. Let $n(x)$ equal the number of zeros (counted with their multiplicities) of $f(z)$ inside the circle $|z| \leq x$ one defines:

$$N(x) = \int_0^x n(t)\,\frac{dt}{t} \tag{4}$$

[note that $n(t)$ has to vanish for t small enough]. Then

$$N(x) \leqslant \log \mathcal{M}(x) \tag{5}$$

If furthermore $f(z)$ is of order smaller than 1 then one has also a bound in the other direction

$$N(x) \leqslant \log \mathcal{M}(x) \leqslant x \int_0^\infty dt\, \frac{N(t)}{(t+x)^2} \tag{6}$$

Since for all $\epsilon > 0$ there exist constants C_ϵ and C'_ϵ such that

$$\log \mathcal{M}(x) < C_\epsilon x^{\rho+\epsilon} + C'_\epsilon$$

for $\rho < 1$, the right-hand side in (6) will converge.

Let z_k denote the zeros of an entire function f of finite order. The smallest positive quantity α for which the series

$$\sum_k |z_k|^{-\alpha}$$

converges will be called ρ_1. The smallest <u>positive</u> integer for which the series converges is called $p+1$, $p = 0, 1, \ldots$ If ρ_1 is not an integer $p = [\rho_1]$, if ρ_1 is a positive integer $p = \rho_1 - 1$, if $\rho_1 = 0$ $p = 0$.

Define furthermore

$$E(u,p) = (1-u)\, e^{u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}} \qquad (7)$$

with the exponential replaced by 1 if $p = 0$.

Then $\prod_k E((z/z_k), p)$ is an entire function of order ρ_1 and

$$f(z) = e^{P_n(z)} \prod_k E\left(\frac{z}{z_k}, p\right) \qquad (8)$$

with P_n a polynomial of degree $n \le \rho$ where ρ is the order of f and $\rho_1 \le \rho$. Summarizing

$$\begin{array}{llll} \rho = \rho_1 & p = [\rho_1] & n \leq p & \rho_1 \text{ not integer} \\ \rho \text{ integer} = \operatorname{Sup}(\rho_1, n) & p = \rho_1 - n & & \rho_1 \text{ integer} \geq 1 \\ \quad \text{or } \rho = n > \rho_1 & p = [\rho_1] & & \rho_1 \text{ not integer} \\ \rho \text{ integer} = \operatorname{Sup}(\rho_1, n) & p = 0 & & \rho_1 = 0 \end{array} \qquad (9)$$

If among the zeros z_k, we delete a number of them, then ρ_1 can only decrease and the product $\prod' E((z/z_k), p)$ will at most be a function of order ρ_1, a fact used in the text.

Let us prove the following lemma

Lemma 1 If Θ is some fixed angle and the entire function f is not a constant

$$\overline{\lim_{x\to\infty}} \left| f(x e^{i\theta}) \right| \mathcal{M}_f(\lambda x) = \infty \tag{10}$$

for every $\lambda > 1/(3-2\sqrt{2})$.

It is possible that our estimate of λ is not the best possible and as would be suggested by the behaviour of polynomials (10) holds for any $\lambda > 1$. In any case the proof goes as follows. By rotation $z \to ze^{-i\Theta}$ we can assume $\Theta = 0$. Assume (10) to be wrong then for x large enough, dropping the index f on $\mathcal{M}_f(x)$

$$f(x)\, \mathcal{M}(\lambda x) \leq C \qquad\qquad x \geq x_0$$

or

$$f(x) \leq \frac{C}{\mathcal{M}(\lambda x)} \leq \frac{C}{\mathcal{M}(\lambda x_0)} \qquad x \geq x_0 \ .$$

Inside the circle $|z| = \lambda x_0$ the function $f(z)$ has now the property that on the boundary circle $|f(z)| \leq \mathcal{M}(\lambda x_0)$ and on the segment $[x_0, \lambda x_0]$, $|f(x)| \leq C/\mathcal{M}(\lambda x_0)$. Consider the map

