

PHYSICAL BROWNIAN MOTION IN A MAGNETIC FIELD AS A ROUGH PATH

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ABSTRACT. The indefinite integral of the homogenized Ornstein-Uhlenbeck process is a well-known model for physical Brownian motion, modelling the behaviour of an object subject to random impulses [L. S. Ornstein, G. E. Uhlenbeck: On the theory of Brownian Motion. In: Physical Review. 36, 1930, 823-841]. One can scale these models by changing the mass of the particle, and in the small mass limit one has almost sure uniform convergence in distribution to the standard idealized model of mathematical Brownian motion. This provides one well-known way of realising the Wiener process. However, this result is less robust than it would appear, and important generic functionals of the trajectories of the physical Brownian motion do not necessarily converge to the same functionals of Brownian motion when one takes the small mass limit. In the presence of a magnetic field the area process associated to the physical process converges - but not to Lévy's stochastic area. As this area is felt generically in settings where the particle interacts through force fields in a non-linear way, the remark is physically significant and indicates that classical Brownian motion, with its usual stochastic calculus, is not an appropriate model for the limiting behaviour.

We compute explicitly the area correction term and establish convergence, in the small mass limit, of the physical Brownian motion in the rough path sense. The small mass limit for the motion of a charged particle in the presence of a magnetic field is, in distribution, an easily calculable, but “non-canonical” rough path lift of Brownian motion. Viewing the trajectory of a charged Brownian particle with small mass as a rough path is informative and allows one to retain information that would be lost if one only considered it as a classical trajectory. We comment on the importance of this point of view.

1. INTRODUCTION

Newton's second law for a particle in \mathbb{R}^3 with mass m , position $x = x(t)$, and (for simplicity: constant) frictions $\alpha_1, \alpha_2, \alpha_3 > 0$ in the coordinate axis, subject to a (3-dimensional) white noise in time, ξ , where $\dot{x} = \xi(t)$ is the distributional derivative of W , a (mathematical) Brownian motion or Wiener process, reads

$$(1.1) \quad m\ddot{x} = -A\dot{x} + \xi,$$

Received by the editors February 4, 2013 and, in revised form, August 20, 2013.

2010 *Mathematics Subject Classification*. Primary 60H99.

Key words and phrases. Physical Brownian motion, homogenization, Lévy's area.

The research of the first and second authors was supported by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement nr. 258237. The research of the third author was supported by the Oxford-Man Institute and the European Research Council under the European Union's Seventh Framework Programme (FP7-IDEAS-ERC) / ERC grant agreement nr. 291244.

where $A = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$. An orthonormal change of coordinates implies that the “correct” assumption for A is to be symmetric with strictly positive spectrum,

$$\sigma(A) \subset (0, \infty).$$

The process $x(t)$ describes what is known as *physical Brownian motion*. It is well-known that in small mass regime, $m \ll 1$, of obvious physical relevance when dealing with particles, a good approximation is given by (mathematical) *Brownian motion*; to see this formally, it suffices to take $m = 0$ in (1.1) in which case Ax is a standard Brownian motion in \mathbb{R}^3 .

Let us now assume that our particle (with position x and momentum $m\dot{x}$) carries an electric charge $q \neq 0$ and moves in a magnetic field \mathbb{B} which we assume to be constant. Recall that such a particle experiences a sideways force (“Lorentz force”) that is proportional to the strength of the magnetic field, the component of the velocity that is perpendicular to the magnetic field and the charge of the particle:

$$F_{\text{Lorentz}} = q\dot{x} \times \mathbb{B}.$$

When \mathbb{B} is constant, which we assume for simplicity, the Lorentz force experienced by the particle (at time t) can be written as a linear function of velocity $\dot{x} = \dot{x}(t)$, namely $qB\dot{x}$ for some anti-symmetric matrix B . In other words, the dynamics for *physical Brownian motion in a magnetic field* are given by

$$\begin{aligned} m\ddot{x} &= -A\dot{x} + qB\dot{x} + \xi \\ &\equiv -M\dot{x} + \xi, \end{aligned}$$

where $M = (A + qB)$ is such that all eigenvalues of $M \in \mathbb{R}^{n \times n}$ have a strictly positive real part (one may still think $n = 3$, but the subsequent analysis works for any dimension n). Note that these second order dynamics can be rewritten as an evolution equation for the *momentum*

$$p(t) = m\dot{x}(t),$$

indeed,

$$\dot{p} = -M\dot{x} + \xi = -\frac{1}{m}Mp + \dot{W},$$

and we shall take this point of view when rewriting the dynamics in terms of standard stochastic differential equations. As before, when $m = 0$, the process $Mx = W$ is a bona fide (i.e. mathematical) 3-dimensional Brownian motion, and one may think that little has changed, apart from the covariance matrix of the resulting Brownian motion. Indeed, writing $\xi = \dot{W}$, and assuming $x_0 = W_0 = 0$, it is easy to see that

$$W_t - Mx_t = \int_0^t (-M\dot{x}_s + \xi_s) ds = \int_0^t \dot{p} = (p_t - p_0) \rightarrow 0 \text{ as mass } m \rightarrow 0.$$

Note that

$$p_0 = m\dot{x}(0) \rightarrow 0$$

as $m \rightarrow 0$, whenever initial velocity remains uniformly bounded, so the statement here is that $p_t \rightarrow 0$ as $m \rightarrow 0$ and one can easily see that this convergence is uniform (we are only interested in a fixed time horizon, say $[0, T]$). However, the momentum may have quite non-trivial effects as *control*. Recalling that a controlled system (differential equations, integrals, etc.) behaves in a robust fashion precisely under rough path metrics, the essence of which has been briefly summarized for

the reader’s convenience in the appendix, the following proposition tells us that momentum does not at all converge to a trivial limit.

Proposition 1. *One has (the iterated integrals are understood in Itô sense)*

$$(P_{s,t}, \mathbb{P}_{s,t}) := \left(\int_s^t dp, \int_s^t \int_s^r dp \otimes dp \right) \rightarrow \left(0, \frac{MC - CM^*}{2} (t - s) \right) \text{ as } m \rightarrow 0,$$

where C is the (symmetric) $n \times n$ matrix defined by

$$C = \int_0^\infty e^{-Ms} e^{-M^*s} ds.$$

More precisely, the convergence holds in the following strong sense: for any $\alpha \in (1/3, 1/2)$,

$$(1.2) \quad \sup_{s,t \in [0,T]} \frac{|P_{s,t}|}{|t-s|^\alpha} + \sup_{s,t \in [0,T]} \frac{\left| \mathbb{P}_{s,t} - \frac{MC - CM^*}{2} (t-s) \right|}{|t-s|^{2\alpha}} \rightarrow 0 \text{ in } L^q, \text{ as } m \downarrow 0.$$

(This is precisely what is meant by convergence in an α -Hölder rough path metric.)

