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Phase Transition of Second Kind in Hadronic Matter

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We consider hadronic matter to be composed of quarks and self-interacting bosons and calculate the effective potential in the one-loop approximation in renormalizable field theory. We also calculate the relevant thermodynamical quantities and show that
a) a phase transition of the second kind can arise at temperature $T=0.17$ GeV,
b) the level density of hadronic matter agrees with the one of the statistical bootstrap model below the critical temperature and
c) our model may predict good fits to the experimental transverse momentum spectra in particle production if based on a fireball model.

§ 1. Introduction

It is well known that the inclusive transverse momentum spectra $d\sigma/dP_T^2$ for the reaction

$$a+b\rightarrow c+\text{anything}$$

has the form

$$\frac{d\sigma}{dP_T^2}\sim\exp\{-(P_T^2+m_c^2)^{1/2}/T\} \tag{1·1}$$

at small $P_T(P_T\lesssim 1\,\text{GeV}/c)$ and that T is roughly constant independently of the incident energy. This is very surprising, and so we can expect that a thermodynamical equilibrium (we refer it to a hadronic matter or a fireball) is instantaneously reached after the collision. The thermodynamical model proposes a two-step picture—the production and subsequent decay of fireballs. The statistical bootstrap model[1),2)] gives an explanation for the constant temperature. The bootstrap condition (a fireball is a statistical equilibrium of an undetermined number of all kinds of fireballs) predicts the existence of a maximum temperature T_0 and also the divergence of the thermodynamical potential in $T \geqq T_0$, i.e., a phase transition of the zero kind at T_0.

On the other hand, remarkable deviation from the form (1·1) has been observed at large $P_T(P_T \gtrsim 1\ \mathrm{GeV}/c)$.[3)~5)] One can get a first approximation for the inclusive transverse momentum spectra at large P_T by assuming the form for the effective temperature

$$T \propto E_{\mathrm{cm}}^{1/4} \tag{1·2}$$

in Eq. (1·1),[6)] where E_{cm} is the incident energy in the center of mass system. In the inclusive reaction, we observe that the hadrons are produced from fireball of various masses even at a definite incident energy, and so we can get the form (1·1) at small P_T if we assume that multi-fireballs with small mass are also produced peripherally. Then the constancy of the temperature at small mass and the increase at large mass must be ensured by

$$\lim_{T \to T_c - 0} \frac{dM_f}{dT} \gg \lim_{T \to T_c + 0} \frac{dM_f}{dT}, \tag{1·3}$$

where M_f is the mass of a fireball. This means a phase transition of the second kind. It was suggested by Eliezer and Weiner[7)] that a hadronic matter might undergo a phase transition of the second kind based on Weinberg model.[8)] Dolan and Jackiw derived the critical temperature by a functional-diagrammatic evaluation, however, their theory is renormalized at zero temperature and so draws an effective potential of complex number.

In this paper, we derive an effective potential of a hadronic matter by Dolan and Jackiw method, where the renormalization is carried out at finite temperature and the hadronic matter is regarded as being composed of quarks and self-interacting bosons. We also show that the predictions of our model are consistent with the condition (1·3) and the level density of the hadronic matter has the form $\exp(-M_f/T_c)$ in $T < T_c$.

In the next section, we calculate the effective potential renormalized at finite temperature in the one-loop approximation. In § 3 we compute entropy, energy, specific heat and physical masses of a quark and a boson from the thermodynamical potential. Finally we give concluding remarks.

§ 2. Effective potential

In this paper we restrict ourselves to a simple model involving only a spinor and a scalar fields. At finite temperature the generating functional is defined as follows:

$$Z(\eta, \bar{\eta}, J) = \mathrm{Tr}\left(e^{-H/T} \left\{\exp\left(\int d^4x\,(\bar{\psi}\eta + \bar{\eta}\psi + J\phi)\right)\right\}_+\right), \tag{2·1}$$

where H is the Hamiltonian of the system, T is the temperature in energy unit, $\eta, \bar{\eta}$ and J are the external sources, $\int d^4x = \int_0^{1/T} du \int_{-\infty}^{\infty} d\boldsymbol{r}$ and the suffix + denotes that the time ordering has been performed. It should be noted that u denotes the imaginary time. The generating functional $Z(\eta, \bar{\eta}, J)$ defined by Eq. (2·1) corresponds to the partition function in the statistical thermodynamics. The simple derivation of it and the related notes are given in Appendix A. By taking the functional differentiations of the generating functional we can get the time ordered n-point Green's functions. In order to investigate the connected Green's functions it is more convenient to define the connected generating functional Z^c by $Z = \exp(-Z^c/T)$.

