PICARD-LEFSCHETZ THEOREM FOR FAMILIES OF NONSINGULAR ALGEBRAIC VARIETIES ACQUIRING ORDINARY SINGULARITIES

BY

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1. **Introduction.** The purpose of this paper is to expand the classical results of Picard [6] and Lefschetz [3, p. 23] concerning a one-complex-parameter family of algebraic varieties S_t acquiring singularities for isolated values of t. The question to be answered is: What action on $H_*(S_t)$ is induced by motion of the parameter t? ($H_*(\ )$ will denote (compact) singular homology with real coefficients.)

More precisely, let B be a complex analytic line bundle over a nonsingular complex projective variety X of (complex) dimension $(n+1)$. Let s_0 and s_∞ be any two linearly independent holomorphic sections of $B \to X$ such that $Z(s_\infty) \cap Z(ds_\infty) = \varnothing$. ($Z(\ )$ will indicate "zeros of ( ) in X.") Let $S_t = Z(s_0 + ts_\infty)$ and $S_\infty = Z(s_\infty)$. Then (s_∞/s_0) is a meromorphic and hence algebraic function on X, and so S_t is a projective hypersurface of X for all t. Since S_∞ is assumed nonsingular, standard arguments show that

$$C_0 = \{t \in C : Z(s_0 + ts_\infty) \cap Z(ds_0 + t\, ds_\infty) \neq \varnothing\}$$

is finite.

For $t \in (C - C_0)$, S_t has a C^∞-normal bundle W in X whose fibres are ordinary two-real-dimensional discs. Let the fibration be given by $\omega: W \to S_t$. For t' near t, $S_{t'}$ is also nonsingular, and for each $P \in S_t$ the set $(\omega^{-1}(P) \cap S_{t'})$ contains exactly one point. Thus ω defines a diffeomorphism $F(t, t'): S_{t'} \to S_t$. Given any path γ in $(C - C_0)$, beginning at t_0 and ending at t, we can define an associated diffeomorphism $F(\gamma) = F(t_k, t_{k-1}) \circ \cdots \circ F(t_2, t_1) \circ F(t_1, t_0)$ where $\{t_0, t_1, \ldots, t_k = t\}$ is a sufficiently fine partition of γ. The homotopy type of $F(\gamma)$ does not depend on the choice of normal bundles [4, pp. 19–26], nor on the partition if it is sufficiently fine, but only on the homotopy class of γ in $(C - C_0)$.

One asks, then, what is the associated homology isomorphism $F(\gamma)_*: H_*(S_{t_0}) \to H_*(S_t)$? From the discussion above, it clearly suffices to consider the case $0 \in C_0$, $\{t : 0 < |t| < 2\} \subseteq (C - C_0)$, and $\gamma = \{\exp \xi : 0 \leqq \xi \leqq \theta\}$ where $\exp \xi = e^{2\pi i \xi}$. For such situations, define $F(\theta) = F(\gamma)$. In [3, p. 23], Lefschetz describes $F(q)_*: H_*(S_1) \to H_*(S_1)$ for $q \in Z$, when S_0 has isolated double points as its singular locus. This result is variously called the Picard-Lefschetz Theorem or the Poincaré formula. Pham [5] extended these results to much more general types of isolated singularities. In the following we will generalize in another direction and allow $\{S_t\}$ to acquire an

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arbitrary ordinary singularity at S_0. Our general philosophy will be to fibre the action of $F(\theta)$ over the singular locus of S_0. (This approach also yields results in some simple cases of nonordinary, nonisolated singularities [1].) We will assume throughout that $\{0<|t|<2\}\subseteq(C-C_0)$.

Theorem 4.4 of this paper will give, from the topological point of view, an appropriate generalization of the classical Picard-Lefschetz Theorem to the case in which we acquire an arbitrary singularity (with arbitrary multiplicities) at S_0. A. Landman, in his Berkeley thesis, treated this same problem from an algebraic point of view. To see the relation between these two approaches, we might note here a corollary of the homological dimensions of the subspaces of S_1 in Theorem 4.4 and Corollary 4.15:

For $0\leqq a\leqq 2n$ and $m=$least common multiple of the multiplicities of the components of S_0 and

$$f_a = F(1)_*\colon H_a(S_1)\to H_a(S_1),$$

the minimal polynomial of the linear transformation f_a is a factor of the polynomial

$$(\lambda^m-1)^{n+1-|a-n|}.$$

2. **Ordinary singularities and the fibration of** $F(\theta)$. For notational convenience, we will set $S_\infty=A(0)$. Then

DEFINITION 2.1. The family $\{S_t\}$ will be said to acquire an ordinary singularity at $t=0$ if:

(1) $S_0=A(1)\cup\cdots\cup A(h)$ where the $A(i)$ are nonsingular hypersurfaces of X;

(2) for $0\leqq i<j\leqq h$, $A(i)$ and $A(j)$ intersect normally (including the possibility that $A(i)\cap A(j)=\varnothing$).

