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Physics of risk and uncertainty in quantum decision making

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Abstract. The Quantum Decision Theory, developed recently by the authors, is applied to clarify the role of risk and uncertainty in decision making and in particular in relation to the phenomenon of dynamic inconsistency. By formulating this notion in precise mathematical terms, we distinguish three types of inconsistency: time inconsistency, planning paradox, and inconsistency occurring in some discounting effects. While time inconsistency is well accounted for in classical decision theory, the planning paradox is in contradiction with classical utility theory. It finds a natural explanation in the frame of the Quantum Decision Theory. Different types of discounting effects are analyzed and shown to enjoy a straightforward explanation within the suggested theory. We also introduce a general methodology based on self-similar approximation theory for deriving the evolution equations for the probabilities of future prospects. This provides a novel classification of possible discount factors, which include the previously known cases (exponential or hyperbolic discounting), but also predicts a novel class of discount factors that decay to a strictly positive constant for very large future time horizons. This class may be useful to deal with very long-term discounting situations associated with intergenerational public policy choices, encompassing issues such as global warming and nuclear waste disposal.

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1 Introduction

The concept of risk is widely used in economics, finance, psychology, as well as in everyday life. Respectively, there exist several definitions of risk and different ways of evaluating it. In any application, the notion of risk is always related to the necessity of taking decisions under uncertainty. It is impossible to achieve optimal results in any science without correct decisions, leading to optimal consequences following from the taken decision. This is why the notion of risk and the problem of its evaluation has, first of all, to be understood in the frame of decision theory. It is precisely the aim of the present paper to formulate a novel approach for taking into account the risk in decision making and to demonstrate in concrete examples, related to temporal effects in making decisions, that this new approach is free of defects and paradoxes plaguing the application of standard decision theory.

Classical decision theory is based on expected utility theory, which was advanced by Bernoulli [1] and was shaped into a rigorous mathematical theory by von Neumann and Morgenstern [2]. In this theory, a decision maker chooses between several lotteries, or gambles, each being composed of a set of outcomes, equipped with a probability measure. Initially [2], the probabilities were assumed to be objective. Savage [3] extended utility theory to the case of subjective probabilities. Savage's generalization has been demonstrated to be tremendously flexible in representing the attitude of decision makers towards risk and uncertainty. Starting with Pratt [4] and Arrow [5], different measures of risk have been proposed. Extensions and modern developments are covered, e.g., in [6–8].

Notwithstanding a remarkable breadth of successful applications, classical decision theory, when applied to real humans, leads to a variety of paradoxes that remain unsolved in its framework. The first such anomaly was described by Allais [9], which is now known as the Allais paradox. Other well known paradoxes are Ellsberg's paradox [10], Kahneman-Tversky's paradox [11], the conjunction fallacy [12,13], the disjunction effect [14], and Rabin's paradox [15]. These and other paradoxes are reviewed in references [16,17].

There has been many attempts to modify expected utility theory in order to get rid of the paradoxes that plague its application to the processes involving decision making of real human beings. One of these approaches is the cumulative-prospect theory or reference-point theory

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[18], which assumes that decision making is not based on the absolute evaluation of payoffs but depends on a reference point that is specific to the present state of the decision maker. Because the reference point is shifted as a result of the consequences emerging from a first decision, the subsequent decision performed, according to the reference-point theory, is therefore sensitive to the difference between subsequent payoffs rather than to the absolute payoff deriving solely from the second decision.

One of the main problems encountered when using reference-point theory is that the reference point of a decision maker is not uniquely defined: for a similar payoff history, each decision maker can possess (and actually does possess) his/her own specific reference point, which is generally unobservable. Moreover, reference-point theory is more suited to address those anomalies that arise in gambles involving at least two-steps, in which the reference point can be expected to be shifted after each outcome. But, the majority of paradoxes appear in single-step gambles, where reference-point theory is not applicable. In the hope of explaining the paradoxes mentioned above, many other variants of the so-called non-expected utility theories have been suggested. A review of a variety of such non-expected utility theories can be found in Machina [19–21]. A rigorous analysis of these theories has been recently performed by Safra and Segal [22], who concluded that the non-expected utility theories cannot explain all paradoxes. Though it is possible to invent a modification of utility theory that will fit one or a few paradoxes, the problem is that many others will remain unexplained at best, or new inconsistencies will arise at worst.

The basic difficulty in taking into account and evaluating risk, when deciding under uncertainty, is that the usual approaches assume that decision makers are rational. However, real human beings are only partially rational [23], as is well documented by numerous empirical data in behavioral economics and neuroeconomics [24–27]. Risk is always related to emotions. But how could one describe emotions within a quantitative framework suitable for decision making?

A new approach to decision making, called Quantum Decision Theory (QDT), has been advanced in references [16,17,28]. The main idea of this approach is to take into account that realistic decision-making problems are composite, consisting of several parts intimately interconnected, intricately correlated, and entangled with each other. Several intended actions can interfere with each other, producing effects that cannot be simply measured by ascribing a classical utility function. The complexity involved in decision making reflects the interplay between the decision maker's underlying emotions and feelings and his/her attitude to risk and uncertainty accompanying the decision making process. In order to take account of these subtle characteristics in the most self-consistent and simple way, we suggest to use the mathematical techniques based on the quantum theory of measurement of von Neumann [29] and developed by other authors (see, e.g., Refs. [30,31]). This is the reason for referring to this new approach under the name Quantum Decision Theory (QDT). It is important to stress that we do not assume that human brains are quantum objects. It should just be understood that we use the techniques of complex Hilbert spaces, as a convenient mathematical toolbox that provides a parsimonious and efficient description of the complex processes involved in decision making.

In our previous papers [16,17,28], we formulated the mathematics of QDT and showed that this approach provides a straightforward explanation of practically all known paradoxes of classical decision making. However, we have not yet considered the class of so-called dynamical inconsistencies that arise in decisions (under risks and/or uncertainty) that compare different time horizons. The aim of the present paper is to analyze this class of inconsistency in the frame of QDT, explaining those effects that have remained unexplained in the standard theory.

Our theory should not be confused with the approach that is called "quantum probabilities from decision theory", where one attempts to derive the rule of defining the probability in quantum mechanics from classical decision theory. To be more precise, let us recall that probabilities enter quantum mechanics via the Born rule, according to which the probability of each outcome of a measurement is prescribed by the squared amplitude of the corresponding term in the given quantum-mechanical state [32]. Deutsch [33] argued that the Born rule could be derived from the notion of rational preferences of standard classical decision theory. This argument was reconsidered by Wallace [34–36] who showed that the Deutsch way of reasoning, first, necessarily requires the Everett [37] many-word interpretation of quantum mechanics and, second, needs additional assumptions that have nothing to do with classical decision theory. A very detailed analysis of the Deutsch-Wallace arguments has recently been given by Lewis [38], who has persuasively demonstrated that there are several serious drawbacks in the Deutsch-Wallace picture. First of all, the Everett many-word interpretation has its own problem related to its basic assumption that, after each measurement, the observer branches into a number of successors living in different words. The number of such branches is not well defined and even can be infinite. According to Lewis [38], "the number of branches associated with an outcome is unknowable, undefined, and uncountable, and hence branch-counting rules are simply unusable". Lewis also showed that there are other gaps in the mathematics of Deutsch and Wallace, which invalidate the proof that the Born rule could be derived form classical decision theory [38].

In our theory, we adopt the quantum-mechanical rules as its very foundation, never trying to derive them from some other assumptions. The mathematics we employ is in complete agreement with the von Neumann axiomatics [29]. Using the techniques of quantum theory, we develop the quantum decision theory that can be applied to real alive beings.

The theory of quantum measurement considers only passive quantum systems subject to a measurement procedure imposed by an external observer. A principal difference is that our theory describes an active decision maker. Mathematically, an active decision maker is characterized by his/her own strategic states describing his/her main personal preferences. In contrast, in quantum measurements performed over a passive system, there is no preferred quantum states, and any basis can be employed.