The situation is reminiscent of a well-known deterministic example, taken from [20], where the path

$$Z_t := \frac{1}{\sqrt{m}} \exp(2\pi i t m) \in \mathbb{C} \simeq \mathbb{R}^2,$$

converges to a non-trivial “pure area” rough path as $m \rightarrow 0$; in this example, one has

$$\left(\int_s^t dZ, \int_s^t \int_s^r dZ \otimes dZ \right) \rightarrow \left(0, \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix} (t-s) \right).$$

In view of this proposition one suspects (correctly) that the convergence of physical Brownian motion to Brownian motion is also non-trivial if one thinks how Brownian particles act as controls, i.e. as rough paths. More specifically, one expects a limit in which Lévy’s stochastic area is perturbed by a term proportional to

$$(MC - CM^*)/2,$$

the (anti-symmetric) matrix which effectively described the pure area (rough path) limit of the previous proposition. Let us insist, however, that such a statement (i.e. the content of the following theorem) is *not* a corollary of the above since, in general,

$$\int (Z - Z^n) d(Z - Z^n) \rightarrow 0 \not\Rightarrow \int Z^n dZ^n \rightarrow \int Z dZ.$$

Indeed, if one thinks of Z^n, Z as rough paths, their (formal) iterated integrals are meaningful by the definition of a rough path. In contrast, the iterated integrals of Z against Z^n will not even be, in general, well-defined. We are now ready to state our main result.

Theorem 1. *Let M be a square matrix in dimension n such that all its eigenvalues have a strictly positive real part. Let W be an n -dimensional standard Brownian*

motion, m (“mass”) as a strictly positive scalar and consider the stochastic differential equations

$$(1.3) \quad dX = \frac{1}{m} P dt \quad (\text{position}),$$

$$(1.4) \quad dP = -\frac{1}{m} M P dt + dW \quad (\text{momentum})$$

with (for simplicity only) zero initial position and momentum. Let $\mathbf{W} = (W, \mathbb{W})$ be the natural rough path lift of W , where $\mathbb{W}_{s,t} = \int_s^t (W_r - W_s) \otimes \circ dW_r$. Define also $\widehat{\mathbf{W}} = (W, \widehat{\mathbb{W}})$, where

$$(1.5) \quad \widehat{\mathbb{W}}_{s,t} = \mathbb{W}_{s,t} + (t - s) \frac{1}{2} (MC - CM^*),$$

and C is as in the previous proposition. Then, as $m \rightarrow 0$, MX converges to $\widehat{\mathbf{W}}$ in L^q and α -Hölder rough path topology, for any $q \geq 1$ and $\alpha \in (1/3, 1/2)$. More precisely, we have

$$(1.6) \quad \sup_{s,t \in [0,T]} \frac{|MX_{s,t} - W_{s,t}|}{|t - s|^\alpha} + \sup_{s,t \in [0,T]} \frac{\left| \int_s^t MX_{s,r} \otimes d(MX)_r - \widehat{\mathbb{W}}_{s,t} \right|}{|t - s|^{2\alpha}} \rightarrow 0 \text{ in } L^q$$

as $m \rightarrow 0$ and this convergence is of a rate arbitrarily close to $1/2 - \alpha$.

Remark 1. In view of the tensorial transformation behaviour of iterated integrals, (1.6) is plainly equivalent to

$$\begin{aligned} X_{s,t} &\rightarrow M^{-1}W_{s,t}, \\ \int_s^t X_{s,u} \otimes dX_u &\rightarrow M^{-1}\mathbb{W}_{s,t}(M^{-1})^* + (t - s) \frac{1}{2} (C(M^{-1})^* - M^{-1}C) \end{aligned}$$

in α -Hölder rough path metric.

Remark 2. One has

$$\widehat{\mathbf{W}} = \mathbf{W}$$

if and only if M is symmetric. To see this, note that MC is symmetric (hence equal to CM^*) if and only if M is symmetric, using the symmetry and invertibility of C .

Remark 3. The framework of Gaussian rough paths [9] and [11, Ch. 15] (which plays a key role in non-Markovian Hörmander theory [3, 4] and some recent breakthroughs in non-linear SPDE theory [12, 13]) is only applicable if M is symmetric, since then one can diagonalize the dynamics so as to have Gaussian driving signals with independent components. In this case, upon checking some uniform variation bounds on the covariance, it could be used to see the convergence (1.6) to \mathbf{W} , the “natural” rough path lift of Brownian motion; [11, Ch. 15]. But since $\widehat{\mathbf{W}}$ is not natural when $Anti(M) \neq 0$, here we also provide an explicit example which illustrates the necessity of the assumptions put forward in [9, 11].

Remark 4. If M is normal, i.e.

$$[M, M^*] = 0,$$

then the difference between $\widehat{\mathbb{W}}$ and \mathbb{W} has a somewhat simpler expression. Indeed, we compute

$$C = \frac{1}{2} \text{Sym}(M)^{-1},$$

and since C commutes with M , we get

$$(1.7) \quad \frac{1}{2}(MC - CM^*) = \frac{1}{2} \text{Anti}(M) \text{Sym}(M)^{-1}.$$

Similarly, the area correction for X is seen to simplify to

$$(1.8) \quad \frac{1}{2}(C(M^{-1})^* - M^{-1}C) = -\frac{1}{2} \text{Anti}(M^{-1}) \text{Sym}(M)^{-1}.$$

Remark 5. It would be possible to “unit” the above proposition and theorem by saying that the physical Brownian motion is a “good” approximation in the sense of [7] to the limiting rough path \widehat{W} .

Remark 6. Incidentally, the rate “anything less than $1/2 - \alpha$ ” is also the rate of convergence (in an α -Hölder rough path metric) for piecewise linear “Wong–Zakai” approximations to Brownian motion obtained in [16], and optimality of this rate is known. In both cases, these rates are obtained as a rather mechanical consequence of good moment estimates (cf. [11, Thm A.13]) so that we also believe our rates to be optimal.

