Phonons as Goldstone bosons

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Abstract

The implications of the hidden, spontaneously broken symmetry for the properties of the sound waves of a solid are analyzed. Although the discussion does not go beyond standard wisdom, it presents some of the known results from a different perspective. In particular, I argue that, as a consequence of the hidden symmetry, the equations of motion for a sound wave necessarily contain nonlinear terms, describing phonon-phonon-scattering and emphasize the analogy with the low energy theorems valid for $\pi\pi$-scattering.

Dedicated to Klaus Hepp and Walter Hunziker

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1 Introduction

Solids represent configurations that spontaneously break translation invariance [1, 2, 3]. The purpose of the present paper is to discuss some of the consequences of this fact, extending the analysis given in ref. [4]. Although none of the statements made below goes beyond what is known about the physics of a solid, it is of interest to view the low energy structure from the point of view of general effective field theory. As I will point out, some of the properties of the sound waves, in particular the fact that phonons necessarily interact with one another, are intimately related to the structure of the spontaneously broken symmetry group.

The hidden symmetry manifests itself through the occurrence of conserved currents. In the present case, the relevant charges are the generators of the translations, i.e. the total momentum P^a. The corresponding currents are the components $\theta^{\mu a}$ of the energy-momentum-tensor,

$$P^a = \int d^3x \, \theta^{0a}(x) \ .$$

In local form, the corresponding conservation law reads

$$\partial_0 \theta^{0a}(x) + \partial_b \theta^{ba}(x) = 0 \ ,$$

where θ^{ab} is the stress tensor. The analogue of current algebra is the commutation relation

$$[\theta^{0a}(x), \theta^{0b}(x)]_{x^0=y^0} = -i\hbar\partial_a \delta^3(x-y)\theta^{0b}(x) - i\hbar\partial_b \delta^3(x-y)\theta^{0a}(y) \ .$$

In the following, current algebra plays an important role, but I do not need it in this local form. Instead, I only make use of the commutation relation

$$[P^a, \theta^{\mu\nu}(x)] = \hbar i \partial_a \theta^{\mu\nu}(x) \ ,$$

which expresses the fact that the momentum generates translations.

Translation symmetry is peculiar in that, up to a factor of c^2, the time components of the corresponding currents, $\theta^{0a} = \theta^{a0}$, at the same time also represent the energy flow and thus occur in the energy conservation law[1]

$$\partial_0 \theta^{00}(x) + \partial_a \theta^{a0}(x) = 0 \ .$$

[1] In the notation used here, the time variable is denoted by $x^0 = t$ (no factor of c). Accordingly, the energy density is given by $c^2\theta^{00}$.

The operator of total energy is the Hamiltonian,

$$H = \int d^3x \, c^2 \theta^{00}(x) \ ,$$

which generates translations in the time direction,

$$[H, \theta^{\mu\nu}(x)] = -\hbar i \partial_0 \theta^{\mu\nu}(x) \ .$$

The spontaneous breakdown of a symmetry implies that the spectrum of the Hamiltonian does not contain an energy gap. In the present case, where the symmetry group of interest is the translation group, the Goldstone bosons associated with the spontaneous symmetry breakdown are the phonons and the Goldstone theorem reduces to the well-known statement that phonons of sufficiently large wavelength carry arbitrarily little energy. There may be other degrees of freedom without an energy gap. In the case of a conductor, for instance, the excitation of electrons near the fermi surface also requires arbitrarily little energy. For a general discussion of effective field theories related to the electronic degrees of freedom, I refer to [5]. In the following, I concentrate on the phonons (more precisely, on the acoustic branch of the dispersion curve) and study their behaviour at large wavelength: $\lambda \gg a$, where a is the lattice spacing.

The low energy properties of the Goldstone bosons have been analyzed in considerable detail for the case where the spontaneous symmetry breakdown preserves Lorentz invariance (see [6, 7, 8] and the references therein). In particular, it is known that the Goldstone bosons generated by the spontaneous breakdown of an exact nonabelian symmetry are subject to an interaction that grows with the square of the momentum, the strength being determined by the transition matrix elements of the currents between the vacuum and states containing one Goldstone particle. In the case of an abelian symmetry group, on the other hand, symmetry implies that the coefficient of the term of order p^2 vanishes, so that the Taylor series expansion of the elastic scattering amplitude only starts at order p^4. Since the translations form an abelian group, one might expect that the phonon-phonon-interaction belongs to this second category. The general analysis mentioned above does, however, not apply here, because it makes essential use of the assumption that the ground state is Lorentz invariant, which drastically simplifies the structure of the effective theory. More importantly, the form of the Ward identities is controlled by the *local* version of the symmetry group, not by the global one [4]. The local form of the translation group is the set of general coordinate transformations and is not abelian. The intrinsic difference in the structure of the global and local versions of the symmetry group shows up in the current algebra commutation rules: while the generators P^a of the translation group commute among themselves, the momentum densities do not. The case under consideration thus corresponds to a nonabelian local symmetry. As we will see, the fact that the commutator of two momentum densities does not vanish requires the phonons to interact among themselves.