Moreover, our approach is completely different from the theory of quantum games, suggested by Meyer [39] (see the review articles [40,41]). What is common for both these theories is merely the use of the quantum theoretical techniques, but their mathematical structure is very different. The general setup of a quantum game is as follows. One considers a passive quantum system (gamble source), several observers (players), and an external machine (judge). The system is prepared in a quantum state. The machine acts on this state entangling it. Each of the players in turn acts on the resulting entangled state by a unitary transformation. In this process, the players can exchange information between themselves and with the machine. Then the machine again acts on the obtained state disentangling it and producing the final product state. The payoffs are calculated according to the classical rule with additive probabilities, hence, there is no interference in this final stage. This scheme can be considered as a variant of quantum computation and communication. Contrary to this scheme, in our approach, we consider only a single decision maker and not several ones. Of course, the single decision maker can represent a group of people that act as a single person. There is no passive quantum system, but the decision maker represents himself/herself an active system acting according to quantum rules. The decision maker does not produce unitary transformations on the given states. There are no external judges or machines. Since the calculations are made by quantum rules, this involves nonadditive quantum probabilities and the related interference terms that are of crucial importance for the analysis of what constitutes the optimal decision. Thus, the overall structure of our theory is principally different from the setup of quantum games.

In Section 2, we provide a brief summary of the architecture of QDT that is needed for our analysis. Section 3 dissects the three classes of dynamic inconsistency (time inconsistency, planning paradox and discounting effects) and applies QDT to them. Section 4 presents a quantitative formulation of the dynamics of prospects, in which hyperbolic discounting is derived from simple principles. Section 5 concludes. Let us stress once more that the dynamic effects have not been treated in our previous articles [16,17,28].

2 Quantum decision theory

In this section, we give a brief formulation of the theory to be used. We follow the scheme of references [17,28], employing Dirac's notation [32,42] for the states belonging to the Hilbert spaces. To be precise, we recall below the basic definitions and axioms of QDT.

2.1 Main definitions

Definition 1. *Action ring.* The set of intended actions A_n, enumerated with an index n, forms an action ring

$$\mathcal{A} = \{A_n : n = 1, 2, \ldots, N\}. \qquad (1)$$

The ring is equipped with the binary operations, namely the addition and multiplication: for each A_m and A_n belonging to $\mathcal{A}$, $A_m + A_n$ and $A_m A_n$ also belong to $\mathcal{A}$. The addition is associative, so that $A_1 + (A_2 + A_3) = (A_1 + A_2) + A_3$, and reversible, in the sense that $A_1 + A_2 = A_3$ yields $A_1 = A_3 - A_2$. The multiplication is distributive, $A_1(A_2 + A_3) = A_1A_2 + A_1A_3$, and idempotent, $A_nA_n = A_n^2 = A_n$. But, generally, it is not commutative, so that A_mA_n does not necessarily equal A_nA_m when m and n are different. There exists an empty action, such that $A_n0 = 0A_n = 0$. Two actions A_m and A_n are disjoint when $A_mA_n = A_nA_m = 0$.

Definition 2. *Action modes.* The elements of the action ring, the actions, can be composite

$$A_n = \bigcup_{\mu=1}^{M_n} A_{n\mu} \qquad (M_n > 1), \qquad (2)$$

being composed of several representations, called modes, labelled by μ. Different modes are assumed to be disjoint,

$$A_{n\mu}A_{n\nu} = \delta_{\mu\nu}A_{n\mu},$$

where $\delta_{\mu\nu}$ is the Kronecker delta. An action is composite if $M_n > 1$, in the other case, it is simple.

Definition 3. *Action prospects.* A more complex structure is an action prospect

$$\pi_j = \bigcap_n A_{j_n} \qquad (A_{j_n} \in \mathcal{A}), \qquad (3)$$

which is a conjunction of several actions. The prospect is composite if it includes composite actions, while it is simple if all actions in (3) are simple. Generally, the number of factors in the intersection can be different, depending on the definition of the prospects. Since the products of actions pertaining to the ring, by the ring structure, also pertain to the ring, then the prospects are also members of the same ring.

Definition 4. *Elementary prospects.* A prospect is called elementary if all actions in its definition (3) are simple, being represented by single modes. The elementary prospects

$$e_\alpha = \bigcap_{n=1}^{N} A_{i_n\mu_n} \qquad (4)$$

are labelled by the binary multi-index

$$\alpha = \{i_n, \mu_n : n = 1, 2, \ldots, N\}_\alpha.$$

The set $\{\alpha\}$ has cardinality $\mathrm{card}\{\alpha\} = \prod_{n=1}^{N} M_n$. All elementary prospects are disjoint with respect to each other,

$$e_\alpha e_\beta = \delta_{\alpha\beta}e_\alpha.$$

Here and in what follows, the cardinality N is the same as in Definition 1.

Definition 5. *Prospect lattice.* A particular family of prospects composes a prospect lattice

$$\mathcal{L} = \{\pi_j : j = 1, 2, \ldots, N_L\}, \tag{5}$$

where the binary operations $\geq$ and $\leq$ are assumed to be defined, ordering the prospects so that, for each pair π_i and π_j, either $\pi_i \geq \pi_j$ or $\pi_i \leq \pi_j$. For a while, it is sufficient to keep in mind that the prospects can be ordered. The explicit ordering procedure will be prescribed below in Definition 16.

Definition 6. *Mode states.* To each mode $A_{n\mu}$ there corresponds a complex function

$$|A_{n\mu}\rangle : \mathcal{A} \to \mathbb{C} \tag{6}$$

called the mode state. The fact that each mode is idempotent and different modes are disjoint is expressed through the orthonormality condition for the scalar product $\langle A_{n\mu}|A_{n\nu}\rangle = \delta_{\mu\nu}$.

Definition 7. *Mode space.* The closed linear envelope

$$\mathcal{M}_n = \text{Span}\{|A_{n\mu}\rangle : \mu = 1, 2, \ldots, M_n\}, \tag{7}$$

spanning all mode states, equipped with a scalar product, is the mode space. This is a Hilbert space of dimensionality $\dim\mathcal{M}_n = M_n$.

Definition 8. *Basic states.* To each elementary prospect (4), there corresponds a complex function

$$|e_\alpha\rangle = |A_{i_1\mu_1} A_{i_2\mu_2} \ldots A_{i_N\mu_N}\rangle = \bigotimes_{n=1}^{N} |A_{i_n\mu_n}\rangle, \tag{8}$$

called a basic state. Since an elementary prospect (4) is a conjunction of single modes, and different modes are disjoint with each other, this is expressed as the orthonormality condition for the scalar product $\langle e_\alpha|e_\beta\rangle = \delta_{\alpha\beta}$.

Definition 9. *Mind space.* The closed linear envelope

$$\mathcal{M} = \text{Span}\{|e_\alpha\rangle : \alpha \in \{\alpha\}\} = \bigotimes_{n=1}^{N} \mathcal{M}_n, \tag{9}$$

spanning all basic states, endowed with a scalar product, is the mind space. This is a Hilbert space of dimensionality $\dim\mathcal{M} = \prod_{n=1}^{N} M_n$.

Definition 10. *Prospect states.* To each prospect (3), there corresponds a complex function $|\pi_j\rangle$ belonging to the mind space $\mathcal{M}$. Because the prospects, generally, are composite, they are not necessarily normalized and orthogonal to each other.

Definition 11. *Strategic states.* In the mind space (9), there exist fixed reference states $|\psi_s\rangle \in \mathcal{M}$, which characterize the features typical of a given decision maker. These states are orthonormal, such that $\langle\psi_s|\psi_{s'}\rangle = \delta_{ss'}$. But they do not necessarily form a basis. The existence of the strategic states is the principal point distinguishing QDT from the usual theory of quantum measurements.

Definition 12. *Mind strategy.* The collection of all strategic states $|\psi_s\rangle$, equipped with their weights w_s, forms the mind strategy

$$\Sigma = \{|\psi_s\rangle, w_s : s = 1, 2, \ldots, S\}, \tag{10}$$

where

$$\sum_{s=1}^{S} w_s = 1, \qquad 0 \leq w_s \leq 1. \tag{11}$$

The mind strategy describes the decision-maker character, his/her main beliefs and principles, according to which he/she makes decisions.

Definition 13. *Prospect operators.* Each prospect state $|\pi_j\rangle$ defines the prospect operator

$$\hat{P}(\pi_j) = |\pi_j\rangle\langle\pi_j|, \tag{12}$$

where $\langle\pi_j|$ is the Hermitian conjugate to $|\pi_j\rangle$. The prospect operators, by definition, are self-adjoint. The family of all prospect operators forms the involutive bijective algebra

$$\mathcal{P} = \{\hat{P}(\pi_j) : \pi_j \in \mathcal{L}\}.$$

This algebra is analogous to the algebra of local observables in quantum theory.

Definition 14. *Operator averages.* The average of a prospect operator (12) is the sum

$$\langle\hat{P}(\pi_j)\rangle = \sum_{s=1}^{S} w_s \langle\psi_s|\hat{P}(\pi_j)|\psi_s\rangle \tag{13}$$

of its matrix elements over the strategic states.

