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PI SEMIGROUP ALGEBRAS OF LINEAR SEMIGROUPS

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ABSTRACT. It is well-known that if a semigroup algebra $K[S]$ over a field K satisfies a polynomial identity then the semigroup S has the permutation property. The converse is not true in general even when S is a group. In this paper we consider linear semigroups $S \subseteq \mathscr{M}_n(F)$ having the permutation property. We show then that $K[S]$ has a polynomial identity of degree bounded by a fixed function of n and the number of irreducible components of the Zariski closure of S.

A semigroup S is said to have the property $\mathscr{P}_m$, $m \geq 2$, if for every $a_1, \dots, a_m \in S$, there exists a non-trivial permutation σ such that $a_1 \dots a_m = a_{\sigma(1)} \dots a_{\sigma(m)}$. S has the permutation property $\mathscr{P}$ if S satisfies $\mathscr{P}_m$ for some $m \geq 2$. The class of groups of this type was shown in [3] to consist exactly of the finite-by-abelian-by-finite groups. For the recent results and references on this extensively studied class of groups, we refer to [1]. The above description of groups satisfying $\mathscr{P}$ was extended to cancellative semigroups in [11], while a study of regular semigroups with this property was begun in [6].

In connection with the corresponding semigroup algebras $K[S]$ over a field K, the problem of the relation between the property $\mathscr{P}$ for S and the PI-property for $K[S]$ attracted the attention of several authors. It is straightforward that S has $\mathscr{P}$ whenever $K[S]$ satisfies a polynomial identity. However the converse fails even for groups in view of [3] and the characterization of PI group algebras, cf. [15]. On the other hand, $K[S]$ was shown to be a PI-algebra whenever S is a finitely generated semigroup (satisfying $\mathscr{P}$) of one of the following types: periodic [20], cancellative [11], 0-simple [3, 5], inverse, or a Rees factor semigroup of free semigroup, cf. [12]. However, a finitely generated regular semigroup S with two non-zero $\mathscr{J}$-classes having $\mathscr{P}$ but with $K[S]$ not being PI was constructed in [12].

The main result of this paper is that if S is a linear semigroup satisfying $\mathscr{P}$, then $K[S]$ is PI for any field K. In the course of the proof, we obtain a structural description of a strongly π-regular semigroup of this type. The basic technique is to consider the Zariski closure $\overline{S}$ of S. Then $\overline{S}$ is a linear

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algebraic semigroup in the sense of [18].

We refer to [2] for basics of semigroup theory. In particular $\mathfrak{M}^\circ(H, I, M, P)$ denotes the completely 0-simple semigroup over a group H with sandwich matrix $P : M \times I \to H \cup \{0\}$. Let S be a semigroup. If $X \subseteq S$, then we let $E(X) = \{e \in X \mid e^2 = e\}$. S is *strongly* π*-regular* if some power of each element lies in a subgroup of S. Let F be an algebraically closed field. We consider $\mathscr{M}_n(F)$ with its Zariski topology, [22]. A subset X of $\mathscr{M}_n(F)$ is *closed* if it is the zero set of a collection of polynomials in n^2 variables. A closed set X is *irreducible* if it is not a union of two proper closed subsets. A closed subsemigroup S of $\mathscr{M}_n(F)$ is called a (*linear*) *algebraic semigroup*. If the underlying closed set S is irreducible, then S is said to be a *connected semigroup*. We refer to [18] for the theory of linear algebraic semigroups. In particular, an algebraic semigroup S is strongly π-regular [18, Theorem 3.18]. Our starting point is the following observation.

Lemma 1. *If* $S \subseteq \mathscr{M}_n(F)$ *is a semigroup having* $\mathscr{P}_m$ *then so does its Zariski closure* $\overline{S}$.