The classical fields are defined by

$$\bar{\psi}_c = -\frac{1}{T}\frac{\delta Z^c}{\delta \eta}, \quad \psi_c = -\frac{1}{T}\frac{\delta Z^c}{\delta \bar{\eta}}, \quad \phi_c = -\frac{1}{T}\frac{\delta Z^c}{\delta J} \tag{2·2}$$

and the functional Legendre transformation:

$$\Gamma\{\bar{\psi}_c, \psi_c, \phi_c\} = Z^c\{\eta, \bar{\eta}, J\} + T\int d^4x (\bar{\psi}_c \eta + \bar{\eta}\psi_c + J\phi_c) \tag{2·3}$$

is called the effective action. One particle irreducible Green's functions are given by taking the functional differentiations of Γ. Here Z^c and Γ correspond to Gibbs' and Helmholtz's free energies in the thermodynamics respectively. We expand the effective action in powers of the external momenta as[10]

$$\Gamma\{\bar{\psi}_c, \psi_c, \phi_c\} = -T\int d^4x \{-P_c(\bar{\psi}_c, \psi_c, \phi_c) + O(\partial_\mu \bar{\psi}_c, \partial_\mu \psi_c, \partial_\mu \phi_c)\}. \tag{2·4}$$

In the tree approximation $-P_c$ is the sum of all non-derivative terms in the Lagrangian density. To investigate the properties of the spontaneously broken theory, we define the effective potential by

$$P(\hat{\phi}) = P_c(\bar{\psi}_c = \psi_c = 0, \phi_c = \hat{\phi}), \tag{2·5}$$

where $\hat{\phi}$ is a constant. The above definition shows that the effective potential is the generating functional for single particle irreducible Green's functions of the scalar field at zero momentum. The purpose of this section is to get the explicit temperature dependence of the effective potential in the one-loop approximation.

Now we begin with the following Lagrangian:

$$\begin{aligned}\mathcal{L} &= -\bar{\psi}\gamma\cdot\partial\psi + g\bar{\psi}\psi\phi - \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4!}\lambda\phi^4 \\ &= -\bar{\psi}\gamma\cdot\partial\psi + g_0\bar{\psi}\psi\phi - \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu_0^2\phi^2 - \frac{1}{4!}\lambda_0\phi^4 \\ &\quad + \left(\frac{1}{Z_g} - 1\right)g_0\bar{\psi}\psi\phi - \frac{1}{2}\delta\mu_0^2\phi^2 - \frac{1}{4!}\left(\frac{1}{Z_\lambda} - 1\right)\lambda_0\phi^4,\end{aligned} \tag{2·6}$$

where

$$\mu^2=\mu_0{}^2+\delta\mu_0{}^2\,,\quad g=\frac{g_0}{Z_g}\,,\quad \lambda=\frac{\lambda_0}{Z_\lambda}\,. \tag{2·7}$$

In Eqs. (2·6) and (2·7) μ_0^2, g_0 and λ_0 are temperature-independent parameters and $\delta\mu_0^2$, Z_g and Z_λ are in general temperature-dependent in the finite temperature system. We have not explicitly written the renormalization constants of field operators as they are not necessary for our estimation of the effective potential. Following to Jackiw[11] the effective potential for our Lagrangian is given at zero temperature by

$$P(\hat\phi)=\frac{1}{2}\mu^2\hat\phi^2+\frac{1}{4!}\lambda\hat\phi^4-\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(k^2+\mu^2+\frac{1}{2}\lambda\hat\phi^2\right)$$
$$+i\int\frac{d^4k}{(2\pi)^4}\ln\det(ik\cdot\gamma+g\hat\phi)$$
$$+i\left\langle\exp\left(i\int d^4x\mathcal{L}_I(\overline{\psi},\psi,\phi,\hat\phi)\right)\right\rangle. \tag{2·8}$$

