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Phase Space Cell Expansion and Borel Summability for the Euclidean φ_3^4 Theory

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Abstract. The stability of the free energy is proved for complex values of the coupling constant by the way of a convergent expansion. As a consequence, one obtains the Borel summability of the perturbation series.

I. Introduction

I.1.

Since the proof by Glimm and Jaffe [1] of the positivity of the φ^4 Hamiltonian in three dimensions, many other results have enlarged our knowledge on this model. Its present status is quite similar to that of $P(\varphi)_2$ theories ([2, 3]): the Wightman axioms with mass gap were proven by Feldman and Osterwalder [4] and Magnen and Sénéor [5] for the weak coupling region, the existence of phase transitions were shown by Fröhlich et al. [6], also Park has shown the convergence of the lattice approximation [7], and Burnap has investigated the particle structure [12]. To complete the analogy with two dimensional theories and in particular with the corresponding φ_2^4 theory, one would like to know the connection of the φ_3^4 theory with its perturbation expansion. From this arises the question of extending the proof of [1] to complex values of the coupling constant. On the other hand, a simplification of the proof given in [1] can allow a better understanding of the ideas introduced and their application to other types of problems. This article presents a contribution in these two directions. In particular, we solve the first problem and prove as a by-product the Borel summability of the theory and its uniqueness with respect to perturbation theory. It is left to the reader to judge if our modified proof of [1] is simple enough!

We want at this point to emphasize the fact that all the results quoted above were obtained using Euclidean space methods. The control of the φ_3^4 theory goes

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through a bound on the partition function $Z_{\lambda,\Lambda,\kappa}$ associated with an interaction $V_{\lambda,\Lambda,\kappa} \equiv \lambda \int_{\Lambda} :\varphi_\kappa^4: +$ counterterms where λ is the coupling constant, φ_κ a cutoff Euclidean field and Λ the volume of localization of the interaction. In [1] it was proved that for suitable counterterms, $\lambda \geqq 0$

$$|Z_{\lambda,\Lambda,\kappa}| \leqq e^{C|\Lambda|} \tag{1.1.1}$$

for some constant C independent of κ, $|\Lambda|$ being the volume of Λ. From the methods involved in proving such an estimate, Feldman [8] proved bounds on the finite volume Green's functions (the so-called Schwinger functions), which were extended (in [4] and [5]) to the infinite volume case. The restriction λ real is a deep consequence of the methods used in [1] and of the nonlinearity in λ of the counterterms in the interaction.

The modification we propose keeps in essence the arguments used in [1] in the framework of an algebraic convergent expansion. The expansion contains no intermediate bounds. This allows the extension of the proof to complex λ.

I.2. Schematic Description of the Expansion

The introduction of functional techniques simplifies considerably the exposition. The Euclidean field φ is considered as a (generalized) Gaussian random variable and the partition function takes the form

$$Z_{\lambda,\Lambda,\kappa} \equiv \int_{\mathscr{S}'} e^{-V(\lambda,\Lambda,\kappa)} d\mu_C$$

where V is the Euclidean interaction and $d\mu_C$ is a measure associated with the field φ (see the next chapter for more details). The formal expansion of the exponential generates the (formal) Euclidean perturbation expansion which is known to diverge.

We now give a schematic description of the expansion. The expansion is based upon a truncated perturbation expansion of the exponential. The truncations are necessary to keep control over the numbers of terms produced (to be compared with their increasing convergence: a feature proper to the superrenormalizable character of φ_3^4). These expansions are performed in cells of phase space. These cells are defined by giving momentum intervals defining upper and lower cutoffs for the fields and cubes Δ of smaller and smaller sizes forming uniform covers of $\mathbb{R}^3$. For a given size of the cubes, the perturbation expansion tends to lower the upper cutoff of the fields in the exponent. The cubes for which the upper cutoff in the exponent has not been lowered enough are then subdivided into cubes from another cover and then the perturbation expansions are applied in the smaller cubes.

For fields of upper momentum cutoff M_u and localized in Δ, the interaction $V(\Delta, M_u)$ is bounded from below by $-O(1)M_u^2|\Delta|$ for some positive constant $O(1)$ and thus

$$e^{-V(\Delta,M_u)} \leqq e^{O(1)M_u^2|\Delta|}.$$

The purpose of the expansion is to arrive at cells such that $M_u^2|\Delta|\leqq 1$ (saturation of the Wick bound) so that the exponential is bounded by a constant.

The control of the resummation of the expansions is obtained by the fact that each high momentum perturbation of the exponential produces, after some renormalization cancellations, a small factor. But this only occurs if the perturbated terms produced during the expansion (for instance, $a:\varphi^4:$ term) have at least one of their fields which is properly localized (i.e. $M_\ell^3 \geqq |\Delta|^{-1}$, M_ℓ being the lower cutoff and Δ the localization cube: see [1] and Chapter II.2 for the importance of this notion). Therefore the perturbation expansion has to be stopped for an upper cutoff somewhere between $|\Delta|^{-1/2}$ and $|\Delta|^{-1/3}$ (for φ_4^4 both reduce to $|\Delta|^{-1/4}$).

A general term in the expansion is then estimated using the Wick bound to control the exponential and using graphs estimates to bound what remains. However each perturbed term contains localized fields with some range of momenta corresponding to an improper localization. Control of the number of terms produced by the Gaussian integration of these fields is lost if the cubes in which they are localized are too small. To avoid that, one has to use the fact that the integration measure (the free measure $d\mu_C$ times the exponential) converges faster than the Gaussian one (it corresponds to a quartic and not a quadratic interaction), i.e. one dominates the improperly localized parts of the field using the formal positivity of the φ^4 interaction. It is really after this procedure that one uses the Wick bounds and the estimates on properly localized fields after a Gaussian integration.

On the technical level, this expansion is only a reordering of the methods and of the concepts introduced in [1]. Most of the estimates are proved in [1], and we do not repeat them.

As a result of the modification introduced, we have been able to eliminate all the regularity conditions of [1]. This allows us to give a purely *algebraic* definition of the expansion, the main difficulty being the proof of a bound in the general term ensuring the convergence. At this level, it seems to us that the method of the combinatoric factors is the only one practicable. We apply it following the lines of [1]. In the same spirit we have extensively used the notation $O(1)$ for any finite positive constant larger than 1 and independent (if not specified otherwise) of the parameters of the theory (the mass m of the free scalar field is not considered a parameter).

I.3. The Main Results

Our results are

Theorem I. *Let* $\lambda\in\mathbb{C}$, $|\mathrm{Arg}\,\lambda|<\pi/2$, *then there exists* $C(\lambda)\geqq 0$ *such that*

$$\left|\int e^{-V(\lambda,\Lambda,\kappa)}d\mu\right|\leqq e^{C(\lambda)|\Lambda|}.$$

Theorem II. *In the region weak coupling region (λ/m small enough) the Schwinger functions of the theory satisfy the Osterwalder-Schrader axioms with an exponential* *clustering. They can be analytically continued in λ (in the domain of Theorem I) and are C^∞ at $\lambda=0$.*

Theorem III. *Under the above conditions, the perturbation series of these Schwinger functions at $\lambda=0$ are Borel summable.*

Theorem II has been proved from Theorem I in [4] and [5], and we do not repeat the proof. A proof of Theorem III is sketched in the Appendix (a more detailed proof is given in [9]). Theorem I is proved in the next two chapters. The cutoff theory is defined in Section II.1. The phase cell expansion is given by formula (2.2.4) and is explained in Sections II.2 and II.4. A general term (2.3.6) of the expansion is bounded by Proposition 2.3.2 which is the main proposition. The proof of Theorem I follows directly from it (Section II.3). Proposition 2.3.2 is proved in Chapter III. In Sections III.1 and III.2 we bound the general term as in formula (3.3.1) using positivity arguments. The number of terms produced by the expansion is given in Sections III.2 and III.4 by Lemma 3.3.1 and Proposition 3.4.1. The final bound on each term of the expansion is given in Section III.5 by Proposition 3.5.2, and we show in Section III.6, Proposition 3.6.1, that this bound implies a small factor for each perturbation step. Proposition 2.3.2 then follows from Propositions 3.6.1 and 3.4.1.

II. The Expansion

II.1. Introduction

Let φ (the "field") be the Gaussian process of mean zero and covariance

$$C(x-y)=(-\Delta+m^2)^{-1}(x,y)=\frac{1}{(2\pi)^3}\int\frac{e^{ik(x-y)}}{k^2+m^2}\,d^3k$$

for $x,y\in\mathbb{R}^3$. Let $d\mu$ be the associated Gaussian measure and let $\kappa\in\mathcal{S}(\mathbb{R}^3)$ be some cutoff and $\varphi_{(\kappa)}=\varphi*\kappa$ be the associated cutoff field.

The renormalized Euclidean interaction (action) in the volume Λ is given by

$$\begin{aligned}V(\lambda;\Lambda;\varphi_{(\kappa)})\equiv{}&\lambda\int{:}\varphi^4_{(\kappa)}{:}(x)\chi_\Lambda(x)d^3x-\frac{\lambda^2}{2}\int\delta m^2(x){:}\varphi^2_{(\kappa)}{:}(x)\chi_\Lambda(x)d^3x\\&+\frac{\lambda^2}{2}\int\left(\int{:}\varphi^4_{(\kappa)}{:}(x)\chi_\Lambda(x)d^3x\right)^2d\mu\\&-\frac{\lambda^3}{3!}\int\left({:}\varphi^4_{(\kappa)}{:}(x)\chi_\Lambda(x)d^3x\right)^3d\mu\end{aligned}\tag{2.1.1}$$

with χ_Λ the characteristic function of the volume Λ,

$$\delta m^2(x)=-4^2\int\left({:}\varphi^3_{(\kappa)}{:}(x)\int{:}\varphi^3_{(\kappa)}{:}(y)d^3y\right)d\mu$$

and

$$:\varphi^4_{(\kappa)}:(x) = (\varphi_\kappa(x))^4 - 6\langle\varphi_\kappa(x)\varphi_\kappa(x)\rangle(\varphi_{(\kappa)}(x))^2 + 3\langle\varphi_\kappa(x)\varphi_\kappa(x)\rangle^2$$

$$:\varphi^3_{(\kappa)}:(x) = (\varphi_\kappa(x))^3 - 3\langle\varphi_\kappa(x)\varphi_\kappa(x)\rangle\varphi_\kappa(x)$$

$$:\varphi^2_{(\kappa)}:(x) = (\varphi_\kappa(x))^2 - \langle\varphi_\kappa(x)\varphi_\kappa(x)\rangle$$

$$\langle\varphi_\kappa(x)\varphi_\kappa(y)\rangle = \int \varphi_\kappa(x)\varphi_\kappa(y)d\mu\,.$$

The partition function $Z_{\lambda,\Lambda,\kappa}$ is then given by

$$Z_{\lambda,\Lambda,\kappa} \equiv \int e^{-V(\lambda;\Lambda;\varphi_{(\kappa)})}d\mu\,.$$

The structure of phase space cells is given by two sequences of nonnegative numbers

$$M_0 = 0, M_1, M_2, \ldots, M_i, \ldots \qquad M_{i-1} < M_i \qquad i \in \mathbb{Z}^+$$

$$|\Delta_0| = 1, |\Delta_1|, \ldots, |\Delta_j|, \ldots \qquad |\Delta_{j-1}| > |\Delta_i| \qquad j \in \mathbb{Z}^+\,.$$

The first sequence is associated with a partition of the momentum range and a decomposition of the field φ as

$$\varphi = \sum \varphi_i\,,$$

where φ_i is, roughly speaking, a field of high momentum M_i and low momentum M_{i-1}.

More precisely, let $\eta \in C^\infty(R)$ be such that

$$\begin{aligned} \eta(\varrho) &= 1 \quad \text{for} \quad |\varrho| \leqq \tfrac{1}{2} \\ 0 \leqq \eta(\varrho) &\leqq 1 \quad \text{for} \quad \tfrac{1}{2} < |\varrho| \leqq 2 \\ \eta(\varrho) &= 0 \quad \text{for} \quad 2 < |\varrho|\,. \end{aligned}$$

Then one defines

$$\eta_j(k) = \prod_{\ell=0}^{2} \eta(k^{(\ell)}/M_j) \quad \text{for} \quad k \equiv (k^{(0)}, k^{(1)}, k^{(2)}) \in \mathbb{R}^3$$

and

$$\varphi_i \equiv \varphi * (\tilde{\eta}_i - \tilde{\eta}_{i-1})$$

where $\tilde{}$ indicates the Fourier transform.

The second sequence is associated with a set of partitions $\{\mathcal{D}_\ell\}$ of $\mathbb{R}^3$. Each $\mathcal{D}_\ell$ is a cover of $\mathbb{R}^3$ with cubes Δ_ℓ of volume $|\Delta_\ell|$ and is obtained as a refinement of the "earlier" cover $\mathcal{D}_{\ell-1}$. For simplicity we choose the cubes of these partitions to be given by $2^{-j}\bar{\Delta}_0 + 2^{-j}z$ for $z \in \mathbb{Z}^3$, $\bar{\Delta}_0$ being the unit cube centered at the origin. It follows from this assumption that for each $\ell \in \mathbb{Z}^+$, there is an integer $n(\ell)$ such that $|\Delta_\ell| = 8^{-n(\ell)}$.

From now on in the definition of the Model (2.1.1) we can assume without loss of generality that

1) the volume Λ is a union of cubes of $\mathscr{D}_0$;
2) the cutoff field $\varphi_{(\kappa)}$ is of the form

$$\varphi_{(\kappa)} = \sum_{i=1}^{\kappa} \varphi_i$$

for some integer $\kappa \geqq 1$. The upper cutoff is therefore M_κ.

One now introduces a mapping $a : \mathbb{Z}^+ \to \mathbb{Z}^+$ which associates the index ℓ of a cover $\mathscr{D}_\ell$ with the index $a(\ell)$ of a momentum cutoff. It will be used as a stopping index for the expansion in the cubes of $\mathscr{D}_\ell$. As a consequence, for the expansion, one needs only to know the following two finite subsequences:

$$M_0 = 0, M_1, \ldots, M_\kappa$$
$$|\Delta_0| = 1, |\Delta_1|, \ldots, |\Delta_{\ell_{\max}}|$$

with $\ell_{\max}$ being the smallest integer such that $a(\ell_{\max}) \geqq \kappa$. The finest cover of $\mathbb{R}^3$ to be considered is thus $\mathscr{D}_{\ell_{\max}}$ and we define $\mathscr{D} = \bigcup_{0 \leqq \ell \leqq \ell_{\max}} \mathscr{D}_\ell$.

Finally let us introduce for each cell of phase space $[M_{i-1}, M_i] \times \Delta$, $\Delta \in \mathscr{D}_\ell$ an interpolating variable $\mathscr{S}_{i,\Delta}$, $0 \leqq \mathscr{S}_{i,\Delta} \leqq 1$, and define

$$\varphi(\{\mathscr{S}\}; x) \equiv \sum_{i=1}^{\kappa} \sum_{\substack{\Delta_j \in \mathscr{D}_j \cap \Lambda \\ a(j) = i}} \prod_{\substack{\Delta_\ell \supset \Delta_j \\ \Delta_\ell \in \mathscr{D}_\ell \cap \Lambda}} (\mathscr{S}_{i,\Delta_\ell} \chi_{\Delta_\ell}(x)) \varphi_i(x). \tag{2.1.2}$$

Replacing $\varphi_{(\kappa)}$ in (2.1.1) by $\varphi(\{\mathscr{S}\})$, one defines an interaction $V(\{\mathscr{S}\}) \equiv V(\lambda; \Lambda; \varphi(\{\mathscr{S}\}))$.

Before going on to the next section, let us remark that due to the local nature of the interaction, if one writes $\varphi = \sum_{\Delta \in \mathscr{D}_\ell} \varphi \chi_\Delta \equiv \sum_{\Delta \in \mathscr{D}_\ell} \varphi_\Delta$ for some cover $\mathscr{D}_\ell$, one has

$$V\Big(\lambda; \Lambda; \sum_{\Delta \in \mathscr{D}_\ell} \varphi_\Delta\Big) = \sum_{\Delta \in \mathscr{D}_\ell \cap \Lambda} V(\lambda; \Delta; \varphi_\Delta) \equiv \sum_{\Delta \in \mathscr{D}_\ell \cap \Lambda} V(\lambda; \varphi_\Delta) \equiv \sum_{\Delta \in \mathscr{D}_\ell \cap \Lambda} V_\Delta$$

with an obvious definition of the counterterms (see [1]).