Definition 15. *Prospect probability.* The probability of a prospect $\pi_j \in \mathcal{L}$ is the average

$$p(\pi_j) = \langle\hat{P}(\pi_j)\rangle \tag{14}$$

of the prospect operator (12), with the normalization condition

$$\sum_{j=1}^{N_L} p(\pi_j) = 1, \tag{15}$$

where the summation is over the whole prospect lattice $\mathcal{L}$.

Definition 16. *Prospect ordering.* A prospect π_1 is indifferent to a prospect π_2 if and only if their probabilities coincide,

$$p(\pi_1) = p(\pi_2) \qquad (\pi_1 = \pi_2) \tag{16}$$

and a prospect π_1 is preferred to π_2 if and only if

$$p(\pi_1) > p(\pi_2) \qquad (\pi_1 > \pi_2). \tag{17}$$

The ordering of prospects through the relation between their probabilities defines the explicit ordering in the prospect lattice (5). The prospect π^* with the largest probability $p(\pi^*) = \sup_j p(\pi_j)$ is called optimal.

Definition 17. *Partial probabilities.* The probability

$$p(\pi_j e_\alpha) = \langle \hat{P}(e_\alpha)\hat{P}(\pi_j)\hat{P}(e_\alpha)\rangle \tag{18}$$

of a conjunction prospect $\pi_j e_\alpha$ defines the partial probability of realizing an elementary prospect e_α when deciding on the prospect π_j. The partial probabilities are normalized as

$$\sum_{j,\alpha} p(\pi_j e_\alpha) = 1, \tag{19}$$

where the sum is over all $\pi_j \in \mathcal{L}$ and all e_α.

Definition 18. *Attraction factor.* The variable

$$q(\pi_j) = \sum_{\alpha \neq \beta} \langle \hat{P}(e_\alpha)\hat{P}(\pi_j)\hat{P}(e_\beta)\rangle \tag{20}$$

quantifies the attractiveness of the prospect π_j for a decision maker with respect to risk, uncertainty, and biases. It arises due to the interference between the intended actions of a given prospect π_j, which occurs during the decision process.

Definition 19. *Attraction ordering.* The prospects are ordered with respect to their attractiveness for a decision maker. A prospect π_1 is more attractive than a prospect π_2 if and only if

$$q(\pi_1) > q(\pi_2). \tag{21}$$

The prospects π_1 and π_2 are equally attractive if and only if

$$q(\pi_1) = q(\pi_2). \tag{22}$$

The impact in decision making of emotions and feelings, which are known to be important and practically inseparable from logical deliberation [43], are quantified by the attraction factor. The ordering of prospects with respect to their attractiveness, quantified by the attraction factor (20), is a principal ingredient of QDT.

Definition 20. *Attraction conditions.* The distinction between more or less attractive prospects is formalized by the following rule. A prospect π_1 is more attractive than a prospect π_2, when it is connected with:

(a) more certain gain;
(b) less certain loss;
(c) higher activity under certainty;
(d) lower activity under uncertainty.

These characteristics describe the aversion of a decision maker to risk, uncertainty, and presumed loss.

2.2 A few theorems

The above definitions constitute the basis of QDT [16,17,28]. They allow us to derive the following theorems proved in reference [17], which will be needed below.

Proposition 1 (*Prospect probability*). The probability of a prospect $\pi_j \in \mathcal{L}$ is

$$p(\pi_j) = \sum_\alpha p(\pi_j e_\alpha) + q(\pi_j), \tag{23}$$

where the summation is over the elementary prospects e_α.

Proposition 2 (*Attraction alternation*). The sum of all attraction factors (20) is equal to zero:

$$\sum_{j=1}^{N_L} q(\pi_j) = 0, \tag{24}$$

where the summation is performed over all $\pi_j \in \mathcal{L}$.

Proposition 3 (*Preference criterion*). A prospect $\pi_1 \in \mathcal{L}$ is preferred to a prospect $\pi_2 \in \mathcal{L}$ if and only if

$$\sum_\alpha [p(\pi_1 e_\alpha) - p(\pi_2 e_\alpha)] > q(\pi_2) - q(\pi_1). \tag{25}$$

Remark. From the form of prospect probability (23), together with condition (19) and property (24), it is immediately seen that the normalization condition (15) is always valid.

These theorems imply that the probability of taking a given decision is controlled by the levels of attraction of the different competing prospects, thus emphasizing the emotional component of the decision process. Indeed, the choice of a specific prospect among several alternatives depends not solely on its value given by the first term in the right-hand side of equation (23), but also on its attractiveness quantified by the attraction factor (20). In classical decision theory, only values measured by a utility function are considered, but emotions and feelings are not taken into account. In QDT, the later are embodied in the new ingredients, the attraction factors.

Two essential characteristics distinguish QDT from classical utility theory:

(i) QDT is a probabilistic theory, in which each prospect is associated with its probability, which has a subjective component captured by the attraction factor. The prospect probability can be measured experimentally, by interpreting it as a relative frequency, that is, it corresponds to the relative ratio of decision makers accepting the given prospect. This probabilistic framework accounts for the observations that, under the same conditions, different people endowed with a priori the same preferences may make different decisions. In contrast, classical utility theory is deterministic, with its prescription to the decision maker forcing him/her to accept the unique alternative which corresponds to the maximal expected utility.
(ii) In addition to the payoff values, QDT takes also into account the attractiveness of the analyzed prospects, quantified by their attraction factors (20). These attraction factors are absent in utility theory. Therefore, a partial reduction of QDT to classical decision theory is obtained by setting the attraction factor to zero.

2.3 Binary mind

To make the structure of the theory clearer, it is instructive to consider the particular case of a binary mind. This case is also of intrinsic interest because the majority of paradoxes can be treated and explained in this specific frame.

The binary mind corresponds to considering only two actions, while each of them can possess a number of representation modes. Let these actions be

$$A = \bigcup_{j=1}^{M_1} A_j, \qquad B = \bigcup_{\mu=1}^{M_2} B_\mu. \tag{26}$$

Hence, there are two mode spaces

$$\mathcal{M}_1 = \mathrm{Span}\{|A_j\rangle : j = 1, 2 \ldots, M_1\},$$
$$\mathcal{M}_2 = \mathrm{Span}\{|B_\mu\rangle : \mu = 1, 2 \ldots, M_2\}. \tag{27}$$

The mind space is the tensor product of these two mode spaces

$$\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2, \tag{28}$$

hence its name "binary". This should not be confused with the dimensionality $\dim\mathcal{M} = M_1 M_2$ of the binary mind, which can be large.

The elementary prospects (4) are $e_{j\mu} = A_j B_\mu$, and the basic states (8) become

$$|e_{j\mu}\rangle = |A_j B_\mu\rangle \equiv |A_j\rangle \otimes |B_\mu\rangle. \tag{29}$$

The action prospects (3) can be constructed as $\pi_j = A_j B$, and the conjunction prospects as $\pi_j e_{j\mu} = A_j B_\mu$. According to equation (23), the prospect probabilities are

$$p(\pi_j) = \sum_{\mu=1}^{M_2} p(A_j B_\mu) + q(\pi_j). \tag{30}$$

One can draw the following analogies between the quantities of QDT presented above, and those of classical utility theory. The set B of modes B_μ corresponds to the set of payoffs. Complementing this set by the related weights $p_j(B_\mu)$ defines a lottery L_j. The weights $p_j(B_\mu)$ can be expressed in terms of the conditional probabilities: $p_j(B_\mu) = p(A_j|B_\mu)$. This defines the probability of getting payoff B_μ in lottery L_j. The analog of the expected utility is the sum

$$\sum_{\mu=1}^{M_2} p(A_j B_\mu) = \sum_{\mu=1}^{M_2} p(A_j|B_\mu) p(B_\mu), \tag{31}$$

where $p(B_\mu)$ is a normalized measure of the payoff B_μ.

With QDT, it is possible to explain all paradoxes emerging in classical decision making [16,17]. To give an idea how this is done, we present here a brief account of the resolution of Allais' paradox [9]. Allais' paradox can be described with a binary mind, as defined above. For the sake of brevity, we survey only the mathematical structure of this paradox, omitting the interpretations related to psychological features (see Refs. [16,17] for in-depth analysis). A detailed description of the mathematical structure of the Allais paradox can be found in reference [17].