Proof. For a permutation $\sigma \neq 1$ of $1, \dots, m$, and $i = 1, \dots, m$, let

$$X_i(\sigma) = \{a_i \in \overline{S} \mid a_1 \dots a_m = a_{\sigma(1)} \dots a_{\sigma(m)} \text{ for all } a_j \in \overline{S}\ (j < i) \text{ and } a_k \in S\ (k > i)\}.$$

Then $Y_i = \bigcup_\sigma X_i(\sigma)$ is a closed subset of $\overline{S}$. Since $S \subseteq Y_1 \subseteq \overline{S}$, we see that $Y_1 = \overline{S}$. If $Y_i = \overline{S}$, then $S \subseteq Y_{i+1} \subseteq \overline{S}$. Hence $Y_{i+1} = \overline{S}$. It follows that $Y_m = \overline{S}$. So $\overline{S}$ has $\mathscr{P}_m$.

The next two lemmas will be used in obtaining a bound on the degree of the polynomial identity satisfied by the relevant semigroup algebras.

Lemma 2. *Let* $S \subseteq \mathscr{M}_n(F)$ *be a strongly* π*-regular semigroup. Then for* $i = 0, \cdots, n$,

$$S_i = \{a \in S \mid \textit{rank of } a \le i\}$$
$$T_i = \{a \in S_i \mid \textit{rank of } a < i \textit{ or } a \textit{ is not regular}\}$$

are (*possibly empty*) *ideals of* S *such that*

$$S_0 \subseteq T_1 \subseteq S_1 \subseteq T_2 \subseteq \cdots \subseteq S_n = S.$$

Moreover S_i/T_i *is a zero disjoint union of at most* $\binom{n}{i}$ *completely 0-simple ideals and* T_i/S_{i-1} *is nilpotent of index at most* $\binom{n}{i}$.

Proof. That S_i is an ideal of S is obvious. Let $a \in T_i$, $x \in S$. Suppose $ax \notin T_i$. Then ax has rank i and is regular. So for some $y \in S$, $e = axy$ is an idempotent of rank i. Hence, [17, Lemma 4], $a = ea = axya$ is regular, a contradiction. Thus each T_i is an ideal of S. The rest follows from [9].

We refer to [8] for the basics on linear algebraic groups.

Lemma 3. *Let S be a linear algebraic semigroup with m irreducible components. Then*

(1) *If G is a maximal subgroup of S with identity component G^c, then $|G/G^c| \leq m$.*

(2) *If J is a regular $\mathcal{J}$-class of S, then $E(J)$ is a closed subset of S with at most m^2 irreducible components.*

Proof. (1) Let e denote the identity element of G. Now eSe is the image of S under the morphism $x \to exe$. Hence eSe has at most m irreducible components. Since G is an open subset of eSe, the same is true of G.

(2) That $E(J)$ is a closed subset of S follows from the proof of [16, Theorem 8]. Let $e \in E(J)$ with $\mathcal{H}$-class H. Let

$$U = \{(a, b) \mid a, b \in S, \ ebae \in H\}.$$

Then U is a non-empty open subset of $S \times S$ and by the proof of [16, Theorem 8], the map:

$$(a, b) \to a(ebae)^{-1}b$$

(where the inverse is taken in H) is a morphism from U onto $E(J)$. Since $S \times S$ has m^2 irreducible components, U (being an open subset) has at most m^2 irreducible components. Hence $E(J)$, being an image of U, has at most m^2 irreducible components.

Lemma 4. *Let S be a linear algebraic semigroup having $\mathcal{P}_m$. Then*

(1) *If G is a maximal subgroup of S, then the identity component G^c of G is abelian.*

(2) *If J is a regular $\mathcal{J}$-class of S, then every irreducible component A of $E(J)$ is a rectangular band, i.e. $efe = e$ for all $e, f \in A$.*

Proof. (1) G^c has $\mathcal{P}_m$ and hence, [3], has a normal subgroup H of finite index such that the commutator subgroup (H, H) is finite. By [8, Proposition 7.3], G^c has no closed subgroups of finite index. Hence H is dense in G^c. We have a morphism $\phi : G^c \times G^c \to G^c$ given by $\phi(x, y) = xyx^{-1}y^{-1}$. Now $G^c \times G^c$ is irreducible, $H \times H$ is dense in $G^c \times G^c$ and $\phi(H \times H)$ is finite, so $\phi(G^c \times G^c) = \{1\}$. Hence G^c is abelian.

