Published for SISSA by Springer

Received: September 19, 2014
Revised: November 28, 2014
Accepted: January 10, 2015
Published: February 23, 2015

Phase transition and thermodynamical geometry for Schwarzschild AdS black hole in $AdS_5 \times S^5$ spacetime

Jia-Lin Zhang,[a,b] **Rong-Gen Cai**[a,b] **and Hongwei Yu**[a,c]

[a] *Department of Physics and Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, Hunan Normal University, Changsha, 410081 China*

[b] *State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, 100190 China*

[c] *Center for Nonlinear Science and Department of Physics, Ningbo University, Ningbo, 315211 China*

E-mail: jlzhang1981@gmail.com, cairg@itp.ac.cn, hwyu@hunnu.edu.cn

Abstract: We study the thermodynamics and thermodynamic geometry of a five-dimensional Schwarzschild AdS black hole in $AdS_5 \times S^5$ spacetime by treating the cosmological constant as the number of colors in the boundary gauge theory and its conjugate quantity as the associated chemical potential. It is found that the chemical potential is always negative in the stable branch of black hole thermodynamics and it has a chance to be positive, but appears in the unstable branch. We calculate the scalar curvatures of the thermodynamical Weinhold metric, Ruppeiner metric and Quevedo metric, respectively and we find that the scalar curvature in the Weinhold metric is always vanishing, while in the Ruppeiner metric the divergence of the scalar curvature is related to the divergence of the heat capacity with fixed chemical potential, and in the Quevedo metric the divergence of the scalar curvature is related to the divergence of the heat capacity with fixed number of colors and to the vanishing of the heat capacity with fixed chemical potential.

Keywords: Black Holes, Black Holes in String Theory, AdS-CFT Correspondence, Models of Quantum Gravity

ArXiv ePrint: 1409.5305

doi:10.1007/JHEP02(2015)143

JHEP02(2015)143

Contents

1 Introduction

The thermodynamical properties of black holes in anti-de Sitter (AdS) space are quite different from those of black holes in asymptotically flat or de Sitter space. The main reason is that the AdS space acts as a confined cavity so that black holes in AdS space can be thermodynamically stable. In particular, there exists a minimal Hawking temperature for a Schwarzschild black hole in AdS space, below which there does not exist any black hole solution, instead a stable thermal gas solution exists. For a given temperature above the minimal one, there exist two black hole solutions. The black hole with smaller horizon is thermodynamically unstable with a negative heat capacity, while the black hole with larger horizon is thermodynamically stable with a positive heat capacity. And Hawking and Page find that a phase transition, named Hawking-Page phase transition, will happen between the stable large black hole and thermal gas in AdS space [1]. According to AdS/CFT correspondence, which says that there is an equivalence between a weakly coupled gravitational theory in d-dimensional AdS spacetime and a strongly coupled conformal field theory (CFT) in a $(d-1)$-dimensional boundary of the AdS space [2–5] (for a review, see [6]), thermodynamical properties of black holes in AdS space can be identified with those of dual strongly coupled CFT. The Hawking-Page phase transition for black holes in AdS space is interpreted as the confinement/deconfinement phase transition in gauge theory [5]. Thus it becomes quite interesting to study thermodynamics and phase structure of black holes in AdS space. Indeed, in the past few years there have been a lot of works on thermodynamics and phase transition for black holes in AdS space.

In ordinary thermodynamic systems, a divergence of heat capacity is usually associated with a second order phase transition. For Kerr-Newmann black holes in Einstein-Maxwell theory, some heat capacities diverge at some black hole parameters. Based on this, Davies argued that some second order phase transitions will happen in Kerr-Newmann black holes [7–9]. For a Reissner-Nordström AdS (RN-AdS) black hole, such a phase transition was studied in some details in [10, 11]. In a canonical ensemble with a fixed charge, it was found that there exists a phase transition between small and large black holes. This phase transition behaves very like the gas/liquid phase transition in a Van der Waals system [10, 12]. However, an identification between RN-AdS black hole and Van der Waals system was recently realized in [13], where the negative cosmological constant plays the role as pressure, while its conjugate acts as thermodynamic volume of the black hole in the so-called extended phase space [14, 15] (for a recent review, see [16]). Recently, thermodynamics and phase transition in the extended phase space for black holes in AdS space have been extensively studied in the literature (for an incomplete list see [17–44].)

In the framework of AdS/CFT correspondence, the negative cosmological constant is related to the degrees of freedom of dual CFT. Thus it is an interesting question as to whether the interpretation of the cosmological constant as pressure is applicable to the boundary CFT. Very recently, it was argued that it is more suitable to view the cosmological constant as the number of colors in gauge field and its conjugate as associated chemical potential [45–47]. This interpretation was examined in the case of $AdS_5 \times S^5$, for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at large N in [45]. The chemical potential conjugate to the number of colors, is calculated. It is found that the chemical potential in the high temperature phase of the Yang-Mills theory is negative and decreases as temperature increases. For spherical black holes in the bulk the chemical potential approaches zero as the temperature is lowered below the Hawking-Page temperature and changes its sign at a temperature near the temperature at which the heat capacity diverges.

On the other hand, applying the geometrical ideas to ordinary thermodynamical systems gives us an alternative way to study phase transition in those systems. Weinhold [48] first introduced a sort of metric defined as the second derivatives of internal energy with respect to entropy and other extensive quantities of a thermodynamic system. Soon later, based on the fluctuation theory of equilibrium thermodynamics, Ruppeiner [49] introduced another metric which is defined as the minus second derivatives of entropy with respect to the internal energy and other extensive quantities of a thermodynamic system. It was argued that the scalar curvature of the Ruppeiner metric can reveal the micro interaction behind the thermodynamic system and its divergence is related to some phase transition in the thermodynamical system [50]. In addition, it was shown that the Weinhold metric is conformal to the Ruppeiner metric [51]. However, both of the Weinhold metric and Ruppeiner metric are not invariant under Legendre transformation and sometimes contradictory results will be produced [52, 53]. In order to solve this puzzle, Quevedo et al. [54–57] proposed a method to obtain a new formulism of Geometrothermodynamics whose metric is Legendre invariant in the space of equilibrium states. To the best of our knowledge, applying the thermodynamical geometry to black hole thermodynamics was initiated in [58], there it was found that the Weinhold metric is proportional to the metric on the moduli space for supersymmetric extremal black holes, whose Hawking temperature is zero, and the Ruppeiner metric governing fluctuations naively diverges, which is consistent with the argument that near the extremal limit, the thermodynamical description breaks down. Applying the thermodynamical geometry approach to phase transition of black holes was followed in [59–61], and for more relevant references see the recent review [62] and references therein. In particular, refs. [63, 64] has investigated the relation between the divergence of the scalar curvature of thermodynamical geometry in different ensembles and the singularity of heat capacities.