One considers two actions as in equation (26), with $M_1 = 4$ and $M_2 = 3$ and the mind dimensionality $\dim\mathcal{M} = M_1 M_2 = 12$. The experiment, demonstrating Allais' paradox, is organized in such a way that the balance condition

$$p(A_1 B_\mu) + p(A_3 B_\mu) = p(A_2 B_\mu) + p(A_4 B_\mu) \tag{32}$$

holds for all $\mu = 1, 2, 3$. The goal is to compare the prospects $\pi_j = A_j B$ for different j. Allais' paradox is that most human decision makers prefer the prospect π_1 to π_2, and π_3 to π_4 which, due to the balance condition (32), leads to a contradiction. The fact that π_1 is preferred to π_2 translates in the language of QDT into the inequality $p(\pi_1) > p(\pi_2)$. The fact that prospect π_1 looks more attractive (less uncertain, less risky) than π_2 implies that $q(\pi_1) > q(\pi_2)$. Using (30), this leads to

$$\sum_{\mu=1}^{3} [p(A_2 B_\mu) - p(A_1 B_\mu)] < q(\pi_1) - q(\pi_2). \tag{33}$$

The fact that π_3 is preferred to π_4 translates in the language of QDT into $p(\pi_3) > p(\pi_4)$. The larger attraction of π_3, compared with π_4, implies that $q(\pi_3) > q(\pi_4)$. Again using (30), this gives

$$\sum_{\mu=1}^{3} [p(A_3 B_\mu) - p(A_4 B_\mu)] > q(\pi_4) - q(\pi_3). \tag{34}$$

Then, using the definitions of Section 2.1 and Proposition 2 on the property of attraction alternation, invoking the balance condition (32), and combining inequalities (33) and (34), we get

$$-|q(\pi_3) - q(\pi_4)| < \sum_{\mu=1}^{3} [p(A_2 B_\mu) - p(A_1 B_\mu) < |q(\pi_1) - q(\pi_2)|. \tag{35}$$

Classical decision theory corresponds to the limit of zero attraction factors ($q(\pi_1) = q(\pi_2) = q(\pi_3) = q(\pi_4) = 0$). In this case, the two inequalities (35) result in a contradiction, since the sum in the middle cannot be larger than zero and, at the same time, smaller than zero. Within QDT, this contradiction does not arise. Actually, within QDT, Allais' paradox is explained from the interplay between the attraction factors of different prospects.

3 Dynamic inconsistency

We now use the framework of QDT to study dynamic inconsistency, which has not been treated in our previous articles. In economics, time inconsistency refers, roughly speaking, to a situation when the preference of a decision-maker changes over time, in such a way that what is preferred at one point in time is inconsistent with what is preferred at another point in time. In fact, there are numerous variants of dynamic inconsistency. By being precise, one can distinguish three broad classes of dynamic inconsistency: (i) time inconsistency, (ii) planning paradox, and (iii) discounting effects. We now examine each one in turn.

3.1 Time inconsistency

Time inconsistency is well epitomized by the Strotz's phrase [44]: "the optimal plan of the present moment is generally one which is not obeyed, or that the individual's future behavior will be inconsistent with his optimal plan". Various examples of this inconsistency have been described in the literature [45–47]. Kydland and Prescott [46] went so far as saying that the rational choice for future times "is not an appropriate tool for economic planning" and that "the application of optimal control theory is equally absurd".

The origin of time inconsistency is rather straightforward. When an individual makes a plan for the far future, he/she cannot be conscious of all the detailed circumstances that will arise in that future. New information is likely to appear and, in addition, the already available information may be open for re-evaluation. Since the future situation is likely to be different, it will require making a decision that is likely to differ from the current decision. The current decision for the future action then turns out to be sub-optimal when the future becomes the present.

There is no real paradox in this time inconsistency and its solution can be readily obtained: when making a decision for the distant future, it is necessary to try to predict future changes and include these forecasts in the decision making process. This recipe was suggested for instance by Strotz [44] who gave, as an example, the behavior of Odysseus when his ship was approaching the Sirens. Wishing to hear the Sirens' songs (short-term gratification) but mindful of the possible delayed danger (falling prey to the Sirens), he ordered his men to close their ears with beeswax and to bind him to the mast of the ship. He also ordered his men not to heed his cries while they would pass the Sirens. In that way, Odysseus limited his future agency and binded himself to a restriction (to the mast) to survive the long-term consequences of his decision. Other numerous example are known, related to pension savings, health insurance, and so on. When making plans for the far future, one tries to anticipate the obstacles that may arise and one imposes restrictions and commitments that oppose the change of decision that would result otherwise due to time inconsistency. With the imposed commitments, time inconsistency disappears and the present decision becomes the optimized one for the future state. As experiments show [48], even rats possess the ability of making decisions that take into account an estimation of future events. We conclude that both the origin and the solution for time inconsistency are well understood and do not require invoking additional concepts for their interpretation.

3.2 Planning paradox

Consider a situation in which an individual makes a plan for a short future period of time, such that no novel information will become available and the individual himself/herself does not change over that period. In the absence of any new information and of any change, the decision should be unchangeable as well. The invariance of the decision in that sense is referred to as the principle of dynamic consistency in classical decision theory.

However, it often happens that the decision maker does change the plan, for not apparent reason. A stylized example of this type of planning paradox is a smoker who plans to stop smoking tomorrow, while enjoying the pleasure of smoking today. Making this plan, he/she promises to stop smoking, understanding well that he/she will forgo future pleasures, for the anticipation of higher health benefits. The next day, while the plan and the utility resulting from its consequences have not changed, it is often observed that the human beings change their plan, and continue smoking.

3.2.1 Mathematical formulation of planning paradox and its resolution

Such a behavior poses a real paradox within expected utility theory. Let us formulate this paradox in precise mathematical terms. When deciding to stop smoking in a plan, one keeps in mind the following intended actions:

- planning to stop smoking tomorrow (A_1);
- planning to continue smoking tomorrow (A_2);
- wishing to have good health (B_1);
- paying little attention to health (B_2).

The decision to stop smoking in reality corresponds to the following intended actions:

- stop smoking in reality (A_3);
- continue smoking in reality (A_4);
- wishing to have good health (B_1);
- paying little attention to health (B_2).

The related four action sets are $X_j = \{A_j B_\mu : \mu = 1, 2\}$, with $j = 1, 2, 3, 4$. Following utility theory, and ascribing probabilities to these actions, one gets the corresponding lotteries $L_j = L_j(X_j)$. Note that the utility functions of the actions A_1B and A_3B, where $B = B_1 + B_2$, are the same when expressed for tomorrow, since the two actions of stopping smoking become equivalent. Similarly, the utility functions tomorrow of the actions A_2B and A_4B are equal, since continuing smoking is the same action, with the same consequences. Therefore, the expected utilities of the lotteries L_1 and L_3 are equal: $U(L_1) = U(L_3)$. And analogously, $U(L_2) = U(L_4)$. But many individuals prefer L_1 to L_3, which implies that, for these individuals, $U(L_1) > U(L_3)$. The same individuals also choose L_4 over L_2, which implies that $U(L_2) < U(L_4)$. This leads to a contradiction violating the principle of dynamic consistency of classical decision making.

Let us now show how this paradox can be explained within QDT. As above, we need to consider the intended actions A_j, with $j = 1, 2, 3, 4$ and the set $\{B_\mu\}$. In addition, the decision of stopping smoking in reality is accompanied by the following intended actions:

– getting pleasure from smoking (C_1);
– having no pleasure from smoking (C_2);
– agreeing to suffer because of addiction (D_1);
– refusing to suffer from addiction (D_2).

These additional intentions reflect emotional feelings of the decision maker, which are not taken into account in classical utility theory.