The following example is taken from Pavliotis–Stuart [24, Section 11.7.7]. We note that the PDE argument (based on multiscale techniques) offered in [24, Section 11.7.7] can only give convergence of the finite-dimensional distributions; tightness - especially in rough path topology - will require additional and non-trivial estimates (which are implied for our work below). Of course, strong convergence, available to us because of a natural coupling of physical Brownian motion X and $W = \dot{\xi}$, is out of reach with PDE methods.

Example 1. Take $n = 2, \alpha \in \mathbf{R}$ and

$$M = Id - \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

so that

$$M^{-1} = \frac{1}{1 + \alpha^2} \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}.$$

Note $\text{Sym}(M) = Id$. Then, from (1.8), the area correction of X , in the $m \rightarrow 0$ limit, equals

$$(1.9) \quad \frac{-1}{2(1 + \alpha^2)} \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix} = \frac{\alpha}{2(1 + \alpha^2)} (e_1 \otimes e_2 - e_2 \otimes e_1),$$

where e_1, e_2 denotes the standard basis of \mathbf{R}^2 . This agrees precisely with [24, (11.7.28c)].

Let us conclude this introduction by noting that the example of physical Brownian motion under influence of a magnetic field as a rough path has some history. Indeed, it appeared as motivation (but without much detail) in several presentations, including some by the last-named author in early 2000.

2. PROOF OF THEOREM 1

We first give a proof of the rough path convergence without rates, based on the ergodic theorem. The adaptations which lead to the announced rates (and also bypass the use of the ergodic theorem!) are then explained in detail in a separate proposition following the proof.

Proof. In order to exploit Brownian scaling, it is convenient to set

$$m = \varepsilon^2$$

and then Y^ε as rescaled momentum,

$$Y^\varepsilon = P/\varepsilon.$$

We shall also write $X^\varepsilon = X$ to emphasize dependence on ε . We then have

$$(2.1) \quad dY^\varepsilon = -\varepsilon^{-2}MY^\varepsilon dt + \varepsilon^{-1}dW,$$

$$(2.2) \quad dX^\varepsilon = \varepsilon^{-1}Y^\varepsilon dt.$$

By assumption, there exists $\lambda > 0$ s.t. the real part of every eigenvalue of M is (strictly) bigger than λ . For later reference, we note that this implies the estimate

$$|\exp(-\tau M)| = O(\exp(-\lambda\tau))$$

as $\tau \rightarrow \infty$. For fixed ε , define the Brownian motion

$$\tilde{W}_\cdot = \varepsilon W_{\varepsilon^{-2}\cdot},$$

and consider the SDEs

$$d\tilde{Y} = -M\tilde{Y}dt + d\tilde{W},$$

$$d\tilde{X} = \tilde{Y}dt;$$

note that the law of the solutions does not depend on ε . Furthermore, when solved with identical initial data as (2.1)-(2.2), we have the pathwise equality

$$(2.3) \quad (Y^\varepsilon, X^\varepsilon/\varepsilon) = \tilde{Y}_{\varepsilon^{-2}\cdot}, \tilde{X}_{\varepsilon^{-2}\cdot}.$$

Thanks to our assumption on M , \tilde{Y} is ergodic; the stationary solution has (zero mean, Gaussian) law

$$\nu \sim \mathcal{N}(0, C)$$

for some covariance matrix C . To compute it, write the stationary solution

$$\tilde{Y}_t^{\text{stat}} = \int_{-\infty}^t e^{-M(t-s)} dW_s;$$

for each t (and in particular $t = 0$) the law of $\tilde{Y}_t^{\text{stat}}$ is precisely ν . We then see that

$$C = \mathbb{E} \left(\tilde{Y}_0^{\text{stat}} \otimes \tilde{Y}_0^{\text{stat}} \right) = \int_{-\infty}^0 e^{-M(-s)} e^{-M^*(-s)} ds = \int_0^\infty e^{-Ms} e^{-M^*s} ds.$$

Since

$$\sup_{0 \leq t < \infty} \mathbb{E}|\tilde{Y}_t^2| < \infty,$$

it is clear that

$$\varepsilon \tilde{Y}_{\varepsilon^{-2}t} = \varepsilon Y_t^\varepsilon \rightarrow 0$$

in L^2 uniformly in t (and hence in L^q for any $q < \infty$). Noting that

$$(2.4) \quad MX_t^\varepsilon = W_t - \varepsilon Y_{0,t}^\varepsilon,$$

the pointwise convergence of MX is now obvious. Moreover, by the ergodic theorem,¹

$$(2.5) \quad \int_0^t f(Y_t^\varepsilon) dt \rightarrow t \int f(y)\nu(dy), \quad \text{in } L^q, \text{ any } q < \infty,$$

¹See e.g. [25] (p. 421) or [19] (p. 409).

for all reasonable test functions f . We shall only use it for quadratics.² Using successively (2.4) and (2.2), we can then write

$$\begin{aligned}
 \int_0^t MX_s^\varepsilon \otimes d(MX^\varepsilon)_s &= \int_0^t MX_s^\varepsilon \otimes dW_s - \varepsilon \int_0^t MX_s^\varepsilon \otimes dY_s^\varepsilon \\
 &= \int_0^t MX_s^\varepsilon \otimes dW_s - MX_t^\varepsilon \otimes (\varepsilon Y_t^\varepsilon) + \varepsilon \int_0^t d(MX^\varepsilon)_s \otimes Y_s^\varepsilon \\
 &= \int_0^t MX_s^\varepsilon \otimes dW_s - MX_t^\varepsilon \otimes (\varepsilon Y_t^\varepsilon) + \int_0^t MY_s^\varepsilon \otimes Y_s^\varepsilon ds \\
 &\rightarrow \int_0^t W_s \otimes dW_s - 0 + t \int My \otimes y\nu(dy) \\
 &= \int_0^t W_s \otimes dW_s + tMC \\
 &= \mathbb{W}_{0,t} + t \left(MC - \frac{1}{2}I \right),
 \end{aligned}$$

where the convergence is in L^q for any $q \geq 2$. By considering the symmetric part of the above equation,

$$\frac{1}{2}(MX_t^\varepsilon) \otimes (MX_t^\varepsilon) \rightarrow \frac{1}{2}W_t \otimes W_t + \text{Sym} \left(MC - \frac{1}{2}I \right),$$

we see that

$$MC - \frac{1}{2}I$$

has symmetric part 0, i.e. is anti-symmetric, and hence also equals

$$\frac{1}{2}(MC - CM^*).$$

This settles pointwise convergence, in the sense that

$$\left(MX_t^\varepsilon, \int_0^t MX_s^\varepsilon \otimes d(MX^\varepsilon)_s \right) \rightarrow \left(W_t, \widehat{\mathbb{W}}_{0,t} \right).$$

