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Phase Transition of Quasi-Two Dimensional Planar System

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The layered spin system with planar rotator symmetry is considered in the case of the small interplane coupling. It is argued that vortex pair excitations are important in this system, and their effect on the critical behavior is studied by the renormalization group equation. In particular, the temperature dependence of the spontaneous magnetization is investigated for the characteristic two dimensional behavior. The effect of the spin anisotropy on the transition temperature is also considered. The layered magnet K_2CuF_4 is discussed as an example of quasi-two dimensional planar system.

§ 1. Introduction

The planar rotator model or equivalent superfluid model in two dimensions is known to have a critical line in the low temperature region where the scale invariance may hold.[1),2)] Kosterlitz and Thouless[3)] have considered the vortex pair excitation mechanism in the low temperature region and found that in this planar model the transition to the high temperature region is determined by the dissociation of the vortex pair.

The important prediction of the theory is the universal jump[4)] of the two dimensional superfluid density ρ_s at the transition, which has been confirmed experimentally in liquid ^{4}He films.[5)] As for the magnetic systems, there is no ideal two dimensional material which shows the planar behavior. However, there are several quasi-two dimensional planar systems which have layered structure with very small interplane coupling. Experimentally these materials are known to show interesting properties.[6)]

Berezinskii and Blank,[7)] and Pokrovskii and Uimin[8)] have considered the layered planar magnetic system with the interplane coupling constant $J_\perp$ which is much smaller than the intraplane coupling constant $J_\parallel$. They derived the scaling law with respect to the ratio $\Delta = J_\perp/J_\parallel$ and discussed in particular the spontaneous magnetization, which should vanish in the case of the vanishing interplane coupling.[9)] Their arguments depend upon the self-consistent spin wave theory in two dimensions and do not incorporate the vortex-pair mechanism. In this paper, we discuss the quasi-two dimensional planar system by considering both the spin wave and vortex-pair excitations. Recently, Kosterlitz[10)] and subsequently Jose, Kadanoff, Kirkpatrick and Nelson[11] developed the renormalization group treatment for the two dimensional planar rotator model. We use the same renormalization group equation of two dimensions in a modified manner because of the presence of the small interplane coupling, which gives rise to the shift of the critical temperature T_c of three dimensional ordering away from the two dimensional critical point T_K.

The scaling law with respect to the ratio of $J_\perp/J_\parallel$ is explained in § 2. The spin wave calculation gives the result consistent with the scaling law in the low temperature region and the spontaneous magnetization is discussed in § 3. In § 4, we discuss possible vortex excitations in the layered system and compare the energy of a vortex ring extending over many sheets and that of an independent vortex pair existing in a single sheet. The result is used in the renormalization group treatment and the curve of the spontaneous magnetization is derived in § 5. In § 6, we consider the effect of spin anisotropy on the transition temperature and from the classical solution of a vortex in the presence of spin anisotropy we derive a condition of quasi-two dimensional planar model. As an example of quasi-two dimensional planar material, the ferromagnet K_2CuF_4 is discussed in § 7.

§ 2. Scaling law for the small interplane coupling

The layered planar rotator model is described by the following Hamiltonian:

$$\mathcal{H} = -J_\parallel \sum_{\langle i,j\rangle} (S_i^x S_j^x + S_i^y S_j^y) - J_\perp \sum_{\langle i,j\rangle} (S_i^x S_j^x + S_i^y S_j^y), \tag{2·1}$$

where $J_\parallel$ and $J_\perp$ denote the intraplane coupling and the interplane coupling, respectively, and the spin is assumed as classical. The summation is taken over the nearest neighbor lattice sites. The two dimensional planar rotator model to which our model reduces to when $J_\perp = 0$, has no spontaneous magnetization but below a certain temperature, which we denote as T_c, it shows the critical behavior with the scale invariance. The critical exponent η for the spin-spin correlation function changes continuously in the region below T_c which can be thought as infinite critical points. That the value of the spontaneous magnetization is zero is consistent with the recent calculation of the non-linear σ model, which gives the infinite value for the exponent β.[12),13)]