(2) For every positive integer i, $\overline{A^i}$ is a closed irreducible set (being the closure of an image of $A \times \cdots \times A$). Hence we have an ascending chain of closed irreducible sets:

$$A \subseteq \overline{A^2} \subseteq \overline{A^3} \subseteq \cdots.$$

Comparison of dimensions shows that this series stabilizes. So for some i, $T = \overline{A^i} = \overline{A^j}$ for all $j > i$. Now $\bigcup_{k \geq 1} A^k$ is a semigroup with closure T. Hence T is a connected semigroup satisfying $\mathcal{P}_m$. Now the irreducible set $T \times \cdots \times T$ (m times) is a finite union of the closed sets

$$B_\sigma = \{(a_1, \dots, a_m) \mid a_1, \dots, a_m \in T, \ a_1 \cdots a_m = a_{\sigma(1)} \cdots a_{\sigma(m)}\}, \ \sigma \neq 1.$$

Hence $T \times \cdots \times T = B_\sigma$ for some $\sigma \neq 1$. Thus T satisfies a fixed permutation identity. Therefore by [19, Theorem 1], T^k satisfies the identity $xyzw = xzyw$ for some k. It follows that $E(T)$ is a subsemigroup of T. Hence, if $J_1, \ldots, J_t$ are the regular $\mathscr{J}$-classes of T, then by [18, Theorem 5.9], each $E(J_r)$ is a rectangular band. Also by [16, Theorem 8], each $E(J_r)$ is closed and irreducible. Since $A \subseteq E(T) = \bigcup E(J_r)$, we see that $A \subseteq E(J_p) \subseteq E(J)$ for some p. So $A = E(J_p)$ is a rectangular band.

We now proceed to obtain some information about rectangular bands in complete 0-simple semigroups. First recall that by [2, Chapter 1], a rectangular band is a direct product of a right zero semigroup and a left zero semigroup. In particular it satisfies the identity $xyzw = xzyw$.

Lemma 5. *Let $S = \mathfrak{M}^0(G, I, M, P)$ be a completely 0-simple semigroup and $E \subseteq S$ be a rectangular band. Then*

(1) *For some $I_1 \subseteq I$, $M_1 \subseteq M$ and Q the $M_1 \times I_1$ matrix induced by P, E is the idempotent set of $\mathfrak{M}(G, I_1, M_1, Q)$.*

(2) *Any two columns of Q are G-proportional, i.e. for every $i, j \in I_1$, there exists $g \in G$ such that $p_{mi} = p_{mj} g$ for all $m \in M_1$.*

Proof. (1) We can let $I_1 = \{i \in I \mid (g, i, m) \in E$ for some $g \in G,\ m \in M\}$ and let $M_1 = \{m \in M \mid (g, i, m) \in E$ for some $g \in G,\ i \in I\}$.

(2) Let $i, j \in I_1$, $n \in M_1$, $g = p_{nj}^{-1} p_{ni} \in G$. Let $m \in M_1$. Then $e = (p_{mi}^{-1}, i, m)$, $f = (p_{nj}^{-1}, j, n) \in E$. Since $efe = e$ we obtain

$$p_{mi}^{-1} p_{mj} p_{nj}^{-1} p_{ni} p_{mi}^{-1} = p_{mi}^{-1}.$$

Hence $p_{mi} = p_{mj} g$.