In view of [11], Proposition A.15, the rough path convergence (1.6) will follow once we have checked that for any $q < \infty$,

$$\begin{aligned}
 \sup_{\varepsilon \in (0,1]} E \left[|X_{s,t}^\varepsilon|^q \right] &\leq C_q |t - s|^{\frac{q}{2}}, \\
 \sup_{\varepsilon \in (0,1]} E \left[\left| \int_s^t X_{s,\cdot}^\varepsilon \otimes dX^\varepsilon \right|^q \right] &\leq C_q |t - s|^q.
 \end{aligned}$$

First, since X is Gaussian, it follows from integrability properties of the first two Wiener-Ito chaos that it is enough to show it for $q = 2$. Secondly, we note that the

²The ergodic theorem in the references we have cited only applies to bounded f , but it is easy to see by a truncation argument that (2.5) still holds as long as $(f(Y_s))_{s \geq 0}$ is bounded in any L^q and ν has finite moments of all order.

desired estimates are a consequence of the following :

$$(2.6) \quad \mathbb{E} \left[\left| \tilde{X}_{s,t} \right|^2 \right] \leq (\text{const.})|t - s|,$$

$$(2.7) \quad E \left[\left| \int_s^t \tilde{X}_{s,\cdot} \otimes d\tilde{X} \right|^2 \right] \leq (\text{const.})|t - s|^2,$$

where the constants must be uniform over $t, s \in (0, \infty)$. Indeed, this follows directly from writing

$$\mathbb{E} \left[\left| X_{s,t}^\varepsilon \right|^2 \right] = \mathbb{E} \left[\left| \varepsilon \tilde{X}_{\varepsilon^{-2}s, \varepsilon^{-2}t} \right|^2 \right] \leq (\text{const.})\varepsilon^2 |\varepsilon^{-2}t - \varepsilon^{-2}s| = (\text{const.})|t - s|$$

(note uniformity in ε), and similarly for the second moment of the iterated integral.

In order to check (2.6), it is enough to note $M\tilde{X}_{s,t} = \tilde{W}_{s,t} - \tilde{Y}_{s,t}$, combined with the estimate

$$\begin{aligned} \mathbb{E} \left[\left| \tilde{Y}_{s,t} \right|^2 \right] &= \mathbb{E} \left[\left| (e^{-M(t-s)} - I)\tilde{Y}_s \right|^2 \right] + \int_s^t Tr(e^{-Mu}e^{-M^*u})du \\ &\leq C(M)|t - s|, \end{aligned}$$

using the fact that $Re(\sigma(M)) \subset (0, +\infty)$. For the second one, since $d\tilde{X}_t = \tilde{Y}_t dt$, we can write

$$\begin{aligned} E \left[\left| \int_s^t (\tilde{X}_{s,u})^i d(\tilde{X}_u)^j \right|^2 \right] &= \mathbb{E} \left[\left| \int_s^t \int_s^u \tilde{Y}_r^i \tilde{Y}_u^j dr du \right|^2 \right] \\ &= \int_{[s,t]^4} \mathbb{E} \left[\tilde{Y}_r^i \tilde{Y}_u^j \tilde{Y}_q^i \tilde{Y}_v^j \right] \mathbf{1}_{\{r \leq u; q \leq v\}} dr du dq dv \\ &\leq \int_{[s,t]^4} \left(\left| \mathbb{E} \left[\tilde{Y}_r^i \tilde{Y}_u^j \right] \right| \left| \mathbb{E} \left[\tilde{Y}_q^i \tilde{Y}_v^j \right] \right| + \left| \mathbb{E} \left[\tilde{Y}_r^i \tilde{Y}_q^i \right] \right| \left| \mathbb{E} \left[\tilde{Y}_u^j \tilde{Y}_v^j \right] \right| \right. \\ &\quad \left. + \left| \mathbb{E} \left[\tilde{Y}_r^i \tilde{Y}_v^j \right] \right| \left| \mathbb{E} \left[\tilde{Y}_u^j \tilde{Y}_q^i \right] \right| \right) dr du dq dv \\ &\leq C_0 \left(\int_{[s,t]^2} \left| \mathbb{E} \left[\tilde{Y}_r \otimes \tilde{Y}_u \right] \right| dr du \right)^2 \\ &\leq C_1 \left(\int_{[s,t]^2} \left| \mathbb{E} \left[\tilde{Y}_r \otimes \tilde{Y}_u \right] \right| \mathbf{1}_{\{r \leq u\}} dr du \right)^2, \end{aligned}$$

where we have used the fact that \tilde{Y} is Gaussian (“Wick’s formula”). But for $r \leq u$, $\mathbb{E} \left[\tilde{Y}_u \tilde{Y}_r \right] = e^{-M(u-r)}\tilde{Y}_r$, so that

$$\begin{aligned} &\left(\int_{[s,t]^2} \left| \mathbb{E} \left[\tilde{Y}_r \otimes \tilde{Y}_u \right] \right| \mathbf{1}_{\{r \leq u\}} dr du \right)^2 \\ &= \left(\int_{[s,t]^2} \left| \mathbb{E} \left[\tilde{Y}_r \otimes e^{-M(u-r)}\tilde{Y}_r \right] \right| \mathbf{1}_{\{r \leq u\}} dr du \right)^2 \\ &\leq C_2 \left(\int_s^t \left(\int_r^t e^{-\lambda(u-r)} du \right) \mathbb{E} \left[\left| \tilde{Y}_r \right|^2 \right] dr \right)^2 \leq C_3(t - s)^2, \end{aligned}$$

recalling that $|\exp(-\tau M)| = O(\exp(-\lambda\tau))$, and (2.7) is proved. □

With a little more care, one can prove the following rates of convergence in Theorem 1.

Proposition 2. *For any $\alpha \in (1/3, 1/2)$, and $q \geq 1$, and any $\delta > 0$, there exists some constant C s.t.*

$$\left\| \left\| \sup_{s,t \in [0,T]} \frac{|MX_{s,t} - W_{s,t}|}{|t-s|^\alpha} + \sup_{s,t \in [0,T]} \frac{\left| \int_s^t MX_{s,r} \otimes d(MX)_r - \widehat{W}_{s,t} \right|}{|t-s|^{2\alpha}} \right\| \right\|_{L^q(\mathbf{P})} \leq C\varepsilon^{1-2\alpha-\delta}.$$

In other words, given $\alpha \in (1/3, 1/2)$ we have a rate arbitrarily close to $1 - 2\alpha$; by a Borel–Cantelli argument this is also the almost-sure rate, say, along $\varepsilon = 1/n$. Note that by working in a stronger topology ($\alpha \uparrow 1/2$), one loses on the rate of convergence. Also, since “level-2” rough path theory imposes $\alpha > 1/3$, an upper bound for the best possible rate (in a rough path metric!) is given by $1 - 2/3 = 1/3$. (The situation is very similar to the rate of convergence of piecewise linear approximations to Brownian motion (as rough path); see [18].)