JHEP02(2015)143

In this paper, we will study thermodynamics and thermodynamical geometry for a five-dimensional Schwarzschild AdS black hole in $AdS_5 \times S^5$ by viewing the number of colors as a thermodynamical variable from the view of point of dual CFT. In next section, we will review some basic thermodynamic properties of a black hole in $AdS_5 \times S^5$ spacetime by treating the cosmological constant in the bulk as the number of colors [45]. In section 3 we will calculate the thermodynamical curvatures of the Weinhold metric, Ruppeiner metric and Quevedo metric, respectively, for the thermodynamical system, in order to see the relations between the thermodynamical curvature and phase transition. Note that such calculations can not be done if one views the cosmological constant as a true constant, or even in the case of the extended phase space in the sense [14, 15], because in the latter case, the heat capacity C_V always vanishes. We end the paper with conclusions in section 4.

2 Thermodynamics of Schwarzschild AdS black hole in $AdS_5 \times S^5$

In this section, we will review the main results obtained by Dolan in ref. [45]. In $AdS_5 \times S^5$ spacetime, the line element for a five-dimensional Schwarzschild AdS black hole reads [10]

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 h_{ij} dx^i dx^j + L^2 d\Omega_5^2, \tag{2.1}$$

where $d\Omega_5^2$ is the metric of a five-dimensional sphere with unit radius, $h_{ij} dx^i dx^j$ is the line element of a three-dimensional Einstein space Σ_3 with constant curvature $6k$, and the metric function f is given by

$$f = k - \frac{m}{r^2} + \frac{r^2}{L^2}\,, \tag{2.2}$$

where L is the AdS radius and m is an integration constant. The cosmological constant is $\Lambda = -6/L^2$. Without loss of generality, one can take the scalar curvature parameter k of the three-dimensional space Σ_3 as $k = 1$, 0, or -1, respectively. The ten-dimensional spacetime (2.1) can be viewed as the near horizon geometry of N coincident $D3$-branes in type IIB supergravity. In that case, the AdS radius L has a relation to the number N of D3-branes [2]

$$L^4 = \frac{\sqrt{2} N \ell_{\rm p}^4}{\pi^2}, \tag{2.3}$$

where $\ell_{\rm p}$ is the ten-dimensional Planck length. According to AdS/CFT correspondence, the spacetime (2.1) can be regarded as the gravity dual to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Then N is nothing, but the rank of the gauge group of the supersymmetric SU(N) Yang-Mills Theory. In the large N limit, the number of degrees of freedom of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is proportional to N^2 (in fact, it is that of $8N^2$ massless bosons and fermions in the weak coupling limit [65]).

The event horizon $r_{\rm h}$ of the black hole is determined by the equation $f = 0$. Then according to eq. (2.2), the mass of the black hole can be expressed as

$$M = \frac{3\omega_3}{16\pi G_5} m = \frac{3\omega_3 {r_{\rm h}}^2}{16\pi G_5 L^2} (kL^2 + r_{\rm h}^2), \tag{2.4}$$

JHEP02(2015)143

where ω_3 is the volume of Σ_3. Using the Bekenstein-Hawking entropy formula of the black hole, we have

$$S = \frac{A}{4G_5} = \frac{\omega_3 r_\mathrm{h}^3}{4G_5}\,. \tag{2.5}$$

Note that $G_5 = G_{10}/(\pi^3 L^5)$ and $G_{10} = \ell_\mathrm{p}^8$. Therefore, the mass of the black hole can be rewritten as a function of N and S

$$M(S,N) = \frac{3\tilde{m}_\mathrm{p}}{4}\left[k\left(\frac{S}{\pi}\right)^{\frac{2}{3}} N^{\frac{5}{12}} + \left(\frac{S}{\pi}\right)^{\frac{4}{3}} N^{-\frac{11}{12}}\right]\,, \tag{2.6}$$

where $\tilde{m}_\mathrm{p} = \sqrt{\pi}\ell_\mathrm{p}^7/(2^{1/8}G_{10})$ is associated with the 10-dimensional Planck mass. According to the standard thermodynamic relation $dM = TdS + \mu dN^2$, the temperature can be obtained

$$T = \left.\frac{\partial M}{\partial S}\right|_N = \frac{\tilde{m}_\mathrm{p}}{2\pi}\left[k\left(\frac{S}{\pi}\right)^{-\frac{1}{3}} N^{\frac{5}{12}} + 2\left(\frac{S}{\pi}\right)^{\frac{1}{3}} N^{-\frac{11}{12}}\right]\,, \tag{2.7}$$

which is nothing but the Hawking temperature of the black hole. The chemical potential μ conjugate to the number of colors is

$$\mu = \left.\frac{\partial M}{\partial N^2}\right|_S = \frac{\tilde{m}_\mathrm{p}}{32}\left[5k\left(\frac{S}{\pi}\right)^{\frac{2}{3}} N^{-\frac{19}{12}} - 11\left(\frac{S}{\pi}\right)^{\frac{4}{3}} N^{-\frac{35}{12}}\right]\,, \tag{2.8}$$

which is the measure of the energy cost to the system when one increases the number of colors.

The Gibbs free energy can be calculated as

$$G(T,N^2) = M - TS = \frac{\tilde{m}_\mathrm{p}}{4}\left[k\left(\frac{S}{\pi}\right)^{\frac{2}{3}} N^{\frac{5}{12}} - \left(\frac{S}{\pi}\right)^{\frac{4}{3}} N^{-\frac{11}{12}}\right]\,. \tag{2.9}$$

For the cases of $k = 0$ or $k = -1$, it is easy to see from eq. (2.7) for a fixed N^2 that the Hawking temperature increases monotonically with the entropy S. Besides, we see from eq. (2.8)and eq. (2.9) that when $k = 0$ or $k = -1$, the chemical potential is always negative, no phase transition happens. However, when $k = 1$, the situation is quite different.

In the case of $k = 1$, the Hawking temperature is not a monotonic function but has a minimum at

$$S_1 = N^2\pi/2^{3/2}\,, \tag{2.10}$$

or equivalently, at $r_\mathrm{h} = L/\sqrt{2}$. We plot the behavior of temperature with respect to entropy in figure 1. The corresponding minimal temperature is $T_\infty = \sqrt{2}\tilde{m}_\mathrm{p}/(\pi N^{1/4}) = \sqrt{2}/(\pi L)$. Namely under the minimal temperature there is no black hole solution. Above the minimal temperature, there exist two branches, as we will see shortly, the branch with small entropy (horizon radius) is thermodynamically unstable, while the branch with large entropy (horizon radius) is thermodynamically stable.