From now on we assume that $\{S_t\}$ is as in §1 and acquires an ordinary singularity at $t=0$.

For any open set $U\subseteq X$, let $I(U)=\{i\in\{0,\ldots,h\} : U\cap A(i)\neq\varnothing\}$. Then locally we can normalize our defining equation $s_0+ts_\infty=0$ for S_t:

LEMMA 2.2. *There is an open covering $\{U\}$ of $S_0\cup S_\infty=A(0)\cup\cdots\cup A(h)$ in X and complex analytic functions $x_i\colon U\to C$ for each $U\in\{U\}$, $i\in I(U)$ such that*

(1) *$(x_i)_{i\in I(U)}$ are part of an analytic coordinate system for U in X;*

(2) *for each $U\in\{U\}$, $Z(x_i)=A(i)\cap U$;*

(3) *$-((s_0/s_\infty)+t)=(\pi\{x_i^{m(i)} : i\in I(U)\}-t)$ on $U\in\{U\}$.*

The $m(i)$ are integers whose values depend only on i and not on the open set U. $m(0)=-1$, and $m(i)$ is a positive integer for $i>0$ and is called the multiplicity of $A(i)$ at $t=0$.

Proof. Theory of functions of several complex variables. See [2]. ■

DEFINITION 2.3. A finite open covering $\mathscr{V}$ of $S_0\cup S_\infty$ in X will be called a regular covering if

(1) each $V\in\mathscr{V}$ is a simply connected subset of a coordinate disc in X;

(2) Lemma 2.2 holds for $\{U\}=\mathscr{V}$;

(3) for $V \in \mathscr{V}, j \notin I(V)$, $\bar{V} \cap A(j) = \varnothing$;

(4) for $V, V' \in \mathscr{V}$ with $V \cap V' \neq \varnothing$, $V \cap V'$ is simply connected.

Next define $A(I) = \bigcap \{A(i) : i \in I\}$ for $I \subseteq \{0, \ldots, h\}$ and

$$B(q) = \bigcup \{A(I) : |I| = q\}.$$

THEOREM 2.4. *There exists a regular covering* $\mathscr{V}$ *of* $S_0 \cup S_\infty = B(1)$ *and* C^∞-*mappings*:

$$y(i, V): (A(I(V)) \cap V) \times V \to \boldsymbol{C} \quad \textit{for } i \in I(V),$$
$$z(V): (A(I(V)) \cap V) \times V \to \boldsymbol{C}^{(n+1-|I(V)|)}$$

for each $V \in \mathscr{V}$ *such that*:

(1) $((y(i, V)(P, \quad))_{i \in I(V)}, z(V)(P, \quad))$ *forms a complex analytic coordinate system for V centered at P for each* $P \in (A(I(V)) \cap V)$;

(2) *on* $V \in \mathscr{V}$, $-(s_0/s_\infty + t) = (\pi\{y(i, V)(P, \quad)^{m(i)} : i \in I(V)\} - t)$;

(3) *for* $V \cap V' \neq \varnothing$ *and* $I(V) = I(V')$:

(a) $(y(i, V)(P_0, Q)/y(i, V')(P_0, Q))$ *is a constant with absolute value* 1 *for each fixed* $P_0 \in V \cap V' \cap A(I(V))$, $i \in I(V)$,

(b) $Z(z(V)(P_0, Q)) = Z(z(V')(P_0, Q))$ *for each* P_0;

(4) *for* $V \cap V' \neq \varnothing$ *and* $I(V) \subseteq I(V')$:

(a) *for* $i \in I(V)$, $(y(i, V)(Q_0, Q)/y(i, V')(P_0, Q))$ *is a nonzero constant for* $z(V)(Q_0, Q) = 0$ *for all* P_0, Q_0 *fixed such that* $Q_0 \in V \cap V' \cap A(I(V))$, $P_0 \in V' \cap A(I(V'))$, *and* $z(V')(P_0, Q_0) = 0$,

(b) $$Z(z(V)(Q_0, Q)) = Z(z(V')(P_0, Q)) \cap \bigcap \{Z(y(i, V')(P_0, Q) - y(i, V')(P_0, Q_0)) : i \in I(V') - I(V)\}$$

for all P_0, Q_0 *as in* (a);

(5) *for all* $V \in \mathscr{V}$, *V is the disjoint union of* $Z(z(V)(P, Q))$ *over* $P \in (A(I(V)) \cap V)$.