Proof. The first step is to check “directly” (without using the ergodic theorem) that

$$\int_0^t Y_s^\varepsilon \otimes Y_s^\varepsilon ds \rightarrow tC = t\mathbb{E} \left(\tilde{Y}_0^{\text{stat}} \otimes \tilde{Y}_0^{\text{stat}} \right)$$

in $L^q \forall q < \infty$, which can be done with a proof similar to the proof of the inequality (2.7). Assume for simplicity that $Y_s^\varepsilon = \tilde{Y}_{s/\varepsilon^2}$ is started with $Y_0^\varepsilon \sim \nu$; note then that $\mathbb{E} \int_0^t Y_s^\varepsilon \otimes Y_s^\varepsilon = tC$. Furthermore:

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^t Y_s^{\varepsilon,i} Y_s^{\varepsilon,j} ds \right)^2 \right] &= \int_{[0,t]^2} \mathbb{E} [Y_s^{\varepsilon,i} Y_s^{\varepsilon,j} Y_r^{\varepsilon,i} Y_r^{\varepsilon,j}] ds dr \\ &= \int_{[0,t]^2} \mathbb{E} [Y_s^{\varepsilon,i} Y_s^{\varepsilon,j}] \mathbb{E} [Y_r^{\varepsilon,i} Y_r^{\varepsilon,j}] ds dr \\ &\quad + 2 \int_{[0,t]^2} \mathbb{E} [Y_r^{\varepsilon,i} Y_s^{\varepsilon,j}] \mathbb{E} [Y_s^{\varepsilon,i} Y_r^{\varepsilon,j}] ds dr \\ &\leq C_{ij}^2 t^2 + \int_{[0,t]^2} \left(\mathbb{E} [Y_r^{\varepsilon,i} Y_s^{\varepsilon,j}]^2 + \mathbb{E} [Y_s^{\varepsilon,i} Y_r^{\varepsilon,j}]^2 \right) ds dr \\ &\leq C_{ij}^2 t^2 + 4 \int_{[0,t]^2} \mathbb{E} [Y_r^\varepsilon \otimes Y_s^\varepsilon]^2 \mathbf{1}_{\{s \leq r\}} ds dr. \end{aligned}$$

Now note that for $s \leq r$,

$$\mathbb{E} [Y_r^\varepsilon | Y_s^\varepsilon] = e^{-\varepsilon^{-2}M(r-s)} Y_s^\varepsilon,$$

so that

$$\mathbb{E} [Y_r^\varepsilon \otimes Y_s^\varepsilon]^2 = O(e^{-\frac{2\lambda}{\varepsilon^2}(r-s)}),$$

and we finally obtain

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^t Y_s^{\varepsilon,i} Y_s^{\varepsilon,j} ds - tC_{ij} \right)^2 \right] &= \mathbb{E} \left[\left(\int_0^t Y_s^{\varepsilon,i} Y_s^{\varepsilon,j} ds \right)^2 \right] - (tC_{ij})^2 \\ &\leq (\text{const.}) \int_0^t \left(\int_s^\infty e^{-\frac{2\lambda}{\varepsilon^2}(r-s)} dr \right) ds = (\text{const.})\varepsilon^2. \end{aligned}$$

We have thus proved that the L^2 -convergence (and by Gaussian properties L^q -convergence for any $q < \infty$) $\int_0^t Y_s^\varepsilon \otimes Y_s^\varepsilon ds \rightarrow tC$ is in fact of order ε . From here on it is not difficult to establish convergence rates of (1.6). From [11, Thm A.13] and the work already done in the previous proof (reduction from q^{th} moments to second moments is immediate by Gaussian chaos) it suffices to check

$$(2.8) \quad \mathbb{E} \left[|W_{s,t} - MX_{s,t}^\varepsilon|^2 \right]^{\frac{1}{2}} \leq C\varepsilon^{1-2\beta} |t-s|^\beta,$$

$$(2.9) \quad \mathbb{E} \left[\left| \widehat{W}_{s,t} - \int_s^t MX_{s,r}^\varepsilon \otimes d(MX^\varepsilon)_r \right|^2 \right]^{\frac{1}{2}} \leq C\varepsilon^{1-2\beta} |t-s|^{2\beta}$$

for fixed $1/3 < \beta \leq 1/2$. A short computation shows that

$$\begin{aligned} \mathbb{E} \left[|W_{s,t} - MX_{s,t}^\varepsilon|^2 \right]^{\frac{1}{2}} &\leq C |t-s|^{1/2} \text{ and } \leq C\varepsilon, \\ \mathbb{E} \left[\left| \widehat{W}_{s,t} - \int_s^t MX_{s,r}^\varepsilon \otimes d(MX^\varepsilon)_r \right|^2 \right]^{\frac{1}{2}} &\leq C |t-s| \text{ and } \leq C\varepsilon, \end{aligned}$$

and we then obtain immediately (2.8)-(2.9) by (geometric) interpolation, with exponents $2\beta \leq 1$ and $1 - 2\beta$. □

3. APPLICATIONS AND REMARKS

We conclude this note with a number of applications and remarks.

First, as a consequence of our main theorem, we have the following convergence result for ODEs driven by “physical” Brownian motion, in the zero mass limit.