The last term of Eq. (2·8) summarizes the following operation. Calculate the vacuum expectation value of $(\exp(i\int d^4x\mathcal{L}_I))_+$ where $\mathcal{L}_I$ is the interaction Lagrangian after the replacement $\phi\to\phi+\hat\phi$ in Eq. (2·6); keep only the connected single particle irreducible graphs; delete an over all factor of space-time volume $\int d^4x$. At finite temperature we have only to replace (see Appendix B)

$$\int d^4k\to 2i\pi T\sum_n\int d^3k$$
$$k_4\to\begin{cases}2n\pi T & \text{for bosons,}\\ (2n+1)\pi T & \text{for fermions,}\end{cases} \tag{2·9}$$

where n denotes the integer. Then the effective potential at finite temperature is given in the one-loop approximation by

$$P(\hat\phi)=\frac{1}{2}\mu^2\hat\phi^2+\frac{1}{4!}\lambda\hat\phi^4$$
$$+\frac{1}{2}T\sum_n\int\frac{d^3k}{(2\pi)^3}\ln\left(4\pi^2n^2T^2+k^2+\mu_T{}^2+\frac{1}{2}\lambda_T\hat\phi^2\right)$$
$$-T\sum_n\int\frac{d^3k}{(2\pi)^3}\ln\det(i\boldsymbol{k}\cdot\boldsymbol{\gamma}+i(2n+1)\pi T\gamma_4+g_T\hat\phi)\,. \tag{2·10}$$

Note that we have replaced μ^2, λ and g by the renormalized temperature-dependent parameters μ_T^2, λ_T and g_T in the last two terms of Eq. (2·10). If we employ μ_0^2, λ_0 and g_0 instead of the temperature-dependent parameters, there appeares a complex thermodynamical potential as can be seen in Ref. 9). This is because the scalar meson mass on the third term in Eq. (2·10) is given by $\mu_0^2+\frac{1}{2}\lambda_0\langle\hat{\phi}\rangle^2$ where $\langle\hat{\phi}\rangle$ is the value of $\hat{\phi}$ when the effective potential takes its minimum. For the requirement of spontaneous symmetry breaking μ_0^2 must be negative. So the scalar meson mass takes the negative value when the temperature tends near to the critical temperature because then $\langle\hat{\phi}\rangle$ approaches zero. To avoid a complex thermodynamical potential, we adopt the loop expansion in terms of the temperature-dependent parameters μ_T, λ_T and g_T as in Eq. (2·10). Then the scalar meson mass is given by $\mu_T^2+\frac{1}{2}\lambda_T\langle\hat{\phi}\rangle^2$ which is found to be always positive. The temperature dependences of our parameters are decided by the renormalization conditions. The sum on n in Eq. (2·10) diverges; we evaluate them by the trick used in Ref. 9). Then it follows

$$\sum_n \ln(4\pi^2n^2T^2+E_B^2)=\frac{2}{T}\left\{\frac{E_B}{2}+T\ln\left[1-\exp\left(\frac{1}{T}E_B\right)\right]\right\}$$
$$+\text{(terms independent of } E_B), \qquad (2\cdot11)$$

$$\sum_n \ln\det(i\boldsymbol{k}\cdot\boldsymbol{\gamma}+i(2n+1)\pi T\gamma_4+g_T\hat{\phi})$$
$$=2\sum_n \ln\{(2n+1)^2\pi^2T^2+E_F^2\}$$
$$=\frac{4}{T}\left\{\frac{E_F}{2}+T\ln\left[1+\exp\left(-\frac{1}{T}E_F\right)\right]\right\}$$
$$+\text{(terms independent of } E_F), \qquad (2\cdot12)$$

where

$$E_B^2=k^2+M^2=k^2+\mu_T^2+\tfrac{1}{2}\lambda_T\hat{\phi}^2,$$
$$E_F^2=k^2+m^2=k^2+g_T^2\hat{\phi}^2. \qquad (2\cdot13)$$