Proof. Starting from $B(n+2) = \varnothing$, we shall assume that we have a finite cover $\mathscr{V}_{q+1}$ of $B(q+1)$ which satisfies (2.3), (1)–(4), and (2.4), (1)–(5), and shall construct a cover $\mathscr{V}_q$ with the corresponding properties over $B(q)$. Since $B(q) - \bigcup \{V : V \in \mathscr{V}_{q+1}\}$ is a disjoint union of subspaces, one for each I with $|I| = q$, it will clearly suffice to demonstrate the construction of the cover for $A(I) \cup B(q+1)$ for some I with $|I| = q$. Thus let $\mathscr{V}'$ be a finite open covering of the compact set $A(I) - \bigcup \{V : V \in \mathscr{V}_{q+1}\}$ in X. By refining $\mathscr{V}_{q+1}$ if necessary we can assume that

(1) $I(V) = I$ for all $V \in \mathscr{V}'$ and $|I(V)| > q$ for all $V \in \mathscr{V}_{q+1}$,

(2) $\mathscr{V}' \cup \mathscr{V}_{q+1}$ satisfies (2.3), (1)–(4) as a finite open covering of $A(I) \cup B(q+1)$ in X. (For (2.3)(4), use a covering of X by open stars with respect to a sufficiently fine triangulation.)

Let $\mathscr{V}'' = \{V \in \mathscr{V}_{q+1} : I \subseteq I(V)\}$. For $V \in \mathscr{V}'$, let $(x(i, V))_{i \in I}$ be the coordinates as in Lemma 2.2. Let $w(V): V \to \boldsymbol{C}^{(n+1-q)}$ be such that $((x(i, V))_I, w(V))$ form a set of coordinates for V in X. Define $x(i, V)(P, Q) = x(i, V)(Q)$ for all $P \in A(I) \cap V$. Define $w(V)(P, Q) = w(V)(Q) - w(V)(P)$.

For $V \in \mathscr{V}''$, define

$$x(i, V)(P, Q) = y(i, V)(P_0, Q)$$

where $P \in V \cap A(I)$ and $z(V)(P_0, P)=0$ and $i \in I$;

$$w(V)(P, Q) = (z(V)(P_0, Q), (y(j, V)(P_0, Q)-y(j, V)(P_0, P))_{j \in I(V)-I})$$

with P, P_0 as before.

Next for $V \in \mathscr{V}'$, $V' \in \mathscr{V}' \cup \mathscr{V}''$ and $V \cap V' \neq \varnothing$, define

$$u(i, V, V')(P, Q)$$
$$= (x(i, V')(P, Q) \cdot \pi\{y(j, V')(P_0, Q)^{m(j)/q(m(i))} : j \in I(V')-I\})/x(i, V)(P, Q)$$

for $i \in I$ with $P \in A(I) \cap V \cap V'$, $P_0 \in A(I(V')) \cap V'$ and $z(V')(P_0, P)=0$. Further, for $V, V' \in \mathscr{V}' \cup \mathscr{V}''$, let $\Delta(V, V')(P)=$the Jacobian matrix

$$(\partial w(V)(P, Q)/\partial w(V')(P, Q))|_P.$$

Now if $\{\rho_V\}$ is a C^∞ partition of unity of $A(I)$ subordinate(1) to the cover $\mathscr{V}' \cup \mathscr{V}''$ define:

$$y(i, V)(P, Q) = x(i, V)(P, Q) \cdot \pi\{u(i, V, V'')(P, Q)\rho_{V''}(P) : V'' \in \mathscr{V}' \cup \mathscr{V}''\}$$

for $V \in \mathscr{V}'$ and $i \in I$. Further define

$$z(V)(P, Q)$$
$$= w(V)(P, Q) - \sum\{\rho_{V''}(P)(w(V)(P, Q) - \Delta(V, V'')(P)w(V'')(P, Q)) : V'' \in \mathscr{V}' \cup \mathscr{V}''\}.$$

With appropriate restrictions of various open sets of the coverings, these mappings give the induction step and hence the theorem. ■

The above theorem leads us to consider what happens locally with respect to normalized coordinates.