2 Effective Lagrangian

The consequences of the hidden, spontaneously broken symmetry may be analyzed in terms of an effective field theory. The relevant effective field is a space-time-dependent element of the translation group, which I denote by $\xi_a(x) = \xi_a(t, \vec{x}\,)$, $a = 1, 2, 3$. The effective field represents the displacement of the material from the position in the ground state. The corresponding effective Lagrangian $\mathcal{L}_{\mathit{eff}} = \mathcal{L}_{\mathit{eff}}(\xi, \dot{\xi}, \partial\xi, \ddot{\xi}, \partial\dot{\xi}, \partial\partial\xi, \ldots)$ may be analyzed by means of an expansion in powers of ξ:

$$\mathcal{L}_{\mathit{eff}} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \ldots$$

The first term is an irrelevant constant, while all others contain a string of contributions with an increasing number of derivatives. The derivative expansion of $\mathcal{L}_2$, for example, starts with:

$$\mathcal{L}_2 = a^1_{ab}\xi_a\xi_b + a^2_{ab}\xi_a\dot{\xi}_b + a^3_{abc}\xi_a\partial_b\xi_c + a^4_{ab}\dot{\xi}_a\dot{\xi}_b + a^5_{abc}\dot{\xi}_a\partial_b\xi_c + a^6_{abcd}\partial_a\xi_b\partial_c\xi_d + \ldots$$

The expression accounts for all terms with less than three derivatives (contributions of the type $\xi\times\ddot{\xi}$ and $\xi\times\partial\partial\xi$ may be eliminated with an integration by parts). The conservation laws and commutation relations associated with the hidden symmetry strongly constrain the coefficients $a^1_{ab}, a^2_{ab}\ldots$ In fact, the properties of the effective field theory that describes the low energy structure in a model-independent way follow from these constraints.

The lattice of a solid does not possess any continuous symmetries, except for the translations in the direction of the time axis. Accordingly, the various coupling constants occurring in the effective Lagrangian for the phonons are only restricted by discrete symmetries, such as reflections on lattice planes [9]. I assume that the system is invariant under space reflections and time reversal. This implies that time derivatives only enter pairwise and that the Lagrangian only contains terms for which the number of space derivatives plus the number of fields is even. The quantity $\mathcal{L}_1$ then exclusively involves total derivatives and may thus be dropped. In the case of $\mathcal{L}_2$, reflection symmetry implies that the coefficients a^2_{ab}, a^3_{abc} and a^5_{abc} vanish, etc.

For a solid with cubic symmetry, the terms $a^1_{ab}\,\xi_a\xi_b$ and $a^4_{ab}\,\dot{\xi}_a\dot{\xi}_b$ are of the form $a_1\,\xi_a\xi_a$ and $a_2\,\dot{\xi}_a\dot{\xi}_a$, respectively, so that, for these contributions, reflection symmetry implies invariance under rotations. In the case of the coefficient a^6_{abcd}, cubic symmetry permits the following terms:

$$a^6_{abcd}\partial_a\xi_b\partial_c\xi_d = a_3\,\partial_a\xi_a\partial_b\xi_b + a_4\partial_a\xi_b\partial_a\xi_b + a_5\partial_a\xi_b\partial_b\xi_a + a_6\sum_a \partial_a\xi_a\partial_a\xi_a \quad .$$

Only the first three are invariant under rotations. In the following, I simplify the bookkeeping by restricting myself to those terms in the effective Lagrangian that are invariant under rotations, thus ignoring the term $\propto a_6$. The same line of reasoning should go through also in the general case, but I did not carry it out. Hence the following discussion does not immediately apply to a real solid, but concerns a situation where the ground state spontaneously breaks translation symmetry while preserving rotation invariance, as for fluids or gases.

With an integration by parts, the term $\propto a_5$ may be absorbed in a_3, so that only two effective couplings remain, which represent the torsion and compression modules of the system, denoted by μ and K, respectively. Also, with a suitable normalization of the field ξ, the coefficient of the term $\dot{\xi}^a\dot{\xi}^a$ may be identified with the mass density ρ_0, so that $\mathcal{L}_2$ takes the form

$$\mathcal{L}_2 = \tfrac{1}{2}\rho_0\,\dot{\xi}_a\dot{\xi}_a - \tfrac{1}{2}\mu\,\partial_a\xi_b\partial_a\xi_b - \tfrac{1}{6}(\mu+3K)\partial_a\xi_a\partial_b\xi_b + l_0\,\xi_a\xi_a + O(p^4)$$

The general expression up to and including two derivatives thus agrees with the standard form of the Lagrangian describing the sound waves, except for the term $l_0\,\xi_a\xi_a$, which does not occur in the standard analysis.

