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Phase Transition and Fractals*)

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A geometrical interpretation of critical phenomena, particularly of the scaling relations on critical exponents is given by the use of statistical fractal dimensionality. The relation between this geometrical interpretation and the finite size scaling theory is also discussed.

The concept of connectivity dimensionality (or generalized fractal topological dimensionality) is introduced to discuss the relation between critical exponents and fractal dimensionalities of some explicit lattices.

§ 1. Introduction

Various concepts and techniques have been developed for dealing very complicated figures such as the Koch curve. In particular Hausdorff[1] introduced a fractal dimensionality to characterize such complicated figures. Mandelbrot[2] has found many possibilities of its application to nature.

The Hausdorff fractal dimensionality D is defined as follows. We consider systems which have the *similarity law* that they have similar structures for different scales of length, namely for different resolutions of measurement of length. That is, we assume that the volume V_d of the relevant system in the ordinary topological dimensionality d is related to the scale of length, r, as

$$V_d \propto (1/r)^D . \tag{1·1}$$

The exponent D in (1·1) is the Hausdorff or similarity dimensionality. A more formal definition of D will be given in § 2 in order to discuss the analogy of fractal dimensionality to the renormalization group approach.

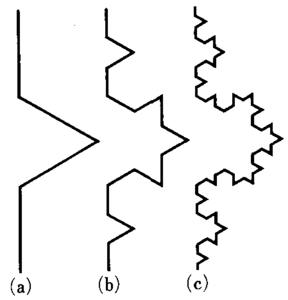

Fig. 1. Koch curve: (a) $\mathcal{F}_0$, (b) $\mathcal{F}_1$, (c) $\mathcal{F}_2$.

For example, the Koch curve which is constructed as in Fig. 1 has the Hausdorff dimensionality $D = \log 4/\log 3$, as is well-known.[2] Namely, the Koch curve in Fig. 1 is a little more complicated in structure than figures such as a straight line in the ordinary topological dimensionality $d = 1$, and the

*) This was reported at the workshop of fractals organized by Dr. B. B. Mandelbrot and held at Courchevel, France, July 18∼25, 1982.

small fraction $D-d=0.26\cdots$ expresses the degree of a deviation of the Koch curve from a straight line in structure qualitatively.

The above definition of the fractal similarity dimensionality D is also valid to irregular systems. This random case is practically more useful. The fractal similarity dimensionality of random or statistical systems may be called *statistical fractal dimensionality.* This concept is discussed in detail in § 3, and we clarify some essential aspects of the scaling law of critical phenomena from a geometrical viewpoint. The concept of connectivity dimensionality is introduced in § 4. In § 5, the dependence of critical exponents upon the connectivity dimensionality is discussed in some lattices with fractal dimensions.

§ 2. Relation between the renormalization group approach and fractals

First we give here a formal description of fractals. We denote a set of figures by $\{\mathcal{F}_n\}$. The initial figure is written as $\mathcal{F}_0$. We transform $\mathcal{F}_0$ into a more detailed figure $\mathcal{F}_1$. The resolution of measurement of length for $\mathcal{F}_1$ is better than that of $\mathcal{F}_0$ by the factor b. We describe this mapping by $\mathcal{T}$ in the same scale (a unit), namely

$$\mathcal{T}\mathcal{F}_0=\mathcal{F}_1\,,\quad \mathcal{T}\mathcal{F}_1=\mathcal{F}_2\,,\cdots,\ \mathcal{T}\mathcal{F}_n=\mathcal{F}_{n+1}\,,\cdots. \tag{2·1}$$