Consequently the third and the fourth term of Eq. (2·10) can be written apart from unimportant terms as

$$\text{(3rd term)}=\int\frac{d^3k}{(2\pi)^3}\left\{\frac{E_B}{2}+T\ln\left[1-\exp\left(-\frac{1}{T}E_B\right)\right]\right\}$$
$$=P_B^1+P_B^2, \qquad (2\cdot14)$$
$$\text{(4th term)}=-4\int\frac{d^3k}{(2\pi)^3}\left\{\frac{E_F}{2}+T\ln\left[1+\exp\left(-\frac{1}{T}E_F\right)\right]\right\}$$
$$=P_F^1+P_F^2. \qquad (2\cdot15)$$

As can be seen from Eqs. (2·13) ~ (2·15), the fourth term is always real but the third term takes a complex value for $M^2<0$. In our perturbation scheme, M and m accord with the relevant physical masses at the value of $\hat{\phi}$ giving the minimum of $P(\hat{\phi})$ which corresponds to the thermodynamical potential density. The third term, consequently, can be real around such a value of $\hat{\phi}$ in contrast to Dolan and Jackiw model, in which M can accord with the physical mass at zero temperature only.

Then we can evaluate $P_B{}^i$ and $P_F{}^i$ as

$$P_B{}^1 = \int \frac{d^3k}{(2\pi)^3}\frac{E_B}{2} = -\frac{\Gamma(-2)}{32\pi^2}M^4 + \frac{M^4}{64\pi^2}\ln(\pi M^2), \tag{2·16}$$

$$P_B{}^2 = T\int\frac{d^3k}{(2\pi)^3}\ln\left\{1-\exp\left(-\frac{1}{T}E_B\right)\right\}$$
$$= -\frac{\pi^2}{90}T^4 + \frac{M^2}{24}T^2 - \frac{M^4}{64\pi^2}\ln(\pi M^2) + O(M^3), \tag{2·17}$$

$$P_F{}^1 = -4\int\frac{d^3k}{(2\pi)^3}\frac{E_F}{2} = \frac{\Gamma(-2)}{8\pi^2}m^4 - \frac{m^4}{16\pi^2}\ln(\pi m^2), \tag{2·18}$$

$$P_F{}^2 = -4T\int\frac{d^3k}{(2\pi)^3}\ln\left\{1+\exp\left(-\frac{1}{T}E_F\right)\right\}$$
$$= -\frac{7\pi^2}{180}T^4 + \frac{m^2}{12}T^2 + \frac{m^4}{16\pi^2}\ln(\pi m^2) + O(m^4), \tag{2·19}$$

where the last terms of Eq. (2·17) and (2·19) are finite. In § 3 M and m will be shown to be very small parameters at $\hat{\phi}$ giving the minimum of $P(\hat{\phi})$.

Owing to Eqs. (2·16) ~ (2·19) the effective potential is given by

$$P(\hat{\phi}) = \frac{1}{2}\mu_0{}^2\hat{\phi}^2 + \frac{1}{4!}\lambda_0\hat{\phi}^4 + \frac{1}{2}\delta\mu_0{}^2\hat{\phi}^2 + \frac{1}{4!}\left(\frac{1}{Z_\lambda}-1\right)\lambda_0\hat{\phi}^4$$
$$-\frac{\Gamma(-2)}{32\pi^2}M^4 + \frac{\Gamma(-2)}{8\pi^2}m^4 - \frac{\pi^2}{20}T^4 + \frac{M^2}{24}T^2 + \frac{m^2}{12}T^2 + O(M^3, m^4). \tag{2·20}$$

We can see from Eqs. (2·13) and (2·20) that the effective potential can be renormalized by the mass and the coupling constant renormalizations. We define the renormalized parameters by the conditions:

$$\left.\frac{d^2P}{d\hat{\phi}^2}\right|_{\hat{\phi}=0} = \mu_T{}^2, \quad \left.\frac{d^4P}{d\hat{\phi}^4}\right|_{\hat{\phi}=0} = \lambda_T, \tag{2·21}$$

and require $\delta\mu_0{}^2$ and $\delta\lambda_0 = (Z_\lambda{}^{-1}-1)\lambda_0$ cancel the divergent terms in Eq. (2·20). Then it follows:

$$\mu_T{}^2 = \mu_0{}^2 + \frac{T^2}{6}\left(\frac{1}{4}\lambda_T + g_T{}^2\right), \quad \lambda_T = \lambda_0. \tag{2·22}$$