In the case of the small nonvanishing interplane coupling $J_\perp$, the expression for the spontaneous magnetization may be written by a scaling function f as

$$M_s = t^\beta f\left(\frac{\Delta}{t^\phi}\right), \tag{2·2}$$

where

$$\Delta = J_\perp/J_\parallel, \qquad t = 1 - \frac{T}{T_c}.$$

The exponent ϕ is the crossover exponent. In our case, the crossover is in the spatial dimensions and is identical to the exponent γ of the susceptibility in two dimensions.[14] Noting that the all region below T_c is critical, we have from (2·2) by eliminating the temperature,

$$M_s = C\Delta^{\beta/\gamma}. \tag{2·3}$$

The exponents β and γ are divergent for two dimensional planar model. However, the ratio β/γ is finite as

$$\beta/\gamma = (d-2+\eta)/(4-2\eta). \tag{2·4}$$

Therefore, we have the scaling law for the spontaneous magnetization of the quasi-two dimensional planar system as

$$M_s = \Delta^{\eta/4-2\eta}. \tag{2·5}$$

Near the critical temperature, we cannot apply the equation and we will discuss this problem in § 5. The exponent η is temperature dependent and in the low temperature limit is given by

$$\eta = \frac{kT}{2\pi J_{\parallel}}. \tag{2·6}$$

Pokrovskii and Uimin[8] obtained (2·5) for the planar rotator model. Berezinskii and Blank[7] discussed quantum spin system by the spin wave argument and obtained the slightly different result from the classical one of (2·5)

$$M_s = \left(\frac{J_{\perp}}{T}\right)^{\eta/4-2\eta}. \tag{2·7}$$

For the quantum spin system, the expression (2·7) is valid in the low temperature region, which then goes over to the classical scaling of (2·5) at high temperature. This quantum to classical crossover may be important for the layered magnet K_2CuF_4 which has spin 1/2. We will discuss this problem in § 7.

§ 3. Spin wave calculation

We consider the two point spin-spin correlation function for the planar rotator model;

$$\langle \boldsymbol{S}(0)\cdot\boldsymbol{S}(r)\rangle = \langle\cos[\theta(0)-\theta(r)]\rangle. \tag{3·1}$$

The average is taken for the following Hamiltonian:

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}J_{\parallel}\sum[\theta(x+a)-\theta(x)] + \frac{1}{2}J_{\perp}\sum[\theta(x+b)-\theta(x)]\\ &\approx \frac{1}{2}J_{\parallel}\int(\nabla\theta(x))^2 + \frac{1}{2}J_{\perp}\int(\nabla\theta(x))^2.\end{aligned} \tag{3·2}$$

Since the Hamiltonian of (3·2) is free field, we can write

$$\langle \mathbf{S}(0)\cdot\mathbf{S}(r)\rangle = \exp\left\{-\frac{1}{2}\langle(\theta(0)-\theta(r))^2\rangle\right\}. \tag{3·3}$$

Therefore, we write the spin-spin correlation function as

$$\langle \mathbf{S}(0)\cdot\mathbf{S}(r)\rangle = \exp\left\{-\frac{1}{2\pi K}G'(r)\right\} \tag{3·4}$$

with

$$G'(r) = 2\pi[G(0)-G(r)],$$
$$K = J_{\parallel}/kT\ , \tag{3·5}$$

where the function $G(r)$ is the lattice Green function. In two dimensions, $J_{\perp}=0$, we have for large r

$$G'(r) = \frac{1}{(2\pi)}\iint\frac{1-e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2}d^2k = \ln r \tag{3·6}$$

and it leads to

$$\langle \mathbf{S}(0)\cdot\mathbf{S}(r)\rangle \simeq \frac{1}{r^{\eta}}, \tag{3·7}$$

where

$$\eta = \frac{kT}{2\pi J_{\parallel}}. \tag{3·8}$$

The spontaneous magnetization is obtained by taking the infinite limit of r in the spin-spin correlation function as

$$M_s = [\lim_{r\to\infty}\langle \mathbf{S}(0)\cdot\mathbf{S}(r)\rangle]^{1/2}. \tag{3·9}$$

We obviously obtain zero spontaneous magnetization[9] in two dimensions from (3·7).