3 Energy-momentum-tensor

Indeed, the conservation laws that follow from the presence of a hidden symmetry imply $l_0 = 0$. To verify this statement, consider the representation of the energy-momentum-tensor in terms of the effective field,

$$\theta^{\mu\nu} = \theta_0^{\mu\nu} + \theta_1^{\mu\nu} + \theta_2^{\mu\nu} + O(\xi^3) \ .$$

The field independent term $\theta_0^{\mu\nu}$ represents the energy density and the pressure in the ground state. Since the derivatives thereof vanish, this contribution is conserved by itself. The expansion of the momentum density starts with the familiar term proportional to the velocity:

$$\theta_1^{0a} = \rho_0 \, \dot{\xi}_a + O(p^3) \ .$$

Energy conservation then requires that the energy density contains a corresponding contribution proportional to the divergence of the effective field,

$$\theta_1^{00} = -\rho_0 \partial_a \xi_a + O(p^3) \ ,$$

which describes the change in the rest energy caused by the deformation $\xi_a(x)$. To first order in ξ, the general expression for the stress-tensor is of the form

$$\theta_1^{ab} = k_1(\partial_a \xi_b + \partial_b \xi_a) + k_2 \delta_{ab} \partial_c \xi_c + O(p^3) \ ,$$

and momentum conservation requires

$$\rho_0 \, \ddot{\xi}_a + k_1 \partial_b \partial_b \xi_a + (k_1 + k_2) \partial_a \partial_b \xi_b = O(\xi^2, p^4) \ .$$

This is consistent with the equation of motion that follows from the effective Lagrangian of the preceding section if and only if

$$l_0 = 0 \ , \quad k_1 = -\mu \ , \quad k_2 = \tfrac{2}{3}\mu - K \ .$$

As claimed above, the coefficient l_0 thus vanishes.

It is clear, however, that these contributions do not represent the full energy-momentum tensor. In particular, the energy of the sound waves must give rise to a contribution in θ^{00} that is quadratic in the field $\xi_a(x)$. In his diploma work [10], Moritz Willers investigated the energy-momentum-tensor belonging to the Lagrangian $\mathcal{L}_2$, using the Noether theorem and run into difficulties rather immediately: although Noether's construction yields a conserved energy-momentum-tensor $\theta_N^{\mu\nu}$, the result fails to be symmetric under an interchange of μ and ν. The conservation laws fix the form of the energy-momentum-tensor only up to

$$\theta^{\mu\nu} = \theta_N^{\mu\nu} + \partial_\lambda \chi^{\lambda\mu\nu} \ ,$$

with $\chi^{\lambda\mu\nu} = -\chi^{\mu\lambda\nu}$. Indeed we may exploit this freedom to arrive at a symmetric tensor, but only at a price: the relevant expression for $\chi^{\lambda\mu\nu}$ contains the field ξ_a itself, not only its derivatives – unless $\mu = \rho_0\, c^2$, $K = -\frac{1}{3}\rho_0\, c^2$, in which case sound propagates with the velocity of light. For realistic values of the torsion and compression modules, the energy-momentum-tensor thus fails to be translation invariant.

The symmetry requirement $\theta^{\mu\nu} = \theta^{\nu\mu}$ is perfectly physical: the energy-momentum-tensor represents the source of gravity and can only enter Einstein's equations if it is symmetric. Also, the expectation that the result should be translation invariant is well justified – otherwise the distribution of energy and momentum depends on the absolute position of the body. The only way out is to conclude that the Lagrangian $\mathcal{L}_2$ cannot be the full story – the equation that describes the propagation of sound must contain nonlinear terms.

4 Covariant formulation

In the standard analysis of elastic deformations, the phenomenon manifests itself as follows. It is convenient not to work with a set of three variables specifying the displacement of the various points, but to use three scalar fields $z_a(x)$ with the property that the world lines of the body-fixed points are characterized by $z_a(x) = \text{constant}$. The variables $\xi_a(x)$ used previously may be converted into this language by setting

$$z_a(t,\vec{x}\,) = x_a - \xi_a(t,\vec{x}\,) \ \ .$$

To first order in the deformation, the body-fixed point with coordinate z_a is then described by the world line $x_a(t) = z_a + \xi_a(t,\vec{z}\,) + \ldots$, so that $\xi_a(x)$ indeed represents the displacement vector. Note that, in the framework used here, the precise physical significance of the variable $\xi_a(x)$ is left open – with the above interpretation, $\xi_a(x)$ differs from the displacement vector at higher orders of the deformation.