Corollary 1. *Write the (anti-symmetric) matrix $MC - CM^*$ as $\sum_{i < j} \gamma_{i,j}[e_i, e_j]$. Let $V_0 \in \text{Lip}^1$, $V = (V_1, \dots, V_n) \in \text{Lip}^{2+\delta}$ for some $\delta > 0$, be vector fields on \mathbb{R}^e . Let Y^ε be the solution to the SDE (actually, random ODE)*

$$dY^\varepsilon = V_0(Y^\varepsilon)dt + \sum_{i=1}^d V_i(Y^\varepsilon)dX^{\varepsilon;i}, \quad Y_0^\varepsilon = x \in \mathbb{R}^e.$$

Then as $\varepsilon \rightarrow 0$, Y^ε converges (almost surely with respect to sup-norm on $[0, T]$, with rate arbitrarily close to $\frac{1}{3}$) to the solution Y of the following Itô SDE:

$$dY = \tilde{V}_0(Y) dt + \sum_{i=1}^n \tilde{V}_i(S^0)dB_s^i,$$

where $(\tilde{V}_1, \dots, \tilde{V}_n) = (V_1, \dots, V_n)M^{-1}$ and

$$\tilde{V}_0 = V_0 + \frac{1}{2} \sum_{i=1}^d \tilde{V}_i \tilde{V}_i + \frac{1}{2} \sum_{1 \leq i < j \leq n} \gamma_{i,j}[\tilde{V}_i, \tilde{V}_j].$$

Proof. Given our main result, this is a simple consequence of local Lipschitzness of the Itô-Lyons map (with respect to the rough path metric which appears in Proposition 2) and the impact of higher order perturbations on RDEs (see e.g. [11, Ch. 12]), combined with the usual Itô–Stratonovich correction. □

Remark 7 (Magnetohydrodynamics). A (physical) system - as described by the above differential equations - which is driven/controlled by a single, charged Brownian particle in a magnetic field may not appear to be a standard situation in applied science. However, it is not hard to think of a system being influenced by a cloud of such particles. If N such particles move independently, our main theorem applies immediately in dimension $n = 3N$. Of course, for $N \gg 1$ one needs to incorporate interactions between the particles. In fact, the movement of a cloud of charged particles will effect very much the magnetic field itself. In our model, the matrix M should then be allowed to depend on the (bulk) behaviour of the N particles. Much more work will be necessary to give a proper rough path analysis of this situation; we do remark however that first mean field results for rough differential equations have been obtained by Cass–Lyons [5]. It is quite conceivable that our rough path perspective then provides a very novel point of view for magnetohydrodynamics.

Remark 8. Similar correction terms appear when one considers limits of one-forms integrated against X^ε . The rough path correction matters, for instance, if one wants to make a change of measure and represent the stochastic integral in the Girsanov factor in terms of X^ε , rather than in terms of the underlying Brownians. (This type of representation plays an important role in “robust” filtering, path sampling of conditioned diffusions [13, 14] and related topics.)

Remark 9. The *signature* of a path is the sequence of its iterated integrals against itself. For (deterministic) paths of bounded variation, it fully characterizes the path up to “tree-like” equivalence [15]. In a similar spirit, the *expected signature* of a processes (in the sense below) characterizes the essence of its law, at least when it comes to the solution of stochastic differential equations; the so-called cubature method is based on this [22]. In [17] expected signatures of many stochastic processes are considered. By either specializing these considerations or a direct computation one can see that, as $\varepsilon \rightarrow 0$, the expected signature of MX^ε converges to

$$\mathbb{E} \left[S(\widehat{\mathbf{W}})_{0,T} \right] = \exp \left(\frac{T}{2} \left(\sum_i e_i \otimes e_i + \sum_{i < j} \gamma_{i,j} [e_i, e_j] \right) \right).$$

Remark 10. In this paper we have only considered the case where M (and the diffusion matrix $Q = I$) were constant. It is however natural to consider the case where the dynamics (i.e. friction, magnetic field and diffusion matrix of the Brownian term) depend on the position x of the particle, i.e.

$$(3.1) \quad m\ddot{x} = -M(x)\dot{x} + Q(x)\xi.$$

Further generalizations are of course possible; for instance Freidlin and coworkers [6, 8] consider (3.1) subject to an additional potential of the form $\nabla H(x)$ (their aims however are quite different, in particular they do not consider area-type corrections).

Returning to (3.1), this leads us to consider the coupled system of SDEs,

$$\begin{aligned} dY^\varepsilon &= -\varepsilon^{-2}M(X^\varepsilon)Y^\varepsilon dt + \varepsilon^{-1}dW_t, \\ dX^\varepsilon &= \varepsilon^{-1}Q(X^\varepsilon)Y^\varepsilon dt. \end{aligned}$$

It is then not difficult to show that the paths $X^\varepsilon \rightarrow X^0$ pointwise, where X^0 is a solution to

$$dX^0 = [D_x(QM^{-1}) : C](X^0)dt + (QM^{-1})(X^0)dW,$$

where $[D_x(QM^{-1}) : C]^i := \sum_{k,l} \partial_k(QM^{-1})^i C^{kl}$ and $C = C(x)$ is still defined by the same formula.

As for the iterated integral, we obtain a similar correction except that it is now state dependent (and thus random):

$$\int_0^t X^\varepsilon \otimes dX^\varepsilon \rightarrow \int_0^t X^0 \otimes \circ dX^0 + \frac{1}{2} \int_0^t Q(C(M^{-1})^* - M^{-1}C) Q^*(X_s^0) ds.$$

The computation of these pointwise convergences is close to the beginning of the proof of Theorem 1, but instead of using the ergodic theorem one should notice that for small ε

$$\int_0^t f(X_s^\varepsilon, Y_s^\varepsilon) ds \approx \int_0^t \left(\int f(X_s^\varepsilon, y) \nu_C(X_s^\varepsilon, dy) \right) ds,$$

as the process Y^ε evolves at a much faster time-scale than X^ε (here \approx should mean that the difference is small in L^2 -norm). The detailed verification of convergence in rough path sense is technical (mainly, because one loses the Gaussian and Markovian structure of Y^ε) and left to subsequent work.

Remark 11 (On the role of Brownian roughness). Let us return to (1.1), but now with Brownian motion replaced by a (deterministic) path γ defined on $[0, T]$. That is, we consider the evolution

$$(3.2) \quad m\ddot{x} = -M\dot{x} + \dot{\gamma}.$$

Of course, even when γ fails to be differentiable this equation is well-defined in the distributional sense, thanks to the additive structure of the noise. As before we assume that M has an anti-symmetric part which therefore triggers rotation and thus effects the area. One may wonder how “rough” the driving force (now assumed deterministic) needs to be to see some non-trivial area correction in the limit. As we now show, the roughness of Brownian motion -with (almost) $1/2$ -Hölder sample paths - is crucial and *no area corrections* arise when γ is Hölder with exponent greater than $1/2$.