In the presence of the anisotropy $\Delta = J_{\perp}/J_{\parallel}$, the Green function becomes

$$\begin{aligned}\frac{1}{2\pi}G'(r, Z; \Delta) &\equiv G(0)-G(r)\\ &= \frac{1}{(2\pi)^2\Lambda}\int_0^{\Lambda}\int_0^{\Lambda}\frac{1-e^{i(\mathbf{k}\cdot\mathbf{r}+k_z\cdot z)}}{k^2+\Delta k_z^2}d^2k\,dk_z\\ &= \frac{1}{(2\pi)\Lambda}\int_0^{\Lambda}e^{ik_z z}dk_z\int_0^{\Lambda}\frac{\Lambda[1-J_0(kr)]}{k^2+\Delta k_z^2}kdk\\ &\quad -\frac{1}{(2\pi)\Lambda}\int_0^{\Lambda}\int_0^{\Lambda}\frac{e^{ik_z z}-1}{k^2+\Delta k_z^2}kdkdk_z.\end{aligned} \tag{3·10}$$

This quantity is calculated as

$$\frac{1}{2\pi}G' = -\frac{1}{2\pi\Lambda}\int_0^\Lambda e^{ik_z z}K_0(r\sqrt{\Delta}\,k_z)\,dk_z + \frac{1}{4\pi\Lambda}\int_0^\Lambda \ln\left(\frac{\Lambda^2+\Delta k_z^2}{\Delta k_z^2}\right)dk_z$$

$$\simeq -\frac{\pi}{(2\pi)^2\sqrt{r^2\Delta+z^2}} - \frac{1}{2\pi}\ln\sqrt{\Delta}\ . \tag{3·11}$$

From (3·4), (3·9) and (3·11), we have the expression for the spontaneous magnetization as

$$M_s = \Delta^{kT/8\pi J_\parallel}\ . \tag{3·12}$$

This result is consistent with (2·5) since the exponent η is given by (3·8) in the low temperature region.

§ 4. Independent vortex pair versus vortex ring

In the previous section we have discussed the spin wave contribution to the spontaneous magnetization. In the low temperature region the system is discussed by the spin wave excitations. However when the temperature is increased, the non-perturbative topological excitation may become important in two dimensions and especially in the planar system the phase transition occurs due to the dissociation of the vortex pair.[3),10)]

Before discussing the renormalization group treatment of vortex-pairs and of spin waves, we investigate the vortex-pair excitation in the quasi-two dimensional planar system. In two dimensions the energy of a vortex is given by

$$E_{\text{pair}} = J_\parallel \ln r + 2\mu\ , \tag{4·1}$$

where r is the distance of two vortices of opposite sign and μ is the energy necessary to make a vortex in one layer. If the interplane interaction $J_\perp$ is strong enough and becomes the same order as $J_\parallel$, the vortices in adjacent layers pile up and make a string which closes itself as a vortex ring. The energy of a vortex ring may be written as

$$E_{\text{ring}} = \frac{r}{a}(2\mu + J_\parallel \ln r), \tag{4·2}$$

where the quantity a is the lattice constant.

For the small interplane coupling $J_\perp$, it may be more favourable for vortex pairs in different layers to become independent, instead of making a vortex ring. The energy of the independent vortex pair is given by

$$E_{\text{i.v.p.}}=J_{\parallel}\ln r+J_{\perp}\left(\frac{r}{a}\right)^2+2\mu\,. \tag{4·3}$$

The first term is the vortex-vortex interaction and the second comes from the energy between adjacent layers since the mismatch of the spin configurations of the layers extends over an area of size r. Comparing (4·2) and (4·3), we find that the energy of an independent vortex pair becomes smaller than the energy of a vortex ring when r is smaller than the critical radius r_0 given by

$$\frac{J_{\perp}}{J_{\parallel}}\sim\frac{a\ln r_0}{r_0}\,. \tag{4·4}$$

For more accurate comparison we must consider the entropy as well, but we believe it will favor the independent vortex pairs.

We note that the energy of the vortex pairs depends on r logarithmically for r smaller than $\sqrt{J_{\parallel}/J_{\perp}}\cdot a$, which is in turn smaller than r_0. As the important excitations in our system, therefore, we only have to consider the independent vortex pairs of radius smaller than $\sqrt{J_{\parallel}/J_{\perp}}\cdot a$. It must be remarked that the three dimensional ordering takes place when the intraplane correlation length ξ becomes of the same order.