Lemma 6. *Let $S = \mathfrak{M}(G, I, M, P)$ be a completely simple semigroup over an abelian-by-finite group G such that $E(S)$ is a rectangular band. Then for every field K, the semigroup algebra $K[S]$ satisfies a polynomial identity.*

Proof. It is well known [2, §3.2, Exercise 2] that S is a direct product $G \times E(S)$. Therefore $K[S] \simeq K[G] \otimes_K K[E(S)]$. Now $K[G]$ is a PI-algebra, cf. [15, Theorem 5.3.7]. Also $K[E(S)]$ is a PI-algebra since $E(S)$ satisfies the multilinear identity $xyzw = xzyw$. Hence by [21, Theorem 6.1.1], $K[S]$ is a PI-algebra.

In view of Lemma 5, Lemma 6 can also be derived from a criterion for the PI-property of semigroup algebras of completely 0-simple semigroups, cf. [10].

Finally we need the following lemma concerning the rank of (possibly infinite) matrices. Here the rank $\mathrm{rk}(P)$ of an infinite matrix P is defined to be the supremum of the ranks of all the finite submatrices of P.

Lemma 7. *Let M, I be non-empty sets, and let P be an $M \times I$ matrix over a field L. Assume that P can be covered by $k < \infty$ submatrices $P_1, \ldots, P_k$ such that $\mathrm{rk}(P_i) \leq t < \infty$ for some t. (That is, for every $m \in M$, $i \in I$, the (m, i) entry of P lies in some P_j if it is non-zero.) Then $\mathrm{rk}(P) \leq (2^k - 1)t$.*

Proof. Induction on k. The case where $k = 1$ is clear. Let $k > 1$. We can assume that P_1 does not pass through all rows of P or P_1 does not pass through all columns of P. By symmetry we consider the former case only. Define A as the submatrix of P consisting of all the entries lying in the rows of P through which P_1 does not pass. It is clear that A can be covered by at most $k-1$ submatrices of ranks not exceeding t. By the induction hypothesis $\mathrm{rk}(A) \leq (2^{k-1}-1)t$. Since $\mathrm{rk}(P_1) \leq t$, then clearly $\mathrm{rk}(B) \leq t + (2^{k-1}-1)t = 2^{k-1}t$ where B is the submatrix of P consisting of all columns passing through P_1. If $B \neq P$, then the submatrix C of P consisting of all columns not passing through B also satisfies the induction hypothesis. Therefore $\mathrm{rk}(C) \leq (2^{k-1}-1)t$, and so

$$\mathrm{rk}(P) \leq \mathrm{rk}(B) + \mathrm{rk}(C) \leq 2^{k-1}t + (2^{k-1}-1)t = (2^k - 1)t$$

proving the assertion.

We are now ready to prove our first theorem concerning strongly π-regular linear semigroups. This class includes all linear algebraic semigroups and also all regular linear semigroups. In view of Lemma 1, this result characterizes a linear semigroup S with the permutation property in terms of its closure $\overline{S}$.

Theorem 1. *Let $S \subseteq \mathscr{M}_n(F)$ be a strongly π-regular semigroup. Then the following conditions are equivalent*:

(1) *S has the permutation property.*

(2) *$E(S)$ is a finite union of rectangular bands and every subgroup of S is abelian-by-finite.*

(3) *$K[S]$ is a PI-algebra for some field K.*

(4) *$K[S]$ is a PI-algebra for every field K.*

Proof. That (4) $\Rightarrow$ (3) $\Rightarrow$ (1) is obvious. That (1) $\Rightarrow$ (2) follows from Lemmas 1, 3 and 4. So we need to prove that (2) $\Rightarrow$ (4). As is well known it suffices to consider the case when K is of characteristic 0. If J is an ideal of S, then the contracted semigroup algebra $K_0[S/J] \simeq K[S]/K[J]$. Moreover the class of PI-algebras is closed under ideal extensions. Hence by Lemma 2, it suffices to show that $K[T]$ is a PI-algebra for every completely 0-simple principal factor T of S.