One can see easily from the Gibbs free energy eq. (2.9), the Hawking-Page phase transition happens at $r_\mathrm{h} = L$ with the phase transition temperature $T_* = 3\tilde{m}_\mathrm{p}/(2\pi N^{1/4}) =$

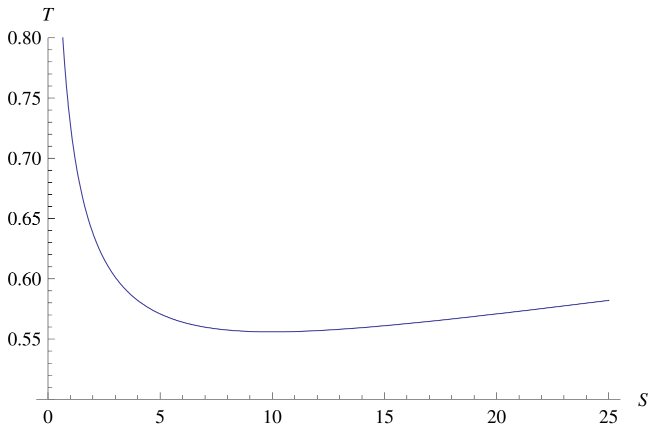

Figure 1. The temperature with respect to entropy. Here we take $\ell_{\rm p} = 1,\ k = 1$ and $N = 3$. The temperature arrives at the minimal value when $S = S_1 = N^2\pi/2^{3/2} \approx 9.9965$.

$3/(2\pi L)$, which is larger than T_∞. And the corresponding entropy at the Hawking-Page transition is

$$S_2 = N^2\pi\ . \tag{2.11}$$

We can see that $S_2 > S_1$. In figure 2, we show the Gibbs free energy with respect to the Hawking temperature T for some fixed N.

In figure 3 we show the chemical potential as a function of entropy S for a fixed N. We see that the chemical potential is positive when S is small, while it changes to be negative when S is large. The chemical potential changes its sign at

$$S_3 = N^2\pi(5/11)^{3/2}\ . \tag{2.12}$$

We see that

$$S_3 < S_1 < S_2. \tag{2.13}$$

As we will see that the vanishing of the chemical potential appears in the unstable branch. This implies that the vanishing of the chemical potential does not make any sense from the point of view of dual supersymmetric Yang-Mills theory. In figure 4 we plot the chemical potential as a function of temperature T for a fixed N, while in figure 5 the chemical potential is plotted as a function of N in the case with a fixed entropy S.

In the following section, we will study thermodynamical geometry of the Schwarzschild AdS black hole in the extended phase space by viewing the cosmological constant as the number of colors. We pay attention to the case of $k = 1$, since the cases of $k = 0$ and $k = -1$ are trivial.

3 Thermodynamical geometry of the Schwarzschild AdS black hole

When the corresponding number of colors N^2 is kept fixed, this corresponds to the case in a canonical ensemble. In this case, the heat capacity for a fixed N^2 can be obtained as

$$C_{N^2} = T\left(\frac{\partial S}{\partial T}\right)_{N^2} = \frac{3S(N^{4/3}\pi^{2/3} + 2S^{2/3})}{2S^{2/3} - N^{4/3}\pi^{2/3}}\ . \tag{3.1}$$

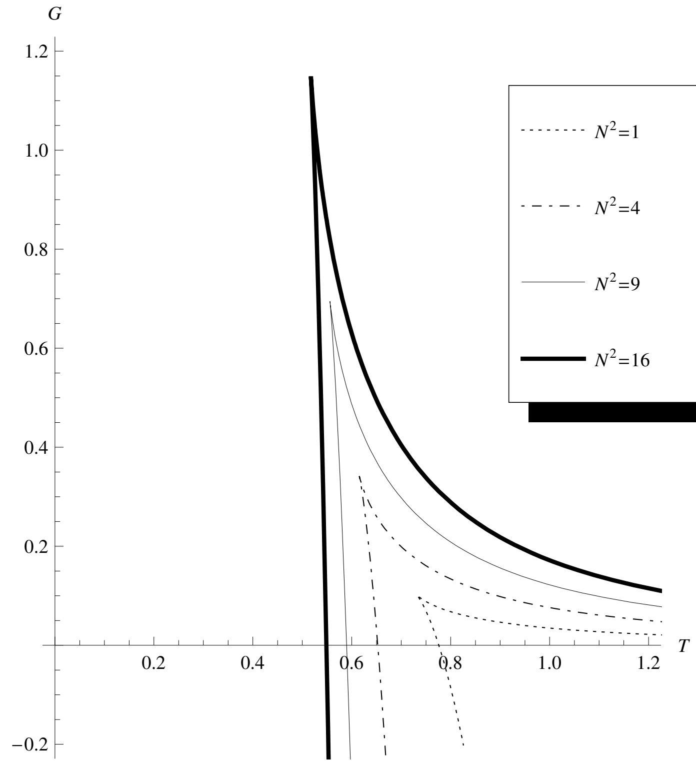

Figure 2. The Gibbs free energy as a function of the temperature for various numbers of colors. Here we take $\ell_{\rm p} = 1$ and $k = 1$. The down branch Gibbs free energy for a fixed N changes its sign at the point $S = S_2 = N^2\pi$, which corresponds to the Hawking-Page transition point.

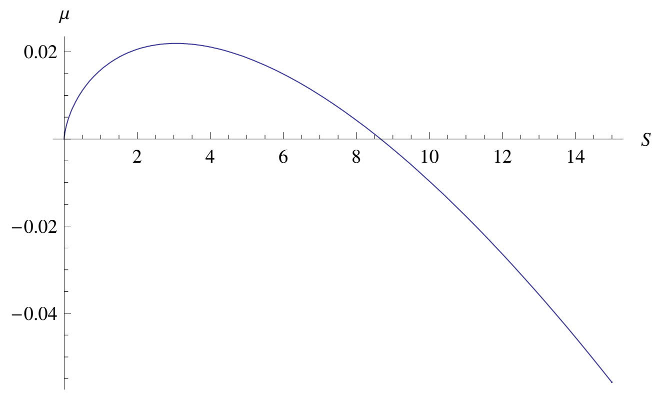

Figure 3. The chemical potential as a function of entropy for a fixed $N = 3$. Here we take $\ell_{\rm p} = 1$ and $k = 1$. The chemical potential changes its sign at $S = S_3 = N^2\pi(5/11)^{3/2} \approx 8.6648$.

The heat capacity diverges at the point of $S_1 = N^2\pi/2^{3/2}$ (i.e., $r_{\rm h} = L/\sqrt{2}$) which just coincides with the point corresponding to the minimal Hawking temperature for a fixed N^2. When $S < S_1$, the heat capacity is negative, indicating the thermodynamical instability, while it is positive as $S > S_1$. We show the behavior of C_{N^2} as a function of S in figure 6.