§ 5. Renormalization group equation

The renormalization group equation for the two dimensional planar rotator model has been developed by Kosterlitz[10] and subsequently by José et al.[11] Here we describe the results of the renormalization group treatment for two dimensional planar model following José et al[11] which is useful to analyze the quasi-two dimensional planar model. The spin-spin correlation function is given by

$$\langle \boldsymbol{S}(0)\cdot\boldsymbol{S}(r)\rangle=\exp\left\{-\frac{1}{2\pi}\left(\frac{1}{K}-\pi^2\sum_{r_0}r_0{}^2\langle m(0)\,m(r_0)\rangle\right)G'(r)\right\}, \tag{5·1}$$

where the lattice Green function $G'(r)$ is given by (3·5) and m is the vortex field. The vortex-vortex correlation is given by

$$\langle m(0)\,m(r)\rangle=-2y^2e^{-2\pi KG'(ar)}, \tag{5·2}$$

where y is introduced as

$$y=e^{-\mu/T} \tag{5·3}$$

which indicates the density of vortex. In the original Hamiltonian, y is unity, from which it deviates due to fluctuations. In two dimensions, from (3·6) we have

$$\langle m(0)\,m(r)\rangle=-2y^2r^{-2\pi K}. \tag{5·4}$$

Therefore, the spin-spin correlation function is written, as we have discussed in

§ 3 by the effective K which is defined by

$$K_{\text{eff}}^{-1}=K^{-1}+4\pi^3y^2\int_a^\infty \frac{dr}{a}\left(\frac{r}{a}\right)^{3-2\pi K_{\text{eff}}}, \tag{5·5}$$

where K in (5·4) is also replaced by K_{eff}. Then, from the scale invariance one derives the renormalization equations for K and y which reduce to

$$K'^{-1}=K^{-1}+4\pi^3y^2\ln b\,,$$
$$y'=y\{1+(2-\pi K)\ln b\}, \tag{5·6}$$

where

$$\pi K\approx 2\,. \tag{5·7}$$

We have from (5·6)

$$\frac{dx}{d\ln b}=-16\pi^2y^2,$$
$$\frac{dy}{d\ln b}=-xy\,. \tag{5·8}$$

The quantity x is defined by

$$x=\pi K_{\text{eff}}-2\,. \tag{5·9}$$

From (5·8) we have a fixed line for $x>0$ and at $x=0$, which corresponds to $K_{\text{eff}}^{-1}=\pi/2$, the dissociation of the vortex pair occurs. Hereafter, we denote this temperature as T_{K}. The solution of the renormalization equation is given by

$$x^2-16\pi^2y^2=-ct\,, \tag{5·10}$$

where t means the deviation from T_{K};

$$t=T-T_{\text{K}}\,. \tag{5·11}$$

The scaling variable b can be solved for $T>T_{\text{K}}$

$$b=\exp\left\{\frac{1}{\sqrt{T-T_{\text{K}}}}\left(\tan^{-1}\frac{\sqrt{T-T_{\text{K}}}}{x}-\tan^{-1}\frac{\sqrt{T-T_{\text{K}}}}{x_i}\right)\right\}, \tag{5·12}$$

where the quantity x_i is the initial value.

Up to now, we have described the renormalization equation for the two dimensional case ($J_\perp=0$). As shown in the previous section, in the quasi-two dimensional planar system the independent vortex pairs can be realized and their energy depends on the size r logarithmically when $r<a/\sqrt{\Delta}$. This means that the two dimensional renormalization group equations are valid as far as the vortex pairs

of size smaller than $a/\sqrt{\Delta}$ are concerned. In other words, the length scale trans formation is now limited by the cutoff

$$b_{\max}=1/\sqrt{\Delta} \tag{5·13}$$

and the upper limit of the integral in (5·5) must be replaced by $1/\sqrt{\Delta}$.