The similarity law is assured by the existence of the fixed point

$$\lim_{n\to\infty}\mathcal{F}_n=\mathcal{F}^*\,. \tag{2·2}$$

Next we change the scale of length at each state by the factor $(1/b)$, according to the resolution of measurement. We denote this transformation by S_b. Now, an arbitrary quantity such as the total length, area or volume associated with the figure $\mathcal{F}_n$ in the scale $S_b{}^n$ is denoted by $Q(S_b{}^n\mathcal{F}_n)$. Then, we have the relation

$$S_b{}^n\mathcal{F}_n=(S_b\mathcal{T})(S_b^{n-1}\mathcal{F}_{n-1})=(S_b\mathcal{T})^n\mathcal{F}_0\,, \tag{2·3}$$

by definition. The similarity law is now expressed more directly by

$$Q(S_b\mathcal{F}_{n+1})=b^{d_Q}Q(\mathcal{F}_n), \tag{2·4}$$

where d_Q denotes the fractal dimensionality of this fixed point figure $\mathcal{F}^*$, by definition. Thus, it should be noted that the fractal dimensionality depends on what quantity is chosen to describe the feature of the corresponding figure. In the case when (2·4) holds only for large n, the fractal dimensionality is defined by the limit

$$d_Q=\lim_{n\to\infty}\frac{\log\{Q(S_b\mathcal{F}_{n+1})/Q(\mathcal{F}_n)\}}{\log b}\,. \tag{2·5}$$

In any case, under the condition of the existence of the fixed point $\mathcal{F}^*$, we may also write

$$d_Q = \frac{\log\{Q(S_b\mathcal{F}^*)/Q(\mathcal{F}^*)\}}{\log b}, \qquad (2\cdot6)$$

or equivalently

$$Q(S_b\mathcal{F}^*) = b^{d_Q}Q(\mathcal{F}^*). \qquad (2\cdot7)$$

Thus, it should be remarked that the similarity law is the direct consequence of the facts that there exists some recursion formula to construct figures and that there exists a fixed point of the recursion procedure.

This feature is quite the same as the relation between the scaling law and the renormalization group approach. In particular, the real space renormalization group R_b can be regarded as a combined transformations $S_b\mathcal{T}$. Up to now, the effective Hamiltonian has been studied recursively in the renormalization group approach. However, the present author proposes in § 3 a geometrical interpretation of critical phenomena and consequently the similarity between the fractal theory and the renormalization group approach becomes quite clear.

§ 3. Statistical fractal dimensionality and geometrical interpretation of the scaling law and critical exponents

As was briefly mentioned in § 1, the concept of statistical fractal dimensionality is very useful in random systems and also in regular lattices with stochastic variables such as the Ising model in thermal equilibrium. In these cases, it is convenient to discuss the scaling property of the ensemble average Q of the relevant physical quantity Q_{op} $(Q = \langle Q_{op} \rangle)$. Such a physical quantity Q depends on the volume $V = L^d$ of the system. If it is extensive, it is proportional to $V = L^d$ by definition. Consequently, the *statistical fractal dimensionality* or *scaling exponent* φ is equal to d for an extensive physical quantity.

In general, such a quantity Q is not extensive at the critical point, due to critical fluctuations. Since the volume $V = L^d$ depends on the scale factor r as

$$V' = (1/r)^d V, \qquad (3\cdot1)$$

the relevant physical quantity Q may depend generally on the scale factor r. Thus, we denote this dependence of the scale factor of Q by $Q[r]$. Then, the statistical fractal dimensionality or scaling exponent φ of the quantity Q is defined by the scaling property

$$Q[r] = r^{-\varphi}Q[1] \qquad (3\cdot2)$$

for the scale transformation of length, $L' = L/r$. Equivalently, if the physical quantity Q for a finite system of length L takes the asymptotic form

$$Q(L) \sim L^{\phi}, \tag{3·3}$$

then ϕ is also a scaling exponent and it is equal to φ, i.e., $\phi=\varphi$, because we have

$$Q(L')=Q(L/r) \sim r^{-\phi}Q(L). \tag{3·4}$$

The first definition of the scaling exponent in (2·2) is also valid for systems infinite from the beginning. This corresponds to the C^*-algebra. The second definition of the scaling exponent (3·4) is justified by the finite-size scaling law[4),5)]

$$Q(\varepsilon, h, L)=L^{\varphi_0}f(h\varepsilon^{-\beta\delta}, \varepsilon L^{1/\nu}), \tag{3·5}$$

where $\varepsilon=(T-T_c)/T_c$ and h denotes the symmetry breaking field. Here, β, δ and ν are critical exponents defined in the ordinary way.[3)]