We replace g_T by g_0 as it is not necessary here to distinguish them. Finally the renormalized effective potential in our approximation is given apart from irrelevent divergent constant by

$$P(\hat{\phi})=\frac{1}{2}\mu_T{}^2\hat{\phi}^2+\frac{1}{4}\lambda_T\hat{\phi}^4-\frac{\pi^2}{20}T^4+\frac{T^2}{24}\mu_T{}^2\,. \tag{2·23}$$

§ 3. Thermodynamical consideration

We will apply the results in the previous section to the system of quarks and scalar bosons (the hadronic matter). We assume that the quarks and the scalar bosons are confined in a constant volume V_h and calculate the relevant thermodynamical quantities. The minimum of $P(\hat{\phi})$ with respect to $\hat{\phi}$ is realized thermodynamically and corresponds to the thermodynamical potential density. We obtain the value of $\hat{\phi}$ giving the minimum of $P(\hat{\phi})$ by the equation:

$$\left.\frac{\partial P(\hat{\phi})}{\partial\hat{\phi}}\right|_{\hat{\phi}=\langle\hat{\phi}\rangle}=\left(\mu_T{}^2+\frac{\lambda_0}{6}\langle\hat{\phi}\rangle^2\right)\langle\hat{\phi}\rangle=0\,, \tag{3·1}$$

which leads to

$$\langle\hat{\phi}\rangle^2=-\frac{6\mu_T{}^2}{\lambda_0}\theta(T_c-T), \tag{3·2}$$

where

$$\theta(x)=\begin{cases}1 & \text{for} \quad x>0\,,\\ 0 & \text{for} \quad x<0\,.\end{cases}$$

Then the thermodynamical potential density $\tilde{P}(T)$ is given by

$$\tilde{P}(T)=-\frac{3}{2}\frac{\mu_T{}^4}{\lambda_0}\theta(T_c-T)-\frac{\pi^2}{20}T^4+\frac{T^2}{24}\mu_T{}^4\,. \tag{3·3}$$

On the other hand, we can get the critical temperature of the hadronic matter from Eq. (2·23) as a function of the renormalized parameters as follows:

$$T_c{}^2=\frac{-\mu_0{}^2}{A}\,, \tag{3·4}$$

$$A=\frac{1}{12}\left(2g_0{}^2+\frac{\lambda_0}{2}\right), \tag{3·5}$$

by the condition:

$$\mu_T{}^2\equiv\left.\frac{\partial^2P(\hat{\phi})}{\partial\hat{\phi}^2}\right|_{\hat{\phi}=0}=0\,. \tag{3·6}$$

Then $\mu_T{}^2$ is represented as

$$\mu_T{}^2=A(T^2-T_c{}^2), \tag{3·7}$$

and Eq. (3·3) is rewritten as follows:

$$\tilde{P} \equiv P(\langle\hat{\phi}\rangle) = -\frac{3}{2\lambda_0}A^2(T^2-T_c^2)^2\theta(T_c-T) + \frac{A}{24}T^2(T^2-T_c^2) - \frac{\pi^2}{20}T^4. \qquad (3\cdot8)$$

The physical masses of a quark and a scalar boson are given by

$$M_q = -\frac{gA}{\lambda_0}(T^2-T_c^2)\theta(T_c-T), \qquad (3\cdot9)$$

$$M_s^2 = -A(T^2-T_c^2)\{3\theta(T_c-T)-1\}. \qquad (3\cdot10)$$

From Eq. (3·8) we can get easily the entropy density s:

$$s = -\frac{\partial\tilde{P}(T)}{\partial T} = \frac{6A^2}{\lambda_0}T(T^2-T_c^2)\theta(T_c-T) - \frac{A}{12}T(2T^2-T_c^2) + \frac{\pi^2}{5}T^3, \qquad (3\cdot11)$$

the energy density ε:

$$\varepsilon = \tilde{P} + Ts = \frac{3A^2}{2\lambda_0}(3T^2+T_c^2)(T^2-T_c^2)\theta(T_c-T) - \frac{A}{24}T^2(3T^2-T_c^2) + \frac{3}{20}\pi^2T^4, \qquad (3\cdot12)$$

and the specific heat c_v:

$$c_v \equiv \frac{\partial\varepsilon}{\partial T} = \frac{6A^2}{\lambda_0}T(3T^2-T_c^2)\theta(T_c-T) - \frac{A}{12}T(6T^2-T_c^2) + \frac{3}{5}\pi^2T^3. \qquad (3\cdot13)$$