II.2. Preliminary Definition of the Expansion

The expansion is generated by repeated applications of the following two formulae:

1) The perturbation formula

$$e^{-V(\{\mathscr{S}\})} \equiv (I_{(i,\Delta)} + P_{(i,\Delta)}) e^{-V(\{\mathscr{S}\})}, \quad i \in \mathbb{Z}^+, \quad \Delta \in \mathscr{D}_\ell \tag{2.2.1}$$

where

$$I_{(i,\Delta)} e^{-V(\{\mathscr{S}\})} = e^{-V(\{\mathscr{S}\})}\big|_{\mathscr{S}_{i,\Delta} = 0}$$

$$P_{(i,\Delta)} e^{-V(\{\mathscr{S}\})} = \int_0^{\mathscr{S}_{i,\Delta}} - \frac{dV(\{\sigma\})}{d\sigma_{i,\Delta}} e^{-V(\{\sigma\})}\Big|_{\substack{\sigma_{i',\Delta'} = \mathscr{S}_{i',\Delta'} \\ (i',\Delta') \neq (i,\Delta)}} d\sigma_{i,\Delta}.$$

The operation $I_{(i,\Delta)}$ has the effect of eliminating from the exponent the field φ_i localized in Δ. The operation $P_{(i,\Delta)}$ has the effect of creating a $P_{i,\Delta}$-vertex: $-\frac{dV(\{\sigma\})}{d\sigma_{i,\Delta}}$. A $P_{(i,\Delta)}$-vertex is a φ^4-vertex and its counterterms. It is localized in Δ and one of the fields is a φ_i-field (we will say a φ_i-leg).

Notice the following properties of I and P

1) $I_{(i,\Delta')}I_{(i,\Delta)}=I_{(i,\Delta)}$ if $\Delta'\subseteqq\Delta$.

2) $P_{(i,\Delta')}I_{(i,\Delta)}=0$ if $\Delta'\subseteqq\Delta$.

From 1) and 2) follows in particular that (applying the identity only on the exponential and omitting indices)

$$\begin{aligned}
e^{-V(\mathscr{S})}&=(I+P)^n e^{-V(\mathscr{S})}=\left(\sum_{i=1}^{n-1} IP^i+P^n\right)e^{-V(\mathscr{S})}\\
&=\sum_{i=1}^{n-1}\int_0^{\mathscr{S}}\left(-\frac{dV(\sigma_1)}{d\sigma_1}\right)\int_0^{\sigma_1}\left(-\frac{dV(\sigma_2)}{d\sigma_2}\right)\dots\int_0^{\sigma_{i-1}}\left(-\frac{dV(\sigma_i)}{d\sigma_i}\right)e^{-V(0)}d\sigma_1\dots d\sigma_i\\
&\quad+\int_0^{\mathscr{S}}\left(-\frac{dV(\sigma_1)}{d\sigma_1}\right)\int_0^{\sigma_1}\left(-\frac{dV(\sigma_2)}{d\sigma_2}\right)\dots\int_0^{\sigma_{n-1}}\left(-\frac{dV(\sigma_n)}{d\sigma_n}\right)e^{-V(\sigma_n)}d\sigma_1\dots d\sigma_n.
\end{aligned}\tag{2.2.2}$$

2) The contraction formula

$$\int :\varphi^n:(x)Re^{-V}d\mu=\int\int\langle\varphi(x)\varphi(y)\rangle :\varphi^{n-1}:(x)\left\{\frac{\delta R}{\delta\varphi(y)}-R\frac{\delta V}{\delta\varphi(y)}\right\}e^{-V}dyd\mu\tag{2.2.3}$$

for $n=1,2,3,4$ and R a Wick polynomial in the fields. This formula will be used to exhibit the renormalization cancellations and to perform the Gaussian integration of Wick polynomials.

In (2.2.3) we say that for the first term of the right hand side, a leg of the φ^n-vertex has contracted to an old leg, and for the second term, that it has contracted to the exponent and created a new vertex: a C-vertex. Now we introduce an order relation on $\mathbb{Z}^+\times\tilde{\mathscr{D}}$ by

$$\begin{aligned}
(i,\Delta)\geqq(i',\Delta')\quad &\text{if}\quad |\Delta|=|\Delta'|\quad\text{and}\quad i\geqq i'\\
&\text{or if}\quad |\Delta|>|\Delta'|
\end{aligned}$$

and specify completely the algebraic character of the expansion by giving us a mapping A from $\mathbb{Z}^+\times\tilde{\mathscr{D}}$ to the nonnegative integers. We choose A to be constant on the cubes of $\mathscr{D}_\ell$

$$A(i,\Delta)=A(i,\ell)\quad\text{if}\quad\Delta\in\mathscr{D}_\ell.$$

The expansion is now formally given by

$$\int e^{-V(\lambda;\Lambda;\varphi(\{1\}))}d\mu \equiv \int \prod_{i,\Delta} [I_{i,\Delta}+P_{i,\Delta}]^{A(i,\Delta)} e^{-V(\lambda;\Lambda;\varphi(\{1\}))}d\mu\,. \tag{2.2.4}$$

This formula needs the following comments:

1) The product $\prod_{(i,\Delta)}$ runs over all coupled $(i,\Delta)\in\mathbb{Z}^+\times\tilde{\mathscr{D}}$ and has to be understood as a successive application of the identity $I_{(i,\Delta)}+P_{(i,\Delta)}$ as defined in (2.2.2) in *decreasing* order in (i,Δ).

2) After each application of operation P, one has to apply the contraction formula a certain number of times in order to perform some renormalization cancellations.

3) Because of the order convention, each time one deals with a couple (i,Δ), $\Delta\in\mathscr{D}_\ell$, one has to consider the interaction V_Λ as $\sum_{\Delta\in\mathscr{D}_\ell} V_\Delta$. In particular if a field φ is contracted, because of the renormalization conditions, during the "step" (i,Δ), $\Delta\in\mathscr{D}_\ell$, one has to localize its contraction in $\mathscr{D}_\ell$, replacing the right hand side of (2.2.3) by

$$\int\int :\varphi^{n-1}:(x)\left\{\langle\varphi(x)\varphi(y)\rangle\frac{\delta R}{\delta\varphi(y)} - R\sum_{\Delta\in\mathscr{D}_\ell}\langle\varphi(x)\varphi(y)\rangle\chi_\Delta(y)\frac{\delta V_\Delta}{\delta\varphi(y)}\right\}e^{-\Sigma V_\Delta}dy\,d\mu\,.$$

We remark that if R is a term resulting from previous steps of the expansion, then all fields in R are localized in cubes belonging to $\mathscr{D}_\ell$ or earlier covers.

II.3. Choice of the Parameters of the Expansion

In this section we discuss in a heuristic way what are the relevant choices of sequences $\{M_i\}$, $\{|\Delta_j|\}$ and of mappings a and A defining an absolutely convergent expansion. This part is to show what are the degree of freedom one has in the definition of the expansion. It can be omitted in a first lecture.

Remember that the principle of the expansion is to apply a truncated perturbation expansion to the exponential in smaller and smaller cubes in such a way that, if at the end of the expansion a field localized in Δ and of high momentum M remains in the exponent, then one has $M^2|\Delta|\leqq O(1)$. The possible choices of parameters defining the expansion have to ensure the control and the convergence of the series of produced terms. We will first discuss the problem of controlling a generic term of the expansion, postponing the question of the convergence.

Anticipating the next chapters, the control of a generic term is mainly governed by the following facts:

1) A "convergent" $P_{i,\Delta}$-vertex will produce a convergent factor $M_{i-1}^{-\varepsilon_0}$ or $|\Delta|^{\varepsilon_0}$ for some $\varepsilon_0>0$.

2) The contraction of a field φ localized in Δ and with low momentum M with a field φ' localized in Δ' with low momentum M' gives a convergent factor

$c_n d(\Delta, \Delta')^{-n}$, $n \in \mathbb{Z}^+$ being as large as we want (see [1]), where $d(\Delta, \Delta')$, the scaled distance is defined as

$$d(\Delta, \Delta') = \sup(1, \mathrm{dist}(\Delta, \Delta'), M\,\mathrm{dist}(\Delta, \Delta'), M'\,\mathrm{dist}(\Delta, \Delta')) \tag{2.3.1}$$

$\mathrm{dist}(\Delta, \Delta')$ being the Euclidean distance between Δ and Δ'.

In 1) the field, let us say φ, is such that

$$M^{-1} \leqq |\Delta|^{1/3} \tag{2.3.2}$$

and if $\Delta \in \mathscr{D}_\ell$ and $\Delta' \in \mathscr{D}_{\ell'}$, $\ell' \leqq \ell$, the factor $d(\Delta, \Delta')^{-n}$ allows us to take account of all possible choices of $\Delta' \in \mathscr{D}_{\ell'}$ since for $n \geqq 4$ (see [1], Lemma 4.2a))

$$\sum_{\Delta' \in \mathscr{D}_{\ell'}} d(\Delta, \Delta')^{-n} \leqq O(1). \tag{2.3.3}$$

3) The interaction $V(1; \Delta; \varphi)$ is bounded from below

$$V(1; \Delta; \varphi) \geqq -O(1) M^2 |\Delta|$$

if the high momentum is M.

Moreover, the dominant part of the interaction is its quartic part φ^4 and one has

$$\frac{1}{|\Delta|} \int \varphi^4(x) \chi_\Delta(x) dx - \left(\frac{1}{|\Delta|} \int \varphi(x) \chi_\Delta(x) dx \right)^4 \geqq 0.$$

4) Among all possible "elementary" graphs produced during contraction steps, there are logarithmically divergent ones which cannot be eliminated by renormalization cancellation rules.

5) The number of terms produced by the contraction of n φ^4-vertices is of order of $2n!$

6) In 2), if none of the fields satisfies Condition (2.3.2) the choices of $\Delta' \in \mathscr{D}_{\ell'}$ lead to a factor $|\Delta|^{-1/2}$ per field. This factor cannot be compensated by a convergent $|\Delta|^{\varepsilon_0}$ (see [1]).

We first discuss the arbitrariness in the choice of the mapping a.

As explained in the introduction, the size of a cube Δ being given, $\Delta \in \mathscr{D}_\ell$, we have to lower the high momentum in Δ at least up to M_u such that $M_u^2 |\Delta| \leqq 1$ [from now on the $O(1)$ of this condition is taken to be 1]. On the other hand, in order to control the phenomena described in 6), it is necessary for at least one field of a P-vertex to be properly localized, i.e. $M_\ell^3 \geqq |\Delta|^{-1}$ (here M_ℓ is the highest lower cutoff) in V_Δ.

The possible choices for a are then among

a) $a(\ell)$ is the largest integer for which $M_{a(\ell)}^2 |\Delta_\ell| \leqq 1$[1]; the condition on proper localization meaning that $M_{a(\ell)-1}^3 \geqq |\Delta_\ell|^{-1}$.

b) $a(\ell)$ is the smallest integer for which $M_{a(\ell)-1}^3 \geqq |\Delta_\ell|^{-1}$; but in order to saturate the Wick bound, one has to have $M_{a(\ell)}^2 |\Delta_\ell| \leqq 1$.

c) Any intermediate situation between a) and b).

[1] In [9] it is shown that using the $|\Delta|^{\varepsilon_0}$ convergent factor per P-vertex, it is possible to extend the control of localization up to momenta such that $M_{a(\ell)}^3 \leqq |\Delta_\ell|^{-1}$, that is to say for $M_\ell \sim \Delta^{-1/3(1-\varepsilon)}$, ε being small enough with respect to ε_0

We remark that all these conditions imply that $M_{a(\ell)-1} \geqq M_{a(\ell)}^{2/3}$ whatever the sequence of $|\Delta_\ell|$'s. It follows that the most rapidly increasing regular[2] sequence of M_i is given by $M_i = M_1^{(1+\nu)^{i-1}}$, $i=1, \ldots$, for $M_1 > 1$, $\nu < \frac{1}{2}$.

From now on we assume that the sequence $\{M_i\}$ satisfies

$$M_{i-1} \geqq M_i^{2/3}, \quad i=2, 3, \ldots. \tag{2.3.4}$$

Let us make a remark depending only on (2.3.4) and independent of the specific choice of a.

Suppose as in Case 6) we have to control a factor $|\Delta|^{-1/2}$ coming from the contraction of improperly localized fields, and suppose that the vertex of one of these fields is a "convergent" $P_{i,\Delta}$-vertex. Then, if the volume of the localization cube is not too small with respect to the high momentum M_i, we can dominate this divergent factor using the convergent factor $M_i^{-\varepsilon_0}$ of the P-vertex. This is what happens if

$$|\Delta|^{-1/2} \leqq M_i^{\xi} \tag{2.3.5}$$

for some ξ, $\xi > 0$, small enough with respect to ε_0.

Let us now discuss the choice of A.

The a priori best choice for A would have been $A(i, \Delta) = 1$ [for all allowed coupled (i, Δ)] since, as remarked above in 5), the number of terms produced by contractions increases factorially but the convergent factor (see 1) above) follows only a power law. However, the vertices produced by the perturbation expansions can, by contraction, generate uncancelled logarithmically divergent graphs. If these logarithmic divergences are produced in cubes small enough they can be compensated by the convergent factor due to the smallness of these cubes (see 1) above). If this is not the case, one has to use a more subtile argument to compensate them, i.e. to transform each vertex into a convergent vertex. A natural condition to separate these two cases is given by inequality (2.3.5). Indeed if $M_i^{\xi} \leqq |\Delta|^{-1/2}$, then given $\varepsilon > 0$, as small as we want, $\ln M_i \leqq |\Delta|^{-\varepsilon}$ if M_1 is chosen large enough in order that $\ln M_i \leqq M_i^{2\xi}$. The argument to control the logarithmic divergences in the other case, i.e. when $M_i^{\xi} \geqq |\Delta|^{-1/2}$ is based on an accumulation effect. The expansion is defined in such a way that to each convergent P-vertex is attached at most a finite number of divergent vertices. Suppose that in a given cube Δ many uncancelled logarithmic divergences have been created (more than the maximum number of P-vertices allowed by the expansion), then this means that some of these divergences have to be associated with P-vertices created far away from Δ. The idea then, to compensate the divergences, is to use the decrease of the propagators with the distance (see 2) above). To work out this argument, one needs two conditions

a) the number of uncancelled divergent graphs has to be less than the number of convergent elementary graphs,

b) enough high momentum P-vertices have to be produced in large cubes; roughly speaking, more than $|\Delta|^{-1}$ (to have a distance effect), i.e. more than $M_i^{2\xi}$.

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[2] A sequence is said to be regular if $M_i = f(M_{i-1})$, $i=2, \ldots$ for some f

Condition a) is satisfied as explained in [1] because of the renormalization conditions (see Section 4 below). As a consequence, equally distributed logarithmic divergences are easily controlled and one has only to take care of local accumulation (see Proposition 3.6.1).

Condition b) will be imposed through the definition of A.

Finally, in choosing A we have to take account of the fact that we need to control the number of terms produced during the perturbation expansions (see 5) above) i.e. these expansions have to be stopped at some order $n \simeq M_i^\varepsilon$ for some ε, $\varepsilon > 0$, $\varepsilon < \varepsilon_0$.

We then set

Definition 2.3.1. Let $\varepsilon, \varepsilon > 0$, be small with respect to ε_0. Then the mapping A is defined by

1) $A(i, \Delta) = A(i; \ell)$ for $\Delta \in \mathscr{D}_\ell$;

2) $A(i; \ell) \neq 0$, only for $0 \leqq \ell \leqq \ell_{\max}$ and $i \geqq \sup(2, a(\ell))$

for $|\Delta_\ell|^{-1} \leqq M_i^{\varepsilon/2}$ (Case α)

$$A(i; \ell) = [M_i^\varepsilon];$$

for $|\Delta_\ell|^{-1} > M_i^{\varepsilon/2}$ (Case β)

$$A(i; \ell) = 1.$$

Here $[a]$ means the entire part of a (we omit this symbol from now on). For simplicity, we have taken $2\xi = \varepsilon/2$.

Remarks. 1) The expansion applies only on fields with high momentum at least M_2 (in order to get a convergent factor, at least $M_1^{-\varepsilon_0}$. independent of the value of the mass).

2) Supposing from now on $M_1 > 1$, notice that all the couples (i, Δ_0), $\Delta_0 \in \mathscr{D}_0$ are in Case α.

The last part of this section is devoted to the conditions to impose on the parameters in order to insure the convergence of the expansion.

A general term is given formally by (according to Section II.2)

$$\int \prod_{(i,\Delta)} I_{(i,\Delta)}^{A(i,\Delta) - a(i,\Delta)} P_{(i,\Delta)}^{a(i,\Delta)} e^{-V(\{1\})} d\mu \equiv \int R_a e^{-V_a} d\mu \tag{2.3.6}$$

for some $a(i, \Delta)$, $0 \leqq a(i, \Delta) \leqq A(i, \Delta)$.