More explicitly we consider the magnetic phase transition. The magnetization per unit volume of a ferromagnet with a finite size L is described[4),5)] by the following equation of state:

$$m(L) \simeq L^{-\beta/\nu}m(h\varepsilon^{-\beta\delta}, \varepsilon L^{1/\nu}) \tag{3·6}$$

for small h and for large L near the critical point. In particular, the magnetization for $h=0$ is defined by $m(L)=\langle|M|\rangle/N$ for the total magnetization variable M and it satisfies the following finite-size scaling law[4),5)]

$$m(L) \simeq L^{-\beta/\nu}g(\varepsilon L^{1/\nu}). \tag{3·7}$$

Consequently the magnetization at the critical point $\varepsilon=0$ is given by[4)]

$$m(L) \simeq g(0)L^{-\beta/\nu}. \tag{3·8}$$

The total magnetization $M(L)=Vm(L)=L^d m(L)$ is then given by[6)]

$$M(L) \sim L^{d-\beta/\nu} \sim L^{(d+2-\eta)/2} \tag{3·9}$$

at the critical point, where we have used the scaling relation $\beta=\nu(d-2+\eta)/2$ to derive the second relation of (3·9). This power-law dependence with respect to the size L of the system corresponds to the similarity law with respect to the scaling transformation $L'=L/r$. This is the most essential feature of critical phenomena. This also reflects the non-existence of a characteristic length at the critical point, namely the correlation length ξ is infinite at the critical point. According to the philosophy of Mandelbrot[2)] that any complex phenomenon with the similarity law may be described by the fractal dimensionality of some geometrical reality corresponding to the phenomenon, we may expect to find some geometrical interpretation of critical phenomena with the use of the fractal dimensionality.

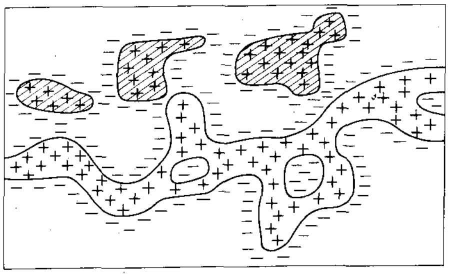

Fig. 2. Resultant dominant clusters which are left by eliminating smaller clusters until the number of eliminated spins becomes $N/2$.

For this purpose, we propose here a concept of *resultant dominant clusters* (RDC) at the critical point. For simplicity, we consider the Ising model. There are many spin clusters with different scales from an atomic distance to the size L of the system, at the critical point. Larger clusters are more important to describe critical phenomena. On the other hand, the number of up (or down) spins, N^+ (or N^-) is almost equal to $N/2$ with a fluctuation ΔM (or $-\Delta M$), namely

$$N^+=\frac{1}{2}N+\Delta M \quad \text{and} \quad N^-=\frac{1}{2}N-\Delta M\,, \tag{3·10}$$

for a specific configuration. The corresponding total magnetization M of the system is given by

$$M=N^+-N^-=2\Delta M\,. \tag{3·11}$$

In order to find a physical geometrical interpretation of the critical magnetization M, we eliminate smaller clusters of up (or down) spins until the total number of these eliminated spins becomes $N/2$ when $\Delta M>0$ (or $\Delta M<0$), as is shown in Fig. 2. The number of the remaining larger clusters may be relatively small. We call these remaining (up spin) clusters (positive) 'resultant dominant clusters' (RDC). The size of these RDC becomes infinite in the thermodynamic limit $L\to\infty$ at the critical point, namely they are percolated clusters. The number of the total spins of positive RDC is ΔM by definition.