$$z \to \eta = \sqrt{\frac{z - x_0}{x_0\lambda - z/\lambda}}$$

The interior of the circle $|z| \leq \lambda x_0$ cut along the real axis from x_0 to $\lambda\, x_0$ is mapped upon the interior of the half circle $|\eta| \leq 1$ $\mathrm{Im}\,\eta > 0$. The points M_0, M_1, M_2 from Fig. 4a are sent on $\widetilde{M}_0$, $\widetilde{M}_1$, $\widetilde{M}_2$ on Fig. 4b. Set $\tilde{f}(\eta) = f(z)$, then on the real axis $|\tilde{f}| < C/\mathcal{M}(\lambda x_0)$ and on $|\eta| = 1$, $|\tilde{f}| < \mathcal{M}(\lambda x_0)$. An easy generalization of the Hadamard three circles theorem says then that if we consider the region bounded by the real axis and an arc of circle passing through the points $\eta = \pm 1$ and making an angle $\sigma\,\pi/2$ with the real axis one has

$$\left| \tilde{f}(\eta) \right| \leq \mathcal{M}(\lambda x_0)^{\sigma} \left(\frac{C}{\mathcal{M}(\lambda x_0)} \right)^{1-\sigma} \tag{11}$$

Choosing $\sigma = \frac{1}{2}$ one will find $|\tilde{f}(\eta)| \leq \sqrt{C}$ inside the shaded area (Fig. 4b) which intersects the imaginary η axis at the point $\sqrt{2}-1$. Assume that $\sqrt{2}-1 > 1/\sqrt{\lambda}$ corresponding to the ordinate of $\tilde{M}_0$. Then in the original plane the region where $|f(z)| \leq \sqrt{C}$ will enclose the origin. As x_0 can be made to increase without bound this region will finally cover the whole z plane. That would say that $f(z)$ is bounded by a constant and from Liouville theorem be a constant. If it is not then we have found a contradiction and (10) must hold. The condition $\sqrt{2}-1 > 1/\sqrt{\lambda}$ also reads

$$\lambda > \frac{1}{(\sqrt{2}-1)^2} = \frac{1}{3-2\sqrt{2}}$$

For functions of order zero we need a further result which can be stated as follows

<u>Lemma 2</u> If f and g, and hence fg are entire functions of order zero such that for x large enough, $x > x_0$

$$\mathfrak{M}_{fg}(x^p) \leq K\left(\mathfrak{M}_f(x)\,\mathfrak{M}_g(x)\right)^q \qquad q > p > 1 \tag{12}$$

then for $x > x_0$ for any $\epsilon > 0$ there exist constants A_ϵ, B_ϵ such that

$$\frac{\mathfrak{M}_f(x)\,\mathfrak{M}_g(x)}{\mathfrak{M}_{fg}(x)} \leq A_\epsilon\, e^{B_\epsilon [\log x]^{\left(\frac{\log q}{\log p} + \epsilon - 1\right)}} \tag{13}$$

In Section 4 this is used for $f = \Delta D$, $g = \Sigma D$, $p = 2$, $q = 3$ [see Eq. (7), Section 4] and Eq. (13) yields the inequality quoted in Eq. (8), Section 4.

Set

$$f(z) = a z^n F(z)$$
$$g(z) = b z^m G(z)$$

n, m being the order of a possible zero of f and g respectively at the origin and A and B being chosen in such a way that $F(0) = G(0) = 1$. Then

$$m_{FG}(x^p) \leq K\,|ab|^{q-1}\, x^{(n+m)(q-p)} \left(m_F(x)\, m_G(x)\right)^q$$

With $K' = \log K\,|ab|^{q-1}$ and taking the logarithms:

$$\log m_{FG}(x^p) \leq K' + (n+m)(q-p)\log x + q\left(\log m_F(x) + \log m_G(x)\right) \tag{14}$$

it is understood in the sequel that all inequalities are valid for $x > x_0$. The set of zeros of F, G being the union of the ones pertaining to F and G the functions n, N defined above are additive [see Eq. (4)]

$$n_{FG}(x) = n_F(x) + n_G(x) \equiv n(x)$$
$$N_{FG}(x) = N_F(x) + N_G(x) \equiv N(x)$$

Since we deal with entire functions of order zero we can combine the inequalities in (6) and (14):

$$N(x^p) \leq \log m_{FG}(x^p) \leq K' + (m+n)(q-p)\log x + q\int_0^\infty \frac{dt}{(1+t)^2} N(xt) \tag{15}$$