Suppose that for a suitable choice of the parameters defining the expansion one has

Proposition 2.3.2. *Independently of the numbers $\{a(i, \Delta)\}$, there exist constants $O(1)$ and ε_0 such that*

$$\left|\int R_a e^{-V_a} d\mu\right| \leqq O(1)^{|\Lambda|} \prod_{\substack{(i,\Delta) \\ \text{of Case } \alpha}} (M_i^{-\varepsilon_0})^{a(i,\Delta)} \prod_{\substack{(i,\Delta) \\ \text{of Case } \beta}} (|\Delta|^{\varepsilon_0})^{a(i,\Delta)}. \tag{2.3.7}$$

Then, Theorem I follows if, in addition, the parameters satisfy the following (sufficient) conditions:

$$\begin{aligned}&1)\ |\Delta_{\ell-1}|\,|\Delta_\ell|^{\varepsilon_0-1}e\leqq 1\,, \qquad \ell_{\max}\geqq\ell\geqq 2\,;\\ &2a)\ (M_i^{-\varepsilon_0/2})^{n_i}n_i!\leqq 1\,, \qquad i=1,\dots,\varkappa, \qquad 0\leqq n_i\leqq[M_i^\varepsilon]\,;\\ &2b)\ eM_i^{-\varepsilon_0/2}(|\Delta_{\ell-1}|/|\Delta_\ell|)\leqq 1 \quad \textit{for} \quad (i,\Delta_\ell)\ \textit{in Case}\ \alpha), \qquad \ell\geqq 1\,;\\ &2c)\ M_i^{-\varepsilon_0/2}\leqq \mathrm{const}\, i^{-2}\,, \qquad i=1,2,\dots,\kappa\,.\end{aligned} \tag{2.3.8}$$

Proof of Theorem I. Define for $i=2,\dots,\kappa$

$$b(i)=\{\sup \ell/|\Delta_\ell|^{-1}\leqq M_i^{\varepsilon/2}\}$$

then by Proposition 2.3.2, Definition 2.3.1, and Formula 2.2.4,

$$\begin{aligned}\int e^{-V(\lambda;\Lambda)}d\mu| \leqq O(1)^{|\Lambda|}\prod_{i\geqq 2}\Bigg\{\prod_{\Delta_0\in\mathscr{D}_0\cap\Lambda}(1+M_i^{-\varepsilon_0}+\dots+(M_i^{-\varepsilon_0})^{A(i;0)}\prod_{\substack{\Delta_1\in\mathscr{D}_1\\ \Delta_1\subset\Delta_0}}\\ \cdot(1+M_i^{-\varepsilon_0}+\dots+(M_i^{-\varepsilon_0})^{A(i;1)}\dots\prod_{\substack{\Delta_{b(i)}\in\mathscr{D}_{b(i)}\\ \Delta_{b(i)}\subset\Delta_{b(i)-1}}}(1+M_i^{-\varepsilon_0}+\dots+(M_i^{-\varepsilon_0})^{A(i;b(i))}\\ \cdot\prod_{\substack{\Delta_{b(i)+1}\subset\mathscr{D}_{b(i)+1}\\ \Delta_{b(i)+1}\subset\Delta_{b(i)}}}\cdot(1+|\Delta_{b(i)+1}|^{\varepsilon_0}\prod_{\substack{\Delta_{b(i)+2}\subset\mathscr{D}_{b(i)+2}\\ \Delta_{b(i)+2}\subset\Delta_{b(i)+1}}}(1+|\Delta_{b(i)+2}|^{\varepsilon_0}\prod(1+\dots\,.\end{aligned} \tag{2.3.9}$$

Now

$$\prod_{\substack{\Delta_\ell\in\mathscr{D}\\ \Delta_\ell\subset\Delta_{\ell-1}}}(1+|\Delta_\ell|^{\varepsilon_0})\leqq\prod_{\substack{\Delta_\ell\in\mathscr{D}\\ \Delta_\ell\subset\Delta_{\ell-1}}}(1+e|\Delta_\ell|^{\varepsilon_0})\leqq\prod_{\substack{\Delta_\ell\in\mathscr{D}\\ \Delta_\ell\subset\Delta_{\ell-1}}}e^{e|\Delta_\ell|^{\varepsilon_0}}$$

$$=e^{e|\Delta_\ell|^{\varepsilon_0}|\Delta_{\ell-1}|/|\Delta_\ell|}\leqq e \quad\text{for}\quad \ell\geqq 2\,,$$

using (2.3.8), 1).

Also for $i\geqq 2$

$$\begin{aligned}\prod_{\substack{\Delta_j\in\mathscr{D}_j\\ \Delta_j\subset\Delta_{j-1}}}(1+M_i^{-\varepsilon_0}+\dots+(M_i^{-\varepsilon_0})^{[M_i^\varepsilon]})&\leqq\prod_{\substack{\Delta_j\in\mathscr{D}_j\\ \Delta_j\subset\Delta_{j-1}}}\left(1+M_i^{-\varepsilon_0/2}+\dots+\frac{(M_i^{-\varepsilon_0/2})^{[M_i^\varepsilon]}}{[M_i^\varepsilon]!}e\right)\\ &\leqq\prod_{\substack{\Delta_j\in\mathscr{D}_j\\ \Delta_j\subset\Delta_{j-1}}}e^{eM_i^{-\varepsilon_0/2}}\,, \qquad\text{by (2.3.8), 2a)}\\ &\leqq e^{eM_i^{-\varepsilon_0/2}|\Delta_{j-1}|/|\Delta_j|}\leqq e\,, \qquad\text{by (2.3.8), 2b)}\end{aligned}$$

and

$$\begin{aligned}&\prod_{i\geqq 2}\Bigg\{\prod_{\Delta_0\in\mathscr{D}_0\cap\Lambda}(1+M_i^{-\varepsilon_0}+\dots+(M_i^{-\varepsilon_0})^{[M_i^\varepsilon]}e)\Bigg\}\\ &\leqq\prod_{i\geqq 2}\Bigg\{\prod_{\Delta_0\in\mathscr{D}_0\cap\Lambda}e^{eM_i^{-\varepsilon_0/2}}\Bigg\}, \qquad\text{by (2.3.8), 2a)}\\ &\leqq\prod_{i\geqq 2}e^{|\Lambda|eM_i^{-\varepsilon_0/2}}\leqq e^{e|\Lambda|\sum_2^\infty M_i^{-\varepsilon_0/2}}\leqq e^{O(1)|\Lambda|}\,, \qquad\text{by (2.3.8), 2c)}\,,\end{aligned}$$

which proves the theorem.

II.4. The Definition of the Expansion Completed

The formula (2.2.4) defining the expansion will be meaningful only after one has described the contractions which have to follow each application of the P-operation. The procedure, similar to the one of [1], depends on whether the couple (i, Δ) of localization of the P-vertex belongs to case α or β.

Case α*:* Each time a $P_{(i,\Delta)}$-vertex (P_α-vertex) is created, one contracts all its legs, creating possibly C_α-vertices. One then contracts all legs of these new C_α-vertices with the following exceptions:

α_1) If the P_α-vertex is a mass counterterm, then one does not contract the legs of the new C_α-vertices it has created.

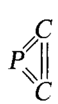
α_2) If the P_α-vertex contracts three times to one new C_α-vertex and once to a second new C_α-vertex forming a mass sub-diagram, then we contract the fourth leg of the first new C_α-vertex but do not contract the three remaining legs of the second new C_α-vertex.

α_3) If the P_α-vertex contracts three times to old vertices and once to a new C_α-vertex, then we undo the last contraction using formula (2.2.2) in reverse.

Case β*:* Each time one has produced in $\Delta \in \mathscr{D}_\ell$ a $P_{(i,\Delta)}$-vertex (P_β-vertex), one decomposes each of its legs φ as $\varphi = \varphi_\ell + \varphi_p$, where φ_ℓ is the part of φ which contains only fields φ_i with $i < a(\ell)$ and φ_p, the "properly localized" part, is the part of φ which contains only fields φ_i with $i \geqq a(\ell)$. The rule is now to contract all the φ_p legs. However one does not contract the legs of the new C_β-vertices.

Let us now comment on the renormalization cancellations. In Case α), as in [1], the contractions are to exactly cancel the vacuum energy diagrams formed from a P_α-vertex and one or two new C_α-vertices

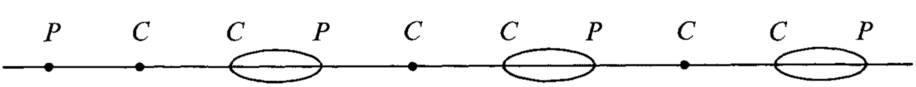
$P \equiv C$



with the energy counterterms. In the same way, one combines the mass counterterm with the mass diagram formed from a P_α-vertex which contracts three times to a new C-vertex:

$$—P \equiv C— + —\delta m^2—.$$

Condition α_3) is to prevent the formation of chains



Remark. As noticed before, case α, i.e. $|\Delta|^{-1} \leqq M_i^{\varepsilon/2}$, is defined in such a way one can compensate the factors $|\Delta|^{-1/2}$ necessary for the sum over the localizations of improperly localized legs. In Case β), the contractions are just introduced to compensate the linearly divergent vacuum diagram generated by contraction of properly localized legs (the logarithmic divergences are compensated by the size of the cubes).

To conclude this chapter we propose a specific choice for the parameters of the expansion. The next chapter is entirely devoted to the proof of Proposition 2.3.2 under this choice.

Let us first see what is imposed by Conditions 1) and 2) following Proposition 2.3.2. If we look for a regular sequence $\{|\Delta_j|\}$, Condition 1) imposes that $|\Delta_j|^{-1}$ cannot grow faster than

$$|\Delta_j|^{-1}=|\Delta_1|^{-\left(\frac{1}{1-\varepsilon_0}\right)^{\ell-1}} e^{\left[1-\left(\frac{1}{1-\varepsilon_0}\right)^{\ell-1}\right]\varepsilon_0^{-1}}$$

for any $|\Delta_1|<1$. Condition 2a) is satisfied for $\varepsilon<\varepsilon_0/2$ and $M_1>1$ sufficiently large depending on ε, whatever is the increasing sequence $\{M_i\}$. Condition 2b) is a consequence of these two conditions for

$$\begin{aligned} eM_i^{-\varepsilon_0/2}\frac{|\Delta_{\ell-1}|}{|\Delta_\ell|} &\leqq M_i^{-\varepsilon_0/2}|\Delta_\ell|^{-\varepsilon_0}, \quad \text{for} \quad \ell\geqq 2\\ &\leqq eM_i^{-\varepsilon_0/2}|\Delta_\ell|^{-1}, \quad \text{for} \quad \ell=1. \end{aligned}$$

Now being in Case α), $|\Delta_\ell|^{-1}\leqq M_i^{\varepsilon/2}$, thus

$$M_i^{-\varepsilon_0/2}|\Delta_\ell|^{-\varepsilon_0}\leqq M_i^{-\varepsilon_0/2(1-\varepsilon)}\leqq 1$$

and

$$eM_i^{-\varepsilon_0/2}|\Delta_\ell|^{-1}\leqq eM_i^{-\frac{\varepsilon_0-\varepsilon}{2}}\leqq 1,$$

for M_1 large enough.

The growth of the sequence $\{M_i\}$ has to be, by Condition 2c), quick enough:

$$M_i\geqq O(1)i^{O(1)}$$

for some constants $O(1)$ depending on ε_0. The last condition imposed on the parameters depends on the choice of the mapping a, but as explained in any case, the main limitation implied is a limit on the growth of the sequence $\{M_i\}$.

In order to simplify the presentation of the next chapter, we summarize these conditions in the following definition:

Definition 2.4.1. 1) The sequence $\{M_j\}$ is given by $M_j=M_1^{(1+\tau)^{j-1}}$, $M_1>1$.

2) The sequence $\{|\Delta_j|\}$ is given by $|\Delta_j|=8^{-n(j)}$, $n(j)\in\mathbb{Z}^+$ defined by

$$\tfrac{1}{8}M_j^{-3}<8^{-n(j)}\leqq M_j^{-3}.$$

3) a is any integer valued increasing function on $\mathbb{Z}^+$ such that

$$\ell+2\leqq a(\ell)\leqq\ell+\frac{\ln 3/2}{\ln(1+\tau)}.$$

4) τ is small enough and M_1 is large enough depending on τ.

Comments. With the above definition of the relation between the two sequences, if $M_1^{3\tau} \geqq 8$ then a field $\varphi_{j,\Delta}$, $\Delta \in \mathscr{D}_\ell$ will be properly localized if $j \geqq \ell + 2$ and improperly localized otherwise. From now on we suppose that this is the case. The condition on $a(\ell)$ in 3) comes from the necessity for $a(\ell)$ to be such that $M_{a(\ell)}^2 |\Delta_\ell| \leqq 1$ for $\ell \geqq 1$ and at some time to correspond to a proper localization. If $\tau \leqq 1 - \sqrt{3/2}$ (we will take τ much smaller later) this condition can be satisfied. Finally, we remark that $\sup(2, a(\ell)) = a(\ell)$ for $\ell \in \mathbb{Z}^+$.

III. Estimates

The purpose of this chapter is to prove Proposition 2.3.2, i.e. to bound the general term

$$\int R_a e^{-V_a} d\mu$$

of the expansion. This is done in three steps. In the first step, one bounds the improperly localized legs of R_a using the formal positivity of the interaction. One then applies the Wick bound to estimate the exponential e^{-V_a}. It remains then in the last step to bound a standard Gaussian integral.

These steps bound a single term in $\int R_a e^{-V_a} d\mu$ (which is a sum of terms) by a sum of terms, and we use the method of combinatoric factors to perform the computation, as applied in [1]. In the same way, the technical estimates we need are essentially those of [1]. We therefore do not repeat them, and refer the reader to the aforementioned reference for more details.

III.1. The Control of the Improperly Localized Legs

Improperly localized legs appear in Case β during the contraction procedure following a perturbation step. They cannot be controlled by the final Gaussian integration[3]. They will be dominated by what remains of the exponent at the end of the expansion. That this is possible is due to the fact that the level of proper localization is below the level we choose to stop the application of the perturbation expansion [superrenormalizability; see the discussion on $a(\ell)$ above]. In other words, if in a given step of the expansion corresponding to a cover $\mathscr{D}_\ell$, the improperly localized part of the field φ in $\Delta \in \mathscr{D}_\ell$ is $\varphi_{\ell,\Delta}$, then it will remain unchanged by the following steps of the expansion. We can therefore hope to dominate such $\varphi_{\ell,\Delta}$, using the formal positivity of what remains from the interaction at the end of the expansion (quartic "integration").

Let φ be a leg of a β-vertex (P_β or C_β) at a step (step ℓ) of the expansion corresponding to a cover $\mathscr{D}_\ell$ and let $\Delta \in \mathscr{D}_\ell$ be its localization cube. One writes

$$\varphi(x)\chi_\Delta(x) = \varphi_\Delta(x) = \varphi_{\ell,\Delta}(x) + \varphi_{\not{\ell},\Delta}(x) \tag{3.1.1}$$

[3] We remark that a contraction is a partial Gaussian integration

where the improperly localized part $\varphi_{\ell,\Delta}$ is given by

$$\varphi_{\ell,\Delta}(x)=\sum_{i=1}^{\ell+1}\varphi_{\ell,\Delta;i}(x)\quad\text{and}\quad\varphi_{\ell,\Delta;i}(x)=\left[\prod_{\substack{\Delta'\in\mathcal{D}\\ \Delta'\supset\Delta}}\mathcal{S}_{i,\Delta'}\right]\chi_\Delta(x)\varphi_i(x). \tag{3.1.2}$$

At the end of the expansion, the interaction in Δ is

$$\begin{aligned}&\lambda\int :\varphi^4:(x)\chi_\Delta(x)dx+\text{counterterms}\\ &=\lambda\int :(\varphi_{\ell,\Delta}(x)+\varphi_{\hbar,\Delta}(x))^4:dx+\text{counterterms}\end{aligned} \tag{3.1.3}$$

where $\varphi_{\hbar,\Delta}$ is the high momentum part (relative to Δ) of the field φ in Δ.

Let, as in [1]

$$\varphi_\Delta(x)=\bar\varphi_\Delta\chi_\Delta(x)+\delta\varphi_\Delta(x) \tag{3.1.4}$$

with

$$\bar\varphi_\Delta=\frac{1}{|\Delta|}\int\varphi_\Delta(x)dx.$$

We want to use (Hölder's inequality)

$$(\bar\varphi_\Delta)^4\leqq\left(\frac{1}{|\Delta|}|\Delta|^{3/4}\int\varphi_\Delta^4(x)dx\right)^4=\frac{1}{|\Delta|}\int\varphi_\Delta^4(x)dx. \tag{3.1.5}$$

Consider now a β-vertex produced at a step ℓ and localized in $\Delta\in\mathcal{D}_\ell$. Writing each of its legs as (3.1.1), the vertex is decomposed in a corresponding sum of vertices. One then can undo the Wick ordering of the $\varphi_{\ell,\Delta}$ legs in each of these vertices. One has the following bounds (easy to prove) for the Wick ordering terms

$$\begin{aligned}&\left|\int\varphi_{\ell,\Delta}(x)\varphi_{\ell,\Delta}(x)d\mu\right|\leqq O(1)|\Delta|^{-1/3}\leqq O(1)(|\Delta|^{-1/4})^2\\ &\left|\int\varphi_{\ell,\Delta}(x)\varphi_{\hbar,\Delta}(x)d\mu\right|\leqq O(1)|\Delta|^{-1/3}\leqq O(1)(|\Delta|^{-1/4})^2.\end{aligned} \tag{3.1.6}$$

Each β-vertex is decomposed as a sum of vertices. In order to compare the improperly localized leg with the exponent, one performs another decomposition and writes

$$\varphi_{\ell,\Delta}(x)=\bar\varphi_\Delta\,\chi_\Delta(x)-\bar\varphi_{\hbar,\Delta}\chi_\Delta(x)+\sum_{i=1}^{\ell+1}\delta\varphi_{i,\Delta}(x) \tag{3.1.7}$$

where φ_Δ is the field in Δ in the exponent at the end of the expansion, and where one used (3.1.2), (3.1.3), (3.1.4), and $\delta\varphi_{i,\Delta}(x)=\varphi_{\ell,\Delta;i}(x)-\bar\varphi_{\ell,\Delta;i}\chi_\Delta(x)$. Thus each $\varphi_{\ell,\Delta}$ is replaced by $\ell+3$ fields. Using the fact that $\ell+3\leqq|\Delta_\ell|^{-\varepsilon_1}$, ε_1 as small as we want, provided M_1 is large enough[4], and that each β-vertex has at most 4 fields, one obtains the result that each initial vertex is replaced by a sum of at most $O(1)|\Delta_\ell|^{-3\varepsilon_1}$ vertices made from $\delta\varphi_{i,\Delta}$, $\bar\varphi_{\hbar,\Delta}$, and $\bar\varphi_\Delta$ fields. Each new vertex has at least one

[4] Remember that $|\Delta_\ell|\sim M_1^{-3(1+\tau)^{\ell-1}}$

properly localized leg (by construction for a P_β-vertex and by contraction with a properly localized leg for a C_β-vertex).