Therefore, from (5·12) and (5·13), we have the equation;

$$x=\frac{\sqrt{T-T_{\mathrm{K}}}}{\tan[\sqrt{T-T_{\mathrm{K}}}\ln(1/\sqrt{\Delta})]}\,. \tag{5·14}$$

we note that at T_{K}, Kosterlitz point where the phase transition occurs for $J_{\perp}=0$, x is not zero due to the presence of Δ as

$$x=\frac{1}{\ln(1/\sqrt{\Delta})} \tag{5·15}$$

and at this point the effective inverse temperature K_{eff} becomes as

$$K_{\mathrm{eff}}=\frac{2}{\pi}+\frac{1}{\ln(1/\sqrt{\Delta})}\,. \tag{5·16}$$

Therefore, the true phase transition is higher than T_{K}. In the two dimensional case the discontinuous change of K_{eff} with the universal jump appears at T_{K}. In the presence of the interplane coupling $J_{\perp}$, the discontinuous change of K_{eff} is not present and there appears the crossover phenomena from two dimensional like behavior to three dimensional behavior. For small Δ, we can estimate the true transition temperature T_c at the point where the denominator of (5·14) becomes close to zero;

$$\sqrt{T_c-T_{\mathrm{K}}}\ln\frac{1}{\sqrt{\Delta}}=\pi\,. \tag{5·17}$$

Thus the critical temperature is given by

$$T_c=T_{\mathrm{K}}+\left(\frac{\pi}{\ln(1/\sqrt{\Delta})}\right)^2. \tag{5·18}$$

This deviation from T_{K} is consistent with the crossover exponent, which is identical with γ of two dimensional planar model and is known to be infinite. The expression for the transition temperature is also derived as follows.[15] When we approach from the high temperature side to the transition temperatnre, the correlation length becomes large as

$$\xi\sim\exp(\pi/\sqrt{T-T_{\mathrm{K}}})\,. \tag{5·19}$$

Thus the true transition may appear at the point where

$$J_\perp \xi^2 \sim J_\parallel . \tag{5·20}$$

It gives the same expression for T_c as (5·18). The equation (5·14) gives the value of $K_{\rm eff}$ as a function of temperature, which decreases rapidly above $T_{\rm K}$. The value of $K_{\rm eff}$ becomes $2/\pi$ at the following temperature:

$$T = T_{\rm K} + \pi^2/4\left(\ln \frac{1}{\sqrt{\varDelta}}\right)^2 . \tag{5·21}$$

From (5·18) and (5·21), we estimate the decreasing of $K_{\rm eff}$ above $T_{\rm K}$ as

$$\frac{\varDelta K_{\rm eff}}{\varDelta T} = -\frac{4}{\pi^2} \ln \frac{1}{\sqrt{\varDelta}} . \tag{5·22}$$

If the temperature approaches the true critical point T_c, the renormalization group equation of (5·16) becomes invalid and the system shows the three dimensional critical behavior. In this region, the effective interplane coupling becomes the same order as intraplane coupling (5·20). For the isotropic planar model we have

$$\frac{kT_c}{J} \simeq 4 . \tag{5·23}$$

Therefore at least when $K_{\rm eff}$ becomes 1/4, the three dimensional critical behavior may be observed. For very small $\varDelta$, the temperature shift (5·18) becomes very small. However, the existing layered materials such as K_2CuF_4 have the value 10^{-3} for $\varDelta$, so that the three dimensional critical region is expected to be seen.

§ 6. Effect of spin anisotropy

We have discussed the quasi-two dimensional planar model with small space anisotropy $\varDelta$. In this section we investigate the effect of the spin anisotropy.

Anisotropic two dimensional Heisenberg model is described by

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \left(S_i^x S_j^x + S_i^y S_j^y + \left(1 - \frac{\lambda}{2}\right) S_i^z S_j^z \right). \tag{6·1}$$

It is interesting to see how the critical behavior of this anisotropic Heisenberg model depends on the anisotropic parameter λ. When λ is zero, there may be no phase transition.[12] For $0 < \lambda/2 \leq 1$, the system is similar to the planar model. However, when a vortex is created, the spins near its center may tend to align into the direction perpendicular to the easy plane. To investigate this situation, we consider the classical solution corresponding to the presence of a vortex excitation. In the continuum limit the Hamiltonian (6·1) is approximated as

$$H = \frac{J}{2} \int \left\{ \left(1 - \frac{\lambda}{2} \cos^2\theta\right) (\nabla\theta)^2 + \sin^2\theta\, (\nabla\phi)^2 + \lambda \cos^2\theta \right\} \tag{6·2}$$

with

$$S^z = \cos\theta\,,$$
$$S^x = \sin\theta\cos\phi\,,$$
$$S^y = \sin\theta\sin\phi\,.$$