Now the total magnetization M is described by these resultant dominant clusters by definition (3·11). The scaling exponent of M is described by the fractal dimensionality d_M of RDC, namely

$$M(L)\sim L^{d_M} \quad \text{at} \quad T_c\,. \tag{3·12}$$

Then, the total susceptibility χ_0 is assumed to have the form:

$$\chi_0\sim L^{d_\chi} \quad \text{at} \quad T_c\,. \tag{3·13}$$

On the other hand, it is expressed by the fluctuation of the total magnetization M as

$$\chi_0\simeq\langle M^2\rangle/kT_c\,. \tag{3·14}$$

Here it should be noted that the magnetic behavior of the system is described asymptotically by the resultant dominant clusters near (or at) the critical point and consequently that the expectation value $\langle M^2\rangle$ can be replaced by the square of the magnetization of RDC, namely

$$\langle M^2\rangle \simeq M^2(L) \quad \text{at} \quad T_c\,, \tag{3·15}$$

as is easily seen from Fig. 2. This is one of the essential points in our arguments based on the geometrical picture of the resultant dominant clusters (RDC) to describe critical fluctuations. Thus, from Eqs. (3·12), (3·13) and (3·15), we obtain the following scaling relation:

$$d_\chi = 2d_M\,. \tag{3·16}$$

This gives a scaling relation between β and γ defined by

$$M_S \sim V\varepsilon^\beta \quad \text{and} \quad \chi_0 \sim V\varepsilon^{-\gamma}\,; \quad \varepsilon \equiv (T-T_c)/T_c\,, \quad V \equiv L^d \tag{3·17}$$

near the critical point T_c. In fact, the correlation length ξ is cut off at $\xi = L$ for a finite lattice at the critical point. On the other hand, ξ takes a singularity of the form

$$\xi \sim \varepsilon^{-\nu} \quad \text{or} \quad \varepsilon \sim \xi^{-1/\nu}\,. \tag{3·18}$$

That is, the critical point is spread out due to the finiteness of the size as

$$\varepsilon \sim L^{-1/\nu}\,. \tag{3·19}$$

Therefore, the total magnetization and susceptibitlty should have the following size-dependence

$$M(L) \sim VL^{-\beta/\nu} \sim L^{d-\beta/\nu} \quad \text{and} \quad \chi_0 \sim VL^{\gamma/\nu} \sim L^{d+\gamma/\nu} \tag{3·20}$$

at the critical point. Rigorously speaking, Eq. (3·19) has to be regarded as the definition of the critical exponent ν, instead of $\xi \sim \varepsilon^{-\nu}$. Thus, we obtain the relations

$$d_M = d - \frac{\beta}{\nu} \quad \text{and} \quad d_\chi = d + \frac{\gamma}{\nu}\,. \tag{3·21}$$

Therefore, from Eqs. (3·16) and (3·21), we arrive at the well-known scaling relation

$$2\beta + \gamma = d\nu\,. \tag{3·22}$$

As is easily understood from the above argument, the physical basis of the scaling law is that the critical fluctuation is governed only by the resultant dominant clusters at (or near) the critical point, and possibly by the largest cluster effectively. This ansatz of the asymptotic equivalence of RDC to the largest cluster is quite analogous to the ansatz by Stanley et al.[7)~9)] that, at

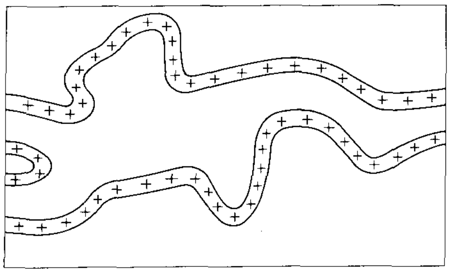

Fig. 3. Resultant dominant clusters in dual configurations.

percolation, magnetic correlations spread through the incipient 'infinite cluster' along a path that is a self-avoiding walk in the percolation problem.