This formalism readily lends itself to a generally covariant formulation of the dynamics. We may consider a curved space-time with metric $g_{\mu\nu}(x)$ and form the covariant derivatives

$$\partial_\mu z_a \ \ , \quad \nabla_\mu \partial_\nu z_a = \partial_\mu \partial_\nu z_a - \Gamma^\lambda_{\mu\nu}\partial_\lambda z_a \ \ , \ \ \ldots$$

If the function $F(\partial z, \nabla\partial z, \ldots)_g$ is a scalar with respect to general coordinate transformations, then the equations of motion that follow from the Lagrangian

$$\mathcal{L}_{\mathit{eff}} = \sqrt{|g|}\, F(\partial z, \nabla\partial z, \ldots)_g \ \ .$$

automatically ensure that the variational derivative of the action with respect to the metric yields a covariantly conserved, symmetric energy-momentum-tensor. In particular, we may consider a Lagrangian that only involves the first derivatives $\partial_\mu z_a$. The matrix

$$H_{ab} = g^{\mu\nu}\partial_\mu z_a \partial_\nu z_b$$

represents a set of scalar fields, so that any expression of the form[2]

$$\mathcal{L}_{\mathit{eff}} = \sqrt{|g|}\, c^2 f(H)$$

yields equations of motion that do admit a translation invariant, conserved, symmetric energy-momentum-tensor, given by

$$\theta^{\mu\nu} = 2\,\partial^\mu z_a \partial^\nu z_b \frac{\partial f(H)}{\partial H_{ab}} - g^{\mu\nu} f(H) \ \ .$$

Free particles, for example, are characterized by the action

$$S = -\sum_i m_i c \int ds_i \ \ ,$$

where ds_i is the Minkowski line element along the world line that describes the motion of one of these particles. For a continuous distribution, the configuation is described by three functions $z_a(x)$ that remain constant along the world lines. The vectors $\partial_\mu z_a(x)$ span a three-dimensional space, orthogonal to the world line passing through x. Together with the tangent vector to the world line, $u^\mu = dx^\mu/ds$, they form a complete set of vectors, so that the difference dx^μ between neighbouring points may be decomposed as $dx^\mu = \partial^\mu z_a dq^a + u^\mu dq^0$. Since the vectors u^μ and $\partial_\mu z_a$ are orthogonal, the line element takes the form $dx^\mu dx_\mu = H_{ab} dq^a dq^b + (dq^0)^2$ and the corresponding expression for the volume element reads $c\sqrt{|g|}\, d^4x = \sqrt{|\det H|}\, d^4q$.

[2] The line element $ds^2 = c^2dt^2 - dx^a dx^a$ corresponds to $g_{00} = c^2$, $g^{00} = c^{-2}$. To ensure that, on flat space, the term $\sqrt{|g|}$ reduces to unity, I set $|g| \equiv \det(-g)/c^2$.

The component dq^0 measures the length in the direction of the world line, $dq^0 = ds$, while the components dq^a represent the projections of dx^μ in the space orthogonal to it, related to

$$dz_a = \partial_\mu z_a dx^\mu = H_{ab} dq^b \quad .$$

Denoting the mass contained in d^3z by $\rho_0\, d^3z$, the action becomes

$$S = -\int \rho_0\, d^3z\, c\, ds = -\rho_0 c \int |\det H| d^4q = -\rho_0 c^2 \int \sqrt{|g|}\,\sqrt{|\det H|}\, d^4x \quad .$$

This shows that the Lagrangian of a continuous distribution of free particles is indeed of the above form, with $f(H) = -\rho_0\sqrt{|\det H|}$. The corresponding energy-momentum-tensor is given by the familiar expression $\theta^{\mu\nu} = \rho\, u^\mu u^\nu$, appropriate for a cloud of dust. The quantity ρ differs from the constant mass density ρ_0 of the ground state by a factor that depends on the deformation: $\rho = \rho_0\sqrt{|\det H|}$.

The free Lagrangian may be generalized to allow for pressure. For the energy-momentum-tensor to take the form $\theta^{\mu\nu} = \rho\, u^\mu u^\nu + c^{-2}(u^\mu u^\nu - g^{\mu\nu})P$, the function $f(H)$ can depend on the matrix H_{ab} only through the determinant,

$$\mathcal{L}_{eff} = -\sqrt{|g|}\,\rho(h) c^2 \quad , \quad h = |\det H| \quad .$$

Accordingly, the system must exhibit invariance with respect to volume-preserving reparametrizations, $z_a \rightarrow \psi_a(z)$, $\det \partial\psi/\partial z = 1$. The corresponding expression for the pressure reads

$$P = c^2\{2h\frac{d\rho(h)}{dh} - \rho(h)\} \quad .$$

The shape of the function $\rho(h)$ thus determines the dependence of the pressure on the mass density and vice versa.