By hypothesis $T \simeq \mathfrak{M}^\circ(G, I, M, P)$ for some abelian-by-finite group G, and $E(T)\setminus\{0\} = E_1 \cup \cdots \cup E_r$ where each E_i is a rectangular band. By Lemma 5, we can construct subsemigroups $T_j = \mathfrak{M}(G, I_j, M_j, P_j)$ with $E(T_j) = E_j$, $j = 1, \ldots, r$. By Lemma 6, each $K[T_j]$ is a PI-algebra. We now use the following characterization of PI semigroup algebras of completely 0-simple semigroups in characteristic zero, cf. [4].

(*) $K[T]$ is PI if and only if G is abelian-by-finite and there exists $q \geq 1$ such that $\mathrm{rk}\,\phi(P) \leq q$ for all irreducible representations ϕ of G over fields of characteristic zero.

Moreover, in this case, $K_0[T]$ satisfies the identity

$$x_{2q+1}S_{2q}(x_1, \dots, x_{2q})x_{2q+2} = 0$$

where S_{2q} is the standard identity of degree $2q$. Now (*) applied to T_j, $j = 1, \dots, r$ implies that there is a positive integer t such that $\operatorname{rk}\phi(P_j) \leq t$ for all j and all irreducible representations of G. Let $a = (g, i, m) \in T$ such that $a \notin T_j$, $j = 1, \dots, r$. Then $a^2 = 0$, so that $p_{mi} = 0$. It is thus clear that P is covered by $P_1, \dots, P_r$ in the sense of Lemma 7. Hence by Lemma 7, $\operatorname{rk}\phi(P) \leq (2^r - 1)t$. Therefore by (*), $K[T]$ is a PI-algebra. This proves the theorem.

Remark. Let S be a strongly π-regular linear semigroup with the permutation property and let T be a completely 0-simple principal factor of S. Then by Theorem 1, $E(T)\backslash\{0\}$ is a union of finitely many rectangular bands $E_1, \dots, E_r$. We can then refine this union to write E as a disjoint union of finitely many rectangular bands. This is a consequence of the fact that $E_i \cap E_j$ is a rectangular band and $E_i \backslash (E_i \cap E_j)$ is a disjoint union of at most 3 rectangular bands. If $T = \mathfrak{M}^0(G, I, M, P)$, then it follows that I, M can be partitioned as $I = I_1 \cup \cdots \cup I_p$, $M = M_1 \cup \cdots \cup M_s$ so that any two columns of the submatrix P_{ij} of P corresponding to the semigroup $T_{ij} = \mathfrak{M}^0(G, I_j, M_i, P_{ij}) \subseteq T$ are G-proportional. Thus T is the 0-disjoint union of the semigroups T_{ij} and each T_{ij} is either null or $T_{ij}\backslash\{0\}$ is a direct product of the abelian-by-finite group G and a rectangular band.

Theorem 2. *Let $S \subseteq \mathscr{M}_n(F)$ be a semigroup with the permutation property. Let m denote the number of irreducible components of $\overline{S}$. Then for any field K, $K[S]$ satisfies a polynomial identity of degree $3^n \cdot \prod_{j=1}^{n} \binom{n}{j} \cdot [2m(2^{m^2} - 1) + 2]$.*

Proof. We can assume that $\operatorname{ch}(K) = 0$. By Lemma 1, we can assume without loss of generality that $S = \overline{S}$ is an algebraic semigroup. In particular S is a strongly π-regular semigroup and by Theorem 1, $K[S]$ is a PI-algebra. Let S_i, T_i, $i = 0, \cdots, n$, be as in Lemma 2. Now S_i/T_i is a zero disjoint union of completely 0-simple ideals. Hence $K_0[S_i/T_i]$ is a direct product of the corresponding contracted semigroup algebras. Therefore by [14, Proposition 1.1], $J(K_0[S_i/T_i])^3 = 0$, where J denotes the Jacobson radical. Hence by Lemma 2, $J(K[S])^r = 0$ where $r = 3^n \cdot \prod_{j=1}^{n} \binom{n}{j}$.