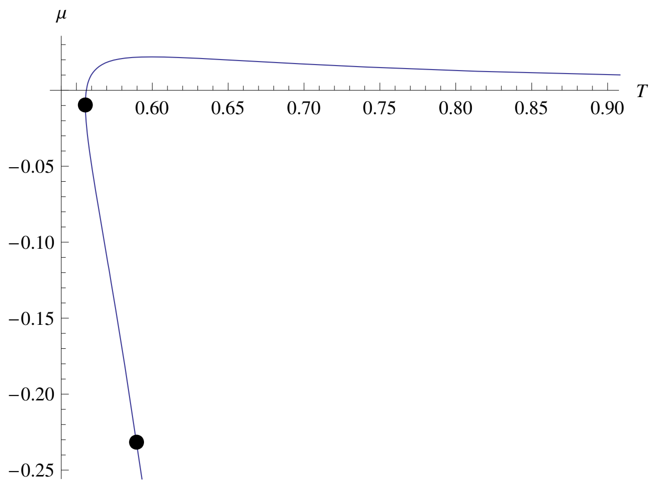

Figure 4. The chemical potential as a function of temperature T for a fixed $N = 3$. Here we take $\ell_{\rm p} = 1$ and $k = 1$. Note that the upper dot denotes the minimal temperature T_∞ and the lower dot denotes the Hawking-Page transition temperature T_*.

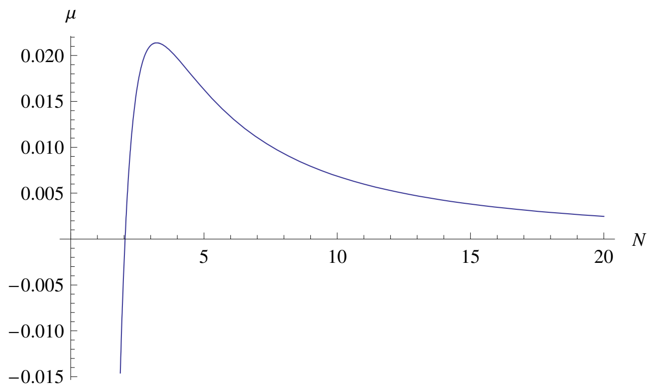

Figure 5. The chemical potential as a function of N for a fixed $S = 4$. Here we take $\ell_{\rm p} = 1$ and $k = 1$. The maximum of the chemical potential corresponds to the point with $S = S_5 = N^2\pi 19^{3/2}/77^{3/2}$, namely, $N \approx 3.2230$.

In the grand canonical ensemble with fixed chemical potential μ, corresponding heat capacity can be obtained as

$$C_\mu = T\left(\frac{\partial S}{\partial T}\right)_\mu = \frac{-95N^{8/3}\pi^{4/3}S + 195N^{4/3}\pi^{2/3}S^{5/3} + 770S^{7/3}}{15N^{8/3}\pi^{4/3} - 45N^{4/3}\pi^{2/3}S^{2/3} - 66S^{4/3}}\,. \tag{3.2}$$

The heat capacity is plotted in figure 7. We see that the heat capacity diverges at

$$S = S_4 = \frac{5\sqrt{2}\pi N^2}{\sqrt{1665 + 67\sqrt{665}}} \tag{3.3}$$

which corresponds to the horizon radius $r_{\rm h} \approx 0.49515L$. Clearly $S_4 < S_1$, namely the divergence happens in the small black hole branch. There exists only a very limited region

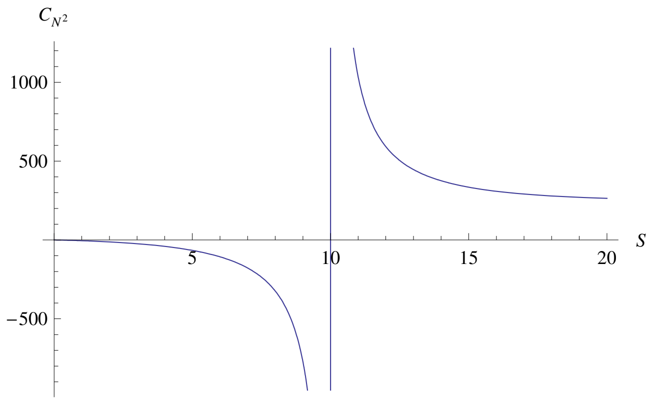

Figure 6. The heat capacity in the case with a fixed $N = 3$ as a function of entropy S. Here we take $k = 1$ and $\ell_\mathrm{p} = 1$. The divergence corresponds to the point $S = S_1 = N^2\pi/2^{3/2} \approx 9.9965$.

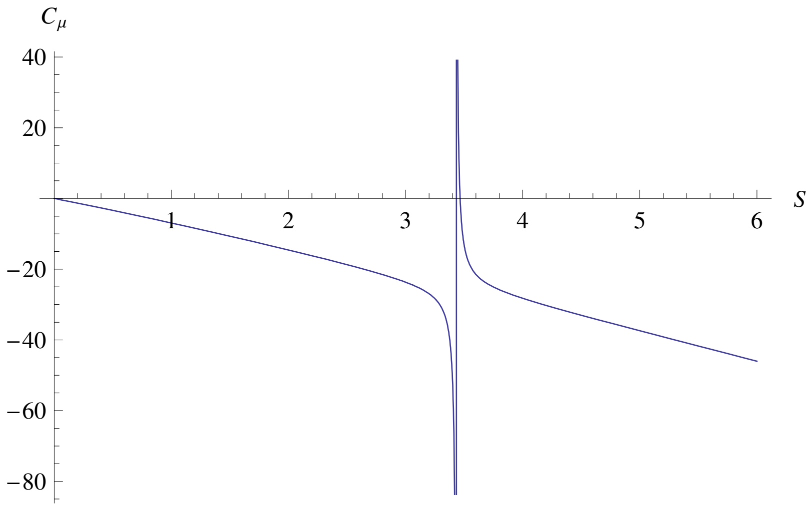

Figure 7. The heat capacity for a fixed μ vs entropy S for $N = 3$,$k = 1$ and $\ell_\mathrm{p} = 1$. The divergence corresponds to the point $S = S_4 = 5\sqrt{2}\pi N^2/\sqrt{1665 + 67\sqrt{665}} \approx 3.4324$. The heat capacity vanishes at a nontrivial point $S = S_5 = N^2\pi 19^{3/2}/77^{3/2} \approx 3.4657$. Note that there is a trivial zero heat capacity at $S = 0$, which will not be considered here.

with a positive heat capacity between $S_4 < S < S_5$, where

$$S_5 = N^2\pi \left(\frac{19}{77}\right)^{3/2}, \tag{3.4}$$

namely, $r_\mathrm{h} \approx 0.49674L$, which has a vanishing heat capacity. Note that S_5 is also less than S_1. This is quite different from the classical gas with negative chemical potential. When the chemical potential approaches zero and becomes positive, quantum effects should come into playing some role [45].