Let now $n_i(\Delta, \Delta')$ be the number of $\bar{\varphi}_\Delta$ legs obtained in this way from $P_{(i,\Delta')}$ vertices of type β. Since $A(i, \Delta')=1$, and since a P_β-vertex generates at most 4 C_β-vertices, one finds that $n_i(\Delta, \Delta')\leqq O(1)$. In order to compare $\bar{\varphi}_\Delta$ with the remaining interaction we bound it by an exponential factor using $y^n\leqq n^n e^{|y|}$ for $n\in Z^+$

$$\begin{aligned}(\bar{\varphi}_\Delta)^{n_i(\Delta,\Delta')}&\leqq\left[O(1)d(\Delta,\Delta')\frac{|\Delta|^{-1/4-\varepsilon_1}}{(\operatorname{Re}\lambda)^{1/4}}\right]^{n_i(\Delta,\Delta')}\left(\frac{n_i(\Delta,\Delta')}{4}\right)^{\frac{n_i(\Delta,\Delta')}{4}}\\&\quad\cdot\exp\left\{\frac{\operatorname{Re}\lambda}{O(1)}d(\Delta,\Delta')^{-4}|\Delta|^{1+4\varepsilon_1}(\bar{\varphi}_\Delta)^4\right\}\\&\leqq\left[O(1)d(\Delta,\Delta')\frac{|\Delta|^{-1/4-\varepsilon_1}}{(\operatorname{Re}\lambda)^{1/4}}\right]^{n_i(\Delta,\Delta')}\exp\left\{\frac{\operatorname{Re}\lambda}{O(1)}d(\Delta,\Delta')^{-4}|\Delta|^{1+4\varepsilon_1}(\bar{\varphi}_\Delta)^4\right\}\end{aligned}\tag{3.1.8}$$

where $d(\Delta, \Delta')$ is the scaled distance of the contraction generating the vertex of $\bar{\varphi}_\Delta$ if it is a C_β vertex and is 1 if the vertex of $\bar{\varphi}_\Delta$ is a P_β vertex; the first $O(1)$ is a bound on $\sum_{\Delta'} d(\Delta, \Delta')^{-4}$ since by definition of the Case β, the legs generating the C_β vertices are properly localized, ε_1 is a small number ($\varepsilon_1 \ll \varepsilon_0$) and the second inequality is obtained by using $n_i(\Delta, \Delta')\leqq O(1)$.

In the expression $\int R_a e^{-V_a}d\mu$, R_a is a sum of terms R'_a each proportional to a power of λ. Before applying the bound we use the Schwarz inequality

$$\int R'_a e^{-V_a}d\mu\leqq\left(\int |R'^2_a| e^{-2\operatorname{Re}V_a}d\mu\right)^{1/2}. \tag{3.1.9}$$

Using inequality (3.1.8), the contribution of the factors $\bar{\varphi}_\Delta$ is

$$\exp\left\{\operatorname{Re}\lambda\sum_{\ell}\sum_{\Delta\in\mathcal{D}_\ell\cap\Lambda}\sum_i\sum_{\Delta'\in\mathcal{D}_\ell}\frac{1}{O(1)}d(\Delta,\Delta')^{-4}|\Delta|^{1+4\varepsilon_1}(\bar{\varphi}_\Delta)^4\right\}$$

the last two sums being the sum over all possible $P_{(i,\Delta')}$ β-vertices which could have generated $\bar{\varphi}_\Delta$.

One bounds it by

$$\exp\left\{\operatorname{Re}\lambda\sum_{\ell}\sum_{\Delta\in\mathcal{D}_\ell\cap\Lambda}|\Delta|^{1+3\varepsilon_1}(\bar{\varphi}_\Delta)^4\right\} \tag{3.1.10}$$

using first inequality (2.3.3) and the fact that in Case β

$$\sum_i|\Delta|^{\varepsilon_1}\leqq\sum_i(M_i^{-\varepsilon/2})^{\varepsilon_1}\leqq 1 \tag{3.1.11}$$

for M_1 large enough depending on ε_1.

Consider now the "pure" φ^4 term in $\operatorname{Re} V_a$. One has, using (3.1.11)

$$\operatorname{Re}\lambda \int_\Lambda \varphi(x)^4 dx \geqq \operatorname{Re}\lambda \sum_\ell \sum_{\Delta\in\mathscr{D}_\ell} |\Delta|^{3\varepsilon_1} \int \varphi_\Delta(x)^4 dx$$

and from (3.1.5)

$$\operatorname{Re}\lambda \int_\Lambda \varphi(x)^4 dx \geqq \operatorname{Re}\lambda \sum_\ell \sum_{\Delta\in\mathscr{D}_\ell\cap\Lambda} |\Delta|^{1+3\varepsilon_1}(\bar{\varphi}_\Delta)^4$$

from which follows the domination of the improperly localized legs. The remaining $\operatorname{Re}\lambda \int_\Lambda \varphi(x)^4 dx$ in $e^{-2\operatorname{Re}V_a}$ will be used to control the Wick ordering (Wick bound). The coefficients in front of the exponential in (3.1.8) are considered as combinatoric factors. We attribute them in the following way:

— the $O(1)|\Delta|^{O(1)\varepsilon_1}$ factor is assigned to the new β-vertices (see below);
— $d(\Delta, \Delta')$ is a combinatoric factor for the contraction generating the C-vertex.

The factor $|\Delta|^{-1/4-\varepsilon_1}$ replacing a $\bar{\varphi}_\Delta$ leg will not be considered as a combinatoric factor. Each β-vertex after the undoing of the Wick ordering and the decomposition of the improper part is replaced by a sum of β-vertices. Including the $|\Delta|^{-1/4-\varepsilon_1}$ factor and the bounds on the Wick counterterms (3.1.6), one gets (see [1], Section III.2) that each new β-vertex is composed of r, $r\leqq 4$, $\varphi_{\not{h},\Delta}$, $\bar{\varphi}_{\not{h},\Delta}$ and $\delta\varphi_{i,\Delta}$ legs with, apart from the original coefficient, a new coefficient bounded by $|\Delta|^{-1+r/4}$.

For the factor $(\operatorname{Re}\lambda)^{-1/4}$, we remark that each vertex has a factor $|\lambda|$ or $|\lambda|^2$, and at most 3 improperly localized legs. The coefficient for a vertex is thus $\frac{|\lambda|^i}{(\operatorname{Re}\lambda)^{j/4}} = a_{ij}(\lambda)$, $i=1,2$, $j=0,1,2,3$ and is bounded in any domain of the form $D_{\varrho,\sigma} = \left\{\lambda\in\mathbb{C} \,\middle|\, |\lambda|\leqq\varrho,\ |\operatorname{Arg}\lambda|\leqq\frac{\pi}{2}-\sigma\right\}$ for $\varrho<\infty$ and $\sigma>0$. Fixing ϱ and σ we attribute $\sup_{i,j} a_{ij}(\lambda)=O(1)$ to the new vertices.

III.2. The Wick Bound

We now explain how to bound what remains in $e^{-2\operatorname{Re}V_a}$ after the domination of the improperly localized legs. Before applying the Wick bound we need to describe the precise structure of V_a in momentum and space localizations. Remember that by definition of the expansion, the perturbation expansion is applied in a cube $\Delta_{\ell+1}$ obtained by subdivision of Δ_ℓ ($\Delta_\ell > \Delta_{\ell+1}$) only if the highest momentum in Δ_ℓ has an index larger than or equal to $a(\ell+1)$. One is thus led to define a partition $\bar{\mathscr{D}}=\{\varrho_\Delta\}_{\Delta\subset\Lambda}$ of Λ inductively by

$$\varrho_{\Delta_\ell} = \bigcup \left\{\Delta_{\ell+1}\in\mathscr{D}_{\ell+1} \,\middle|\, \Delta_{\ell+1}\subset\left[\Lambda\setminus\bigcup_{j=0}^{\ell-1}\varrho_{\Delta_j}\right], \Delta_{\ell+1}\subset\Delta_\ell;\ \mathscr{S}_{i,\Delta_{\ell+1}}=0\,\forall i\geqq a(\ell)\right\} \tag{3.2.1}$$

for $\ell=0,1,\cdots,\ell_{\max}-1$, and

$$\varrho_{\Delta_{\ell_{\max}}} = \left\{\Delta_{\ell_{\max}}\subset\mathscr{D}_{\ell_{\max}} \,\middle|\, \Delta_{\ell_{\max}}\subset\left[\Lambda\setminus\bigcup_{j=0}^{\ell_{\max}-1}\varrho_{\Delta_j}\right]\right\}.$$

In this way $\varrho_{\Delta_\ell} \subset \Delta_\ell$ and $|\varrho_{\Delta_\ell}| \leqq |\Delta_\ell|$ where $|\varrho_{\Delta_\ell}|$ is the volume of ϱ_{Δ_ℓ}. In ϱ_{Δ_ℓ} the upper cutoff $M_{u(\varrho_{\Delta_\ell})}$ is less than or equal to $M_{a(\ell)}$, from which follows that $M^2_{u(\varrho_{\Delta_\ell})}|\varrho_{\Delta_\ell}| \leqq 1$ for $\ell = 1, 2, \cdots$.

We write $2\mathrm{Re}\, V_1 - \mathrm{Re}\,\lambda \int_\Lambda \varphi^4(x)dx$ as a sum over $\bar{\mathscr{D}}$ and bound each term, localized in ϱ_Δ, from below in the standard way (the Wick bound: see [1]) by $O(1)M^2_{u(\varrho_{\Delta_\ell})}|\varrho_{\Delta_\ell}|$. Here $O(1)$ depends only on λ (each $\mathscr{S}_{i,\Delta} \neq 0$ is bounded by 1). One gets

$$e^{-2\mathrm{Re}\, V_a + \mathrm{Re}\,\lambda \int_\Lambda \varphi^4(x)dx} \leqq \prod_{\varrho_\Delta \in \bar{\mathscr{D}}} e^{O(1)M^2_{u(\varrho_\Delta)}|\varrho_\Delta|}. \tag{3.2.2}$$

By Definition (3.2.1) $\varrho_{\Delta_\ell} \neq \emptyset$, $\ell \neq 0$, if there exists a $\Delta_{\ell+1} \subset \Delta_\ell$ and $i \geqq a(\ell)$ such that $\mathscr{S}_{i,\Delta_{\ell+1}} \neq 0$, thus that a P-vertex has been created in Δ_ℓ. Therefore one has

$$\prod_{\substack{\varrho_\Delta \in \bar{\mathscr{D}} \\ \Delta \notin \mathscr{D}_0}} e^{O(1)M^2_{u(\varrho_\Delta)}|\varrho_\Delta|} \leqq \prod_{P\text{-vertex}} O(1). \tag{3.2.3}$$

The contribution of the ϱ_{Δ_0}'s is $\prod_{\substack{\varrho_{\Delta_0} \in \bar{\mathscr{D}} \\ \Delta_0 \in \mathscr{D}_0 \cap \Lambda}} O(1)$ where $O(1)$ is for $e^{O(1)M^2_{a(0)}}$. Since there are at most $|\Lambda|\Delta_0$ in $\mathscr{D}_0 \cap \Lambda$, one gets finally that (3.2.2) is bounded by

$$O(1)^{|\Lambda|} \prod_{P\text{-vertex}} O(1) \tag{3.2.4}$$

the first $O(1)$ depending on λ and M_1. This establishes the Wick bound in its combinatoric form.

III.3. The Gaussian Integration

In R'_a once one has dominated the improperly localized legs and bounded the exponential, one performs the Gaussian integration *starting from the legs localized in the smallest cubes.* In this way each leg contracts to a leg localized in a larger cube. The result of the Gaussian integration is a sum of contraction graphs G.

From Sections III.1 and III.2 one gets

$$|\int R_a e^{-V_a} d\mu| \leqq O(1)^{|\Lambda|} \sum_{R'_a} \left\{ \prod_{P\text{-vertex}} O(1) \prod_{\text{contrac.}} d(\Delta, \Delta')^{O(1)} \sum_G G \right\}^{1/2}. \tag{3.3.1}$$

Our purpose is to replace the sums by a supremum using the method of the combinatoric factors. We first discuss the combinatoric factor due to the contractions, treating on the same foot the renormalization contractions of R'_a (partial Gaussian integration) and the contractions of the final Gaussian integration. One has

Lemma 3.3.1. *The following factors are upper bounds on combinatoric factors associated with the contractions*

— $d^{O(1)}$ *per contraction line;*
— $b^{3(1+\varepsilon_2)/2}$ *per* $\delta\varphi$ *leg;*
— $M_i^{\varepsilon_3}$ *per* $P_{(i,\Delta)}$*-vertex of Type* α*;*
— $|\Delta|^{-\varepsilon_3}$ *per* $P_{(i,\Delta)}$*-vertex of Type* β*;*

$\varepsilon_2>0, \varepsilon_3$ *small (depending on* ε*) and* M_1 *large enough depending on* ε *and* ε_3*. Here b is the localization ration introduced in* [1]:

$$b \equiv \max\{1, |\Delta|^{-1/3} M_j^{-1}\} \equiv b_\Delta \tag{3.3.2}$$

where Δ *is the localization cube and* M_j *is the lower momentum of the leg.*

Proof of Lemma 3.3.1. A field in $\Delta\in\mathcal{D}_\ell$ contracts only, by construction, to a field in $\Delta'\in\mathcal{D}_{\ell'}$, $\ell'\leqq\ell$. The contraction is completely specified if one knows the cover $\mathcal{D}_{\ell'}$, the localization cube Δ' in the cover, the $P_{(i,\Delta'')}$-vertex which has generated the field in Δ', and finally which one among all possible fields having the above specifications. Let us consider these cases:

1) Choice of ℓ': With a factor 2 we choose whether $\ell'=0$ or not. If ℓ' is different from zero we choose it with a factor $|\Delta'|^{-\varepsilon_1}=|\Delta''|^{-\varepsilon_1}$ provided M_1 is large enough, ε_1 fixed as small as we want. The combinatoric factor for the choice is therefore $O(1)|\Delta''|^{-\varepsilon_1}$ attributed to the $P_{(i,\Delta'')}$-vertex, with the remark that if this vertex is a P_α-vertex, then one uses $|\Delta''|^{-1}\leqq M_i^{\varepsilon/2}$ to take $O(1)M_i^{\varepsilon_1}$ as combinatoric factor.

2) Choice of the cube of localization Δ': With a factor 3 we distinguish the following three cases

a) one of the two contracting fields is properly localized. The combinatoric factor is $O(1)d(\Delta,\Delta')^4$ as seen from (2.3.3).

b) The two fields are improperly localized and one of them is not a $\delta\varphi$ field, but then this means it has been obtained from a generating P_α-vertex (of high momentum i). Using (see [1])

$$\sum_{\Delta'\in\mathcal{D}_{\ell'}} d(\Delta,\Delta')^{-4}\leqq O(1)|\Delta'|^{-1} \quad \text{and} \quad |\Delta'|^{-1}\leqq M_i^{\varepsilon/2},$$

one attributes d^4 to the contraction line and $O(1)M_i^{\varepsilon/2}$ to the P_α-vertex.

c) The two fields are improperly localized and are both $\delta\varphi$ legs.