3. **The normalized case.** For the present chapter only, let $S_t = Z(y_1^{m_1} \cdot \cdots \cdot y_q^{m_q} - t) \subseteq \mathbf{C}^q$. Let m_c be the greatest common divisor of $\{m_1, \ldots, m_q\}$. Then $S_t \approx T_t \times R$ for $t \neq 0$ where

$$T_t = \{(y_1, \ldots, y_q) \in S_t : |y_1|^{m_1} = \cdots = |y_q|^{m_q}\},$$
$$R = \{(y_1, \ldots, y_q) \in S_1 : y_i \text{ real, positive for all } i\}.$$

LEMMA 3.1. *For $t \neq 0$, T_t consists of m_c disjoint tori, each of real dimension $(q-1)$.*

Proof. $(y_1^{m_1} \cdots y_q^{m_q}) - t$ factors into

$$\pi\{y_1^{n_1} \cdot \cdots \cdot y_q^{n_q} - t_k : 1 \leq k \leq m_c\}$$

where the t_k are the distinct m_c-roots of t and $n_i = (m_i/m_c)$. T_t has at least one component corresponding to each factor. But for the case $m_c = 1$, it is easily seen that T_t is pathwise connected. Hence in general T_t has exactly m_c components. An induction on q then gives that each factor is a $(q-1)$-torus. ■

(1) See definition in [2, Appendix A].

It is immediate that $R \approx (\boldsymbol{R}^+)^{q-1}$. Let

$$R_J = \{(r_1, \ldots, r_q) \in R : r_i^{m_i} = r_j^{m_j} \text{ for all } i, j \in J, r_i^{m_i} \leqq r_j^{m_j} \text{ for } i \in J, j \notin J\},$$

$$R_{J,\theta} = \{(y_1, \ldots, y_q) : \exists (r_1, \ldots, r_q) \in R_J, \theta_i \geqq 0 \text{ for all } i \in J \text{ such that}$$
$$\sum \{m_i\theta_i : i \in J\} = \theta, y_i = (\exp \theta_i) r_i \text{ for } i \in J, y_i = r_i \text{ for } i \notin J\}.$$

Define $R_\theta = \bigcup \{R_{J,\theta} : J \subseteq \{1, \ldots, q\}, J \neq \varnothing\}$. Then $R_\theta \subseteq S_t$ where $t = \exp \theta$, and R_θ is a continuous perturbation of $R = R_0$.

Further, for $J \subseteq \{1, \ldots, q\}$, and a fixed constant $K > 1$, define

$$\begin{aligned} A(J) &= \bigcap \{Z(y_i) : i \in J\} \subseteq \boldsymbol{C}^\infty, \\ Y(J) &= \{(y_1, \ldots, y_q : |y_i|^{m_i} < K, i \in J\}, \\ Y'(J) &= \{(y_1, \ldots, y_q) \in Y(J) : |y_j|^{m_j} \geqq K, j \notin J\}, \\ A'(J) &= A(J) \cap Y'(J), \end{aligned}$$

and let $\mu(J)$: $\mathrm{Cl}\,(Y'(J)) \to A'(J)$ be the natural projection produced by setting $y_i = 0$ for $i \in J$.

Define

$$\varphi : R \to Q = \left\{(\rho_1, \ldots, \rho_q) \in \boldsymbol{R}^q : \sum \rho_i = 1\right\}$$

by $\rho_i = (\log K - m_i \log r_i)/q \log K$. $\sigma_r = R \cap Y(\{1, \ldots, q\})$ corresponds under φ to $\sigma_\rho = \{\rho \in Q : \rho_i \geqq 0 \text{ for all } i\}$. For $J \subseteq \{1, \ldots, q\}$, define $\sigma_\rho(J)$ to be the subsimplex spanned by the collection of vertices of σ_ρ which have 1 in the jth position for some $j \in J$. Let $\bar{I}$ = the closed unit interval.