The prospects that need now to be compared are

$$\pi_1 = A_1 B, \qquad \pi_2 = A_2 B \qquad \left(B = \bigcup_\mu B_\mu \right) \tag{36}$$

and

$$\pi_3 = A_3 BCD, \qquad \pi_4 = A_4 BCD, \tag{37}$$

where

$$B = B_1 + B_2, \qquad C = C_1 + C_2, \qquad D = D_1 + D_2.$$

The value of quitting smoking, either today or tomorrow, has the same determined value. Respectively, the value of continuing smoking is also determined, being the same either today or tomorrow. In both these cases, the utility of stopping smoking is larger than that of continuing smoking, which can be expressed as the inequality

$$\sum_\mu p(A_1 B_\mu) > \sum_\mu p(A_2 B_\mu). \tag{38}$$

In QDT, the attraction factors are taken into account, which model the subjective emotions associated with different actions. Since the health benefits are evident, stopping smoking in a plan seems to be more attractive than to continue smoking. This looks easy, since the associated pain is not yet felt but the risk for health, associated with the continuation of smoking, seems evident. This is why to stop smoking in a plan is more attractive than to continue smoking. Then the corresponding attraction factors obey the inequality $q(\pi_1) > q(\pi_2)$. In contrast, continuing smoking unconditionally amounts to abandon oneself to the pleasure of addiction, which is preferred in general to the failure of not abiding to a plan to abandon smoking, given that the health benefits are felt to be uncertain. One can summarize these emotions by saying that continuing smoking in reality is more attractive than stopping smoking. This is formulated mathematically by the inequality $q(\pi_4) > q(\pi_3)$ for the corresponding attraction factors. Summarizing, we have

$$q(\pi_1) > q(\pi_2), \qquad q(\pi_4) > q(\pi_3). \tag{39}$$

Writing the prospect probabilities according to equation (30), and taking into account the above discussion, shows that, in reality, the probability of continuing smoking becomes larger than that of stopping smoking when

$$\sum_\mu [p(A_1 B_\mu) - p(A_2 B_\mu)] < q(\pi_4) - q(\pi_3). \tag{40}$$

Then it is implied immediately that the prospect π_1 is preferred to π_2, while π_4 is preferred to π_3. As for other paradoxes, the absence of contradiction in QDT results from the existence of the attraction factors, which are absent in classical utility theory. We have shown that the attraction factors derive intrinsically from the Hilbert space structure of the theory that accounts for interference between prospects. Putting the attraction factors to zero recovers the inconsistency associated with the planning paradox. As QDT is a probabilistic theory, the above conclusion that $p(\pi_1) > p(\pi_2)$ and $p(\pi_4) > p(\pi_3)$ does not mean that no individual can stop smoking. The general subjective preferences embodied in the attraction factors only tell us that the majority of them will not be able to quit smoking.

3.2.2 Generalization to two-step games

To show that the explanation proposed by QDT is general, let us consider another example of the planning paradox, with two-step gambles. In two-step gambles, decision makers are typically confronted sequentially with two successive gambles, with probabilities 1/2 to gain or to loose in each of them. Before playing the first gamble, participants are asked to make a planned choice as whether they would take the second gamble, provided the first one is either won or lost. Then the first gamble is played. After experiencing the actual results of the first gamble, decision makers are asked to make a final choice regarding the second gamble, whether they accept it or not.

A number of experiments have been performed to test the dynamic consistency in the frame of such two-step gambles [49–51]. The experiments showed that the final choices of the participants were frequently inconsistent with their plans, even when the anticipated and experienced outcomes were identical. These inconsistencies are found to occur in a systematic direction: anticipating a gain in the first gamble, decision makers planned to take the second gamble – but after experiencing the gain, some of them changed their minds and rejected the second gamble. And, anticipating a loss in the first gamble, the participants planned to restrain from the second gamble – however, experiencing the actual loss, they often changed their plans and accepted the second gamble. Attempts were made [49,51] to explain this inconsistency within the framework of the reference-point theory [18], arguing that, after the first gamble, the reference point of the decision makers has been shifted. In the introduction Section 1, we have already discussed the weakness of the reference-point approach. These are the ambiguity in defining both the reference point as well as the shift. And, what is more important, the reference-point theory can be applied only to two-step or multi-step gambles. It is not applicable to single-step gambles. But there are numerous cases where the planning paradox occurs in single-step gambles, such as in the above example of the smokers planning to stop smoking. In an earlier publication [52], the authors mentioned that the planning paradox in two-step gambles could be related to quantum effects. Below, we provide a concrete proof in the frame of QDT, by showing how the planning paradox in two-step games finds a natural resolution.

The mathematical structure of the two-step gambles of the type described in references [49–51] can be reduced to a structure that is similar to, though slightly more complicated than, the structure underlying the case described in the previous Section 3.2.1. The two-step game proceeds as follows. The first gamble is obligatory and cannot be refused while the second gamble can be rejected. Specifically, the following alternatives are offered to the decision maker.

– Assuming an anticipated gain (C_1) or loss (C_2) in the first gamble, the second gamble can be accepted (A_1) or rejected (A_2), with the chances of winning (B_1) or loosing (B_2) being equal.
– After experiencing a realized gain (C_3) or an actual loss (C_4) in the first gamble, the second gamble can be accepted (A_1) or rejected (A_2), with the chances of winning (B_1) or to loose (B_2).

The planning stage, before playing the first game, is characterized by the four prospects

$$\pi_1 = A_1BC_1, \qquad \pi_2 = A_2BC_1,$$
$$\pi_3 = A_1BC_2, \qquad \pi_4 = A_2BC_2, \tag{41}$$

where $B = B_1 + B_2$. After having played the first game, the decision maker faces the four new prospects

$$\pi_5 = A_1BC_3, \qquad \pi_6 = A_2BC_3,$$
$$\pi_7 = A_1BC_4, \qquad \pi_8 = A_2BC_4. \tag{42}$$

The four prospects in the planning stage form two binary lattices:

$$\mathcal{L}_1 = \{\pi_1, \pi_2\}, \qquad \mathcal{L}_2 = \{\pi_3, \pi_4\}. \tag{43}$$

The four prospects available after playing the first game form the two other binary lattices

$$\mathcal{L}_3 = \{\pi_5, \pi_6\}, \qquad \mathcal{L}_4 = \{\pi_7, \pi_8\}. \tag{44}$$

Analogously to conditions in Section 3.2.1, it is assumed that the utility of accepting or rejecting the second gamble does not depend on whether the first gamble is assumed to be won or lost in the planning stage or actually won or lost in reality. This means that

$$\sum_\mu p(A_1B_\mu C_1) = \sum_\mu p(A_1B_\mu C_3),$$
$$\sum_\mu p(A_1B_\mu C_2) = \sum_\mu p(A_1B_\mu C_4). \tag{45}$$

Next, we model the subjective beliefs and emotions commonly observed in humans by specifying the attraction factors of each prospect. Many human beings share the gambler's fallacy [53], in which an observed deviation from an expected fair chance of winning or losing is expected to be followed by a reversal. In other words, playing a gamble with equal chances to win or to loose, humans often expect that, after winning one gamble, the chance to win a second gamble is reduced. Reciprocally, after loosing one gamble, the odds to win the next gamble are felt to increase. One can say that, after winning a gamble, a fear to loose the next gamble appears. However, this fear is less intense in imagination than in reality. That is, the perceived risk in the planning stage is weaker than after the realized gain of the first game, since an imaginary gain or loss is less certain than the real one. This makes the prospect π_1 of accepting the second gamble, after an anticipated gain in the first gamble, more attractive than the prospect π_5 of really accepting the second gamble after an actual gain in the first gamble. This translates into

$$q(\pi_1) > q(\pi_5). \tag{46}$$

Similarly, after loosing in the first gamble, the expectation to win in the second gamble increases, but less in imagination than following a realized win, hence

$$q(\pi_3) < q(\pi_7). \tag{47}$$

We thus obtain the probabilities of the prospects π_1 and π_5 as

$$p(\pi_1) = p(A_1B_1C_1) + p(A_1B_2C_1) + q(\pi_1),$$
$$p(\pi_5) = p(A_1B_1C_3) + p(A_1B_2C_3) + q(\pi_5). \tag{48}$$

Similarly, the probabilities of the prospects π_3 and π_7 are

$$p(\pi_3) = p(A_1B_1C_2) + p(A_1B_2C_2) + q(\pi_3),$$
$$p(\pi_7) = p(A_1B_1C_4) + p(A_1B_2C_4) + q(\pi_7). \tag{49}$$

Comparing these probabilities, with taking account of conditions (45), we get

$$p(\pi_1) - p(\pi_5) = q(\pi_1) - q(\pi_5),$$
$$p(\pi_7) - p(\pi_3) = q(\pi_7) - q(\pi_3). \tag{50}$$

From equation (46), we obtain $p(\pi_1) > p(\pi_5)$, that is, the first prospect is preferred to the fifth prospect, $\pi_1 > \pi_5$: individuals choose to play the second game more often when they do not know the outcome of the first game but expect a gain, than after the gain is realized. From equation (47), we see that $p(\pi_7) > p(\pi_3)$, hence the seventh prospect is preferred to the third one, $\pi_7 > \pi_3$: individuals choose more often to play the second game after losing the first game than when imagining that they could lose before playing the first game. Thus, no contradiction arises within QDT.