On flat space, the matrix H_{ab} reduces to

$$H_{ab} = -\delta_{ab} + \overline{H}_{ab} \quad , \quad \overline{H}_{ab} = H_{ab} = \partial_a\xi_b + \partial_b\xi_a - \partial_c\xi_a\partial_c\xi_b + c^{-2}\dot{\xi}_a\dot{\xi}_b \quad .$$

To account for the torsion and compression modules, it suffices to include additional terms in the effective Lagrangian,

$$\begin{aligned}\mathcal{L}_{eff} = \sqrt{|g|}\sqrt{|\det H|}\, &\Big\{-\rho_0 c^2 - \tfrac{1}{8}(K - \tfrac{2}{3}\mu)\,\mathrm{Sp}(\overline{H})^2 - \tfrac{1}{4}\mu\,\mathrm{Sp}(\overline{H}^2) \\ &+ L_1\,\mathrm{Sp}(\overline{H})^3 + L_2\,\mathrm{Sp}(\overline{H})\,\mathrm{Sp}(\overline{H}^2) + L_3\,\mathrm{Sp}(\overline{H}^3) + \ldots\Big\} \quad ,\end{aligned}$$

where $\mathrm{Sp}(\overline{H}^2)$ stands for $H_{ab}H_{ba}$. Note that, for the additional terms, the factor $\sqrt{|\det H|}$ could just as well be dropped, as it merely amounts to a reordering: the expansion in powers of $\overline{H}_{ab}$ yields

$$\sqrt{|\det H|} = \{\det(1-\overline{H})\}^{\frac{1}{2}} = 1 - \tfrac{1}{2}\mathrm{Sp}(\overline{H}) + \tfrac{1}{8}\mathrm{Sp}(\overline{H})^2 - \tfrac{1}{4}\mathrm{Sp}(\overline{H}^2) \\ - \tfrac{1}{48}\mathrm{Sp}(\overline{H})^3 + \tfrac{1}{8}\mathrm{Sp}(\overline{H})\,\mathrm{Sp}(\overline{H}^2) - \tfrac{1}{6}\mathrm{Sp}(\overline{H}^3) + \ldots$$

The general expression does reproduce the terms of order ξ^2 occurring in $\mathcal{L}_2$, but the expansion necessarily also generates higher order contributions, because the matrix $\overline{H}_{ab}$ contains terms quadratic in ξ. Within the covariant framework, the fact that the effective Lagrangian must contain terms of order ξ^3 is so obvious that it is barely mentioned in the textbooks. The reparametrization invariant Lagrangian considered above corresponds to the special case $\mu = 0$, $L_2 = -\frac{1}{8}K$, $L_3 = 0$. At the order of the derivative expansion considered, these systems are characterized by two independent parameters: K, L_1. In particular, the energy-momentum tensor can take the form $\theta^{\mu\nu} = \rho\, u^\mu u^\nu + c^{-2}(u^\mu u^\nu - g^{\mu\nu})P$ only if the torsion module vanishes, as it is the case for a fluid.

Incidentally, the covariant formulation readily accomodates anisotropy terms like $\Sigma_a \partial_a\xi_a\partial_a\xi_a$: the Lagrangian relevant for the general case is obtained by replacing the traces in the above expression with a polynomial formed with the matrix $\overline{H}_{ab}$. In the case of cubic symmetry, for instance, there is one additional contribution of second order, proportional to $\Sigma_a \overline{H}_{aa}\,\overline{H}_{aa}$. Note also that, if the derivative expansion is carried further, the general Lagrangian contains invariants formed with higher derivatives of the scalar fields.

5 General analysis at next-to-leading order

The formulation of the effective theory discussed in the preceding section assumes that the effective Lagrangian may be brought to a form that is manifestly invariant with respect to the local symmetry group. This assumption is sufficient, but not necessary. The ferromagnet represents an interesting example, where the effective Lagrangian fails to be manifestly invariant: under a local symmetry transformation, $\mathcal{L}_{\mathit{eff}}$ changes by a total derivative. In principle, the same phenomenon could also take place here. The Ward identities that express the conservation of energy and momentum on the level of the correlation functions formed with $\theta^{\mu\nu}(x)$ only ensure that the corresponding effective *action* is invariant under general coordinate transformations. If the effective *Lagrangian* is invariant, the corresponding effective action is automatically invariant, too, but the converse is not evident. I now wish to show that the conservation laws and the commutation relations listed in section 1 indeed imply that, with a suitable choice of the effective field, the interaction term $\mathcal{L}_3$ may be brought to manifestly invariant form, at least to first nontrivial order in the derivative expansion. The technique used is brute force: I first write down all possible terms occurring in the relevant expressions for the effective Lagrangian and for the energy momentum tensor and then work out the constraints imposed by the conservation laws and commutation relations.