Suppose the $\delta\varphi$ legs, localized in Δ and Δ', have respectively j and j' as high momentum (remember, Section III.1 states that $\delta\varphi$ legs are of the type $\delta\varphi_{i,\Delta}$ legs, i.e. of high momentum M_i and low momentum M_{i-1}). Then

$$b_\Delta d(\Delta,\Delta')\geqq b_\Delta \tfrac{1}{2}(1+M_{j-1}\operatorname{dist}(\Delta,\Delta'))\geqq\tfrac{1}{2}(1+|\Delta|^{-1/3}\operatorname{dist}(\Delta,\Delta'))$$
$$b_{\Delta'} d(\Delta,\Delta')\geqq b_{\Delta'} \tfrac{1}{2}(1+M_{j'-1}\operatorname{dist}(\Delta,\Delta'))\geqq\tfrac{1}{2}(1+|\Delta'|^{-1/3}\operatorname{dist}(\Delta,\Delta')).$$

Since $|\Delta|\leqq|\Delta'|$, for $\varepsilon_2>0$, small enough

$$\sum_{\Delta'\in\mathcal{D}_{\ell'}} (b_\Delta b_{\Delta'} d(\Delta,\Delta'))^{-3(1+\varepsilon_2)/2}$$
$$\leqq 2^{3(1+\varepsilon_2)} \sum_{\Delta'\in\mathcal{D}_{\ell'}} (1+|\Delta'|^{-1/3}\operatorname{dist}(\Delta,\Delta'))^{-3(1+\varepsilon_2)}\leqq O(1)$$

with $O(1)$ depending on ε_2. One attributes d^4 ($\geqq d^{3(1+\varepsilon_2)}$) to the contraction line, b_Δ and $b_{\Delta'}$ to the $\delta\varphi$ legs, and $O(1)$ to the generating P-vertex (of one of the $\delta\varphi$ legs).

3) choice of the generating $P_{(i,\Delta'')}$-vertex: By a factor 2 we decide whether the field in Δ' has been created by a P_α or a P_β-vertex.

Case α) The localization cube Δ'' is chosen using

$$\sum_{\Delta''\in\mathscr{D}_{\ell'}} d(\Delta',\Delta'')^{-4} \leqq O(1)|\Delta''|^{-1} \leqq O(1)\, M_i^{\varepsilon/2}$$

by giving $O(1)\,M_i^{\varepsilon/2}$ to the $P_{(i,\Delta'')}$ α-vertex and $d(\Delta,\Delta')^4$ to the contraction line (if the field in Δ' belongs to C-vertex created by the contraction of a C-vertex (an inner C-vertex, see [1]) in Δ''' one uses $O(1)\,d(\Delta',\,\Delta''')^4\,d(\Delta''',\Delta'')^4$ and one attributes $d^4(\Delta',\Delta''')$ and $d^4(\Delta''',\Delta'')$ to the contraction lines generating the C-vertices). To choose the momentum i of the $P_{(i,\Delta'')}$-vertex, one uses $i \leqq M_i^{\varepsilon_1}$ for M_1 large enough depending on ε_1 which can be taken as small as we want. Finally we choose by a factor $A(i,\Delta'')=[M_i^\varepsilon]$ which $P_{(i,\Delta'')}$-vertex it is. We attribute both factors to the P-vertex.

Case β) The localization cube Δ'' and the momentum i are chosen using

$$\sum_{\Delta''\in\mathscr{D}_{\ell'}} d(\Delta',\Delta'')^{-4} \leqq O(1) \quad \text{and} \quad i \leqq M_i^{\varepsilon\varepsilon_1/2} \leqq |\Delta''|^{-\varepsilon_1}$$

for M_1 large enough depending on ε and ε_1. There is only one $P_{(i,\Delta'')}\beta$-vertex. The attributions are done as above.

4) Choice of the field, the generating P-vertex being fixed: We consider 3 cases. A factor 3 separates them (attributed to the generating P-vertex).

a) The field in Δ' is in a new C-vertex (the contraction in discussion has created this C-vertex). A factor $O(1)$ ($O(1)=8$) is enough to choose the field.

b) The field in Δ' belongs to a α-vertex. A P_α-vertex creates at most $O(1)$ ($O(1)=96$) free legs.

c) The field in Δ' belongs to a β-vertex. A P_β-vertex creates at most $O(1)$ ($O(1)=24$) free legs.

Thus each P-vertex generates at most $O(1)$ legs. Therefore each combinatoric factor for the contraction attributed in 1)–3) to the generating P-vertex is raised to the power $O(1)$.

Remarking also that $O(1)$ factors by P-vertex can be bounded by $M_i^{\varepsilon_1}$ if it is a $P_{(i,\Delta)}\alpha$-vertex, and by $|\Delta|^{-\varepsilon_1}$ if it is a $P_{(i,\Delta)}\beta$-vertex, we get

- $d^{O(1)}$ per contraction line;
- $b^{3(1+\varepsilon_2)/2}$ per $\delta\varphi$ leg;
- $M_i^{O(1)\varepsilon+O(1)\varepsilon_1}$ per $P_{(i,\Delta)}$-vertex of Type α;
- $|\Delta|^{-O(1)\varepsilon_1}$ per $P_{(i,\Delta)}$-vertex of Type β.

One obtains the lemma with $\varepsilon_3=\sup(O(1)\varepsilon_1, O(1)\varepsilon+O(1)\varepsilon_1)$.

III.4. The Combinatoric Factors

In this part we collect the combinatoric factors introduced in III.1, III.2, and III.3. From III.1, we have a factor $O(1)|\Delta|^{O(1)\varepsilon_1}$ per $P_{(i,\Delta)}\beta$-vertex corresponding to the decomposition of improperly localized legs $\left(\text{this is the part in the } \sum_{R'_a} \text{ in (3.3.1)}\right)$ which is not due to contractions, and a $d^{O(1)}$ per contraction generating a C_β-vertex. From III.2, we get a factor $O(1)$ per P-vertex, and from III.3 the attribution of Lemma 3.3.1.

Including the λ-dependence, in particular the one explained at the end of Section III.1, one can replace [each P-vertex generates at most $O(1)$ C-vertices] the $O(1)$ factors in front of the P-vertices by $M_i^{\varepsilon_3}$ per P_β-vertex and $|\Delta|^{-\varepsilon_3}$ per P_α-vertex provided M_1 is chosen large enough depending on ε_3 and on the parameters ϱ and σ of the domain of variation $D_{\varrho,\sigma}$ of λ.

Noticing that the combinatoric factors have the same form when raised to a power, one gets

Proposition 3.4.1. *Let ε_2 and ε_4 be small enough, $\varepsilon_4 > O(1)\varepsilon$ for some $O(1)$, then there exists $M_1(\lambda, \varepsilon_4, \varepsilon)$ large enough and two constants $C_1(\lambda, M_1)$ and n_2 such that*

$$|\int R_a e^{-V_a} d\mu| \leqq C_1^{|\Lambda|} \sup_G \left\{ \prod_{P_{(i,\Delta)}\alpha\text{-vertex}} M_i^{\varepsilon_4} \prod_{P_{(i,\Delta)}\beta\text{-vertex}} |\Delta|^{-\varepsilon_4} \cdot \prod_{\text{contraction}} d^{n_2} \prod_{\delta\varphi\,\text{leg}} b^{-3(1+\varepsilon_2)/2}\, G \right\}^{1/2}.$$

Here the contraction graph G includes the β-vertices with their new coefficient as described in III.1.

III.5. The Bounds on the Graphs

This part concerns the extraction of the convergent factors from estimates on the graphs. It is very similar to the Sections 5 and 6 of [1], and we will not repeat the proofs. For example, one can obtain, as in [1], a factor $O(1)d^{-n_1}$ per contraction with n_1 as large as we want provided $O(1)$ is large enough.

The method used in [1] to estimate a large graph G is to decompose it as a disjoint union of small subgraphs and then to estimate these subgraphs. A delicate technical point in [1] was the control of chains of divergent subgraphs (P_D). This was done using regularity conditions in the way the covers were defined. We propose here an alternative analysis. Let us call $P_{\alpha,3}$ the P_α-vertices which contract at least three legs to earlier vertices (i.e. produced in an earlier stage as referred to in the order relation defining the expansion). Two $P_{\alpha,3}$-vertices are connected if they are linked by a contraction line (it is possible by condition α_3) of II.4), and consider a connected set of $P_{\alpha,3}$-vertices. The boundary elements of this set are the $P_{\alpha,3}$-vertices which have contractions with C or P_β-vertices; these are initial and final boundary elements (as in [1]). We decompose each such set (in a nonunique way) into a disjoint union of linear chains whose initial elements are the initial boundary elements. In a chain of $P_{\alpha,3}$-vertices we call $P_{I,D}$ the two boundary vertices and $P_{I,C}$ the other intermediate vertices. The subindices D and C are for divergent and convergent. To generate a divergent mass subdiagram a $P_{\alpha,3}$ has to contract three of its legs to the same earlier vertex. Such a fact is impossible for a $P_{I,C}$ vertex by construction.

We complete the enumeration of vertices in divergent and convergent in the following way. Each P_α-vertex contracting less than three legs to earlier vertices is said to be convergent: $P_{\alpha,C}$-vertex. A $P_{(i,\Delta)}\beta$-vertex whose higher momentum is M_j is a divergent $P_{\beta,D}$-vertex if $i<j$ (the size of the localization cube Δ cannot compensate a logarithmic divergence in $\ln M_j$) and a convergent $P_{\beta,C}$-vertex if $i=j$. A C-vertex localized in Δ whose higher cutoff is M_j is a divergent C_D-vertex if $|\Delta|^{-1}<M_j^{\varepsilon/2}$.

Before we give the decomposition of G in small graphs, we describe a procedure associating in a finite way the divergent vertices to convergent vertices. This association will be important in transferring convergent factors to divergent vertices. (It was emphasized in [1] how basic is the fact that the number of P_D subgraphs is bounded by the number of P_C subgraphs.)

A $P_{\alpha,D}$ (a P_α-vertex which is not a $P_{\alpha,C}$-vertex) or a $P_{I,D}$-vertex is associated to the latest P-vertex to which it contracts or to the P-vertex which created the latest C-vertex to which it contracts. If, following this procedure, the $P_{I,D}$-vertex is associated to another $P_{I,D}$ (chain without $P_{I,C}$) then this last $P_{I,D}$ is necessarily associated to a P_β or to a $P_{\alpha,C}$ vertex and we associate also the first $P_{I,D}$-vertex to this P_β or $P_{\alpha,C}$ vertex. In this way each $P_{\alpha,D}$ or $P_{I,D}$-vertex is associated to a P_β or to a $P_{\alpha,C}$-vertex, and to each such vertex is associated a finite number (72 at most) of $P_{\alpha,D}$ or $P_{I,D}$-vertices.

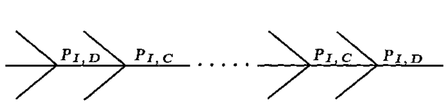
One then decomposes the large graph G in a disjoint union of small graph g_j of the following type:

—a $P_{\alpha,C}$-vertex, the C-vertices it created, and the associated $P_{\alpha,D}$ vertices;

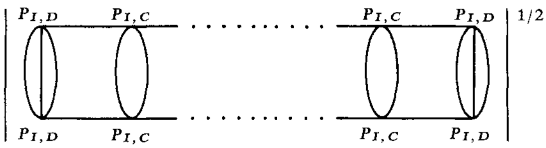
—a P_β-vertex, the C vertices it created, and the associated $P_{\alpha,D}$ vertices;

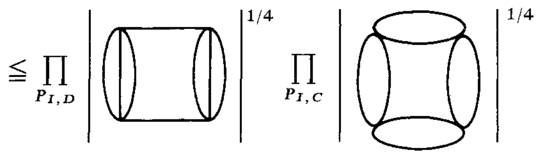
—a chain of $P_{I,C}$-vertices and the $P_{I,D}$ vertices at each end, following the procedure in [1].

The $P_{\alpha,C}$ and P_β subgraphs are numbered in the order they are produced, the chains are numbered in an arbitrary way compatible with the preceding enumeration. The $P_{\alpha,C}$ and P_β subgraphs are decomposed in elementary subgraphs as in [1]. The chains are bounded by their Hilbert-Schmidt norms which themselves are bounded by the Hilbert Schmidt norms of their elementary subgraphs. For example the chain

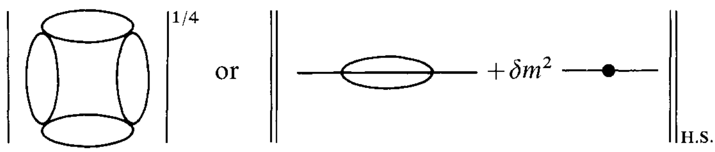


is bounded by

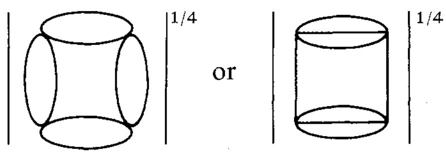




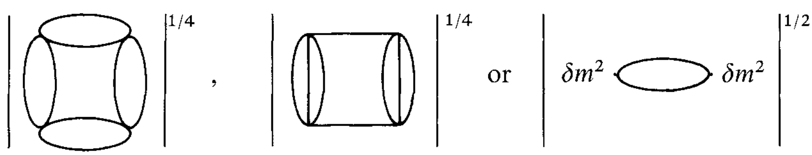
One gets finally a bound for G has products of



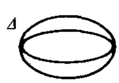
by $P_{\alpha,C}$-vertex



by $P_{\alpha,D}$ or $P_{I,D}$-vertex



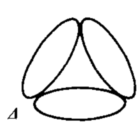
by β-vertex with 4 properly localized legs or by C_α-vertex



with at least one improperly localized leg and with a derived leg (of index $i: M_i^{\varepsilon/2} < |\Delta|^{-1}$)

or

by P_β-vertex which is an energy counterterm



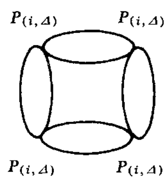
with a derived leg (of index $i: M_i^{\varepsilon/2} < |\Delta|^{-1}$)

$\|\beta\text{-vertex}\|_{\text{H.S.}}$

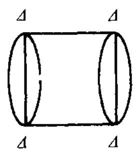
by β-vertex with not all legs properly localized.

These elementary graphs are bounded by

Proposition[5] **3.5.1.** *Let τ be sufficiently small, then there exist $\lambda_1 > 0$ such that*

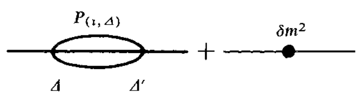
a) 

$$\left| \;\cdot\; \right|^{1/4} \leqq |\Delta|^{\lambda_1} M_i^{-\lambda_1} \leqq |\Delta|^{\lambda_1}$$

b) 

$$\left| \;\cdot\; \right|^{1/4} \leqq |\Delta|^{2\lambda_1} \ln M_i \qquad \textit{if the high momentum is } M_i$$

$$\leqq |\Delta|^{\lambda 1} \qquad \textit{if the vertex is a } P_{\beta, C}\textit{-vertex}$$

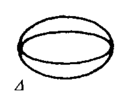
c) 

$$\left\| \;\cdot\; \right\|_{\mathrm{H.S.}} \leqq d(\Delta, \Delta')^{-n_1} |\Delta|^{\lambda_1} M_i^{-\lambda_1}$$

d) 

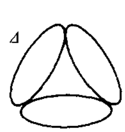
$$\left| \;\cdot\; \right|^{1/2} \leqq |\Delta|^{2\lambda_1} \ln M_i \qquad \textit{if the high momentum is } M_i$$

$$\leqq |\Delta|^{\lambda_1} \qquad \textit{if the vertex is a } P_{\beta, C}\textit{-vertex}$$

e) 

$$\left| \;\cdot\; \right| \leqq O(1) |\Delta|^{1-\delta} |\Delta|^{-1/3} \ln M_i \leqq |\Delta|^{\lambda_1}$$

if in the energy counterterm one of the lines is improperly localized and another one is a derived leg



$$\left| \;\cdot\; \right| \leqq O(1) |\Delta|^{1-\delta} \ln M_i \leqq |\Delta|^{\lambda_1}$$

if in the energy counterterm one of the lines is a derived one

f) $$\|\beta\text{-vertex}\|_{\mathrm{H.S.}} \leqq \ln M_i |\Delta|^{2\lambda_1} \prod_{\delta\varphi \,\mathrm{leg}} b^{-3(1+\tau)/2}.$$

---

5 In the bounds we have systematically replaced a bound in $\ln M(2)$ (see [1]) by a bound in the logarithm of the highest momentum

Proof. a) and b) are Proposition 5.3.3 of [1]; the second inequality in b) comes from the fact that $M_i^{\varepsilon/2} < |\Delta|^{-1}$ for a $P_{\beta,C}$-vertex (definition of $P_{\beta,C}$) and $|\Delta|^{\lambda_1} \ln M_i \leqq 1$ if M_1 is large enough depending on λ_1. c) and d) are Proposition 5.3.4 of [1]; the second inequality in d) comes from the fact that for a $P_{\beta,C}$-vertex $M_i^{\varepsilon/2} < |\Delta|^{-1}$. e) is proven in Proposition 6.3.1 of [1]. Since one leg is improperly localized, $M(1) \leqq |\Delta|^{-1/3}$ (see [1]). Since a leg has been derived (Case β) one has $M_i^{\varepsilon/2} < |\Delta|^{-1}$. f) is Proposition 5.3.5 of [1]. In fact the β-vertices as defined at the end of Section III.1 are as the W-vertices of [1] as soon as one has eliminated the $\bar{\varphi}_{k,\Delta}$, using the fact that

$$\left(\int (\bar{\varphi}_{k,\Delta})^2 d\mu\right)^{1/2} \leqq O(1)|\Delta|^{-1/6} \leqq |\Delta|^{-1/4}.$$

If the β-vertex is a mass counterterm, one has used $|\delta m^2| \leqq O(1) \ln M_i$.