Proposition 3. *Assume γ is α -Hölder on $[0, T]$ for some $\alpha \in (0, 1]$. Then, denoting by x a solution to (3.2), as $m \rightarrow 0$, $Mx_{0,\cdot}$ converges to $\gamma_{0,\cdot}$ in β -Hölder topology for any $\beta < \alpha$. In particular, when $\alpha > 1/2$ it follows that*

$$\int_0^\cdot Mx_{0,t} \otimes d(Mx_t) \rightarrow \int_0^\cdot \gamma_{0,\cdot} \otimes d\gamma \text{ as } m \rightarrow 0,$$

where the integral on the left-hand side is understood in the Young sense, and convergence may be understood uniformly on $[0, T]$ (and actually in the β -Hölder rough path sense).

Proof. First note that by continuity of Young integration in β -Hölder topology, $\beta > 1/2$, the stated convergence of $\int Mx \otimes d(Mx)$ when $\alpha > 1/2$ is indeed a consequence of the first part. We now proceed to prove that $Mx \rightarrow \gamma$ in β -Hölder topology for any $\beta < \alpha$. By interpolation, it is enough to establish pointwise convergence of $Mx_{0,\cdot}$ to $\gamma_{0,\cdot}$ in conjunction with uniform α -Hölder bounds. Equivalently, we want to show that, pointwise and with uniform α -Hölder bounds,

$$z := \gamma_{0,\cdot} - Mx_{0,\cdot} \rightarrow 0 \text{ as } m \rightarrow 0.$$

Note that

$$dz_t = -\frac{M}{m}z_t dt + d\gamma_t,$$

from which we see, writing $z_{s,t} = z_t - z_s$ as usual,

$$z_{s,t} = (e^{-\frac{M}{m}(t-s)} - I)z_s + \int_s^t e^{-\frac{M}{m}(t-r)} d\gamma_r.$$

The last integral is a Young (actually Riemann-Stieltjes) integral, for its integrand has finite variation. To see this, note that for $s \leq t$, $|e^{-tM} - e^{-sM}| \leq Ce^{-\lambda s}((t-s) \wedge 1)$ where λ is, as in previous sections, a lower bound on the real part of the spectrum of M . We may then estimate, for any subdivision $0 \leq s_0 \leq \dots \leq s_N < \infty$,

$$\begin{aligned} \sum_i |e^{-s_{i+1}M} - e^{-s_iM}| &\leq C \sum_i e^{-\lambda s_i} ((s_{i+1} - s_i) \wedge 1) \\ &\leq C \sum_{n=0}^{\infty} e^{-\lambda n} \sum_i \mathbf{1}_{\{n \leq s_i < n+1\}} ((s_{i+1} - s_i) \wedge 1) \\ &\leq 2C \sum_{n=0}^{\infty} e^{-\lambda n} < \infty. \end{aligned}$$

In particular, it follows that

$$\sup_{0 < m \leq 1} \left\| e^{-\frac{M}{m}} \right\|_{1\text{-var};[0,T]} < \infty.$$

We now address pointwise convergence. Since $z_0 = 0$ we can estimate, whenever $0 < \delta \leq t \leq T$,

$$\begin{aligned} |z_t| &= \left| \int_0^t e^{-\frac{M}{m}(t-r)} d\gamma_r \right| \leq e^{-\frac{\lambda}{m}\delta} \left| \int_0^{t-\delta} e^{-\frac{M}{m}((t-\delta)-r)} d\gamma_r \right| + \left| \int_{t-\delta}^t e^{-\frac{M}{m}(t-r)} d\gamma_r \right| \\ &\leq Ce^{-\frac{\lambda}{m}\delta} \|\gamma\|_{\alpha\text{-Hö};[0,T]} T^\alpha \left(1 + \left\| e^{-\frac{M}{m}} \right\|_{1\text{-var};[0,T]} \right) \\ &\quad + C\|\gamma\|_{\alpha\text{-Hö};[0,T]} \delta^\alpha \left(1 + \left\| e^{-\frac{M}{m}} \right\|_{1\text{-var};[0,T]} \right) \\ &\leq C \left(e^{-\frac{\lambda}{m}\delta} + \delta^\alpha \right), \end{aligned}$$

where C is a constant which does not depend on m . Taking

$$\delta = \frac{\alpha m}{\lambda \log(1/m)}$$

one sees that

$$|z_t| \leq Cm^\alpha (1 + |\log m|)^\alpha \leq Cm^\beta,$$

which in particular gives us pointwise convergence. As for uniform Hölder bounds, take $s \leq t$ so that

$$z_{s,t} = \int_s^t e^{-\frac{M}{m}(t-r)} d\gamma_r + (e^{-\frac{M}{m}(t-s)} - I)z_s.$$

As before, the integral term is bounded by $C(t-s)^\alpha$. For the other term, note that

$$\begin{aligned} \left| \left(e^{-\frac{M}{m}(t-s)} - I \right) z_s \right| &\leq C \left(\frac{t-s}{m} \wedge 1 \right) |z_s| \\ &\leq C \left(\frac{t-s}{m} \right)^\beta |z_s| \leq C(t-s)^\beta, \end{aligned}$$

where we have used the previous pointwise estimate on z in the last inequality. This proves that the paths z are uniformly β -Hölder continuous and finishes the proof. \square

The above proposition shows, for instance, that replacing Brownian motion in our main theorem by *fractional* Brownian motion with Hurst parameter $H > 1/2$ will not allow for a similar statement with non-trivial stochastic area correction. (It is recalled that fractional Brownian motion has a Hölder continuous sample path with exponent arbitrarily close to H .) Let us, finally and briefly, discuss a similar statement when Brownian motion in our main theorem is replaced by a finite energy path; that is, a path

$$\gamma : [0, T] \rightarrow \mathbb{R}^n,$$

which may be written as an indefinite integral of some function in $L^2([0, T], \mathbb{R}^n)$, which we shall call $\dot{\gamma}$. By Cauchy-Schwarz, such finite energy paths are guaranteed to be $1/2$ -Hölder, but, in general, one does not have better Hölder regularity. In particular, since the area is not continuous in $1/2$ -Hölder topology, the above proposition just about fails to cover finite energy paths. A direct argument, however, is not difficult. As in the above proof, we set

$$z := \gamma_{0,\cdot} - Mx_0,$$

and note from the previous argument that $|z_t| \leq Cm^{1/2}(1 + |\log m|)^{1/2}$, uniformly over $t \in [0, T]$. We then write

$$\dot{z}_t = -\frac{M}{m}z_t + \dot{\gamma}_t$$

and take the scalar product in \mathbb{R}^n with \dot{z}_t , by integration over $[0, T]$, to see that

$$\begin{aligned} \int_0^T |\dot{z}_t|^2 dt &= -\frac{M}{m} \frac{1}{2} |z_T|^2 + \int_0^T \langle \dot{\gamma}_t, \dot{z}_t \rangle dt \\ &\leq \frac{1}{2} \int_0^T |\dot{\gamma}_t|^2 dt + \frac{1}{2} \int_0^T |\dot{z}_t|^2 dt. \end{aligned}$$

This implies a uniform (in m) L^2 -bound on \dot{z} . This of course implies a uniform L^1 -bound on \dot{z} and thus a uniform 1-variation bound on z . Knowing that z converges to zero uniformly on $[0, T]$, it now follows from interpolation that this convergence also takes place in p -variation, for any $p > 1$. Now, the area is a continuous function of the underlying paths in p -variation as long as $p < 2$, and so we can conclude: replacing Brownian motion in our main theorem by a finite energy (also known as Cameron–Martin) path will not allow for a similar statement with non-trivial stochastic area correction.