The general expression for $\mathcal{L}_3$ contains terms with an odd number of space derivatives and an even number of time derivatives. Consider all contributions up to to and including three derivatives. An integration by parts suffices to eliminate terms containing the third derivative of the field. The same operation also removes contributions containing $\ddot{\xi}_a$. The general expression then contains altogether 9 independent vertices:

$$\begin{aligned}\mathcal{L}_3 = & \, l_1\, \dot{\xi}_a\dot{\xi}_a\partial_b\xi_b + l_2\, \dot{\xi}_a\dot{\xi}_b\partial_a\xi_b + l_3\, \partial_a\xi_a\partial_b\xi_b\partial_c\xi_c \\ & + l_4\, \partial_a\xi_a\partial_b\xi_c\partial_b\xi_c + l_5\, \partial_a\xi_a\partial_b\xi_c\partial_c\xi_b + l_6\, \partial_a\xi_b\partial_a\xi_c\partial_b\xi_c \\ & + l_7\, \xi_a\xi_a\partial_b\xi_b + l_8\, \xi_a\dot{\xi}_a\partial_b\dot{\xi}_b + l_9\, \xi_a\partial_b\xi_b\partial_c\partial_c\xi_a + O(p^5)\ .\end{aligned}$$

Using partial integration, all other terms, such as $\partial_a\xi_b\partial_b\xi_c\partial_c\xi_a$, may be absorbed in the coupling constants $l_1, \dots, l_9$.

The covariant Lagrangian given in the preceding section only contains 3 independent coupling constants at this order of the expansion: L_1, L_2, L_3. I wish to show that the symmetry indeed determines all but 3 of the above 9 couplings – except for the freedom in the choice of the variables. For this purpose, the effective representation of the energy-momentum-tensor is needed to first nonleading order in ξ. In the case of the momentum density, the general expression involves 6 independent terms at this order:

$$\begin{aligned}\theta_2^{0a} = & \, p_1\, \dot{\xi}_a\partial_b\xi_b + p_2\, \dot{\xi}_b\partial_a\xi_b + p_3\, \dot{\xi}_b\partial_b\xi_a \\ & + p_4\, \xi_a\partial_b\dot{\xi}_b + p_5\, \xi_b\partial_a\dot{\xi}_b + p_6\, \xi_b\partial_b\dot{\xi}_a + O(p^4)\ .\end{aligned}$$

Three of these may be eliminated, however, with the following observation. The dynamical variables occurring in the framework of an effective theory represent auxiliary quantities. In the present context, the effective field may be subject to the transformation

$$\xi_a \rightarrow \xi_a + \kappa_1\, \xi_a \partial_b \xi_b + \kappa_2\, \xi_b \partial_a \xi_b + \kappa_3\, \xi_b \partial_b \xi_a \;\; ,$$

without changing the content of the effective theory. Inserted in the above expression for the momentum density, the change of variables modifies the values of the constants occurring therein. We may exploit this freedom and, with a suitable choice of $\kappa_1, \kappa_2, \kappa_3$, remove the coefficients p_4, p_5, p_6. Without loss of generality, the expression for the momentum density then becomes translation invariant:

$$\theta_2^{0a} = p_1\, \dot{\xi}_a \partial_b \xi_b + p_2\, \dot{\xi}_b \partial_a \xi_b + p_3\, \dot{\xi}_b \partial_b \xi_a + O(p^4) \;\; .$$

Next, I observe that energy conservation determines the energy density in terms of $\partial_a \theta^{0a}$, up to an additive constant that represents the energy density of the ground state. In particular, translation invariance of θ^{0a} implies translation invariance of θ^{00}. With this simplification, the explicit expression takes the form

$$\theta_2^{00} = e_1\, \dot{\xi}_a \dot{\xi}_a + e_2\, \partial_a \xi_b \partial_a \xi_b + e_3\, \partial_a \xi_b \partial_b \xi_a + e_4\, \partial_a \xi_a \partial_b \xi_b + O(p^4) \;\; .$$

The coefficients are fixed in terms of those occurring in θ^{0a}:

$$e_1 = -\frac{\rho_0\, p_2}{2\mu} \;\; , \;\; e_2 = -\frac{p_2}{2} \;\; , \;\; e_3 = -\frac{p_3}{2} \;\; , \;\; e_4 = -\frac{p_1}{2} \;\; .$$

The energy is conserved if and only if p_1, p_2, p_3 obey the condition

$$\mu\, p_1 - (K + \tfrac{1}{3}\mu)\, p_2 + \mu\, p_3 = 0 \;\; .$$

In the case of the stress tensor, the general expression for the first non-leading terms is rather voluminous: eliminating second time derivatives with the equation of motion, it takes the form

$$\begin{aligned}
\theta_2^{ab} =\; & s_1\, \dot{\xi}_a \dot{\xi}_b + s_2\, \partial_c \xi_a \partial_c \xi_b + s_3\, (\partial_a \xi_c \partial_c \xi_b + \partial_b \xi_c \partial_c \xi_a) + s_4\, \partial_a \xi_c \partial_b \xi_c \\
& + s_5\, (\partial_a \xi_b + \partial_b \xi_a) \partial_c \xi_c + s_6\, (\xi_a \partial_c \partial_c \xi_b + \xi_b \partial_c \partial_c \xi_a) + s_7\, \partial_c \xi \partial_a \partial_b \xi_c \\
& + s_8\, \xi_c (\partial_a \partial_c \xi_b + \partial_b \partial_c \xi_a) + s_9\, (\xi_a \partial_b \partial_c \xi_c + \xi_b \partial_a \partial_c \xi_c) + s_{10}\, \delta_{ab} \dot{\xi}_c \dot{\xi}_c + s_{11}\, \delta_{ab} \partial_c \xi_d \partial_c \xi_d \\
& + s_{12}\, \delta_{ab} \partial_c \xi_d \partial_d \xi_c + s_{13}\, \delta_{ab} \partial_c \xi_c \partial_d \xi_d + s_{14}\, \delta_{ab} \xi_c \partial_d \partial_d \xi_c + s_{15}\, \delta_{ab} \xi_c \partial_c \partial_d \xi_d + O(p^4) \;\; .
\end{aligned}$$