As a consequence of this proposition, one gets (see [1])

Proposition 3.5.2. *Let τ be sufficiently small. Then there exists $\lambda_2 > 0$, $n_0 > 0$ such that for M_1 large enough depending on λ_2 each graph G is bounded by*

—d^{-n_0} by contraction line;

—$b^{-3(1+\tau)/2}$ by $\delta\varphi$ leg;

—$M_i^{-\lambda_2}$ by $P_{I,C}$ or $P_{\alpha,C}$ $P_{(i,\Delta)}$-vertex;

—$|\Delta|^{\lambda_2}$ by $P_{(i,\Delta)}\beta$-vertex;

—$\ln M_i$ by $P_{\alpha,D}$ and $P_{I,C}$-vertex and by C_D-vertex localized in Δ and with high momentum M_i with $M_i^{\varepsilon/2} > |\Delta|^{-1}$.

n_0 can be taken as large as we want.

To compare these factors with the combinatoric factors of Proposition 3.4.1, one has to eliminate the logarithmic divergences of Proposition 3.5.2. This will be done in the next section.

III.6. Compensation of the Logarithmic Divergences

We want to prove

Proposition 3.6.1. *Let τ be sufficiently small. There exists $\lambda_3 > 0$ and $n_1 > 0$ such that for M_1 large enough depending on λ_3 and n_1, each graph G is bounded by*

—d^{-n_1} by contraction line;

—$b^{-3(1+\tau)/2}$ by $\delta\varphi$ leg;

—$M_i^{-\lambda_3}$ by $P_{(i,\Delta)}\alpha$-vertex;

—$|\Delta|^{\lambda_2}$ by $P_{(i,\Delta)}\beta$-vertex.

n_1 can be chosen as large as we want (provided n_0 of Proposition 3.5.2 was large enough).

In comparison with Proposition 3.5.2, $n_0 > n_1$ and $\lambda_2 > \lambda_3$; we have eliminated the logarithms using a part of the convergent factors. Comparing now with Proposition 3.4.1, one sees that one gets Proposition 2.3.2 by taking $\varepsilon_2 < \tau$, $n_2 < n_1$, ε_4 small enough with respect to λ_3 and then setting $\varepsilon_0 = \lambda_3 - \varepsilon_4$.

Proof of Proposition 3.6.1. This proposition is proved by transferring a part of the convergent factors of (convergent) P-vertices to divergent vertices. The convergent factor $M_i^{-\lambda_2}$ (or $|\Delta|^{-\lambda_2}$) by convergent P-vertex is divided in 4 parts, one $M_i^{-\lambda_2/4}$ remains to this P-vertex, the other three $M_i^{-\lambda_2/4}$ are used to attribute convergent factors successively to $P_{\alpha,D}$ and $P_{I,D}$ vertices, to C_D-vertices, and finally to $P_{\beta,D}$-vertices.

III.6.1. Attribution of Convergent Factors to $P_{\alpha,D}$ and $P_{I,D}$-Vertices

Each of these $P_{(i,\Delta)}\alpha$-vertices, $\Delta\in\mathscr{D}_\ell$, has a logarithmic divergence $\ln M_j$ where M_j is the high momentum of the vertex. We replace it by a divergence depending only on the index i of the $P_{(i,\Delta)}$-vertex. In fact, if $j>i$, then by definition of the expansion, this means one has created $M_j^\varepsilon P_{(j,\Delta)}$-vertices in Δ. Thus j being fixed, all the $\ln M_j$ of later $P_{(i,\Delta)}\alpha$-vertices with high momentum M_j can be replaced by a divergent factor $M_i^{\lambda_4}$ by $P_{(j,\Delta)}$-vertex using

$$\prod_{i<j}(\ln M_j)^{M_i^\varepsilon}\leqq(\ln M_j)^{jM_j^\varepsilon}\leqq(M_j^{\lambda_4})^{M_j^\varepsilon}$$

for λ_4 as small as we want provided M_1 is large enough. For later convenience, we take $\lambda_4<\lambda_2(7\times4\times8\times72)^{-1}$. Such a divergent factor being also attributed to convergent P_α-vertices, we take $(M_j^{-\lambda_2/4})^{1/4}$ to compensate it (using $M_j^{\lambda_4}M_j^{-\lambda_2/16}\leqq1$).

We now compensate the bound $M_i^{\lambda_4}$ per divergent $P_{\alpha,D}$ and $P_{I,D}$-vertex of the type $P_{(i,\Delta)}$, $\Delta\in\mathscr{D}_\ell$. Each divergent $P_{(i,\Delta)}$-vertex is associated with a $P_{(k,\Delta')}$-vertex, $\Delta'\in\mathscr{D}_{\ell'}$, $\ell'\leqq\ell$, which can be a $P_{\alpha,C}$ or P_β vertex. There is a finite number, less than or equal to 72, of divergent vertices associated to a given $P_{\alpha,C}$-vertex.

The attribution is proved inductively starting from earlier covers.

1.1) Case $\ell=0$, i.e. $\ell'=0$. All vertices in $\mathscr{D}_0$ are α-vertices. Since $P_{(k,\Delta')}$ is earlier than $P_{(i,\Delta)}$, $k\geqq i$. We use a factor $(M_k^{-\lambda_2/4})^{1/2}$ by $P_{\alpha,C}$-vertex to attribute convergent factors to the at most 72 divergent associated vertices using

$$M_k^{-\lambda_2/8}\leqq(M_k^{-\lambda_2/8\times72})^{72}\leqq(M_i^{-7\lambda_4})^{72}\,. \tag{3.6.1}$$

One $M_i^{-\lambda_4}$ is used to compensate the divergent $M_i^{\lambda_4}$. Therefore it remains $M_i^{-6\lambda_4}$ per P_α-vertex in $\mathscr{D}_0$. Proceeding inductively we suppose that each P_α-vertex of type (i,Δ), $\Delta\in\mathscr{D}_m$, $m<\ell$, has a factor $M_i^{-6\lambda_4}$ (for a $P_{\alpha,C}$-vertex we use the fact that the remaining $(M_i^{-\lambda_2/4})^{1/4}\leqq M_i^{-6\lambda_4}$). Also, as before, by formula 3.6.1 we can assume that each $P_{\alpha,D}$ and $P_{I,D}$-vertex has obtained a factor $M_k^{-\lambda_2/8\times72}$ from the associated $P_{(k,\Delta')}$-vertex, if it is a P_α-vertex.

1.2) Case $\ell\geqq1$.

1.2.1) The associated $P_{(k,\Delta')}$-vertex is a $P_{\alpha,C}$-vertex

1.2.1.1) $\ell'=\ell$. Then by definition of the expansion $k\geqq i$. Using

$$M_k^{-\lambda_2/8\times72}\leqq M_i^{-7\lambda_4}$$

one has a $M_i^{-6\lambda_4}$ per divergent vertex (after compensation of the M^{λ_4} divergence)

1.2.1.2) $\ell'<\ell$.

a) $M_i \leqq M_k^4$. Using

$$M_k^{-\lambda_2/8\times 72} \leqq M_i^{-\lambda_2/8\times 4\times 72} \leqq M_i^{-7\lambda_4}$$

one proceeds as before.

b) $M_i > M_k^4$. Let $\Delta'' \in \mathscr{D}_{\ell-1}$, $\Delta'' \supset \Delta$. By definition of the expansion, one has created $M_i^\varepsilon P_{(i,\Delta'')}\alpha$-vertices and by the induction hypothesis we have at our disposal a factor $M_i^{-6\lambda_4}$. This gives (using only $M_i^{-\lambda_4}$) a factor

$$(M_i^{-\lambda_4})^{M_i^\varepsilon|\Delta|/|\Delta''|}$$

per cube Δ. From Definition 2.4.1

$$\frac{|\Delta|}{|\Delta''|} = \frac{|\Delta_\ell|}{|\Delta_{\ell-1}|} \geqq \frac{1}{8}|\Delta_\ell|^{\frac{\tau}{1+\tau}}$$

and from definition of α-vertices $|\Delta_\ell| > M_i^{-\varepsilon/2}$, thus

$$(M_i^{-\lambda_4})^{M_i^\varepsilon|\Delta|/|\Delta''|} \leqq (M_i^{-\lambda_4})^{\frac{1}{8}M_i^{\varepsilon(1-(\tau/2(1+\tau)))}} \tag{3.6.2}$$

as factor per cube Δ.

Now let $n_D(i,\Delta)$ be the number of $P_{(i,\Delta)}$-vertices of type $P_{\alpha,D}$ or $P_{I,D}$ in Δ associated with $P_{(k,\Delta')}\alpha$-vertices with $\Delta' \in \mathscr{D}_{\ell'}$, $\ell' < \ell$ and $M_k < M_i^{1/4}$. We consider two cases setting $N(i) = \frac{1}{56} M_i^{\varepsilon(1-(\tau/2(1+\tau)))}$.

α) $n_D(i,\Delta) \leqq N(i)$. Then the factor per cube of (3.6.2) can be distributed to the divergent vertices using

$$(M_i^{-\lambda_4})^{\frac{1}{8}M^{\varepsilon(1-(\tau/2(1+\tau)))}} \leqq (M_i^{-7\lambda_4})^{N(i)} \leqq (M_i^{-7\lambda_4})^{n_D(i,\Delta)} \tag{3.6.3}$$

and this gives a $M_i^{-6\lambda_4}$ per divergent vertex (after compensation of the $M_i^{\lambda_4}$ divergence).

β) $n_D(i,\Delta) > N(i)$. The idea is to use a large distance argument. First let us compute a bound on the number $P_{(k,\Delta')}P_\alpha$-vertices such that $M_k < M_i^{1/4}$. It is

$$\sum_{k:M_k<M_i^{1/4}} M_k^\varepsilon \leqq i M_i^{\varepsilon/4} \leqq M_i^{\varepsilon/2}$$

for M_1 large enough. On the other hand, the smallest possible volume of localization Δ' is such that $|\Delta'|^{-1} \leqq M_k^{\varepsilon/2} \leqq M_i^{\varepsilon/8}$ and the number of cover $\mathscr{D}_{\ell'}, \ell' \leqq \ell$ is $\ell + 1 \leqq |\Delta|^{-\varepsilon_1} \leqq M_i^{\varepsilon\varepsilon_1/2}$ for M_1 large enough. In this way at least half of the $n_D(i,\Delta)$ vertices are associated with a P-vertex localized in Δ' such that[6]

$$\bar{d}(\Delta,\Delta') = \tfrac{1}{2}(1+\mathrm{dist}(\Delta,\Delta')) \geqq \tfrac{1}{2}\left(\frac{1}{\frac{4\pi}{3}}\,\frac{N(i)|\Delta'|}{2M_i^{\varepsilon/2}M_i^{\varepsilon\varepsilon_1/2}}\right)^{1/3}$$

$$\geqq M_i^{\varepsilon/24}$$

---

6 We remark that $d(\Delta,\Delta') \geqq \bar{d}(\Delta,\Delta')$

by taking $\varepsilon_1 \leqq \frac{1}{8}$, $\tau \leqq \frac{1}{8}$ and M_1 large enough depending on ε to bound the numerical factors.

Thus, taking $O(1)$ large enough, for at least half of the P-vertices, one gets

$$\bar{d}(\Delta, \Delta')^{-O(1)} \leqq M_i^{-14\lambda_4}$$

and lead to a factor $M_i^{-6\lambda_4}$ to each $n_D(i, \Delta)$ divergent vertex (after compensation of the usual $M_i^{\lambda_4}$). It remains at least a factor $M_i^{-5\lambda_4}$ by $P_{\alpha,C}$ or $P_{I,D}$-vertices.

1.2.2) The associated $P_{(k,\Delta')}$-vertex is a P_β-vertex. Again let $\Delta'' \in \mathscr{D}_{\ell-1}$, $\Delta'' \supset \Delta$. By definition of the expansion, one has created $M_i^\varepsilon P_{(i,\Delta'')}\alpha$-vertices. As in 1.2.1.2) we will use the fact that one has at his disposal a factor $M_i^{-\lambda_4}$ per P_α-vertex in $\mathscr{D}_{\ell-1}$. Let $n_D(\beta, \Delta)$ be the number of $P_{\alpha,D}$ and $P_{I,D}$ vertices in Δ associated with a P_β-vertex. We consider two cases (as above)

α) $n_D(\beta, \Delta) \leqq N(i)$. It is as in 1.2.1.2b), α).

β) $n_D(\beta, \Delta) > N(i)$. The number of P_β-vertices in Δ is bounded by the number of j such that $M_j^{\varepsilon/2} \leqq |\Delta'|^{-1}$. But this number is bounded by $\ell' + n$ for some integer n depending on ε and τ. Thus given ε_4 as small as we want, there exists M_1 large enough depending on ε_4 and n such that

$$\ell' + n \leqq |\Delta'|^{-\varepsilon_4} \leqq |\Delta|^{-\varepsilon_4} \leqq M_i^{\varepsilon_4 \varepsilon/2}.$$

The smallest volume of localization Δ' is such that

$$|\Delta'|^{-1} \leqq |\Delta|^{-1} \leqq M_i^{\varepsilon/2}$$

and the number of possible covers is bounded by $M_i^{\varepsilon_1 \varepsilon/2}$. As in 1.2.1.1b), β) at least half of the $n_D(\beta, \Delta)$ divergent vertices are associated with a P_β-vertex in Δ' such that

$$\bar{d}(\Delta, \Delta') \geqq M_i^{\varepsilon/24}$$

for $\varepsilon_4 \leqq \frac{1}{16}$, $\varepsilon_1 \leqq \frac{1}{16}$ and $\tau \leqq \frac{1}{8}$. One can therefore do the usual attributions. As a result of 1) we have obtained that a factor $M_i^{-\lambda_2/2}$ per $P_{\alpha,C}$-vertex can be replaced by a factor $M_i^{-4\lambda_4}$ per P_α-vertex without logarithmic divergences. In the last two subsections we will use the remaining $(M_i^{-\lambda_2/4})^2$ per $P_{\alpha,C}$ vertex and $M_i^{-3\lambda_4}$ per P_α-vertex to control the divergences associated with C_D and $P_{\beta,D}$-vertices.

III.6.2. Domination of the Divergent C_D Vertices

Let us consider a C_D vertex localized in $\Delta \in \mathscr{D}_\ell$ with high momentum M_i and created by a $P_{(k,\Delta')}$ vertex $\Delta' \in \mathscr{D}_{\ell'}$. It diverges in $\ln M_i$. We consider several cases as in 1).

2.1) The associated $P_{(k,\Delta')}$-vertex is a $P_{\alpha,C}$-vertex.

a) $M_i \leqq M_k^4$. A P-vertex creates at most 16 C-vertices. Using the factor $M_k^{-\lambda_2/4}$ of the $P_{\alpha,C}$-vertex, one attributes convergent factors to the C-vertices

$$M_k^{-\lambda_2/4} \leqq (M_k^{-\lambda_2/4 \times 16})^{16} \leqq (M_i^{-\lambda_2/4 \times 4 \times 16})^{16} \leqq (M_i^{-\lambda_4})^{16}.$$

Since for M_1 large enough, $M_i^{-\lambda_4} \ln M_i \leqq 1$, one has dominated the logarithmic divergence of the C_D-vertices.

b) $M_i > M_k^4$. This means that (i, Δ) is of type α and therefore since the high momentum in C_D localized in Δ is M_i, one has created $A(i;\Delta) = M_i^\varepsilon P_{(i,\Delta)}\alpha$-vertices. Let $n_D^{(\alpha)}(C,\Delta)$ be the number of C_D-vertices in Δ with high momentum M_i.

α) $n_D^{(\alpha)}(C,\Delta) \leqq M_i^\varepsilon$. One takes a factor $M_i^{-\lambda_4}$ per $P_{(i,\Delta)}\alpha$-vertex (obtained in III.6.1)

$$(M_i^{-\lambda_4})^{M_i^\varepsilon} \leqq (M_i^{-\lambda_4})^{n_D^{(\alpha)}(C,\Delta)}$$

and this factor per C_D-vertex allows us to control the logarithmic divergences (as above).

β) $n_D^{(\alpha)}(C,\Delta) > M_i^\varepsilon$. The number of $P_{(k,\Delta')}$-vertices such that $M_k < M_i^{1/4}$ is bounded by

$$\sum_{k: M_k < M_i^{1/4}} M_k^\varepsilon \leqq M_i^{\varepsilon/2}$$

and the smallest possible volume of localization is bounded as

$$|\Delta'|^{-1} = |\Delta|^{-1} \leqq M_k^{\varepsilon/2} \leqq M_i^{\varepsilon/8}.$$

Thus at least half of the $n_D^{(\alpha)}(C,\Delta)$ vertices are associated with $P_{(k,\Delta')}$ localized in Δ' such that

$$\bar{d}(\Delta,\Delta') \geqq \left(\frac{1}{O(1)} \frac{M_i^\varepsilon}{M_i^{\varepsilon/2} M_i^{\varepsilon/8}}\right)^{1/3} \geqq M_i^{\varepsilon/12}$$

and for at least half of the C_D-vertices one has

$$\bar{d}(\Delta,\Delta')^{-O(1)} \leqq (M_i^{-\varepsilon/12})^{O(1)} \leqq M_i^{-2\lambda_4}$$

which allows us to compensate the logarithmic divergence.