Let R be a simple homomorphic image of $K[S]$. Then R is an image of the semigroup algebra $K_0[T]$ of some completely 0-simple principal factor T of S. By (*) in the proof of Theorem 1, $K_0[T]$ satisfies the identity

$$x_{2p+1}S_{2p}(x_1, \dots, x_{2p})x_{2p+2} = 0$$

for all $p \geq q$, where

$$q = \max_{\phi} \operatorname{rk}\phi(P)$$

as ϕ runs through all irreducible representations (in characteristic zero) of the maximal subgroup G of T. By Lemmas 3, 4, 7, we see as in the proof of Theorem 1 that

$$q \leq (2^{m^2} - 1) \max_{\phi, Q} \{\operatorname{rk} \phi(Q)\}$$

where Q runs through the submatrices of P corresponding to the at most m^2 rectangular bands covering $E(T)$. By Lemma 5, the columns of each Q are G-proportional. Hence $\operatorname{rk}\phi(Q)$ does not exceed the dimension of ϕ. Now by Lemmas 3, 4, G^c is a normal abelian subgroup of G of index $\leq m$. Hence by a standard argument, cf. [15, §5.1], $\dim \phi \leq m$. Therefore $q \leq m(2^{m^2} - 1)$. It follows that every simple homomorphic image of $K[S]$ satisfies a fixed identity of degree $2m(2^{m^2} - 1) + 2$. Since $K[S]$ is a PI-algebra, we see that $K[S]/J(K[S])$ satisfies the same identity. Since $J(K[S])$ is nilpotent of index $r = 3^n \cdot \prod_{j=1}^{n} \binom{n}{j}$, $K[S]$ satisfies the identity

$$[x_{2p+1} S_{2p}(x_1, \dots, x_{2p}) x_{2p+2}]^r = 0$$

where $p = m(2^{m^2} - 1)$.

An essentially weaker result than our Theorem 1 was asserted in [7]. Namely, for every finitely generated and regular $S \subseteq \mathscr{M}_n(F)$ satisfying the permutation property, the semigroup algebra $K[S]$ satisfies a polynomial identity. However, in the proof, the authors claimed that every principal factor of such a semigroup S must be finitely generated. We show that this may not be true in general.

Example. Let F be a field with an element $\alpha \in F$ generating an infinite cyclic subgroup in the multiplicative group F^*. Define S as the subsemigroup generated in $\mathscr{M}_2(F)$ by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $I = \left\{ \begin{pmatrix} \alpha^k & \alpha^n \\ 0 & 0 \end{pmatrix} \middle| \, k, n \in Z \right\}$ is an ideal of S and S/I is an infinite cyclic group G with zero. It is easy to see that $I \simeq \mathfrak{M}(G, 1, Z, P)$ where $p_{m1} = 1$ for all $m \in Z$. Clearly, S is regular with infinitely many $\mathscr{H}$-classes, but $K[S]$ is a PI-algebra by Theorem 1.

Our last aim is to extend Theorem 2 to an important class of non-linear semigroups.

Proposition. *Let S be a semigroup satisfying the permutation property. If $K[S]$ is right noetherian for a field K, then $K[S]$ is a PI-algebra.*

Proof. Let Q be a prime ideal of $K[S]$, and let S_Q denote the image of S in $K[S]/Q$. S_Q has an ideal $J \neq 0$ contained in a semigroup of matrix type $\mathfrak{M}^0(G, I, M, P)$ for a finite set I, cf. [3, the proof of Theorem III.1.6]. If R

is a maximal cancellative subsemigroup of J, then from [13, Lemma II.1.21] it follows that the group ring $K[RR^{-1}]$ is noetherian. $K[RR^{-1}]$ is then a PI-algebra, cf. [13, Section IV.2]. As in [10, Proposition 3] one can show that $K[J]$ is a PI-algebra. Since the image of $K[J]$ in $K[S]/Q$ is a nonzero ideal, the latter also is a PI-algebra. Therefore $K[S]/B(K[S])$ is a PI-algebra where $B(K[S])$ denotes the prime radical of $K[S]$. Since $B(K[S])$ is nilpotent, the result follows.

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