Our above argument is quite general and it is extended to other critical fluctuations. We consider, for example, the singularity of specific heat. For this purpose, we introduce a concept of *resultant dominant clusters in dual configurations* (RDCIDC). First we define a local energy variable E_{ij} by

$$E_{ij} = -JS_iS_j , \tag{3·23}$$

which constructs dual configurations, as shown in Fig. 3. Then we can define resultant dominant clusters in these dual configurations by eliminating smaller clusters until the eliminated energy of E^+ becomes the average $\langle E^+ \rangle$ at the critical point. Thus, the resultant dominant cluster of the energy E^+ is given by

$$E^+ = \langle E^+ \rangle + \Delta E^+ . \tag{3·24}$$

Similarly, for the energy E^-, we have

$$E^- = \langle E^- \rangle + \Delta E^- . \tag{3·25}$$

The total energy fluctuation ΔE is given by

$$\Delta E = E - \langle E \rangle = \Delta E^+ + \Delta E^- . \tag{3·26}$$

Thus, the geometrical picture of the energy fluctuation ΔE is described by the fractal dimensionality d_E of ΔE^+ for large L, namely

$$\Delta E \sim L^{d_E} . \tag{3·27}$$

Therefore, the total specific heat of the system is described as

$$C = \frac{1}{kT_c^2} \langle (\Delta E)^2 \rangle \sim \{\Delta E(L)\}^2 \sim L^{2d_E} \sim L^{d_C} . \tag{3·28}$$

On the other hand, we have the relation

$$\Delta E \sim \varepsilon^{-1} \sim L^{1/\nu} , \tag{3·29}$$

because ΔE is a variable conjugate[10] to the temperature difference ε. Therefore, we obtain

$$d_C = 2d_E = 2/\nu . \tag{3·30}$$

The specific heat has the singularity

$$C\sim V\varepsilon^{-\alpha}\sim VL^{-d/\nu}\sim L^{d-\alpha/\nu}\,. \tag{3·31}$$

Thus, we arrive at Kadanoff's scaling relation

$$2-\alpha=d\nu\,. \tag{3·32}$$

From (3·22) and (3·32), we obtain the well-known scaling relation

$$\alpha+2\beta+\gamma=2\,. \tag{3·33}$$

Our argument can be extended to the following general situation

$$\mathcal{H}=\sum_{j=1}^{m}h_jQ_j\,;\quad Q_j=\text{relevant}\,, \tag{3·34}$$

where $Q_1=\mathcal{H}_0$, $h_1=\varepsilon$, $Q_2=M$, $h_2=h$, Q_3 denotes a symmetry breaking quantity, and h_3 is its conjugate parameter, etc. The fractal dimensionality d_{Q_j} of the quantity Q_j is related to that of the corresponding susceptibility, d_{χ_j}, as

$$d_{\chi_j}=2d_{Q_j} \tag{3·35}$$

through the relation

$$\chi_j\simeq\langle(\Delta Q_j)^2\rangle\simeq\{\Delta Q_j(L)\}^2\sim L^{2d_{Q_j}}\,. \tag{3·36}$$

On the other hand, Q_j and χ_j are assumed to have the following singularities:

$$Q_j\sim V\varepsilon^{\varphi_j}\quad\text{and}\quad\chi_j\sim V\varepsilon^{-\psi_j}\,, \tag{3·37}$$

respectively near the critical point. Therefore, these have the following L-dependence

$$Q_j\sim L^{d-\varphi_j/\nu}\quad\text{and}\quad\chi_j\sim L^{d+\psi_j/\nu} \tag{3·38}$$

at the critical point. Consequently we arrive at the general scaling relation

$$\psi_j=d\nu-2\varphi_j \tag{3·39}$$

from (3·35).