Now we turn to the thermodynamical geometry of the black hole and want to see whether the thermodynamical curvature can reveal the singularity of these two heats capacities. The Weinhold metric [48] is defined as the second derivatives of internal energy

JHEP02(2015)143

with respect to entropy and other extensive quantities in the energy representation, while the Ruppeiner metric [49] is related to the Weinhold metric by a conformal factor of temperature [51]

$$ds^2_{\rm R} = \frac{1}{T} ds^2_{\rm W} \, . \tag{3.5}$$

The Weinhold metric and Ruppeiner metric, which are dependent on the choice of thermodynamic potentials, are not Legendre invariant.

Quevedo et al. [54–57] proposed a method to obtain a thermodynamical metric from a Legendre invariant thermodynamic potential. This method allows one to obtain a new formulism of Geometrothermodynamics whose metric is Legendre invariant in the space of equilibrium states. In what follows, we will first briefly review the formulism of Geometrothermodynamics. Define an $(2n+1)$-dimensional thermodynamic phase space $\mathcal{T}$ which can be described by the coordinates of $\{\phi, E^a, I^a\}$, $a = 1, \ldots, n$, where ϕ denotes the thermodynamic potential, E^a and I^a respectively represent the set of extensive variables and the set of intensive variables. Then the fundamental Gibbs 1-form can be defined on the space $\mathcal{T}$ as $\Theta = d\phi - \delta_{ab} I^a dE^b$ with $\delta_{ab} = {\rm diag}(1, 1, \ldots, 1)$. Under the assumption that $\mathcal{T}$ is differentiable and Θ satisfies the condition of $\Theta \wedge (d\Theta)^n \neq 0$, the pair $(\mathcal{T}, \Theta)$ defines a contact manifold. Considering G as a non-degenerate Riemannian metric on the space $\mathcal{T}$, especially, the geometric properties of metric G do not depend on the choice of thermodynamic potential in its construction because of Legendre invariance, then the set $(\mathcal{T}, \Theta, G)$ can define a Riemannian contact manifold or the phase manifold. As a result, an n-dimensional Riemannian submanifold $\varepsilon \subset \mathcal{T}$ can be defined as the space of thermodynamic equilibrium states (equilibrium manifold) by a smooth map $\varphi : \varepsilon \to \mathcal{T}$, i.e., $\varphi : (E^a) \mapsto (\phi, E^a, I^a)$ where the pullback of the map should satisfy the condition $\varphi^*(\Theta) = 0$. Furthermore, Quevedo metric g can be induced on the equilibrium manifold ε by using $\varphi^*(G)$. The non-degenerate Riemannian metric G can be chosen as [56]

$$G = (d\phi - \delta_{ab} I^a dE^b)^2 + (\delta_{ab} E^a I^b)(\eta_{cd} dE^c dI^d), \qquad \eta_{cd} = {\rm diag}(-1, 1, \ldots, 1). \tag{3.6}$$

Then Quevedo metric reads

$$g = \varphi^*(G) = \left(E^c \frac{\partial \phi}{\partial E^c} \right) \left(\eta_{ab} \delta^{bc} \frac{\partial^2 \phi}{\partial E^c \partial E^d} dE^a dE^d \right) \, . \tag{3.7}$$

Now we calculate the thermodynamical curvature for the Schwarzschild AdS black hole. The Weinhold metric is given by

$$g^W = \begin{pmatrix} M_{SS} & M_{SN^2} \\ M_{N^2S} & M_{N^2N^2} \end{pmatrix} \, , \tag{3.8}$$

where ρ_{ij} stands for $\partial^2 \rho / \partial x^i \partial x^j$, and $x^1 = S$, $x^2 = N^2$. The scalar curvature of this metric can be calculated directly. Substituting eq. (2.6) and eq. (2.7) into eq. (3.8), we can see that the scalar curvature of the Weinhold metric is always vanishing.

On the other hand, considering eq. (3.5), the Ruppeiner metric can be written as

$$g^R = \frac{1}{T} \begin{pmatrix} M_{SS} & M_{SN^2} \\ M_{N^2S} & M_{N^2N^2} \end{pmatrix} \, , \tag{3.9}$$

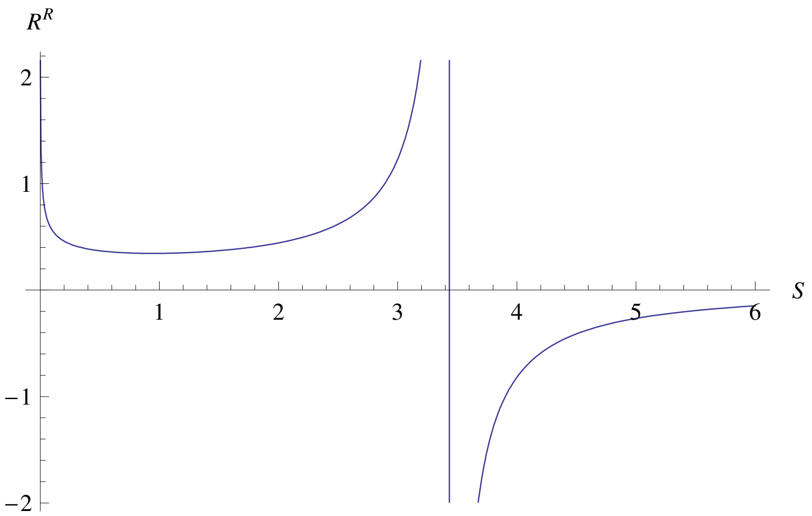

Figure 8. The scalar curvature vs entropy for the Ruppeiner metric case with $N = 3$, $k = 1$ and $\ell_{\rm p} = 1$. The scalar curvatures diverge at $S = S_4 = 5\sqrt{2}\pi N^2/\sqrt{1665 + 67\sqrt{665}} \approx 3.4324$.

and the corresponding curvature of this metric is

$$R^R = \frac{7(5N^{8/3}\pi^{4/3}S^{-1/3} + 2N^{4/3}\pi^{2/3}S^{1/3})}{15N^4\pi^2 - 15N^{8/3}\pi^{4/3}S^{2/3} - 156N^{4/3}\pi^{2/3}S^{4/3} - 132S^2}\,. \tag{3.10}$$

From eq. (3.10), we can conclude that the scalar curvatures of the Ruppeiner metric possess a positive singularity at $S = S_4 = 5\sqrt{2}\pi N^2/\sqrt{1665 + 67\sqrt{665}}$, i.e., $r_{\rm h} \approx 0.49515L$ (see figure 8). This singularity just coincides with the divergence of the heat capacity C_μ for fixed chemical potential (comparing figure 7 with figure 8). Therefore, we may conclude that the Ruppeiner metric can reveal the phase transition of the Schwarzschild AdS black hole in $AdS_5 \times S^5$ in grand canonical ensemble, while the Weinhold metric cannot here.