4. APPENDIX: ELEMENTS OF ROUGH PATH THEORY

A rough path on an interval $[0, T]$ with values in a Banach space V consists of a continuous function $X : [0, T] \rightarrow V$, as well as a continuous “second order process” $\mathbb{X} : [0, T]^2 \rightarrow V \otimes V$, subject to certain (i) algebraic and (ii) analytic conditions. Towards (i), the behaviour of iterated integrals of smooth paths suggests imposing the algebraic relation (“Chen’s relation”),

$$\mathbb{X}_{s,t} - \mathbb{X}_{u,t} - \mathbb{X}_{s,u} = X_{s,u} \otimes X_{u,t} ,$$

which is assumed to hold for every triple of times (s, u, t) . Since $X_{t,t} = 0$, it immediately follows (take $s = u = t$) that we also have $\mathbb{X}_{t,t} = 0$ for every t . One should think of \mathbb{X} as *postulating* the value of the quantity

$$\int_s^t X_{s,r} \otimes dX_r = X X_{s,t} ,$$

where we take the right-hand side as a *definition* for the left-hand side. We insist that knowledge of the path $t \mapsto (X_{0,t}, \mathbb{X}_{0,t})$ already determines the entire second order process \mathbb{X} . In this sense (X, \mathbb{X}) is indeed a path, and not some two-parameter object.

Note that the algebraic relations are by themselves not sufficient to determine \mathbb{X} as a function of X . Indeed, for any $V \otimes V$ -valued function F , the substitution $\mathbb{X}_{s,t} \mapsto \mathbb{X}_{s,t} + F_t - F_s$ leaves the left-hand side of the above algebraic relation invariant. We will see later on how one should interpret such a substitution. The aim of imposing these algebraic relations is to ensure that \mathbb{X} does indeed have the basic additivity properties of any (reasonable) integral when considering it over two adjacent intervals.

For $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, one can denote by $\mathcal{D}^\alpha([0, T], V)$ the space of those rough paths (X, \mathbb{X}) such that

$$\|X\|_\alpha = \sup_{s \neq t \in [0, T]} \frac{|X_{s,t}|}{|t - s|^\alpha} < \infty , \quad \|\mathbb{X}\|_{2\alpha} = \sup_{s \neq t \in [0, T]} \frac{|\mathbb{X}_{s,t}|}{|t - s|^{2\alpha}} < \infty .$$

If one ignores the non-linear, algebraic constraint there is a natural way to think of (X, \mathbb{X}) as an element in the Banach space of such maps with (semi-)norm $\|X\|_\alpha + \|\mathbb{X}\|_{2\alpha}$. However, due to the non-linear algebraic relation \mathcal{D}^α is not a linear space, although it is a closed subset of the aforementioned Banach space.

Definition 1. Given rough paths $\mathbf{X}, \mathbf{Y} \in \mathcal{D}^\alpha([0, T], V)$, we define the (inhomogeneous) α -Hölder rough path metric

$$\rho_\alpha(\mathbf{X}, \mathbf{Y}) := \sup_{s \neq t \in [0, T]} \frac{|X_{s,t} - Y_{s,t}|}{|t - s|^\alpha} + \sup_{s \neq t \in [0, T]} \frac{|\mathbb{X}_{s,t} - \mathbb{Y}_{s,t}|}{|t - s|^{2\alpha}} .$$

Let us note that $\mathcal{D}^\alpha([0, T], V)$ thus becomes a complete, metric space. The perhaps cheapest way to show convergence with respect to this rough path metric is based on interpolation: in essence, it is enough to establish pointwise convergence in conjunction with uniform “rough path” α -Hölder bounds. We conclude this part with two important remarks. First, we can ask ourselves up to which point the algebraic relations are already sufficient to determine \mathbb{X} . Assume that we can associate to a given function X two different second order processes \mathbb{X} and $\tilde{\mathbb{X}}$, and set $G_{s,t} = \mathbb{X}_{s,t} - \tilde{\mathbb{X}}_{s,t}$. It then follows immediately that

$$G_{s,t} = G_{u,t} + G_{s,u}$$

so that in particular $G_{s,t} = G_{0,t} - G_{0,s}$. We conclude that \mathbb{X} is in general determined only up to the increments of some function F with values in $(V \otimes V)$ and Hölder continuous with exponent 2α . The choice of F *does* usually matter, and there is in general no obvious canonical choice.

The second remark is that this construction can possibly be useful only if $\alpha \leq \frac{1}{2}$. Indeed, if $\alpha > \frac{1}{2}$, then a canonical choice of \mathbb{X} is given in terms of the Young integral. Furthermore, it is clear in this case that \mathbb{X} must be unique, since any additional increment should be 2α -Hölder continuous, which is of course only possible if $\alpha \leq \frac{1}{2}$. This is however *not* to say that \mathbb{X} is uniquely determined by X if the latter is smooth, when interpreted as an element of \mathcal{D}^α . Indeed, if $\alpha \leq \frac{1}{2}$, F is any 2α -Hölder continuous function with values in $V \otimes V$ and $\mathbb{X}_{s,t} = F_t - F_s$, then the path $(0, \mathbb{X})$ is a perfectly “legal” element of \mathcal{D}^α , even though one cannot get any smoother than the function 0. The impact of perturbing \mathbb{X} by some $F \in \mathcal{C}^{2\alpha}$ in the context of differential equations and integration is dramatic: additional drift terms in (Lie bracket) directions can appear; the famous Itô-Stratonovich correction is also understood from this picture. The reader may find (much) more in [11, 21] and [10].

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