2.2) The associated $P_{(k,\Delta')}$-vertex is a P_β-vertex. Since $P_{(k,\Delta')}$ is a β-vertex at least one has $\ell' \geqq 1$. Let $\Delta'' \in \mathscr{D}_{\ell-1}$, $\Delta'' \subset \Delta$. By definition of the expansion, one has created $M_i^\varepsilon P_{(i,\Delta'')}\alpha$-vertices. Let $n_D^{(\beta)}(C,\Delta)$ be the number of C_D-vertices localized in Δ associated with P_β-vertices.

One proceeds as in 1.2.1.2b) introducing $N(i)$

α) $n_D^{(\beta)}(C,\Delta) \leqq N(i)$. With a factor $M_i^{-\lambda_4}$ per $P_{(i,\Delta'')}\alpha$-vertex, one gets

$$(M_i^{-\lambda_4})^{N(i)} \leqq (M_i^{-\lambda_4})^{n_D^{(\beta)}(C,\Delta)}$$

which allows us to compensate the logarithmic divergence.

β) $n_D^{(\beta)}(C,\Delta) > N(i)$. The number of P_β-vertices in $\Delta' \subset \mathscr{D}_\ell$ is (see 1.2.2β))

$$\sum_{k: M_k^{\varepsilon/2} \leqq |\Delta'|^{-1}} 1 \leqq |\Delta'|^{-\varepsilon_4} \leqq M_i^{\varepsilon_4 \varepsilon/2}$$

since $|\Delta'|^{-1} \leqq M_i^{\varepsilon/2}$. Thus for at least half of the $n_D^{(\beta)}(C,\Delta)$ C_D-vertice

$$\bar{d}(\Delta,\Delta') \geqq \left(\frac{1}{O(1)} \frac{N(i)|\Delta'|}{M_i^{\varepsilon_4\varepsilon/2}}\right)^{1/3} \geqq M_i^{\varepsilon/24}$$

for $\varepsilon_4 \leqq \frac{1}{8}$, $\tau \leqq \frac{1}{8}$, and M_1 large enough. For at least half of the C_D vertices one gets

$$\bar{d}(\Delta,\Delta')^{-O(1)} \leqq M_i^{-2\lambda_4}$$

and thus one can compensate the logarithmic divergences.

To compensate the logarithmic divergences of the C_D-vertices, one has used a factor $M_i^{-2\lambda_4}$ per $P_{(i,\Delta)}\alpha$-vertex and a factor $M_i^{-\lambda_2/4}$ per $P_{\alpha,C}$-vertex.

III.6.3. Attribution of Convergent Factors to the $P_{\beta,D}$-Vertices

Let us consider a $P_{(i,\Delta)}$ divergent β-vertex with high momentum M_j. By definition of divergent P_β-vertex, this means $M_j > M_i$ and therefore that one has created at least one $P_{(j,\Delta)}$-vertex.

3.1) If $M_j^{\varepsilon/2} < |\Delta|^{-1}$, this is a P_β-vertex. At most $jP_{\beta,D}$ vertices can have divergences in $\ln M_j$; on the other hand, each $P_{(j,\Delta)}\beta$-vertex has a convergent factor $|\Delta|^{\lambda_2}$. We use $|\Delta|^{\lambda_2/2}$ to compensate the $P_{\beta,D}$-vertices attributing

$$(|\Delta|^{\lambda_2/2})^{1/j} \leqq (M_j^{-\varepsilon_2/4})^{1/j}$$

to each such $P_{\beta,D}$-vertex. Now for M_1 large enough depending on λ_2 and τ

$$M_j^{-\varepsilon\lambda_2/4}(\ln M_j)^j \leqq 1 .$$

One has therefore compensated the divergence and it remains a factor $|\Delta|^{\lambda_2/2}$ per P_β-vertex.

3.2) If $M_j^{\varepsilon/2} \geqq |\Delta|^{-1}$, this is a P_α-vertex. The number of $P_{\beta,D}$ vertices, with divergence in $\ln M_j$, is, in this case, also bounded by j. We use one of the factors $M_j^{-\lambda_4}$ attributed to each P_α-vertex to compensate the divergences

$$M_j^{-\lambda_4}(\ln M_j)^j \leqq 1$$

for M_1 large enough (as above).

To summarize: From a factor $(M_i^{-\lambda_2/4})^4$ per $P_{\alpha,C}$ vertex, $|\Delta|^{-\lambda_2/2}$ per P_β-vertex and $\bar{d}^{O(1)}$ per lines of contraction, we have obtained a factor $M_i^{-\lambda_4}$ per P_α-vertex, a factor $|\Delta|^{-\lambda_2/2}$ per P_β-vertex and the compensation of the logarithmic divergences of the $P_{\alpha,D}$, $P_{I,D}$, C_D and $P_{\beta,D}$ vertices. Proposition 3.6.1 follows with $n_1 = n_0 - O(1)$ and $\lambda_3 = \frac{\lambda_4}{2}$ (we use $M_i^{-\lambda_4/2}$ and $|\Delta|^{\lambda_4/2}$ to absorb $O(1)$ factors which could have been attributed to P-vertices after application of the triangular inequality for $\mathrm{dist}(\Delta, \Delta')$ to distribute the distance factors from the divergent vertices to the associated P-vertices through the lines of contraction).

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Appendix: The Borel Summability of φ_3^4

The Schwinger functions of the theory are defined as

$$S_\lambda(f_1, \ldots, f_n) = \lim_{\Lambda\to\infty} \lim_{\kappa\to\infty} \frac{\int \varphi_\kappa(f_1)\ldots\varphi_\kappa(f_n) e^{-V(\lambda,\Lambda,\kappa)} d\mu}{\int e^{-V(\lambda,\Lambda,\kappa)} d\mu} .$$

We want to prove (compare with [10]).

Theorem III′. *Let the mass parameter of the theory m be large enough, then for functions $f_1, \ldots, f_n$ in a suitable class, the perturbation series of the Schwinger functions of the $\lambda\varphi_3^4$ theory are Borel summable.*

The perturbation (renormalized) series of $S_\lambda(f_1, \ldots, f_n)$ is the Taylor series at $\lambda = 0^+$ of $S_\lambda(f_1, \ldots, f_n)$ (see [10] for a proof in the case of φ_2^4).

To prove Theorem III′ it is enough to show that there exists two constants $\lambda_0 > 0$ and $\delta_0 > 0$ such that for $\lambda \in D_{\lambda_0, -\delta_0}$.

1) $S_\lambda(f_1, \ldots, f_n)$ is analytic in λ.

2) For ξ small enough depending on δ_0

$$\left|\frac{d^N}{d\lambda^N} S_\lambda(f_1, \ldots, f_n)\right| \leqq C_1^N (N!)^{2+\xi} C_2(f_1, \ldots, f_n) \tag{A.0}$$

for two constants C_1 and C_2.

The proof of 1) can be made following step by step the argument of [11], using as a starting point the analyticity stated in Theorem II. We sketch the proof of 2), (for more details see [9]).

Following [11], $\frac{d^N}{d\lambda^N} S_\lambda(f_1, \ldots, f_n)$ has the form of a truncated function of $N+1$ points and can be expressed in terms of derivatives of unnormalized Schwinger functions. To control the infinite volume limit, the standard method is to apply the cluster expansion [2]. We recall how the expansion of φ_3^4 is combined with the cluster expansion (see [4] and [5]). One proceeds along three successive steps[7]:

a) one performs the algebraic expansion in $\mathscr{D}_0$;

b) one performs the cluster expansion;

c) one completes the algebraic expansion.

The cluster expansion has the form (see [2])

$$\int \prod \varphi(f_i) e^{-V} d\mu = \sum_{\Gamma \subset Z^{3*}} \int_0^1 \prod_{b \in \Gamma} d\sigma_b \left(\prod_{b \in \Gamma} \frac{d}{d\sigma_b} \int \prod \varphi(f_i) e^{-V(\sigma)} d\mu \right).$$

Let Λ be a connected subset of $\mathscr{D}_0$, suppose that $f_1, \ldots, f_n$ are functions supported in the cubes of $\mathscr{D}_0 \cap \Lambda$, and let $\Gamma \subset Z^{3*}$, $\Gamma \subset \Lambda$, then the combination of the two expansions takes the form

$$\begin{aligned}\prod_{b \in \Gamma} \frac{d}{d\sigma_b} \int \varphi(f_1) \ldots \varphi(f_n) e^{-V(\lambda;\Lambda)} d\mu &= \sum_a \prod_{b \in \Gamma} \frac{d}{d\sigma_b} \int R_a e^{-V_a(\lambda;\Lambda)} d\mu \\ &= \sum_a \sum_b \int R_{b,a} e^{-V_{b,a}(\lambda;\Lambda)} d\mu = \sum_a \sum_b \sum_c \int R_{c,b,a} e^{-V_{c,b,a}(\lambda;\Lambda)} d\mu\end{aligned} \tag{A.1}$$

where the indices a, b, c refer to the three steps a, b, and c.

The action of a derivation $\frac{d}{d\lambda}$ is localized in $\mathscr{D}_0$:

$$\frac{d}{d\lambda} = \sum_{\Delta \in \mathscr{D}_0} \chi_\Delta(x) \frac{d}{d\lambda} = \sum_{\Delta \in \mathscr{D}_0} \frac{d}{d\lambda_\Delta}$$

---

[7] Although the algebraic expansion and the cluster expansions are commutative, this order provides the suitable recombinations

with the meaning

$$\frac{d}{d\lambda_\Delta}\left(\lambda^q \int :\varphi^p:(x)\chi_\Lambda(x)dx\right)^n = n\left(\lambda^q \int :\varphi^p:(x)\chi_\Lambda(x)dx\right)^{n-1}\left(q\lambda^{q-1}\int :\varphi^p:(x)\chi_{\Lambda\cap\Delta}(x)dx\right)$$

for any q, p and $n \in Z^+$.

Following the combinatoric analysis of [10], to prove 2) it is enough to show that there exist constants C_1, C_2, K, and ζ, K large enough depending on the mass m and ζ small enough depending on δ_0 such that for cubes $\Delta_1, \ldots, \Delta_r \in \mathscr{D}_0 \cap \Lambda$ one has

$$\left|\prod_{i=1}^{r} \frac{d}{d\lambda_{\Delta_i}} \prod_{b\in\Gamma} \frac{d}{d\sigma_b} \int \varphi(f_1)\ldots\varphi(f_n) e^{-V(\lambda;\Lambda)} d\mu\right|$$

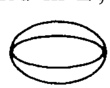
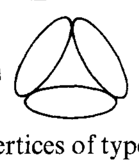
$$\leqq C_2(f_1, \ldots, f_n) e^{-K|\Gamma|} \prod_{j=1}^{r} C_1^{n(\Delta_j)} (n(\Delta_j)!)^{2+\zeta} \tag{A.2}$$

where $|\Gamma| \geqq |\Lambda| + 1$ and $n(\Delta) = \{\text{number of } \Delta_i | \Delta_i = \Delta, i = 1, \ldots, r\}$. According to (A.1), the left hand side of (A.2) is equal to

$$\sum_a \sum_b \sum_c \prod_{j=1}^{r} \frac{d}{d\lambda_{\Delta_j}} \int R_{c,b,a} e^{-V_{cba}(\lambda;\Lambda)} d\mu \,. \tag{A.3}$$

Our purpose is to estimate (A.3) using the method of the combinatoric factors.

The various derivations (with respect to the σ-parameters of the algebraic and cluster expansions or to the λ_Δ's) acting on the initial expression generate three types of vertices: the P-vertices of the algebraic expansion, the E-vertices of the cluster expansion (see [5]) and the D-vertices of the λ-derivations. The P and E vertices are known; we describe the D-vertices. The D-vertices obtained by a derivation $\frac{d}{d\lambda_\Delta}$ acting on the exponential are of four different types:

1) a $:\varphi^4:$ vertex localized in Δ;
2) a $\lambda\delta m^2 :\varphi^2:$ vertex localized in Δ;
3) an energy counterterm λ_Δ , one vertex being localized in Δ;
4) an energy counterterm $\frac{\lambda^2}{2}{}_\Delta$ , one vertex being localized in Δ.

Let $n_i(\Delta)$ be the number of D-vertices of type i generated by derivations in Δ, let $n_0(\Delta)$ be the derivations in Δ acting on vertices created during the steps a, b, and c, and let $n_E(\Delta)$ be the number of E-vertices generated in Δ.

Performing the derivation $\frac{d}{d\lambda}$ on the integrand of (A.3), this last expression is replaced by

$$\sum_{\text{deriv.}} \sum_a \sum_b \sum_c \int R_{d,c,b,a} e^{-V_{c,b,a}(\lambda;\Lambda)} d\mu \tag{A.4}$$

and in (A.4) $R_{d,c,b,a}$ has the form $R'_{d,c,b,a} \prod_{D\text{-vertex}} D$-vertex. We first compute bounds on the combinatoric factors for the sum over the derivations $\frac{d}{d\lambda}$.

Lemma A.1. *The following are upper bounds on the combinatoric factors for the λ-derivations*

— $M_i^{\varepsilon_5}$ *per* $P_{(i,\Delta)}\alpha$*-vertex;*
— $|\Delta|^{-\varepsilon_5}$ *per* $P_{(i,\Delta)}\beta$*-vertex;*
— $d^{O(1)}$ *per line of contraction;*
— $[O(1)n_E(\Delta)]^{O(1)}$ *per* P_E*-vertex localized in* Δ*;*
— $O(1)^{n(\Delta)}n_0(\Delta)!n_2(\Delta)!n_3(\Delta)!(n_4(\Delta)!)^2$ *per* $\Delta\in\mathscr{D}_0\cap\Lambda$,

where ε_5 *is small as we want provided* M_1 *is large enough and the* $O(1)$ *are some constants.*

Proof of Lemma A.1. We just outline the proof. One distinguishes three cases according to whether a λ-derivation acts on

1) a vertex created during the expansions a, b, or c;
2) the exponent;
3) a previously created D-vertex.

Case 1) is treated as in the algebraic and cluster expansions. To control the derived vertex, one refers it to its generating P_α, P_β or P_E vertex. One remarks also that if a λ-derivation acts on a vertex localized in $\Delta'\in\mathscr{D}_\ell\cap\Delta$, then this means (obviously) that this vertex has been created during the expansions and each P vertex generates at most a finite number of such vertices. Case 2) is just to decide which type of D-vertices are created. Case 3) is obvious noticing that a D-vertex of Type 1 has no more λ factor in front of it, that D-vertices of Type 2 and 3 have a λ factor, and a D-vertex of Type 4 has a λ^2 factor. This gives bounds of the type

$$O(1)^{n(\Delta)}[n_1(\Delta)!]^0\,[n_2(\Delta)!n_3(\Delta)!]^1\,[n_4(\Delta)!]^2\,.$$

The combinatoric factors for the sums over steps a, b, c are those of the algebraic and cluster expansions with two exceptions:

1) we treat separately the D-vertices;
2) if a λ derivation acts on β-vertex, it suppresses a power of λ and therefore (see III.1)) one cannot apply the procedure of domination by the exponential to the associated $\varphi_{\ell,\Delta}$.

To separate the D-vertices, we use for each integral of (A.4) the Schwarz inequality

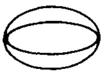
$$\begin{aligned}&\left|\int R_{d,c,b,a}e^{-V_{c,b,a}}d\mu\right|\\&\leqq\left(\int |R'_{d,c,b,a}|^2e^{-2\operatorname{Re}V_{c,b,a}}d\mu\right)^{1/2}\left(\int\left|\prod_{D\text{-vertex}}D\right|^2d\mu\right)^{1/2}.\end{aligned}\tag{A.5}$$

We prove

Lemma A.2. *There exists* $\zeta>0$ *as small as we want and constants* D_1, D_2, *and* $O(1)$ *depending on* ζ *such that*

$$\begin{aligned}&\left(\int\left|\prod_{D\text{-vertex}}D\right|^2d\mu\right)^{1/2}\\&\leqq D_1\prod_{\Delta\in\mathscr{D}_0\cap\Lambda}D_2^{n(\Delta)}[n_1(\Delta)!]^{2+\zeta}[n_2(\Delta)!]^{1+\zeta}[n_3(\Delta)!]\,[n_4(\Delta)!]^{\zeta}\prod_{P\text{-vertex}}O(1)\,.\end{aligned}\tag{A.6}$$

Proof of Lemma A.2. The integral under consideration is a sum over all possible schemes of contractions of vacuum graphs G. We first bound the number of terms. To do that, it is convenient to separate the φ^4 part of the D-vertices in two classes.

A) The φ^4-vertices which contract their 4 legs to another φ^4 vertex forming . Let $n_1^A(\Delta)$ *be the number of such vertices in* Δ.