It is advantageos to use Mathematica for the evaluation of the derivative $\partial_a\theta^{ab}$. The calculation shows that the conservation of momentum fixes all but two of the coefficients $s_1,\ldots,s_{15}$ in terms of those occurring in the effective Lagrangian and in the momentum density. The freedom of choosing two of these arbitrarily arises because we can form two independent expressions for which the divergence $\partial_a\theta^{ab}$ vanishes identically. This is what remains of the ambiguity $\theta^{\mu\nu}\rightarrow\theta^{\mu\nu}+\partial_\lambda\chi^{\lambda\mu\nu}$ mentioned earlier – if we adopt the above convention that leads to a translation invariant expression for θ^{0a}. Remarkably, energy and momentum conservation does not constrain the translation invariant couplings $l_1,\ldots l_6$ at all: there is a conserved and symmetric energy-momentum-tensor for any choice of these constants. The couplings that break translation invariance, l_7,l_8,l_9, on the other hand, are fixed in terms of $p_1,p_2,p_3,l_1,\ldots l_6$. In particular, the conservation laws imply that l_7 vanishes.

6 Current algebra

So far, I have only imposed the conservation laws. Now, I turn to the requirement that the components of the energy-momentum-tensor obey the commutation relations of current algebra: the correlation functions of the energy-momentum-tensor only satisfy the Ward identities if this condition is met.

At leading order of the low energy expansion, only the tree graphs of the effective theory are relevant – the leading term in the low energy expansion of the effective action is the classical action. In the case of the commutator between two operators A,B, the tree graph contributions arise from the exchange of a single particle, generating terms with a single propagator. These are of first order in $\hbar$ and are obtained by replacing the commutator with the Poisson bracket:

$$[A,B]=-i\hbar\{A,B\}_{\mathrm{PB}}+O(\hbar^2)$$

Accordingly, the current algebra conditions of section 1 are satisfied at leading order of the low energy expansion, provided

$$\{P^a,\theta^{\mu\nu}(x)\}_{\mathrm{PB}}=-\partial_a\theta^{\mu\nu}(x)\ \ ,\quad \{H,\theta^{\mu\nu}(x)\}_{\mathrm{PB}}=\partial_0\theta^{\mu\nu}(x)\ \ .$$

To evaluate these conditions, we need to replace the time derivatives of the field by the canonical momentum $\pi^a = \partial\mathcal{L}_{eff}/\partial\dot{\xi}_a$,

$$\pi^a = \rho_0\,\dot{\xi}_a + 2l_1\,\dot{\xi}_a\partial_b\xi_b + l_2\,\dot{\xi}_b(\partial_a\xi_b + \partial_b\xi_a) + l_8\,\partial_b\dot{\xi}_b\xi_a - l_8\,\partial_a\dot{\xi}_b\xi_b - l_8\,\dot{\xi}_b\partial_a\xi_b \;\;,$$

and use the relations that define the Poisson bracket

$$\{\xi_a(x),\xi_b(x)\}_{\mathrm{PB}} = \{\pi^a(x),\pi^b(x)\}_{\mathrm{PB}} = 0 \;\;, \{\pi^a(x),\xi_b(x)\}_{\mathrm{PB}} = \delta^a_b\delta^3(x-y) \;\;.$$

For the commutator of the Hamiltonian with the energy density to coincide with the time derivative of θ^{00}, the constant e_1 must be related to the mass density by

$$e_1 = \tfrac{1}{2}\rho_0\,c^{-2} \;\;.$$

This result agrees with the standard expression $\theta^{00}_2 = \frac{1}{2}\rho_0\,\dot{\xi}_a\dot{\xi}_a + \ldots$ for the kinetic energy density. Similarly, the bracket $\{P^a,\theta^{0b}\}_{\mathrm{PB}}$ coincides with $-\partial_a\theta^{0b}$ only if

$$p_1 = 2l_1 - l_8 \;\;,\quad p_2 = l_2 - \rho_0 \;\;,\quad p_3 = l_2 \;\;.$$

These conditions do impose strong constraints on the coupling constants of the Lagrangian. Together with the conservation laws, they imply that the couplings of those terms that violate translation invariance vanish. Moreover, they permit only three independent translation invariant couplings. In fact, the resulting form of the effective Lagrangian coincides with the manifestly covariant expression given in section 4, so that all of the coupling constants may be expressed in terms of the parameters L_1, L_2, L_3 occurring therein:

$$\begin{aligned}
&l_1 = -\tfrac{1}{2}\rho_0 - \tfrac{1}{2}Kc^{-2} + \tfrac{1}{3}\mu c^{-2}\;,\;\; l_2 = \rho_0 - \mu c^{-2}\;,\;\; l_3 = \tfrac{1}{2}K - \tfrac{1}{3}\mu + 8L_1 - L_3\;,\\
&l_4 = \tfrac{1}{2}K + \tfrac{1}{6}\mu + 4L_2\;,\;\; l_5 = \tfrac{1}{2}\mu + 4L_2 + 3L_3\;,\;\; l_6 = \mu + 6L_3\;\;,\\
&l_7 = 0\;,\;\; l_8 = 0\;,\;\; l_9 = 0\;\;.
\end{aligned}$$

7 Conclusion

This completes the explicit demonstration of the claim made above: current algebra forces the phonons to interact, so that the wave equation for sound necessarily contains nonlinear terms. The Ward identities associated with the hidden symmetry imply that, with a suitable choice of the effective fields, the low energy structure of the system may be described in terms of a manifestly covariant effective Lagrangian that involves three independent coupling constants at the order considered. Ignoring the tiny relativistic corrections of order $(v/c)^2$, where v is the velocity of sound, the contributions from l_1 and l_2 modify the kinetic term according to

$$\tfrac{1}{2}\rho_0\,\dot{\xi}_a\dot{\xi}_a \rightarrow \tfrac{1}{2}\rho_0\,\dot{\xi}_a\dot{\xi}_a - \tfrac{1}{2}\rho_0\,\partial_b\xi_b\dot{\xi}_a\dot{\xi}_a + \rho_0\,\partial_a\xi_b\dot{\xi}_a\dot{\xi}_b \ \ .$$

With the choice of the effective fields adopted here, the symmetry implies that those couplings which break translation invariance vanish: $l_7 = l_8 = l_9 = 0$. Moreover, the symmetry imposes one constraint among the four translation invariant couplings l_3, l_4, l_5, l_6.

In the case of $\pi\pi$-scattering, Lorentz invariance and Bose statistics imply that the leading term in the low energy expansion of the scattering amplitude contains two constants, $A(s,t,u) = a_1 M_\pi^2 + a_2 s + O(p^4)$. The symmetry determines a_1 and a_2 in terms of a single parameter, the pion decay constant [11]: $a_1 = -1/F_\pi^2$, $a_2 = 1/F_\pi^2$. In the present case, the leading term in the derivative expansion of the interaction involves 9 constants $(l_1, \ldots, l_9)$ and the symmetry determines these in terms of the three independent couplings (L_1, L_2, L_3) that occur in the covariant expression for the effective Lagrangian. As announced in the introduction, the rather lengthy calculation described here merely rederives the standard form of the effective Lagrangian and does therefore not add anything to what is known about the physics of the phonons. It demonstrates, however, that the same mechanism that subjects the Goldstone bosons of QCD to a specific interaction at low energies is also at work in a solid, where it implies that the propagation of sound is an intrinsically nonlinear phenomenon.

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References

[1] A detailed discussion of the Goldstone theorem, in particular also in the context of nonrelativistic broken symmetries is given in
G. S. Guralnik, C. R. Hagen and T. W. B. Kibble, in *Advances in particle*

physics, Vol. 2, p. 567, ed. R. L. Cool and R. E. Marshak (Wiley, New York, 1968).

[2] R. V. Lange, *Phys. Rev. Lett.* 14 (1965) 3; *Phys. Rev.* 146 (1966) 301.

[3] H. B. Nielsen and S. Chadha, *Nucl. Phys.* B 105 (1976) 445.

[4] H. Leutwyler, *Phys. Rev.* D 49 (1994) 3033.

[5] J. Fröhlich, *Nonperturbative Quantum Field Theory* (World Scientific, Singapore, 1992);
J. Fröhlich et al., in *Lectures at the Nato Advanced Study Institute on Fluctuating Geometries in Statistical Mechanics and Field Theory*, Les Houches, France, 1994, cond-mat/950862.

[6] S. Weinberg, *Physica* A 96 (1979) 327.

[7] J. Gasser and H. Leutwyler, *Ann. Phys. (N.Y.)* 158 (1984) 142; *Nucl. Phys.* B 250 (1985) 465.

[8] H. Leutwyler, *Ann. Phys. (N.Y.)* 235 (1994) 165.

[9] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, Vol. 7, *Elasticity Theory* (Pergamon, London, 1959).

[10] M. Willers, *Effective Lagrangian for phonons*, Diplomarbeit, phil. nat. Fakultät der Universität Bern, 1995.

[11] S. Weinberg, *Phys. Rev. Lett.* 17 (1966) 616.