We again emphasize that the preference for one prospect at the expense of a second prospect does not imply that all decision makers choose it, but only that the fraction of decision makers preferring that prospect is larger than the fraction of decision makers choosing the second prospect. Depending on the gain prizes and on the loss amounts, the resulting differences between the corresponding prospect probabilities may be small. For example, in the experiment of Barkan and Busemeyer [51] on the planning paradox, the probabilities, measured as the average fractions of decision makers taking the corresponding alternatives are as follows. In the planning stage before playing the first game, one has

$$p(\pi_1) = 0.60, \qquad p(\pi_2) = 0.40,$$
$$p(\pi_3) = 0.63, \qquad p(\pi_4) = 0.37. \qquad (51)$$

After the gain or loss of playing the first game are known, the probabilities of the different prospects are

$$p(\pi_5) = 0.53, \qquad p(\pi_6) = 0.47,$$
$$p(\pi_7) = 0.69, \qquad p(\pi_8) = 0.31. \qquad (52)$$

This gives

$$p(\pi_1) - p(\pi_5) = 0.07, \qquad p(\pi_7) - p(\pi_3) = 0.06. \qquad (53)$$

Thus, while the planning paradox is clear, not all individuals follow it, justifying the probabilistic framework of QDT. Moreover, as is seen from the above equations, the difference between the compared prospect probabilities is rather small, lying on the boundary of statistical errors.

Concluding this section, the planning paradox has been explained away by taking into account the impact of subjective beliefs and emotions in decision making via the attraction factor defined by expression (20). We stress also that the proposed framework remains valid both for single-step as well as for multistep gambles.

3.3 Discounting effects

Generally, the term discounting addresses the problem of translating values from one time period to another. The larger the discount rate, the more weight the decision maker places on costs and benefits in the near term over costs and benefits over the long term. Depending on the specification of the problem, it is possible to distinguish several discounting effects, that we analyze in turn.

3.3.1 Value discounting

According to classical utility theory, the costs and benefits of an action can be evaluated by means of its utility, or its value to the decision maker. The benefits of an action are, for instance, to receive an amount of money or any other useful object at a given time. When an action x is made at time t, it has a utility $u(x,t)$. Assume, we start our analysis at time zero, $t = 0$, when the action utility is $u(x,0)$. But the same action at a later time t is $u(x,t)$, which may be different. The difference comes from the obvious understanding that what we get earlier we can start using earlier, hence, it is more useful than what we would get later, having less time for its use. A typical example is provided by the time value of money. An amount x of money received at time $t = 0$ has a value $u(x,0)$. This money can bring a profit, increasing, after the period of time t_n to the amount $x(1+r)^{t_n}$, where r is an interest rate for a unit time interval. Therefore, the value of money x today is larger than the value of the same amount of money after time t_n. Hence, it is natural to prefer x now, instead of x at a future time t_n.

In QDT, this preference for a receipt now rather than delayed can be framed in the following decision making procedure. We consider the intended actions of getting an amount of money now (A_1) or, the same amount, sometimes later (A_2). The different possible ways of using this money are described by a set $\{B_\mu\}$ of intended actions B_μ. One makes a choice between the prospects

$$\pi_j = A_j B, \qquad B \equiv \bigcup_\mu B_\mu \qquad (j = 1,2). \qquad (54)$$

The prospect probabilities are

$$p(\pi_1) = \sum_\mu p(A_1 B_\mu) + q(\pi_1),$$
$$p(\pi_2) = \sum_\mu p(A_2 B_\mu) + q(\pi_2). \qquad (55)$$

The fact that an amount of money now gives more possibilities than the same amount received later means that

$$\sum_\mu p(A_1 B_\mu) > \sum_\mu p(A_2 B_\mu). \qquad (56)$$

In addition, getting something later is more uncertain, hence, $q(\pi_1) > q(\pi_2)$. Then it is evident that $\pi_1 > \pi_2$.

While the conclusion is the same as in classical utility theory, what QDT brings additionally is the breakdown of the time value into an objective component (the sums of probabilities in (56) quantifying the investment and consumption opportunities) and a subjective component $q(\pi)$ quantifying the emotional cost of various degrees of delaying.

3.3.2 Event uncertainty

Certain paradoxes arise because the problems are not well-posed or are too ill-defined with some features remaining unspecified or vague. Consider the typical example where one has to choose between 50 dollars now or a significantly larger amount, say 100 dollars, in a year. Proposing a larger amount in the future is supposed to account for the discounting effect of the previous subsection. Indeed, given that a given amount now is always preferred to the same amount in the future (assuming a normal growing economy), as explained in the previous subsection, one can expect to find some larger amount tomorrow that would be as attractive as the proposed sum today. The ratio of the two sums defines the discount factor of a given individual, which quantifies the value of his/her time preference. The example comparing \$50 now to \$100 in a year implicitly considers that the rational discount factor cannot be less than 1/2, or in other words, the interest rate that would provide dividends to an investment of \$50 cannot be larger than 100%, so that the sum of \$100 in a year should be more attractive than the sum of \$50 received immediately. It turns out that it is often observed that individuals prefer to get \$50 now instead of \$100 in a year. This seems a priori quite puzzling.

In fact, there is no real mystery, even within classical utility theory: because of the formulation of the problem, the related probabilities are not defined. And decision makers intuitively understand that the receipt of \$50 now is rather certain, while the sum of \$100 in a year is not certain at all. That is, one compares the lottery $L_1 = \{0, 0; \$50, 1; \$100, 0\}$ with the lottery $L_2 = \{0, 1-p; \$50, 0; \$100, p\}$, where p is not known. It can be perceived to be small because of many reasons, e.g., lack of trust in the commitment to deliver \$100 in a year due to uncertainties associated with the possible death, bankruptcy or simply default of the counter party, or uncertainty in the survival of the decision maker who would not be in a position to enjoy the receipt of \$100 in a year. Therefore, the expected utility of the first lottery is $U(L_1) = u(\$50)$, while that of the second lottery is $U(L_2) = (1-p)u(0) + pu(\$100)$. For sufficiently small $p \ll 1$, it happens that $U(L_1) > U(L_2)$, justifying the preference of L_1 to L_2. The effect is referred to in the literature as "uncertainty aversion".

In QDT, this effect is easily described in the same way as in Section 3.3.1. One compares the prospects of getting \$50 now (π_1) or \$100 in a year (π_2). The smaller probability of the second prospect implies inequality (56). The process of waiting is related to anxiety [54], making the delayed event of getting money less attractive. And, by definition, the second prospect is less attractive since it is more uncertain. That is, $q(\pi_1) > q(\pi_2)$. The immediate result is that $\pi_1 > \pi_2$.

It is interesting to compare the two explanations. In classical utility theory, the preference for \$50 now instead of \$100 in a year is accounted for by uncertainty aversion, translating into a small subjective probability for the \$100 payoff to happen. In QDT, the uncertainty aversion is embodied automatically into the attraction factor $q(\pi)$, while the normal discounting effects associated with different opportunities are included in the objective probabilities $\sum_\mu p(A_j B_\mu)$.

3.3.3 Preference reversal

A standard problem in classical decision theory is revealed by a dynamic-inconsistency paradox associated with the inversion of preferences, in which money versus time preferences are inverted as the time horizon is changed. To specify the problem, let us consider the following setup. There is a choice between \$50 now and \$100 in a year. As discussed in Section 3.3.2 above, individuals almost always prefer \$50 now. But when there is a choice between \$50 in ten years and \$100 in eleven years, human beings usually prefer \$100 in eleven years. This reversal of preference occurs notwithstanding the fact that the time difference between ten and eleven years is exactly the same as between zero and one, so that a pure rational discounting mechanism would predict the same consistent choice of the smaller amount at the earlier time. This reversal is usually associated with a trait characterizing human beings, called hyperbolic value discounting or generalized hyperbolic discounting [55–60], such that the near events are characterized by larger discount rates than the events in a more distant future. The problem is that this explanation unavoidably leads to time inconsistency since, when the decision maker reconsiders the same choice after ten years, he/she again would prefer \$50 today to \$100 in a year, thus again reversing the previous preference he/she expressed ten years earlier.