In the classical equation

$$\left(1-\frac{\lambda}{2}\cos^2\theta\right)\nabla^2\theta+\frac{\lambda\sin 2\theta}{4}(\nabla\theta)^2-\frac{\sin 2\theta}{2r^2}+\frac{\lambda}{2}\sin 2\theta=0 \tag{6·3}$$

we put

$$(\nabla\phi)^2 = 1/r^2 \tag{6·4}$$

for the vortex solution with unit circulation. Hence we have

$$\left(1-\frac{\lambda}{2}\cos^2\theta\right)\left(\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}\right)+\frac{\lambda\sin 2\theta}{4}\left(\frac{d\theta}{dr}\right)^2-\frac{\sin 2\theta}{2r^2}+\frac{\lambda}{2}\sin 2\theta=0 \tag{6·5}$$

with the boundary condition

$$\theta=\frac{\pi}{2}\quad(r\to\infty)\quad\text{and}\quad\theta=0\,.\quad(r=0) \tag{6·6}$$

The asymptotic solution becomes

$$\theta\simeq\begin{cases}\dfrac{\pi}{2}-C_\infty e^{-\sqrt{\lambda}r}\,, & (r\to\infty)\\ cr^{1/\sqrt{1-\lambda/2}}\,. & (r\to 0)\end{cases} \tag{6·7}$$

As we see from (6·7), the characteristic length is provided by $1/\sqrt{\lambda}$ which we interpret as the radius of the vortex core: To estimate the energy of a single vortex with finite size R, we use the asymptotic solution for $r\to\infty$ with $C_\infty=\pi/2$ and find

$$\begin{aligned}E&=\frac{J}{2}\int_a^R(\sin^2\theta)\,(\nabla\phi)^2d^2r\\&=\pi J\int_a^R\frac{1-(\pi/2)\,e^{-\sqrt{\lambda}r}}{r}dr\\&=\pi J\left\{\ln\left(\frac{R}{a}\right)-\frac{\pi}{2}(E_1(\sqrt{\lambda}\,a)-E_1(\sqrt{\lambda}R))\right\},\end{aligned} \tag{6·8}$$

where E_1 is exponential integral. Therefore we expect the vortex pair energy still depends logarithmically on the pair distance when it is greater than the core size $1/\sqrt{\lambda}$.

If λ becomes zero, the system becomes the isotropic Heisenberg model and the critical temperature drops to zero. It is interesting to note that the vortex pair turns out to be an instanton in the limit $\lambda=0$ and the energy becomes finite and independent of the pair distance. In the Appendix we consider the classiccal solution for $\lambda=0$ in detail. For small λ case, the transition temperature is estimated by the value of the crossover exponent ϕ of spin anisotropy, which is known to be infinite in two dimensions.[16),17)] However, the ratio ϕ/ν for the two dimensional Heisenberg model becomes two. Thus, we have

$$\lambda\sim\xi^{-\phi/\nu}\sim\xi^{-2}. \tag{6·9}$$

The correlation length of 2d Heisenberg model[12)] is known to be

$$\xi\sim e^{2\pi J/kT}. \tag{6·10}$$

Therefore, by the scaling argument for the crossover phenomena, we have the transition temperature as

$$\frac{kT_c}{J}\simeq\frac{4\pi}{\ln(c/\lambda)}. \qquad (\lambda\to 0) \tag{6·11}$$

Including the anisotropy of spin freedom, we have the expression for the energy of vortex pair as (6·8), which is not essentially different from (4·3) and the independent vortex pair picture of the previous section may be valid when the following condition of the distance of vortex pair r is satisfied:

$$\frac{1}{\sqrt{\lambda}}<r<\frac{1}{\sqrt{\Delta}}. \tag{6·12}$$

If the spin anisotropy parameter λ and the space anisotropy $\Delta=J_{\perp}/J_{\parallel}$ become comparable, we may have a picture which is completely different from the picture of independent vortex pair. In this case, the crossover from the anisotropic Heisenberg to the three dimensional planar behavior occurs and the transition temperature is estimated in the small Δ and small λ case by

$$\frac{kT_c}{J}=-\frac{4\pi}{\ln(c_1\Delta)}+c_2\lambda^{1/\phi}. \tag{6·13}$$

The first term*) is obtained from the correlation length ξ of the two dimensional Heisenberg model (6·10) by a similar argument to (6·9). The second term is due to the crossover from the three dimensional Heisenberg to the three dimensional planar model. The crossover exponent ϕ is known to be 1.22.[18)] When the condition of (6·12) is satisfied, we may have the following expression for the transi-