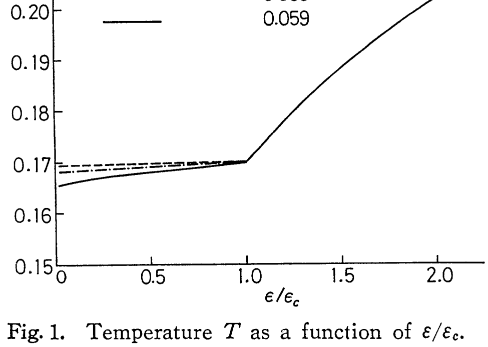

Fig. 1. Temperature T as a function of $\varepsilon/\varepsilon_c$.

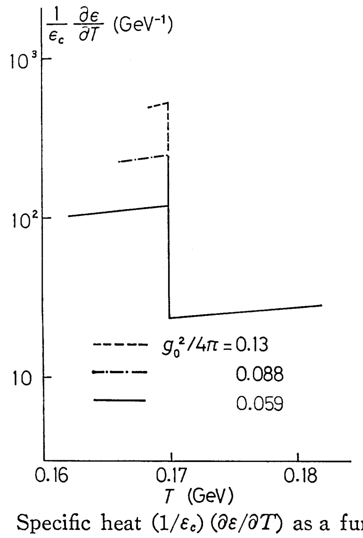

Fig. 2. Specific heat $(1/\varepsilon_c)(\partial\varepsilon/\partial T)$ as a function of T.

We shall need to impose some sort of weak coupling condition in order to justify the use of the perturbation theory. On the other hand stronger coupling is desirable to satisfy the condition (1·3) and to get the low critical temperature, as shown by Eqs. (3·1) and (3·13) respectively. First we settle T_c and λ_0 as follows:

$$T_c = 0.17\ \text{GeV}, \tag{3·14}$$

$$\lambda_0 = \frac{e^2}{4\pi}, \tag{3·15}$$

then we choose the value of g_0^2 which satisfies both the conditions (1·3) and $(g_0^2/4\pi) \ll 1$. For some values of $g_0^2/4\pi$, we calculate the energy density ε and the specific heat c_v as functions of T (Figs. 1 and 2). We regard that $V_h\varepsilon$ corresponds to the mass of hadronic matter, where V_h is the volume of the system. We obtain the critical mass M_c:

$$M_c = \frac{3}{20}\pi^2 T_c^4 V = \frac{3}{20}\pi^2 T_c^4 \frac{4}{3}\pi\left(\frac{1}{m_\pi}\right)^3 \approx 0.87\ \text{GeV}, \tag{3·16}$$

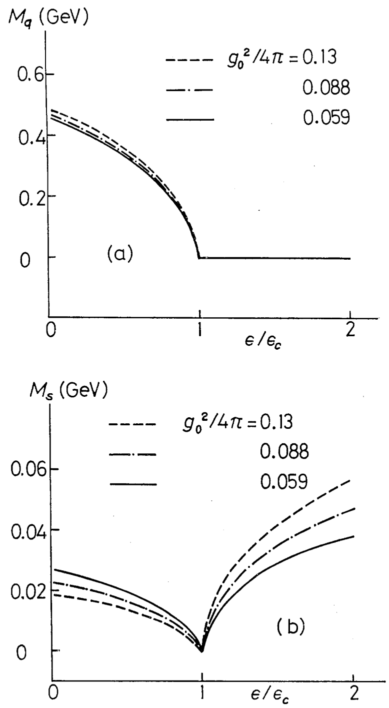

Fig. 3. (a) The quark mass as a function of $\varepsilon/\varepsilon_c$.
(b) The scalar boson mass as a function of $\varepsilon/\varepsilon_c$.