§ 4. Connectivity dimensionality (or generalized topological dimensionality)

In the present section, we define the concept of *connectivity dimensionality**) (or generalized topological dimensionality) in order to discuss the relation between critical exponents and fractal dimensionality of some explicit lattices. For this purpose we consider general lattices which are composed appropriately

*) Dr B. B. Mandelbrot has kindly suggested the present author the terminoly 'connectivity dimensions' instead of 'topological dimensionality' which was originally used in Ref. 6).

of a lattice hyper-surface around an arbitrary point O by a set of lattice points which are connected with the point O by minimum r steps, namely r lattice-spacings. The *numbers of lattice points* on this lattice hyper-surface and inside this lattice hyper-surface are denoted by S_r and V_r, respectively. Now, we define the connectivity dimensionality D_{con} by

$$D_{\text{con}} = \lim_{r\to\infty} (\log V_r)/(\log r). \tag{4·1}$$

We may also define $\hat{D}_{\text{con}}$ by

$$\hat{D}_{\text{con}} = 1 + \lim_{r\to\infty} (\log S_r)/\log r. \tag{4·2}$$

The equality

$$\hat{D}_{\text{con}} = D_{\text{con}} \tag{4·3}$$

holds in the ordinary space, but it may not be valid in general fractal dimensions. It is clear that this connectivity dimensionality is a generalization of the ordinary topological dimensionality d. The above definition of the connectivity dimensionality does not contain the *concept of length in space.* In this sense, it may also be called[6] "generalized topological dimensionality".

Now we show how useful this concept of connectivity dimensionality is, by applying the above definition (4·1) of D_{con} to some explicit lattices.

a) *Koch curve* The "volume" V_r (i.e., lattice points inside the region connected by r steps from a certain point O) of the Koch curve in Fig. 1 is given by $V_r = 2r + 1$. Therefore, we have

$$D_{\text{con}} = \lim_{r\to\infty} \log V_r/\log r = \lim_{r\to\infty} \log(2r+1)/\log r = 1. \tag{4·4}$$

Namely, the connectivity dimensionality is given by $D_{\text{con}} = 1$ in the Koch curve, while the Hausdorff (or similarity) dimensionality is given by $D = \log 4/\log 3$, as was mentioned in § 1. The connectivity dimensionality $D_{\text{con}} = 1$ of the Koch curve describes the non-existence of phase transition in the Ising model for the Koch curve.

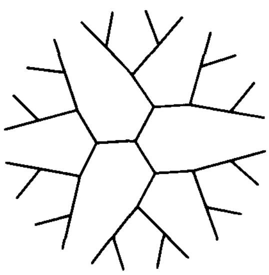

Fig. 4. Cayley tree (the case $z=3$).

b) *Cayley tree* We study the connectivity dimensionality of Cayley tree in Fig. 4. The "volume" V_r inside a lattice hyper-surface of radius r is calculated as

$$V_r = 1 + z\{1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots + (z-1)^{r-1}\}$$
$$= \frac{z(z-1)^r - 2}{z-2}, \tag{4·5}$$

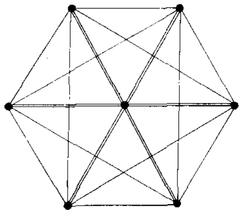

Fig. 5. Completely connected graph.