The Quevedo metric reads

$$g^Q = (ST + N^2\mu)\begin{pmatrix} -M_{SS} & 0 \\ 0 & M_{N^2N^2} \end{pmatrix}\,. \tag{3.11}$$

Calculating its scalar curvature gives

$$R^Q = A_1/B_1, \tag{3.12}$$

where A_1 and B_1 are given by

$$\begin{aligned} A_1 &= 256N^{19/6}\pi^{10/3}\left(3982S^2 - 1741N^4\pi^2 + 2372N^{8/3}\pi^{4/3}S^{2/3} + 17311\pi^{2/3}N^{4/3}S^{4/3}\right)\,, \\ B_1 &= 105\tilde{m}_{\rm p}^2 S^{2/3}\left(N^{4/3}\pi^{2/3} + S^{2/3}\right)^3\left(19N^{8/3}\pi^{4/3} - 115N^{4/3}\pi^{2/3}S^{2/3} + 154S^{4/3}\right)^2\,. \end{aligned}$$

The scalar curvature is plotted in figure 9, we see that there exist two divergent points at

$$S_1 = \frac{N^2\pi}{2\sqrt{2}} \qquad \text{and} \qquad S_5 = N^2\pi\left(\frac{19}{77}\right)^{3/2}\,, \tag{3.13}$$

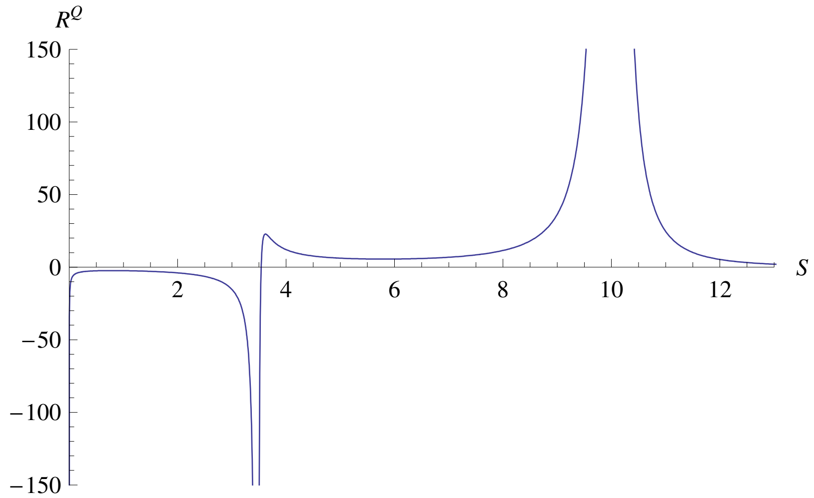

Figure 9. Scalar curvature vs entropy S for the Quevedo metric with $N = 3, \ell_{\rm p} = 1$ and $k = 1$. There exist two divergences at $S = S_5 = N^2\pi 19^{3/2}/77^{3/2} \approx 3.4657$ and $S = S_1 = N^2\pi/2^{3/2} \approx 9.9965$, respectively.

respectively. The first one just coincides with the divergent point of C_{N^2}, while the second one corresponds to $C_\mu = 0$. This result is consistent with the recent study in [67, 68] that the divergences of scalar curvature for the Quevedo metric correspond to divergence or zero for heat capacity. These results are meaningful to further understand the relation between phase transition and thermodynamical curvature.

4 Conclusions

In this paper, we have studied thermodynamics of a Schwarzschild AdS black hole in $AdS_5 \times S^5$ spacetime in the extended phase space where the cosmological constant is viewed as the number of colors in the dual supersymmetric Yang-Mills theory. We calculated and discussed the chemical potential associated with the number of colors, and found that the chemical potential is always negative in the stable branch of black hole thermodynamics. The chemical potential has a chance to be positive, but it appears in the unstable branch.

The heat capacities with fixed number of colors C_{N^2} and with fixed chemical potential C_μ have been calculated, respectively. It is found that C_{N^2} diverges at the minimal temperature of the black hole, while C_μ diverges at a smaller horizon radius.

In the extended phase space, we have a chance to study the thermodynamical geometry associated with the Schwarzschild AdS black hole. By calculating scalar curvatures of the Weinhold metric, Ruppeiner metric and Quevedo metric, we see that in the Weinhold metric the scalar curvature is always zero, no singularity is found. However, in the Ruppeiner metric the scalar curvature diverges at same divergent point of C_μ, and in the Quevedo metric, the scalar curvature diverges at the divergence of C_{N^2}, besides at the point of $C_\mu = 0$. These results indicate that the divergence of thermodynamical curvature indeed is related to some divergence of heat capacities, but the divergence of thermodynamical curvature may be also related to the vanishing points of the thermodynamic potential, temperature, heat capacity, etc. [67, 68]. This is helpful to further understand the relation between phase transition and divergence of thermodynamical curvature. For a further study of this relation, it should be of great interest to discuss thermodynamics and thermodynamical curvature for other black holes in AdS space in the extended phase space.

Acknowledgments

RGC thanks Hunan Normal University for warm hospitality extended to him during his visit. This work was supported in part by the National Natural Science Foundation of China (No. 11005038, No. 10975168, No. 11035008, No. 11375092 and No. 11435006), the Hunan Provincial Natural Science Foundation of China under Grant No. 11JJ7001 and the Program of Excellent Talent of Hunan Normal University No. ET13102.

References

[1] S.W. Hawking and D.N. Page, *Thermodynamics of Black Holes in anti-de Sitter Space*, *Commun. Math. Phys.* **87** (1983) 577 [INSPIRE].

[2] J.M. Maldacena, *The Large-N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [*Adv. Theor. Math. Phys.* **2** (1998) 231] [hep-th/9711200] [INSPIRE].

[3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109] [INSPIRE].

[4] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150] [INSPIRE].

[5] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Int. J. Mod. Phys.* **A 16** (2001) 2747 [*Adv. Theor. Math. Phys.* **2** (1998) 505] [hep-th/9803131] [INSPIRE].

[6] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111] [INSPIRE].