We remark that since $n(x)$ is a non-decreasing function the definition (4) of $N(x)$ implies that it is a convex function of $\log x$. In other words if $0 < \lambda < 1$

$$N(xt) = N\left(\left(x^{\frac{1}{\lambda}}\right)^{\lambda}\left(t^{\frac{1}{1-\lambda}}\right)^{1-\lambda}\right) \leq \lambda N\left(x^{\frac{1}{\lambda}}\right) + (1-\lambda) N\left(t^{\frac{1}{1-\lambda}}\right)$$

Moreover $N(t)$ vanishes in a neighbourhood $0 \leq t \leq t_0$ of $t = 0$ and is bounded by $At^{\epsilon} + B$ with some constants depending on ϵ for every positive ϵ as follows from (3) and the fact that we deal with functions of order zero. Given λ there is always a choice of ϵ such that the integral

$$\int_0^\infty \frac{dt}{(1+t)^2} N(t^{\frac{1}{1-\lambda}})$$

converges. Hence no matter how close λ is from unity there exists a constant $K''(\lambda)$ such that

$$N(x^p) \leq K''(\lambda) + (m+n)(q-p)\log x + \lambda q\, N(x^{\frac{1}{\lambda}})$$

Write

$$N(e^u) = \nu(u) - (m+n)u - \frac{K''}{\lambda q - 1}$$

Expressed in terms of ν the previous inequality (for $u > u_0$) gives

$$\nu(\lambda p u) \leq \lambda q\, \nu(u)$$

Dividing both sides by $\lambda q(u)^{(\log \lambda q/\log \lambda p)}$ this yields

$$\frac{\nu(\lambda p u)}{(\lambda p u)^{\frac{\log \lambda q}{\log \lambda p}}} \leq \frac{\nu(u)}{(u)^{\frac{\log \lambda q}{\log \lambda p}}} \qquad (16)$$

Notice that given $\epsilon > 0$ sufficiently small by choosing λ sufficiently close to one $\log \lambda q/\log \lambda p = (\log q/\log p) + \epsilon > 1$ and $\lambda p > 1$.

Since

$$\frac{\nu(u)}{(u)^{\frac{\log q}{\log p} + \epsilon}}$$

is a continuous function and $\lambda p > 1$ the inequality (16) means that for $u > u_0$

$$\nu(u) \leq K''' u^{\frac{\log q}{\log p} + \epsilon}$$

or

$$N(e^u) \leq K''' u^{\frac{\log q}{\log p}+\epsilon} - (m+n)u - \frac{K''}{\lambda q-1}$$
$$\leq A_\epsilon \, u^{\frac{\log q}{\log p}+\epsilon} + B_\epsilon \tag{17}$$

With this in hand we return to (15), we can majorize

$$\int_0^\infty \frac{dt}{(1+t)^2} N(xt) = \int_0^\infty \frac{dt}{t(1+t)} n(xt)$$

by splitting the last integral in $\int_0^1 + \int_1^\infty$ and replacing in the first term $1/t(1+t)$ by $1/t$ and in the second $1/t(1+t)$ by $1/t^2$

$$\int_0^\infty \frac{dt}{(1+t)^2} N(xt) \leq \int_0^1 \frac{dt}{t} n(xt) + \int_1^\infty \frac{dt}{t^2} n(xt) = N(x) + Q(x) \tag{18}$$

Next we obtain a bound on n by using

$$n(u) \log u = n(u) \int_u^{u^2} \frac{dv}{v} \leq \int_u^{u^2} n(v) \frac{dv}{v} \leq N(u^2)$$

Using (17) this gives

$$n(u) \leq A'_\epsilon (\log u)^{\frac{\log q}{\log p}+\epsilon-1} + B'_\epsilon$$

and

$$Q(x) \leq B'_\epsilon + A'_\epsilon x \int_x^\infty \frac{dt}{t^2} (\log t)^{\frac{\log q}{\log p}+\epsilon-1} = B'_\epsilon + A'_\epsilon x \int_{\log x}^\infty du\, e^{-u} (u)^{\frac{\log q}{\log p}+\epsilon-1}$$

The last integral is an incomplete Euler function

$$\Lambda_\alpha(x) = \int_{\log x}^\infty du\, e^{-u} u^{\alpha-1} \qquad \alpha > 1$$

It can always be assumed that $\log x > 1$ and the reader will readily see that under these conditions there exists a constant C_α such that

$$\Lambda_\alpha(x) \leq C_\alpha \frac{(\log x)^{\alpha-1}}{x}$$

Finally, then

$$Q(x) \leq B'_\epsilon + A''_\epsilon (\log x)^{\frac{\log q}{\log p}+\epsilon-1} \qquad (19)$$