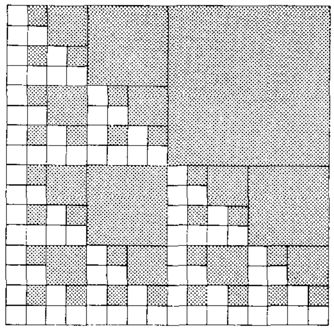

Fig. 6. Modified Sierpiński carpet. $D_{\rm con}=\log 3/\log 2=1.585\cdots$.

where z denotes the number of nearest neighbors. Therefore, we obtain

$$D_{\rm con}=\lim_{r\to\infty}(\log V_r)/\log r=\infty\,. \tag{4·6}$$

On the other hand, the Hausdorff or similarity dimensionality of the Cayley tree, D, is finite and it depends on the ratio of the length of the n-th fragments to that of the $(n+1)$-th fragments, as was discussed in detail by Mandelbrot.[2)]

c) *Completely connected graph* It is well-known that the phase transition of the completely connected graph shown in Fig. 5 is described by the mean field theory. The connectivity dimensionality $D_{\rm con}$ is clearly given by

$$D_{\rm con}=\infty\,. \tag{4·7}$$

d) *Modified Sierpiński carpet* We introduce a non-uniform fractal lattice shown in Fig. 6. The connectivity dimensionality $D_{\rm con}$ is given by

$$D_{\rm con}=\log 3/\log 2=1.585\cdots\,. \tag{4·8}$$

This lattice is quite similar to the Sierpiński carpets discussed in detail by Gefen, Mandelbrot and Aharony.[11)] Thus, we expect the existence of phase transition in this modified Sierpiński carpet. It is also expected that there appear critical phenomena corresponding to fractal dimensionality between one and two dimensions.

§ 5. Dependence of critical exponents upon the connectivity dimensionality

One of the main purposes of studying critical phenomena is to confirm the universality[12),13)] or weak universality,[14)] namely to study how critical exponents depend on some fundamental parameters such as the spatial dimensionality d. In 1972, the present author[15)] argued the following inequalities of critical exponents:

$$\gamma(d)\geq\gamma(d+1),\quad \beta(d)\leq\beta(d+1) \quad\text{and}\quad \nu(d)\geq\nu(d+1). \tag{5·1}$$

From our viewpoint of fractal dimensionality, the above inquality (5·1) may be extended in the form

$$\gamma(d)\geq\gamma(d'),\quad \beta(d)\leq\beta(d') \quad\text{and}\quad \nu(d)\geq\nu(d') \tag{5·2}$$

for $d<d'$ or equivalently

$$\gamma'(d)\leq 0\,,\quad \beta'(d)\geq 0 \quad\text{and}\quad \nu'(d)\leq 0\,, \tag{5·3}$$

where the prime denotes the differentiation with repect to the dimensionality d.

According to the renormalization group theory,[3] the mean field theory is exact for $d>4$. As is well-known, critical phenomena in the Cayley tree and completely connected graph are rigorously described by the mean field theory. This is well understood by the fact that the connectivity dimensionality (or generalized topological dimensionality) D_{con} of these lattices is given by $D_{\text{con}}=\infty$. For critical exponents in fractal similarity dimensions, see papers by Gefen et al.[11]

It will be a quite interesting question in future to study what kind of dimensionality should be used as d in (5·2). One possibility may be $d=\min(D_{\text{con}}, \hat{D}_{\text{con}})$.

§ 6. Summary and discussion

In the present paper, some geometrical interpretation of critical phenomena has been proposed and scaling relations among critical exponents have been rederived on the basis of this geometrical interpretation of critical phenomena namely the statistical fractal dimensionality.

The concept of connectivity dimensionality has been introduced to discuss the relation between the geometrical features of the relevant system and critical exponents. Our alternative definition $\hat{D}_{\text{con}}$ may be related to the connectivity parameter Q introduced by Gefen et al.[11] through the possible relation $D_{\text{con}}=1+Q$. However, our connectivity dimensionality D_{con} is not necessarily equal to $\hat{D}_{\text{con}}$, as was discussed in the text.

We have also clarified the similarity law of the fractal theory and we have argued the complete similarity between the fractal theory and the renormalization group approach.[16]

The present geometrical interpretation of phase transition may be complementary to the previous droplet theory of critical phenomena.[17)~19)]

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