[7] P.C.W. Davies, *Thermodynamics of Black Holes*, *Proc. Roy. Soc. Lond.* **A 353** (1977) 499 [INSPIRE].

[8] P.C.W. Davies, *Thermodynamics of black holes*, *Rept. Prog. Phys.* **41** (1978) 1313 [INSPIRE].

[9] P.C.W. Davies, *Thermodynamic Phase Transitions of Kerr-Newman Black Holes in de Sitter Space*, *Class. Quant. Grav.* **6** (1989) 1909 [INSPIRE].

[10] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys. Rev.* **D 60** (1999) 064018 [hep-th/9902170] [INSPIRE].

[11] C. Niu, Y. Tian and X.-N. Wu, *Critical Phenomena and Thermodynamic Geometry of RN-AdS Black Holes*, *Phys. Rev.* **D 85** (2012) 024017 [arXiv:1104.3066] [INSPIRE].

[12] J.-y. Shen, R.-G. Cai, B. Wang and R.-K. Su, *Thermodynamic geometry and critical behavior of black holes*, *Int. J. Mod. Phys.* **A 22** (2007) 11 [gr-qc/0512035] [INSPIRE].

[13] D. Kubiznak and R.B. Mann, *$P-V$ criticality of charged AdS black holes*, *JHEP* **07** (2012) 033 [arXiv:1205.0559] [INSPIRE].

[14] D. Kastor, S. Ray and J. Traschen, *Enthalpy and the Mechanics of AdS Black Holes*, *Class. Quant. Grav.* **26** (2009) 195011 [arXiv:0904.2765] [INSPIRE].

[15] B.P. Dolan, *Pressure and volume in the first law of black hole thermodynamics*, *Class. Quant. Grav.* **28** (2011) 235017 [arXiv:1106.6260] [INSPIRE].

[16] B.P. Dolan, *Black holes and Boyle's law — the thermodynamics of the cosmological constant*, arXiv:1408.4023 [INSPIRE].

[17] S. Gunasekaran, R.B. Mann and D. Kubiznak, *Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization*, *JHEP* **11** (2012) 110 [arXiv:1208.6251] [INSPIRE].

[18] S.-W. Wei and Y.-X. Liu, *Critical phenomena and thermodynamic geometry of charged Gauss-Bonnet AdS black holes*, *Phys. Rev.* **D 87** (2013) 044014 [arXiv:1209.1707] [INSPIRE].

[19] A. Belhaj, M. Chabab, H. El Moumni and M.B. Sedra, *On Thermodynamics of AdS Black Holes in Arbitrary Dimensions*, *Chin. Phys. Lett.* **29** (2012) 100401 [arXiv:1210.4617] [INSPIRE].

[20] S.H. Hendi and M.H. Vahidinia, *Extended phase space thermodynamics and $P-V$ criticality of black holes with a nonlinear source*, *Phys. Rev.* **D 88** (2013) 084045 [arXiv:1212.6128] [INSPIRE].

[21] S. Chen, X. Liu, C. Liu and J. Jing, *$P-V$ criticality of AdS black hole in $f(R)$ gravity*, *Chin. Phys. Lett.* **30** (2013) 060401 [arXiv:1301.3234] [INSPIRE].

[22] E. Spallucci and A. Smailagic, *Maxwell's equal area law for charged Anti-deSitter black holes*, *Phys. Lett.* **B 723** (2013) 436 [arXiv:1305.3379] [INSPIRE].

[23] R. Zhao, H.-H. Zhao, M.-S. Ma and L.-C. Zhang, *On the critical phenomena and thermodynamics of charged topological dilaton AdS black holes*, *Eur. Phys. J.* **C 73** (2013) 2645 [arXiv:1305.3725] [INSPIRE].

[24] A. Belhaj, M. Chabab, H. El Moumni and M.B. Sedra, *Critical Behaviors of 3D Black Holes with a Scalar Hair*, arXiv:1306.2518 [INSPIRE].

[25] N. Altamirano, D. Kubiznak and R.B. Mann, *Reentrant phase transitions in rotating anti-de Sitter black holes*, *Phys. Rev.* **D 88** (2013) 101502 [arXiv:1306.5756] [INSPIRE].

[26] R.-G. Cai, L.-M. Cao, L. Li and R.-Q. Yang, *$P-V$ criticality in the extended phase space of Gauss-Bonnet black holes in AdS space*, *JHEP* **09** (2013) 005 [arXiv:1306.6233] [INSPIRE].

[27] W. Xu, H. Xu and L. Zhao, *Gauss-Bonnet coupling constant as a free thermodynamical variable and the associated criticality*, *Eur. Phys. J.* **C 74** (2014) 2970 [arXiv:1311.3053] [INSPIRE].

[28] J.-X. Mo and W.-B. Liu, *Ehrenfest scheme for $P-V$ criticality in the extended phase space of black holes*, *Phys. Lett.* **B 727** (2013) 336 [INSPIRE].

[29] D.-C. Zou, S.-J. Zhang and B. Wang, *Critical behavior of Born-Infeld AdS black holes in the extended phase space thermodynamics*, *Phys. Rev.* **D 89** (2014) 044002 [arXiv:1311.7299] [INSPIRE].

[30] J.-X. Mo and W.-B. Liu, *$P-V$ criticality of topological black holes in Lovelock-Born-Infeld gravity*, *Eur. Phys. J.* **C 74** (2014) 2836 [arXiv:1401.0785] [INSPIRE].

JHEP02(2015)143

[31] N. Altamirano, D. Kubiznak, R.B. Mann and Z. Sherkatghanad, *Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume*, *Galaxies* **2** (2014) 89 [arXiv:1401.2586] [INSPIRE].

[32] S.-W. Wei and Y.-X. Liu, *Triple points and phase diagrams in the extended phase space of charged Gauss-Bonnet black holes in AdS space*, *Phys. Rev.* **D 90** (2014) 044057 [arXiv:1402.2837] [INSPIRE].

[33] J.-X. Mo, *Ehrenfest scheme for the extended phase space of $f(R)$ black holes*, *Europhys. Lett.* **105** (2014) 20003 [INSPIRE].

[34] J.-X. Mo, G.-Q. Li and W.-B. Liu, *Another novel Ehrenfest scheme for $P-V$ criticality of RN-AdS black holes*, *Phys. Lett.* **B 730** (2014) 111 [INSPIRE].

[35] J.-X. Mo and W.-B. Liu, *Ehrenfest scheme for $P-V$ criticality of higher dimensional charged black holes, rotating black holes and Gauss-Bonnet AdS black holes*, *Phys. Rev.* **D 89** (2014) 084057 [arXiv:1404.3872] [INSPIRE].