We now have the sequence of inequalities which uses (6) and (18)

$$\log \mathfrak{M}_{FG}(x) \leq N(x)+Q(x) = N_F(x)+N_G(x)+Q(x) \leq \log\left(\mathfrak{M}_F(x)\,\mathfrak{M}_G(x)\right) + Q(x)$$

That is

$$\log \frac{\mathfrak{M}_{FG}(x)}{\mathfrak{M}_F(x)\,\mathfrak{M}_G(x)} \leq Q(x) \leq B'_\epsilon + A''_\epsilon (\log x)^{\frac{\log q}{\log p}+\epsilon-1}$$

Moreover

$$\frac{\mathfrak{M}_{FG}(x)}{\mathfrak{M}_F(x)\,\mathfrak{M}_G(x)} = \frac{\mathfrak{M}_{fg}(x)}{\mathfrak{M}_f(x)\,\mathfrak{M}_g(x)}$$

the common factor $|ab|x^{m+n}$ dropping from the numerator and the denominator. With A_ϵ and B_ϵ denoting new constants we have reached the required inequality

$$\frac{\mathfrak{M}_{fg}(x)}{\mathfrak{M}_f(x)\,\mathfrak{M}_g(x)} \leq A_\epsilon\, e^{B_\epsilon (\log x)^{\frac{\log q}{\log p}+\epsilon-1}}$$

	Polynomials				Entire functions of order zero	Entire functions of non-integer order	Entire functions of order n $n = 1, 2 \ldots$	Entire functions of infinite order	Functions analytic in the neighbourhood of the physical region but with singularities in the complex $\cos\theta$ plane
	$L = 1$	$L = 2$	$L = 3$	$L > 3$					
Possible existence of more than one amplitude	no	yes	yes	yes	yes	no	yes	?	?
More than two amplitudes excluded		yes	yes	?	yes		yes	?	?
Twofold ambiguity on closed non-interesting analytic curves		yes	yes	?	?		?	?	?
cross-section < 1.38 entails no ambiguity		yes	yes	yes	yes		?	?	?
difference between two amplitudes	polynomials				polynomials		non-polynomial	non-polynomial	non-polynomial

Table : Summary of our results and of some remaining problems to be solved.

REFERENCES

1) J.M. Crichton, Nuovo Cimento 45A, 256 (1966); see also D. Atkinson, P.W. Johnson, N. Mehta and M. de Roo, Groningen preprint (1972).

2) F.A. Berends and S.N.M. Ruijsenaars, Leiden preprint (1972).

3) A. Martin, Nuovo Cimento 59A, 131 (1969).

4) R.G. Newton, J.Math.Phys. 9, 2050 (1968).

5) D. Atkinson, P.W. Johnson and R.L. Warnock, Commun.Math.Phys. 28, 133 (1972).

6) D. Atkinson, G. Mahoux and F.J. Yndurain, Saclay and Groningen preprints (1972).

7) H. Burckhardt, Nuovo Cimento 10A, 379 (1972).

8) A. Martin and D. Bessis, Nuovo Cimento 52A, 719 (1967).

9) R.F. Alvarez-Estrada, Ann.Phys. (N.Y.) 68, 222 (1971).

10) R.P. Boas, "Entire Functions", Academic Press, New York (1954).

FIGURE CAPTIONS

Figure 1 :

Relation between the highest partial wave amplitude and the first different partial waves in the case where the amplitudes are polynomial.

Figure 2 :

Graphical display of the notations used in Section 5 to discuss the minimal cross-section.

Figure 3 :

The lemniscate

$$\left(\frac{X^2 - Y^2}{x} + 1 \right)^2 + 4 \frac{X^2 Y^2}{x^2} = 1$$

with foci $\pm \widetilde{M}$.

Figure 4 :

The original z plane and its transform the η plane used to prove Eq. (10), Appendix B.