*) For the exactly solvable anisotropic spherical model, the constant c_1 is 64 and this expression gives quite accurate value for $\Delta=10^{-2}$.

tion temperature,

$$T_c = T'_K(\lambda) + A(\lambda) \Big/ \left(\ln \frac{c}{\Delta}\right)^2, \qquad (6\cdot14)$$

Where T_K' is the transition temperature of two dimensional planar model with the spin anisotropy λ and it is approximated by (6·11) for small λ. From (6·11) and (6·13), we get for $\Delta < \lambda \ll 1$,

$$T_c(\Delta > 0, \lambda = 0) < T_c(\Delta = 0, \lambda > 0). \qquad (6\cdot15)$$

Therefore, Eq. (6·14) is better approximate expression than Eq. (6·13) for $\Delta < \lambda$.

In the next section, we consider a quasi-two dimensional planar ferromagnet K_2CuF_4 which has small space anisotropy Δ and small spin anisotropy λ.

As another spin anisotropy, the dipole interaction is important. The dipole interaction is known to shift the transition temperature upwards. However, the crossover from the planar fixed point to the dipole fixed point is complicated and has not been considered completely.[19),20)]

§ 7. Quasi-two dimensional materials

Recently, quasi-two dimensional materials[6)] have been investigated in experiments. As quasi-two dimensional Ising like materials, K_2NiF_4 and Rb_2MnF_4 are known and show remarkable two dimensional Ising like behavior except in a very narrow region near the transition temperature.

The ferromagnet K_2CuF_4[21),22)] is known as a good quasi-two dimensional planar like material. The interplane interaction is almost Heisenberg like with the small spin anisotropy $\lambda \simeq 2\times10^{-2}$. Due to this spin anisotropy, the system shows the critical behavior of the planar system. The interplane coupling constant is small and the ratio $\Delta \equiv J_\perp/J_\parallel = 6\times10^{-4}$. The transition temperature is 6.25°K. Since the ratio Δ is very small and the spin anisotropy satisfies the condition of (6·12), we apply the independent vortex pair picture to this material. The spontaneous magnetization has been measured and it shows $T^{3/2}$ law up to 4.5°K. The spin wave theory, which is valid in the low temperature region, gives $T^{3/2}$ behavior for the spontaneous magnetization only for $T < J_\perp k = 0.024$°K, and it gives $T \ln T$ term above $J_\perp/k$, which Berezinskii and Blank[7)] exponentiated as (2·7). Therefore we compare the experimental curve of the spontaneous magnetization with (2·7) in the low temperature region and we may have crossover from the quantum scaling to the classical scaling as (2·5). This material K_2CuF_4 has spin 1/2 and the intralayer coupling $J_\parallel/k$ as 20°K. From these values, we estimate the Kosterlitz transition temperature T_K as 7.8°K*) when $\lambda = 1$. Since the actual value of λ is 0.02, the transition temperature T_K is reduced considerably. Equation (6·11)

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*) Here we take T_K as $\pi J/2k$; however more precise value has been investigated as (1.4) J/k,[10)] which gives T_K as 7°K.

gives the same value as T_K, if we take $c=64$ and we need more accurate formula. However, we believe the reduction due to the spin anisotropy is around 30%, for $\lambda=0.02$, which leads to the value 5.5°K as the Kosterlitz transition temperature. Taking the value of Δ as 6×10^{-4}, the theoretical curve of the spontanous magnetization agrees well with the experiment. We note that the spontaneous magnetization (2·5) becomes large even for the small $\Delta=6\times10^{-4}$, and therefore, the experimental sharp neutron Bragg peak observed in neutron scattering can be explained. The recent neutron scattering experiment[23] shows the crossover phenomena near the transition temperature. The value of the magnetization critical exponent β changes from 0.24 to 0.33 at 6°K. The small value 0.24 of β means the rather sharp decreasing of the magnetization. This sharp decreasing may be interpreted by the renormalization group equation in § 5 and is typical of the quasi-two dimensional planar system.