The preference-reversal paradox finds a natural explanation within QDT, since its formulation in terms of prospects implies that choices considered at different times and planned for at different future instants of time are actually different prospects, even though they are associated with equivalent actions. To be more precise, a correct definition of a prospect π_j depends on the point in time t_0 when it is considered, as well as on the point in time t for which it is planned to be realized. That is, strictly speaking, a prospect is a function $\pi_j(t, t_0)$. With this specification, the above setup can be formalized as follows. Let the prospects of getting \$50 or \$100 correspond to the notations π_1 and π_2, respectively. At time $t_0 = 0$, there are four prospects. One is the prospect $\pi_1(0, 0)$ of getting \$50 now. Another is the prospect $\pi_2(1, 0)$ of getting \$100 in a year. The third prospect is $\pi_1(10, 0)$ of getting \$50 in 10 years. And the fourth prospect $\pi_2(11, 0)$ is getting \$100 in 11 years. As discussed above, the odds of getting \$50 now are more certain than those of getting \$100 in a year, hence

$$\pi_1(0, 0) > \pi_2(1, 0). \tag{57}$$

At the same time, both prospects of getting \$50 in ten years or \$100 in eleven years seem almost equally uncertain. However, the stake in the latter case is larger, which results in the preference

$$\pi_2(11, 0) > \pi_1(10, 0). \tag{58}$$

After time elapses to the decision at the point in time $t_0 = 10$, two new prospects become available. One is the prospect $\pi_1(10, 10)$ of getting \$50 at this moment of time and another, $\pi_2(11, 10)$ of getting \$100 one year later after $t_0 = 10$. Using the same arguments, one has

$$\pi_1(10, 10) > \pi_2(11, 10). \tag{59}$$

There is no contradiction between the above decisions, since different prospects are compared.

4 Prospect dynamics

4.1 Definition of the discount factor

The evolution of probabilities in classical decision theory are usually characterized by Markov equations [52,61]. To determine how the probability of a given prospect in QDT evolves as a function of time, let us consider a prospect $\pi_j(t,t_0)$ of deciding at time t_0 for the planned realization at a later time t. The corresponding prospect state is $|\pi_j(t,t_0)\rangle$. Using the definitions of Section 2.1, the corresponding prospect operator is

$$\hat{P}(\pi_j(t,t_0)) = |\pi_j(t,t_0)\rangle\langle\pi_j(t,t_0)|. \tag{60}$$

The prospect probability is defined by the average (13), which we denote

$$p_j(t,t_0) \equiv \langle\hat{P}(\pi_j(t,t_0))\rangle. \tag{61}$$

We may assume that the mind strategy defined by equation (10), which characterizes a given decision maker, does not change during the time during which the decisions are made. In other words, the same decision maker is considered. Then, the prospect probability varies in time as

$$\frac{d}{dt}p_j(t,t_0) = \left\langle \frac{d}{dt}\hat{P}(\pi_j(t,t_0))\right\rangle. \tag{62}$$

Let us define the decay rate $\alpha_j(t,t_0)$ of the prospect state $|\pi_j(t,t_0)\rangle$ through the equation

$$\frac{d}{dt}|\pi_j(t,t_0)\rangle = -\alpha_j(t,t_0)|\pi_j(t,t_0)\rangle. \tag{63}$$

The decay rate $\alpha_j(t,t_0)$ accounts for the possible disappearance of opportunities as the future unfolds. The above equation is the definition of the decay rate. Since the latter depends on time, this definition does not necessarily imply that the time evolution of the prospect state is exponential. And the following consideration will show that, really, there can occur different types of the time evolution.

Accomplishing the differentiation in the right-hand side of equation (62) yields

$$\frac{d}{dt}p_j(t,t_0) = -\gamma_j(t,t_0)p_j(t,t_0), \tag{64}$$

where

$$\gamma_j(t,t_0) \equiv 2\mathrm{Re}[\alpha_j(t,t_0)] \tag{65}$$

can be called the "probability discount rate." Integrating equation (64) gives the prospect probability

$$p_j(t,t_0) = p_j(t_0,t_0)f_j(t,t_0), \tag{66}$$

with the discount factor

$$f_j(t,t_0) \equiv \exp\left\{-\int_{t_0}^{t}\gamma_j(t',t_0)dt'\right\}, \tag{67}$$

obeying the initial value condition $f_j(t_0,t_0) = 1$. Equations (66) and (67) define the probability of a prospect, evaluated at an initial time t_0, which is to be realized at the instant of time t.

In the economic literature, the simplest and standard assumption is to assume a constant discount rate, corresponding to an exponential discount factor. As reviewed by Cochrane [62], the exponential discount factor can be generalized into the concept of the stochastic discount factor which, by capturing the macro-economic risks underlying each security's value, provides a consistent pricing of all assets. Different models, such as the Capital Asset Pricing Model, multifactor models, term structure of bond yields, and option pricing can be derived as different specifications of the discount factor.

4.2 First-principle construction of discount rate

Here in contrast, rather than deriving the form of the discount factor that corresponds to a specific economic model, we construct, by using general symmetry requirements, the possible generic functional dependencies that the discount factor can take to describe the value of delayed payoffs. For this, we use the self-similar approximation theory [63–69]. The idea is to start from an expansion of the discount rate valid for short time, that is believed to be generally valid. Then, particular conditions are implemented to construct the functional forms that can be naturally associated with the initial expansion. The derivation of the corresponding discount factor proceeds through three successive steps. First, to improve the convergence property of a perturbative sequence, control functions, defined by an optimization procedure, are introduced. This idea forms the foundation of the optimized perturbation theory [68,69]. The second pivotal idea is to consider the successive passage from one approximation to the next one as a dynamical evolution on the manifold of approximants, which is formalized by the notion of group self-similarity. The third principal point is the introduction of control functions in the course of rearranging perturbative asymptotic expansions by means of algebraic transforms. We use the variant of the self-similar approximation theory [63–69] employing the self-similar factor approximants [70–74], based on the property that the control parameters entering the self-similar factors can be completely defined from a given asymptotic expansion by the so-called accuracy-through-order matching method. This approach was shown to be essentially more accurate than the method of Padé approximants [75]. Moreover, the latter method, as is well known, does not allow a unique reconstruction of the sought function, but results in a whole table of approximants for each given approximation order. Contrary to this, the factor approximants are uniquely defined. In addition to providing reconstruction with a very good accuracy of rational functions, as the Padé method does, the method of factor approximants determines irrational and transcendental functions with excellent precision [70–74]. These approximants also allow one to reconstruct a wide class of functions exactly.

In its applications to the construction of the functional dependence of the discount factor, we proceed as follows.

First, we note that, in full generality, the probability discount rate $\gamma_j(t, t_0)$ can be positive as well as negative. This is because the prospect probabilities are normalized according to condition (15). Consequently, if there are diminishing probabilities, then there should exist increasing probabilities in order that normalization (15) be always valid. For instance, if the probability of getting something attractive, like money, diminishes with time, then the probability of getting nothing, respectively, increases. Therefore, in what follows, it is sufficient to consider only decreasing probabilities, related to getting something appealing, keeping in mind that there exist as well their increasing counterparts defined through the normalization (15). The condition, that the probability discount rate $\gamma_j(t, t_0)$ is a nonincreasing function of time, reads

$$\frac{d}{dt}\gamma_j(t, t_0) \leq 0. \tag{68}$$

To go further, we assume that the rate $\gamma_j(t, t_0)$ is an analytic function of t in the vicinity of the initial time $t = t_0$. This means that the expansion

$$\gamma_j(t, t_0) \simeq \gamma_j \sum_{n=0}^{k} a_n (t - t_0)^n, \tag{69}$$

where $\gamma_j \equiv \gamma_j(t_0, t_0)$ is the spot rate and $a_0 = 1$, is valid for asymptotically small $t - t_0 \to 0$. The upper limit k of the summation can be taken to infinity.

Then, the method of self-similar factor approximants [70–74] mentioned above is used to construct the general class of functions corresponding to the expansion (69). This amounts to extrapolate the asymptotic series (69), valid for small $t - t_0$, to the region of all $t > t_0$. Extrapolating, by means of the self-similar factor approximants [70–74], the asymptotic series (69) under condition (68) gives

$$\gamma_j(t, t_0) = \gamma_j \left(1 + \frac{t - t_0}{t_j}\right)^{-n_j}, \tag{70}$$

where t_j is a time scale and $n_j \geq 0$. We stress the nontrivial nature of the construction of the function (70) by the self-similar factor approximants, which makes appear the exponent n_j. This exponent plays a key role in structuring the form of the discount factor.

4.3 Four classes of discount factors

Four types of discount factors are predicted, corresponding to the four different sets:

(i) $n_j = 0$;
(ii) $0 < n_j < 1$;
(iii) $n_j = 1$; and
(iv) $1 < n_j$.

(i) $n_j = 0$. The discounting function (67) is the simple exponential

$$f_j(t, t_0) = \exp\{-\gamma_j (t - t_0)\}. \tag{71}$$

This type of discount factor is standard in the value-discounting problems. We may notice that reparametrizing equation (71) with the substitution $\delta \equiv \exp(-\gamma_j)$ yields an equivalent expression $f_j(t, t_0) = \delta_j^{t - t_0}$.