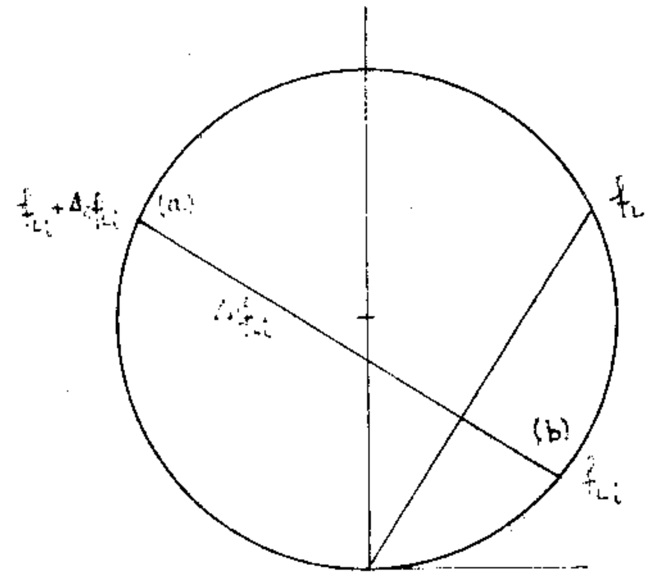


fig1

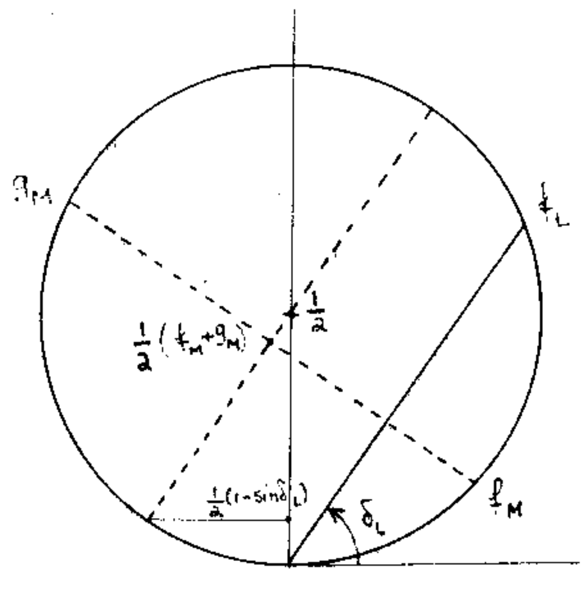


fig2

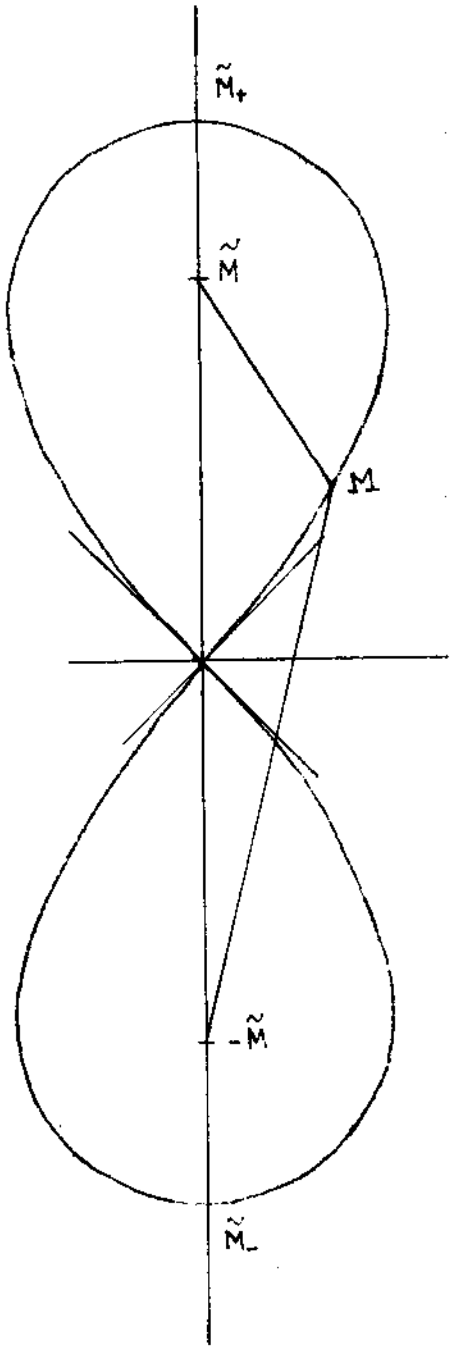
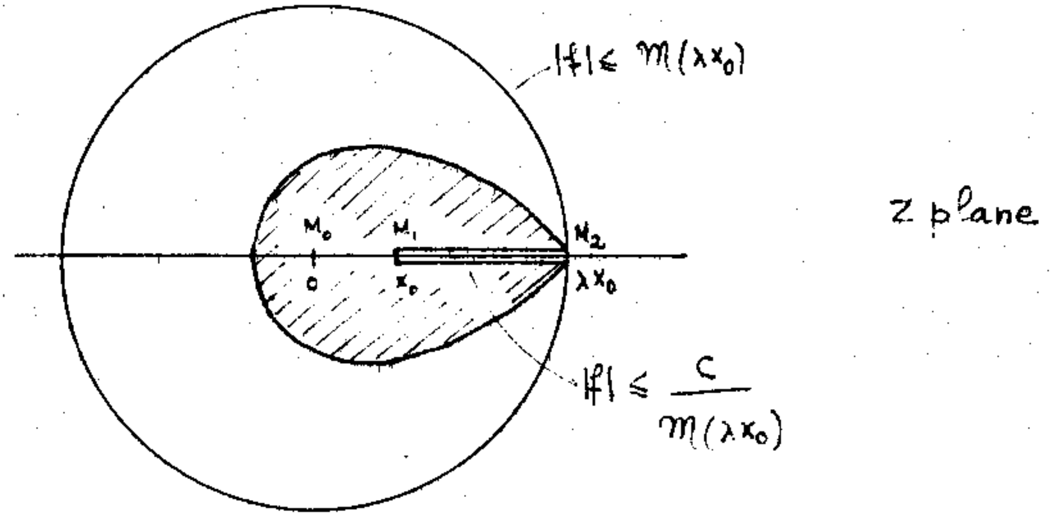
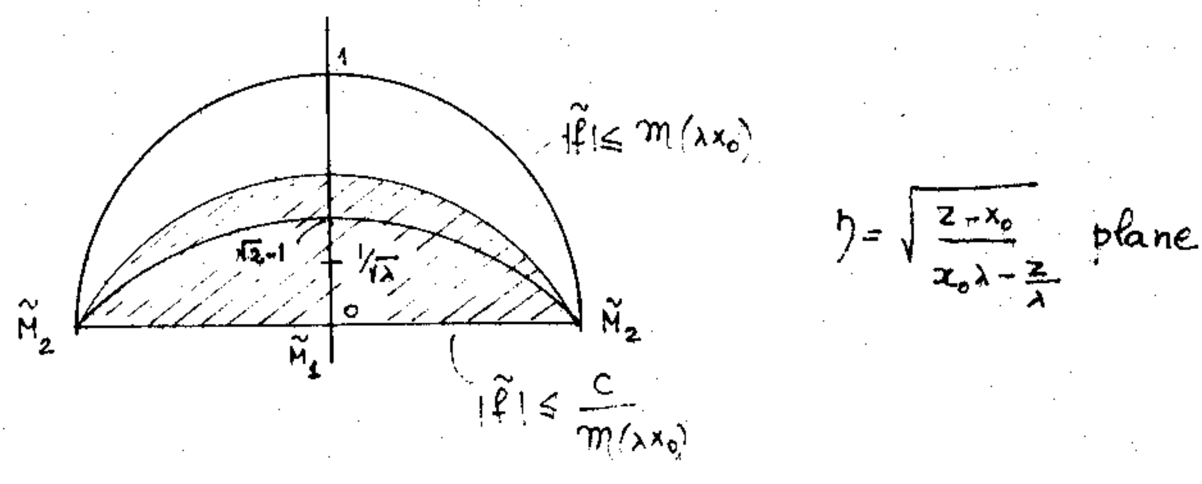


fig3



$|f| \leq m(\lambda x_0)$
M_0
M_1
M_2
0
x_0
λx_0
$|f| \leq \frac{c}{m(\lambda x_0)}$
z plane

fig 4 (a)



1
$|\tilde{f}| \leq m(\lambda x_0)$
$\sqrt{2}-1$
$1/\sqrt{\lambda}$
0
$\tilde{M}_2$
$\tilde{M}_1$
$\tilde{M}_2$
$|\tilde{f}| \leq \frac{c}{m(\lambda x_0)}$
$\eta = \sqrt{\frac{z - x_0}{x_0 \lambda - \frac{z}{\lambda}}}$ plane

fig4(b)