As another interesting example, the layered copper compounds $(C_nH_{2n+1}NH_3)_2CuCl_4$[6),24)] shows quasi-two dimansional behavior. If these series have the planar like condition $\Delta<\lambda$, it is interesting to measure the shape of the spontaneous sublattice magnetization and the shift of the transition temperature.

In liquid crystals, the smectic B phase is regarded as a layered quasi-two dimensional solid. If the interplane coupling is weak enough, we can apply the similar analysis although the two dimensional melting transition[26] is different from the transition of the planar model; for example, the temperature dependences of the exponent η is different. Recent experiment of BBOA (408), measured the transition from smectic B to smectic A phase.[27),28)] The smectic B phase, if the interplane coupling is weak, may be interpreted as a stacking of two dimensional solid.[29),30)] In this case also, the X ray scattering may give sharp Bragg peak as in K_2CuF_4. The typical behavior of quasi-two dimensionality may be seen in the dynamical behavior.

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Appendix

We have considered the single vortex classical solution for the anisotropic Heisenberg model in § 6. Here we discuss in detail the vortex pair solution, and in particular we note that the spin configuration of vortex pair in the 2d anisotropic Heisenberg model changes continuously into an instanton solution in the limit of vanishing anisotropic parameter λ.

We assume that two vortices are located at $(-(a/2),0)$ and $((a/2),0)$ in the two dimensional coordinate space and they are attracting each other. The

Heisenberg spin is described by (6·2) with the parameter θ and ϕ. When the anisotropy λ becomes zero, the system becomes the isotropic Heisenberg model and it has instanton solutions by Belavin and Polyakov.[31] Representing the coordinate (x, y) by the complex plane z

$$z = x + iy\,, \tag{A·1}$$

we have the following one instanton solution:

$$\frac{z + a/2}{z - a/2} = \cot\frac{\theta}{2} e^{i\phi}. \tag{A·2}$$

Along the real axis $(y=0, \phi=0)$, we have

$$\theta = 2\tan^{-1}\left(\frac{x - a/2}{x + a/2}\right). \tag{A·3}$$

In the limit $x \to \pm\infty$, the spin aligns in the x direction. The spin configuration changes continuously from the vortex pair to an instanton in the limit $\lambda = 0$. The configuration still looks like vortex pair even for $\lambda = 0$, but the energy now becomes $4\pi J$ which is independent of the distance between vortices.

It is also interesting to consider the single vortex solution for $\lambda = 0$ (6·5) with the following boundary condition:

$$\theta = \begin{cases} \frac{\pi}{2}\,, & r = R\,, \\ 0\,, & r = 0\,. \end{cases} \tag{A·4}$$

The classical solution of (6·5) is given by

$$\theta = 2\tan^{-1}\left(\frac{r}{R}\right). \tag{A·5}$$

The energy is calculated as

$$E = \frac{J}{2}\int\left[(\nabla\theta)^2 + \frac{\sin^2\theta}{r^2}\right]d^2r$$

$$= \pi J\int_0^R\left((\partial\theta)^2 + \frac{\sin^2\theta}{r^2}\right)r\,dr\,, \tag{A·6}$$

$$= 2\pi J\,. \tag{A·7}$$

Thus the energy is independent of R and just half of the energy of an instanton.

For the very small λ, we may have the following free energy[3] of the vortex pair:

$$\Delta F = 4\pi J - kT\ln\frac{1}{\lambda}\,. \tag{A·8}$$

The first term is the energy of the instanton and the second term of (A·8), is entropy term which is obtained from the fact that the core of vortex is $1/\sqrt{\lambda}$ when anisotropy λ is finite. We have the expression for the transition temperature from (A·8) as

$$kT_c = \frac{4\pi J}{\ln(1/\lambda)}\,. \tag{A·9}$$

This result is obtained by considering the instanton. We obtain the same formula (6·11) from the scaling argument using the correlation length ξ which is obtained by the renormalization group.

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