[36] D.-C. Zou, Y. Liu and B. Wang, *Critical behavior of charged Gauss-Bonnet AdS black holes in the grand canonical ensemble*, *Phys. Rev.* **D 90** (2014) 044063 [arXiv:1404.5194] [INSPIRE].

[37] R. Zhao, M. Ma, H. Zhao and L. Zhang, *The Critical Phenomena and Thermodynamics of the Reissner-Nordstrom-de Sitter Black Hole*, *Adv. High Energy Phys.* **2014** (2014) 124854 [INSPIRE].

[38] H. Xu, W. Xu and L. Zhao, *Extended phase space thermodynamics for third order Lovelock black holes in diverse dimensions*, *Eur. Phys. J.* **C 74** (2014) 3074 [arXiv:1405.4143] [INSPIRE].

[39] W. Xu and L. Zhao, *Critical phenomena of static charged AdS black holes in conformal gravity*, *Phys. Lett.* **B 736** (2014) 214 [arXiv:1405.7665] [INSPIRE].

[40] M.-S. Ma and Y.-Q. Ma, *Critical behaviors of black hole in an asymptotically safe gravity with cosmological constant*, arXiv:1405.7609 [INSPIRE].

[41] A.M. Frassino, D. Kubiznak, R.B. Mann and F. Simovic, *Multiple Reentrant Phase Transitions and Triple Points in Lovelock Thermodynamics*, *JHEP* **09** (2014) 080 [arXiv:1406.7015] [INSPIRE].

[42] G.-Q. Li, *Effects of dark energy on $P-V$ criticality of charged AdS black holes*, *Phys. Lett.* **B 735** (2014) 256 [arXiv:1407.0011] [INSPIRE].

[43] C.O. Lee, *The extended thermodynamic properties of Taub-NUT/Bolt-AdS*, *Phys. Lett.* **B 738** (2014) 294 [arXiv:1408.2073] [INSPIRE].

[44] C.V. Johnson, *The Extended Thermodynamic Phase Structure of Taub-NUT and Taub-Bolt*, arXiv:1406.4533 [INSPIRE].

[45] B.P. Dolan, *Bose condensation and branes*, *JHEP* **10** (2014) 179 [arXiv:1406.7267] [INSPIRE].

[46] C.V. Johnson, *Holographic Heat Engines*, *Class. Quant. Grav.* **31** (2014) 205002 [arXiv:1404.5982] [INSPIRE].

[47] D. Kastor, S. Ray and J. Traschen, *Chemical Potential in the First Law for Holographic Entanglement Entropy*, *JHEP* **11** (2014) 120 [arXiv:1409.3521] [INSPIRE].

[48] F. Weinhold, *Metric geometry of equilibrium thermodynamics*, *J. Chem. Phys.* **63** (1975) 2479.

JHEP02(2015)143

[49] G. Ruppeiner, *Thermodynamics: a Riemannian geometric model*, *Phys. Rev.* **A 20** (1979) 1608.

[50] G. Ruppeiner, *Riemannian geometry in thermodynamic fluctuation theory*, *Rev. Mod. Phys.* **67** (1995) 605 [*Erratum ibid.* **68** (1996) 313] [INSPIRE].

[51] P. Salamon, J.D. Nulton and E. Ihrig, *On the relation between entropy and energy versions of thermodynamic length*, *J. Chem. Phys.* **80** (1984) 436.

[52] P. Salamon, E. Ihrig and R.S. Berry, *A group of coordinate transformations which preserve the metric of Weinhold*, *J. Math. Phys.* **24** (1983) 2515.

[53] R. Mrugala, J.D. Nulton, J.C. Schon and P. Salamon, *Statistical approach to the geometric structure of thermodynamics*, *Phys. Rev.* **A 41** (1990) 3156.

[54] H. Quevedo, *Geometrothermodynamics*, *J. Math. Phys.* **48** (2007) 013506 [physics/0604164] [INSPIRE].

[55] H. Quevedo, *Geometrothermodynamics of black holes*, *Gen. Rel. Grav.* **40** (2008) 971 [arXiv:0704.3102] [INSPIRE].

[56] H. Quevedo, A. Sanchez, S. Taj and A. Vazquez, *Phase transitions in geometrothermodynamics*, *Gen. Rel. Grav.* **43** (2011) 1153 [arXiv:1010.5599] [INSPIRE].

[57] A. Bravetti, D. Momeni, R. Myrzakulov and H. Quevedo, *Geometrothermodynamics of higher dimensional black holes*, *Gen. Rel. Grav.* **45** (2013) 1603 [arXiv:1211.7134] [INSPIRE].

[58] S. Ferrara, G.W. Gibbons and R. Kallosh, *Black holes and critical points in moduli space*, *Nucl. Phys.* **B 500** (1997) 75 [hep-th/9702103] [INSPIRE].

[59] R.-G. Cai and J.-H. Cho, *Thermodynamic curvature of the BTZ black hole*, *Phys. Rev.* **D 60** (1999) 067502 [hep-th/9803261] [INSPIRE].

[60] J.E. Aman, I. Bengtsson and N. Pidokrajt, *Geometry of black hole thermodynamics*, *Gen. Rel. Grav.* **35** (2003) 1733 [gr-qc/0304015] [INSPIRE].

[61] J.E. Aman and N. Pidokrajt, *Geometry of higher-dimensional black hole thermodynamics*, *Phys. Rev.* **D 73** (2006) 024017 [hep-th/0510139] [INSPIRE].

[62] G. Ruppeiner, *Thermodynamic curvature and black holes*, *Springer Proc. Phys.* **153** (2014) 179 [arXiv:1309.0901] [INSPIRE].

[63] S.A.H. Mansoori and B. Mirza, *Correspondence of phase transition points and singularities of thermodynamic geometry of black holes*, *Eur. Phys. J.* **C 74** (2014) 2681 [arXiv:1308.1543] [INSPIRE].

[64] S.A.H. Mansoori, B. Mirza and M. Fazel, *Hessian matrix, specific heats, Nambu brackets and thermodynamic geometry*, arXiv:1411.2582 [INSPIRE].

[65] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *Coupling constant dependence in the thermodynamics of $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 534** (1998) 202 [hep-th/9805156] [INSPIRE].

[66] H.B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, second edition, John Wiley and Sons (1985).

[67] S.H. Hendi, *Thermodynamic properties of asymptotically Reissner-Nordström black holes*, *Annals Phys.* **346** (2014) 42 [arXiv:1405.6996] [INSPIRE].

[68] J. Suresh, R. Tharanath, N. Varghese and V.C. Kuriakose, *The thermodynamics and thermodynamic geometry of the Park black hole*, *Eur. Phys. J.* **C 74** (2014) 2819 [arXiv:1403.4710] [INSPIRE].