B) The other class. Their number is $n_1^B(\Delta)=n_1(\Delta)-n_1^A(\Delta)$. We start computing the combinatoric factors associated with vertices of Type A. We suppose that the contractions were performed in the following order: first the vertices in Δ_1, then those (the ones not previously contracted) in Δ_2, and so on. Here $\Delta_1, \Delta_2, \ldots$ are unit cubes in $\mathscr{D}_0 \cap \Lambda$ such that

$$n_1^A(\Delta_1) \geqq n_1^A(\Delta_2) \geqq \ldots . \tag{A.7}$$

Let us consider an uncontracted vertex of Type A in Δ_i. It contracts with a vertex of Type A in $\Delta_j, j \geqq i$. By a factor $O(1)\, d(\Delta_i, \Delta_j)^4$ we choose the cube Δ_j and by a factor $2n_1^A(\Delta_j)$ (remember the squaring), we choose which vertex in Δ_j. Using

$$n_1^A(\Delta_j) \leqq n_1^A(\Delta_i)^{1/2} n_1^A(\Delta_j)^{1/2}$$

we attribute $[2n_1^A(\Delta_i)]^{1/2}$ to the contracting vertex and $[2n_1^A(\Delta_j)]^{1/2}$ to the contracted one. We obtain, therefore, as a total combinatoric factor for the contractions of Type A

$$\left(\prod_{\Delta} O(1)^{n(\Delta)} [n_1^A(\Delta)!]^{1/2} \right)^2 \prod_{\text{contr.}} d^4 . \tag{A.8}$$

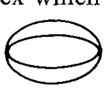
One considers now all the other contractions: Type B and mass counterterms. There are $N(\Delta)=2[4n_1^B(\Delta)+2n_2(\Delta)]$ fields of this type which are contracted. One introduces, as above, an order relation and one follows the same procedure.

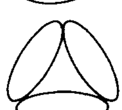
Finally, one gets

$$\left(\int \left| \prod_{D\text{-vertex}} D \right|^2 d\mu \right)^{1/2} \leqq \sup_G |G|^{1/2} \prod_{\text{contr.}} d^{O(1)} \prod_{\Delta} O(1)^{n(\Delta)} (n_1^A(\Delta)!)^{1/2} (n_1^B(\Delta)!)^2 n_2(\Delta)! . \tag{A.9}$$

We then bound a graph G. One first extracts the localization factors $d^{-O(1)}$ with $O(1)$ large enough to compensate the one in (A.9). Then to estimate G one decomposes it as a union of the following elementary graphs:

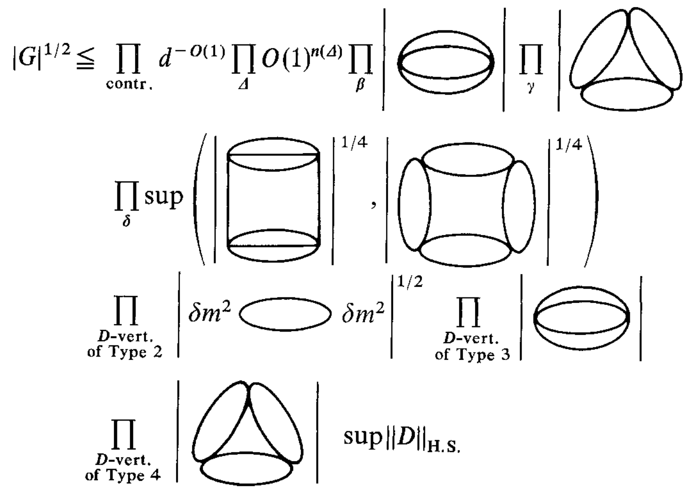
α) a D-vertex which is a vacuum counterterm;

β) a graph  formed by 2 vertices of Type A;

γ) a graph  formed by 3 vertices of Type B;

δ) a graph formed by 2 vertices with at most three contractions between them and eventually a φ^4 vertex if it contracts its 4 legs to these two vertices.

With these definitions, each graph G is a union of such graphs plus, eventually, an isolated D-vertex, and one gets as an obvious bound



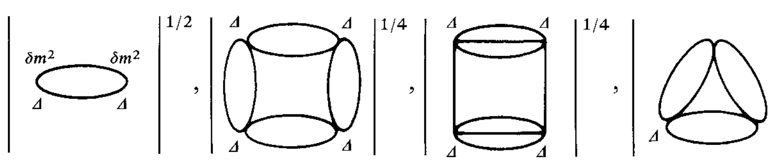
(A.10)

where we have separated the mass counterterms (Type 2 D-vertices) from subgraphs of Type δ, and subgraphs of Type α are written as being or of Type 3 or of Type 4. The last term is a bound on the eventually isolated D-vertex.

One has the following bounds (see III.2))

Lemma A.3.

1)  $\leqq O(1) + \sum_{\ell \neq 0} \sum_{\varrho_{\Delta_\ell} \in \mathscr{D} \cap \Delta} O(1) M^2_{a(\ell)} |\varrho_{\Delta_\ell}|$

2) 

are bounded by

$$O(1) + O(1) \sum_{\ell \neq 0} \sum_{\varrho_{\Delta_\ell} \in \mathscr{D} \cap \Delta} \ln M_{a(\ell)} |\varrho_{\Delta_\ell}|$$

3) $\|D\|_{\text{H.S.}} \leqq O(1) + \sum_{\ell \neq 0} \sum_{\varrho_{\Delta_\ell} \in \mathscr{D} \cap \Delta} O(1) M^2_{a(\ell)} |\varrho_{\Delta_\ell}|$.

The bounds of Lemma A.3 are obvious; they are consequences of Proposition 3.5.1 and of Proposition 6.3.1 of [1] with $M(1) \ln M(2)$ bounded by $M^2_{a(\ell)}$.

We now use the definition of $a(\ell)$ and the fact that, for $\ell \neq 0$, in $\Delta_\ell \supset \varrho_{\Delta_\ell}$ at least one P-vertex has been created:

$$\sum_{\ell \neq 0} \sum_{\varrho_{\Delta_\ell} \in \mathscr{D} \cap \Delta} O(1) M^2_{a(\ell)} |\varrho_{\Delta_\ell}| \leqq \sum_{\substack{P\text{-vertex}\\ \text{in } \Delta}} O(1). \tag{A.11}$$

Using the concavity of the logarithm and the fact that $\{\varrho_{\Delta_\ell} | \Delta_\ell \subset \Delta, \Delta \in \mathscr{D}_0\}$ is a partition of Δ, we have also

$$\sum_{\ell \neq 0} \sum_{\varrho_{\Delta_\ell} \in \bar{\mathscr{D}} \cap \Delta} \ln M_{a(\ell)} |\varrho_{\Delta_\ell}| \leqq \ln \left(\sum_{\ell \neq 0} \sum_{\varrho_{\Delta_\ell} \in \bar{\mathscr{D}} \cap \Delta} M_{a(\ell)} |\varrho_{\Delta_\ell}| + 1 \right)$$
$$\leqq \ln \left(\sum_{\substack{P\text{-vertex} \\ \text{in } \Delta}} O(1) + 1 \right). \tag{A.12}$$

Now from $(pa)^n \leqq n! \, (e^a)^p$ and $[\ln(1+x)]^n \leqq C(\zeta) [n!]^\zeta e^x$ valid for p and $x \geqq 0$, $n \in Z^+$ and $\zeta > 0$, one gets

$$\left[\sum_{\substack{P\text{-vertex} \\ \text{in } \Delta}} O(1) \right]^n \leqq n! \prod_{\substack{P\text{-vertex} \\ \text{in } \Delta}} O(1)$$
$$\left[\ln \left(\sum_{\substack{P\text{-vertex} \\ \text{in } \Delta}} O(1) + 1 \right) \right]^n \leqq C(\zeta) [n!]^\zeta \prod_{\substack{P\text{-vertex} \\ \text{in } \Delta}} O(1). \tag{A.13}$$

Finally one obtains

$$|G|^{1/2} \leqq O(1) \prod_{P\text{-vertex}} O(1) \prod_{\text{contr.}} d^{-O(1)} \prod_{\Delta \in \mathscr{D}_0 \cap \Lambda} O(1)^{n(\Delta)} (n_1^A(\Delta)!)^{1/2}$$
$$\cdot (n_1^B(\Delta)!)^\zeta (n_2(\Delta)!)^\zeta (n_3(\Delta)!) (n_4(\Delta)!)^\zeta. \tag{A.14}$$

Lemma A.2 follows from (A.9) and (A.14).

It remains now to discuss the treatment of the improperly localized legs of the derived β-vertices. For a derived β-vertex, we modify the procedure of III.1. We write the improperly localized legs as

$$\varphi_{\ell,\Delta}(x) = \bar{\varphi}_{\ell,\Delta} \chi_\Delta(x) + \sum_{i=1}^{\ell+1} \delta\varphi_{i,\Delta}(x).$$

We separate the $\bar{\varphi}_{\ell,\Delta}$ legs of $R'_{d,c,b,a}$ from the remaining applying the Schwarz inequality. Let $r(\Delta')$ be the number of $\bar{\varphi}_{\ell,\Delta'}$ legs coming from derived β-vertices and localized in Δ'. One has

$$\sum_{\Delta' \subset \Delta} r(\Delta') \leqq 3 n_0(\Delta) \tag{A.15}$$

since each β-vertex has at least 3 improperly localized legs.

From the results of Chapter III combined with the cluster expansion (see [5]) one has

$$\left| \int |R'_{d,c,b,a}|^2 e^{-2\mathrm{Re}V_{c,b,a}(\lambda;\Lambda)} d\mu \right|$$
$$\leqq e^{-K|\Lambda|} \prod_{P_{(i,\Delta)}\alpha\text{-vert.}} M_i^{-\varepsilon_0} \prod_{P_{(i,\Delta)}\beta\text{-vertex}} |\Delta|^{+\varepsilon_0}$$
$$\cdot \prod_{P_\Delta E\text{-vertex}} [O(1) n_E(\Delta)]^{-O(1)} \prod_{\text{contr.}} d^{-O(1)}$$
$$\cdot \sup_x \left[\int \prod_{\Delta \in \bar{\mathscr{D}}} (\varphi_{\ell,\Delta} \chi_\Delta(x) |\Delta|^{1/4+\varepsilon_1})^{4r(\Delta)} d\mu \right]^{1/4} \tag{A.16}$$

where we have included the factor $|\Delta|^{1/4+\varepsilon_1}$ resulting from the bound on β-vertices of Proposition 3.5.1.

Comparing (A.16), (A.6), and Lemma A.1, one sees that the bound (A.0) is proved if one can show that [because of (A.15)]

$$\int \prod_{\Delta\in\tilde{\mathscr{D}}} (\bar{\varphi}_{\ell,\Delta}\chi_\Delta(x)|\Delta|^{1/4+\varepsilon_1})^{4r(\Delta)} d\mu$$
$$\leqq \prod_{\Delta\in\tilde{\mathscr{D}}} O(1)^{4r(\Delta)}\left(\frac{4r(\Delta)}{3}!\right) \prod_{P\text{-vertex}} O(1) \prod_{\text{contr.}}{}' d^{O(1)}. \tag{A.17}$$

The product over the contractions $\prod'$ excludes the contractions between $\bar{\varphi}_{\ell,\Delta}$ legs.

Proof of A.17. We decompose each leg $\bar{\varphi}_{\ell,\Delta}$ as $\bar{\varphi}_{\ell,\Delta}=\bar{\varphi}_{1,\Delta}+\bar{\varphi}_{2,\Delta}$ where $\bar{\varphi}_{1,\Delta}$ is the part of $\bar{\varphi}_{\ell,\Delta}$ with high momentum less than or equal to $|\Delta|^{-1/6}$ and $\bar{\varphi}_{2,\Delta}$ is the part of $\bar{\varphi}_{\ell,\Delta}$ with low momentum above $|\Delta|^{-1/6}$. This decomposition replaces the left hand side of (A.17) by a sum of terms. We perform the Gaussian integration and bound the sums, using the method of combinatoric factors. We compute them in two different ways and finally take a geometric mean.

Let $r_i(\Delta)$ be the number of $\varphi_{i,\Delta}$, $i=1,2$; one has $r_1(\Delta)+r_2(\Delta)=4r(\Delta)$.

1) We suppose that the contractions (in the Gaussian integration) are made starting from the smallest cubes. Consider $\varphi_{i,\Delta}$, $\Delta\in\mathscr{D}_\ell$ contracting with $\varphi_{j,\Delta'}$, $\Delta'\in\mathscr{D}_{\ell'}$. Using the fact that $\ell'+1\leqq\ell+1\leqq|\Delta_\ell|^{-\varepsilon_1/2}$, a factor $|\Delta|^{-\varepsilon_1/2}$ is enough to choose ℓ'. The cube Δ' is chosen noticing

1) if i or $j=2$

$$\sum_{\Delta'} d(\Delta,\Delta')^{-4}\leqq O(1)|\Delta'|^{-1/2}\leqq O(1)|\Delta'|^{-1/4}|\Delta|^{-1/4};$$

2) if $i=j=3$

$$\sum_{\Delta'} d(\Delta,\Delta')^{-4}\leqq O(1)|\Delta'|^{-1}\leqq O(1)|\Delta'|^{-1/2}|\Delta|^{-1/2}.$$

Finally, by a factor $O(1)d(\Delta',\Delta'')^4|\Delta''|^{-\varepsilon_1/2}$ we can choose the cube Δ'' and the index i of the $P_{(i,\Delta)}\beta$-vertex which has generated the contracted leg (and which leg).

In this way, one gets as combinatoric factor

$$\prod_{\text{contr.}} d^{O(1)} \prod_{P_{(i,\Delta)}\beta\text{-vert.}} O(1)|\Delta|^{-\varepsilon_1} \prod_{\Delta\in\tilde{\mathscr{D}}} O(1)^{4r(\Delta)}|\Delta|^{-(1/2r_1(\Delta)+1/4r_2(\Delta))}. \tag{A.18}$$

2) Let for $\Delta\in\mathscr{D}_0$, $N(\Delta)=\sum_{\Delta'\in\tilde{\mathscr{D}}\cap\Delta} r(\Delta')$. One orders the cubes of $\mathscr{D}_0\cap\Delta$ according to

$$N(\Delta_1)\geqq N(\Delta_2)\geqq\dots.$$

The contractions are supposed to be made starting from legs in Δ_1, then in Δ_2 and so on.

Let $\Delta\in\tilde{\mathscr{D}}\cap\Delta_k$ and $\Delta'\in\tilde{\mathscr{D}}\cap\Delta_{k'}$ and suppose that $\bar{\varphi}_{i,\Delta}$ contracts with $\bar{\varphi}_{j,\Delta'}$, $k'\geqq k$. With $N(\Delta_{k'})$, one chooses the leg in $\Delta_{k'}$ and $N(\Delta_{k'})\leqq N(\Delta_{k'})^{1/2}N(\Delta_k)^{1/2}$. What remains is as in 1), and one gets as a total combinatoric factor

$$\prod_{\text{contr.}} d^{O(1)} \prod_{P_{(i,\Delta)}\beta\text{-vert.}} O(1)|\Delta|^{-\varepsilon_1} \prod_{\Delta\in\mathscr{D}_0} (N(\Delta)^{1/2})^{N(\Delta)}. \tag{A.19}$$

Using the fact that $N(\Delta)\leqq 4\times 3n_0(\Delta)$, (A.19) is bounded by

$$\prod_{\text{contr.}} d^{O(1)} \prod_{P_{(i,\Delta)}\beta\text{-vert.}} O(1)|\Delta|^{-\varepsilon_1} \prod_{\Delta\in\mathscr{D}_0} O(1)^{n_0(\Delta)}[4n_0(\Delta)!]^{3/2}. \tag{A.20}$$

Taking the geometric mean of (A.18) and (A.20) with weight 1/3 and 2/3, one gets that the final combinatoric factor is

$$\prod_{\text{contr.}} d^{O(1)} \prod_{P_{(i,\Delta)}\beta\text{-vert.}} O(1)|\Delta|^{-\varepsilon_1} \prod_{\Delta\in\mathscr{D}_0} O(1)^{n_0(\Delta)} \cdot [n_0(\Delta)!]^4 \prod_{\Delta\in\tilde{\mathscr{D}}} |\Delta|^{-1/3\left(\frac{r_1(\Delta)}{2}+\frac{r_2(\Delta)}{2}\right)}. \tag{A.21}$$

To get (A.17) one has to bound the contraction graphs. One first extracts the localization factors and then one bounds the diagrams, using

$$\int \bar{\varphi}_{i,\Delta}\bar{\varphi}_{j,\Delta'}d\mu|\Delta|^{1/4+\varepsilon_1}|\Delta'|^{1/4+\varepsilon_1} \leqq \left(\int(\bar{\varphi}_{i,\Delta})^2 d\mu\right)^{1/2}|\Delta|^{1/4+\varepsilon_1}\left(\int(\bar{\varphi}_{j,\Delta'})^2 d\mu\right)^{1/2}|\Delta'|^{1/4+\varepsilon_1}$$

and

$$\left|\int(\bar{\varphi}_{1,\Delta})^2 d\mu\right|^{1/2}\leqq O(1)|\Delta|^{-1/12}\left|\int(\bar{\varphi}_{2,\Delta})^2 d\mu\right|^{1/2}\leqq O(1)|\Delta|^{-1/6}.$$

The required bound follows from the fact that $\frac{1}{4}-\frac{1}{12}=\frac{1}{6}$ and $\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$. This completes the proof of the Borel summability.

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