where we have assumed that the radius of hadronic matter is given by the compton length of pion. It is surprising indeed that the order of magnitude is consistent with the mass of a hadron. From our formulation it is clear that the critical mass also depends on the freedom of the boson and the quark field, so that the value itself is not to be taken seriously. The physical masses of the quark and the scalar boson are evaluated as functions of T and ε respectively (Fig. 3). The level density of the hadronic matter is given by the formula:

$$\rho(M_h) = \exp(S(M_h)), \tag{3·17}$$

where $S(M_h)$ is the entropy of the hadronic matter and it is derived as follows:

$$S(M_h) \xrightarrow[T>T_c]{} \frac{\pi^2}{5}\left(\frac{20}{3\pi^2}\right)^{3/4} M_h^{3/4} V_h^{1/4}, \tag{3·18}$$

$$\xrightarrow[\substack{T<T_c \\ M_h \ll M_c}]{} \frac{M_f}{T_c}, \tag{3·19}$$

from Eqs. (3·11) and (3·12). This means that the level density of the hadronic matter below T_c agrees with the one of the statistical bootstrap model.

§ 4. Concluding remarks

We have studied the phase transition of the hadronic matter in renormalizable field theory at finite temperature. It has been shown that one could get low critical temperature even for weak couplings to justify the one-loop approximation for the effective potential. Temperature may turn into an important parameter in the field theory for the strong interaction if the critical temperature is sufficiently low as given in this paper.

Appendix A

When we study the phase transition, the corresponding long range order-parameter (for example, $\hat{\phi}$: the thermal average of ϕ) is introduced and the conjugate external potential J to ϕ is added to the system.[12] In this case the partition function $Z(J)$ is given by

$$Z(J) = \mathrm{Tr}\{e^{-(H-J\phi)/T}\}$$
$$= \mathrm{Tr}\left\{e^{-H/T}\left(\exp\left(\int_0^{1/T} du\, J\phi(u)\right)\right)_+\right\}, \qquad (\mathrm{A}\cdot 1)$$

where $\phi(u) = e^{uH}\phi e^{-uH}$, the u is the imaginary time. The proof of the equality (A·1) is given as follows.

If $e^{-x(H+B)} = e^{-xH}U(x)$, then we get the differential equation $U(x)$ by differentiating both sides by x,

$$\frac{\partial U(x)}{\partial x} = B(x)U(x), \qquad (\mathrm{A}\cdot 2)$$

where $B(x) = e^{xH}Be^{-xH}$. Equation (A·2) is rewritten as

$$U(x) = 1 - \int_0^x B(u)U(u)\,du\,. \qquad (\mathrm{A}\cdot 3)$$

From Eq. (A·3) the $U(x)$ is formally obtained using the time-ordering trick as

$$U(x) = \left\{\exp\left(-\int_0^x B(u)\,du\right)\right\}_+. \qquad (\mathrm{A}\cdot 4)$$

If $B = -J\phi$ and $X = 1/T$, we get Eq. (A·1). Furthermore, we can extend Eq. (A·1) to the system with the time-dependent and space-varing external potential[13] by replacing $\int_0^{1/T} du J\phi(u)$ with

$$\int_0^{1/T} du \int_{-\infty}^{\infty} d\boldsymbol{r}\, J(r,u)\phi(r,u) \equiv \int d^4x\, J(x)\phi(x)\,.$$

Appendix B

From Eq. (A·1) it is found that the time-evolution of ϕ at finite temperature is described with the imaginary time. Therefore. it is convenient to use the temperature-Green's function (Matsubara Green's function), which is defined as

$$G(x, x') = -\frac{\mathrm{Tr}\{e^{-H/T}(\phi(x)\phi^{\dagger}(x))_{+}\}}{\mathrm{Tr}(e^{-H/T})}, \tag{B·1}$$

where $\phi(x)$ is the Bose or Fermi field operator and $\phi(x) = e^{uH}\phi(\boldsymbol{r})e^{-uH}$. $G(x, x')$ obeys the following boundary condition with time,

$$\begin{aligned} G(u, u') &= G(u-u') \\ &= \begin{cases} +G(u-u'+1/T) & \text{for the Bose field,} \\ -G(u-u'+1/T) & \text{for the Fermi field.} \end{cases} \end{aligned} \tag{B·2}$$

Therefore the Fourier transformation of G with respect to time is given by

$$G(u-u') = \sum_{n} G(i\omega_n)\, e^{i\omega_n(u-u')} \tag{B·3}$$

with $\omega_n = 2\pi nT$ for the Bose field and $\omega_n = \pi(2n+1)T$ for the Fermi field (n is the integer).

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