(ii) $0 < n_j < 1$. The discounting function (67) takes the form

$$f_j(t, t_0) = \exp\left\{-\frac{\gamma_j t_j}{1 - n_j}\left[\left(1 + \frac{t - t_0}{t_j}\right)^{1 - n_j} - 1\right]\right\}. \tag{72}$$

At short times $t - t_0 < t_j$, the expression $f_j(t, t_0)$ reduces approximately to the pure exponential form. However, for large times, such that $t \gg t_0, t_j$, this $f_j(t, t_0)$ is approximated by the function

$$f_j(t, t_0) \simeq \exp\left\{-\frac{\gamma_j t_j}{1 - n_j}\left(\frac{t}{t_j}\right)^{1 - n_j}\right\}, \tag{73}$$

called the stretched exponential (see, e.g., Chap. 6 of Ref. [76]). Stretched exponential relaxation of a macroscopic variable to an equilibrium is well-known in physics, such as in "complex" fluids [77], glasses [78–82], porous media, semiconductors, etc., a law known under the name Kohlrausch–Williams–Watts law [78,81]. The stretched-exponential decay of the discount factor as a function of time reflects a decay slower than exponential of the time value of future payoffs. An even slower decay is found for the next case.

(iii) $n_j = 1$. The discounting function (67) reads

$$f_j(t, t_0) = \frac{1}{[1 + (t - t_0)/t_j]^{\gamma_j t_j}}. \tag{74}$$

This recovers the postulated form associated with so-called generalized hyperbolic discounting or, simply, hyperbolic discounting function [55–59], which seems to account better for the observed time-preference of human beings than the standard exponential form (71).

(iv) $n_j > 1$. Equation (67) leads to

$$f_j(t, t_0) = \exp\left\{-\frac{\gamma_j t_j}{n_j - 1}\left[1 - \frac{1}{(1 + (t - t_0)/t_j)^{n_j - 1}}\right]\right\}. \tag{75}$$

At short times, $f_j(t, t_0)$ is again well-approximated by an exponential form. However, at large times, when $t - t_0 \gg t_j$, the factor $f_j(t, t_0)$ tends to a non-zero limit

$$\lim_{t \to \infty} f_j(t, t_0) = \exp\left(-\frac{\gamma_j t_j}{n_j - 1}\right). \tag{76}$$

This is in contrast with the previous cases (71) to (74) and with the standard assumption that $f_j(t, t_0)$ tends to zero at large times because individuals do not care for events that are very-very far in the future. This new regime is a priori unexpected and surprising, because it implies that payoffs or costs that are very far in the future still contribute a finite amount to the likelihood of a given prospect. In common terms, according to (75) leading to (76), extremely far ahead outcomes are not discounted to zero, but provide a finite input to the effective utility of the decision maker. While providing perhaps the most dramatic rupture with standard discounting and decision making theory, we believe that the form (75) leading to the bizarre result (76) is actually formalizing an important element of decision making. Specifically, very low to zero discount rates are presently being discussed for analyzing intergenerational public policy choices [83–85]. These policies encompass issues such as global warming and nuclear waste disposal. Nuclear waste disposal, in particular, involves time scales up to millions of years over which mankind will have to continue to monitor and watch the long-lived radionuclides resulting from the burning of nuclear fuel in nuclear plants. The ongoing challenge is to characterize distant future costs or benefits in a way that is relevant for policy makers, who must evaluate trade-offs today.

4.4 Prospect-dependent discount rates

In full generality, different intended actions can be characterized by different discount functions. Even if, for simplicity, the same discount function is employed, then different actions can have different decay rates γ_j or different time scales t_j. This can lead to a reversal of natural preferences.

For example, let a prospect π_1 be preferred to π_2, if they are realized at the initial time t_0, so that for their probabilities the following inequality holds:

$$\frac{p_1(t_0,t_0)}{p_2(t_0,t_0)} > 1. \tag{77}$$

But, if these prospects are planned to be realized at a later time t, then their probabilities form the ratio

$$\frac{p_1(t,t_0)}{p_2(t,t_0)} = \frac{p_1(t_0,t_0)f_1(t,t_0)}{p_2(t_0,t_0)f_2(t,t_0)}. \tag{78}$$

It may happen that at some moment of time t_{rev}, their probabilities reverse, so that for $t > t_{rev}$,

$$\frac{p_1(t,t_0)}{p_2(t,t_0)} < 1 \qquad (t > t_{rev}), \tag{79}$$

which implies preference reversal. This phenomenon, known as "time inconsistency" in the literature, is usually associated with non-exponential discount factors. It may also occur with exponential discount factors, when the discount rate is different from the risk-adjusted return on saving (see, e.g., Chap. 15 in Ref. [6]). Within QDT, time reversal can also occur for the exponential discount factor when the discount rates of two prospects are different. The reversal time in the case of the exponential discounting (71) is

$$t_{rev} = t_0 + \frac{1}{\gamma_1-\gamma_2}\ln\frac{p_1(t_0,t_0)}{p_2(t_0,t_0)}, \tag{80}$$

which exists for $\gamma_1 > \gamma_2$. In the case of the hyperbolic discounting (74), with $\gamma_j t_j = 1$, the reversal time reads as

$$t_{rev} = t_0 + \frac{p_1(t_0,t_0)-p_2(t_0,t_0)}{\gamma_1 p_2(t_0,t_0)-\gamma_2 p_1(t_0,t_0)}, \tag{81}$$

which exists under the condition

$$\frac{\gamma_1}{\gamma_2} > \frac{p_1(t_0,t_0)}{p_2(t_0,t_0)}. \tag{82}$$

Recall that the initial time t_0 corresponds to the planning time when the decision maker evaluates a prospect that is assumed to be realized at the point in time $t \geq t_0$. Thus, the planning time t_0 is also a variable, which shifts when the decision-maker re-evaluates his/her plans. As a consequence, there is no preference-reversal paradox within QDT, as explained in Section 3.3.3.

5 Conclusion

We have presented a novel approach to decision making, based on the mathematical techniques of complex Hilbert spaces over a lattice of composite prospects. Such techniques are typical for the theory of quantum measurements, which explains the name "Quantum Decision Theory" (QDT). We stress that this does not presuppose that decision makers are assumed to be quantum objects. The employed mathematical methods are just the most convenient tool for taking into account such notions as risk and uncertainty which have strong emotional effects in decision making. QDT makes it possible to explain the paradoxes appearing in the application of classical utility theory to decision making. In the present paper, we have analyzed the stylized effects and paradoxes, associated with dynamic aspects of decision theory, such as time inconsistency, planning paradox, value discounting, event uncertainty, and preference reversal. These temporal effects have not been considered in our previous articles on QDT [16,17,28] and the treatment offered here is original. We have also suggested a constructive approach for deriving the evolution equations for the prospect probabilities. The derived discount functions provide a novel classification of possible discount factors, which include the previously known cases (exponential or hyperbolic discounting), but also predicts a novel class of discount factors that can be applied for very long-term discounting situations.

One of the basic conclusions of QDT is the necessity of taking into account not merely the utility of the considered prospects, as in classical utility theory, but also the attractiveness of the related alternatives. This is accounted for by the attraction factor, whose appearance is due to the use of the quantum techniques. The attraction factor characterizes the level of attractiveness of each prospect with regard to the risk and uncertainty associated with the choice among the related alternatives. In that way, the attraction factor is a new measure of risk in decision making. Mathematically, its appearance is caused by the use of quantum rules in defining the prospect probabilities. And its meaning is the characterization of the perceived level of risk associated with emotions and subconscious processes that influence decision making. In brief, we can say that the physics of risk in decision making, described by the attraction factor, embodies the existence of subconscious feelings, emotions, and biases.

The notion of risk is met in many applications, such as economics, finance, psychology, and so on. In all these applications, it is always connected with the process of taking decisions. Therefore, to elucidate the physics of risk, one needs, first of all, to understand its meaning in decision making. Without such an understanding, it is impossible to properly employ this notion in applications to other fields. As we have shown, the evaluation of risk presents two sides. In addition to a first contribution, measured, e.g., through the risk-aversion coefficients [4,5] and the lottery dispersion, it is necessary to take into account its subjective part caused by emotions. A principal result of our theory is that, despite the subjectivity of the emotional side of risk, it is possible to naturally take it into account in a logical and mathematically self-consistent way. Our QDT is the first mathematically rigorous realization of the old Bohr idea [86] that mental human processes can be described by techniques of quantum theory.

Obviously, taking correct decisions is of paramount importance. This is why the developed theory can find numerous applications. Several illustrations have been analyzed in the present paper. We have concentrated our attention here on temporal effects, related to time inconsistency, which have not been considered in our previous articles.

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