LEMMA 3.2. *There is a C^∞ retraction ψ: $Q \times \bar{I} \to Q$ of Q onto σ_ρ such that*

(1) $\psi(\ , 0)$ = *identity on* Q,

(2) $\psi(\ , \alpha)$ = *identity on a neighborhood of* $(1/q, \ldots, 1/q)$ *for all* $\alpha \in \bar{I}$,

(3) $\psi(\ , \alpha)(\sigma_\rho(J)) = \sigma_\rho(J)$ *for all* $J \subseteq \{1, \ldots, q\}$, $\alpha \in \bar{I}$,

(4) $\psi(\ , 1)$: $\varphi(Y'(J) \cap Q) \to \sigma_\rho(J)$ *and this map is independent of the variables* r_j *for* $j \notin J$.

Proof. Use induction on q. (This retraction is similar to the one used by Pham [5, p. 338].) ∎

Define ρ: $R \to \sigma_\rho$ to be the composition $\psi(\ , 1) \circ \varphi$. Then we can define $F(\theta)$: $S_1 \to S_{\exp \theta}$ by

$$F(\theta)(y) = (\exp (\rho_1\theta/m_1) y_1, \ldots, \exp (\rho_q\theta/m_q) y_q)$$

where $(\rho_1, \ldots, \rho_q) = \rho(|y_1|, \ldots, |y_q|)$. $F(\theta)$ is induced by a C^∞-normal bundle construction as in §1 and its action is fibred over $A'(J)$ by the fibrations $\mu(J)$ defined above. Further $F(\theta)(R)$ is homotopic to R_θ in $S_{\exp \theta}$ in such a way that

(1) the homotopy respects the fibration $\mu(J)$ of $S_{\exp \theta} \cap Y'(J)$;

(2) under the homotopy, $F(\theta)(\sigma_r)$ corresponds to

$$\sigma_\theta = R_\theta \cap \mathrm{Cl}\,(Y(\{1, \ldots, q\})).$$

If we orient $\sigma_r = \sigma_0$ by the natural order of local coordinates $r_1, \ldots, r_{q-1}$, then $F(\theta)$ induces an orientation on each

$$\sigma_{J,\theta} = R_{J,\theta} \cap \mathrm{Cl}\,(Y(\{1, \ldots, q\})) \text{ of } \sigma_\theta.$$

For $J = \{1, \ldots, q\}$, this orientation is equal to $(-1)^{q-1}$ times the orientation induced by the natural order of the local coordinates $\theta_1, \ldots, \theta_{q-1}$. This follows from Lemma 3.2(2) and the fact that under $F(\theta)$ we have $(\partial\theta_i/\partial r_i) < 0$ at $(1/q, \ldots, 1/q)$.

Let m be any common multiple of $\{m_1, \ldots, m_q\}$. For $J = \{1, \ldots, q\}$, we consider σ_θ as a $(q-1)$-homology chain with orientation and coefficients of its simplicial decomposition given by considering σ_θ as the image under $F(\theta)$ of the oriented simplex σ_0. Then $\sigma_{J,m(k-1)}$ is parametrized by

$$\left\{(\theta_1, \ldots, \theta_{q-1}) : \theta_i \geqq 0 \text{ for all } i, \sum \{m_i\theta_i : 1 \leqq i \leqq q-1\} \leqq mk \text{ and } m_{i_0}\cdot\theta_{i_0} \geqq m\right\}$$

for any fixed $1 \leqq i_0 \leqq q-1$. Using this fact and direct calculation one obtains that the chain

$$\eta_{m,q-1} = \sigma_{m(q-1)} - \binom{q-1}{1}\sigma_{m(q-2)} + \binom{q-1}{2}\sigma_{m(q-3)} - \cdots \pm \binom{q-1}{q-1}\sigma_{m\cdot 0}$$

is a $(q-1)$-chain parametrized by $0 \leqq m_i\theta_i \leqq m$, $1 \leqq i \leqq q-1$.

Let T be the $(q-1)$-torus in T_1 which has nonempty intersection with R. T is a topological group with a component T' of $\{(y_1, \ldots, y_q) \in T : y_{q-1} = 1\}$ as a maximal connected subgroup, and

$$\{(y_1, \ldots, y_q) \in T : y_1 = \cdots = y_{q-2} = 1 \text{ and } y_{q-1} = \exp \theta_{q-1},$$
$$0 \leqq \theta_{q-1} \leqq (\text{g.c.d. } \{m_1, \ldots, m_{q-2}, m_q\}/m_c)\}$$

a transverse set for T/T'. Thus by our remarks on orientation above and induction on q we have

FORMULAE 3.3. $\eta_{m,q-1} = (-1)^{q-1}(m^{q-1}m_c/\pi\{m_i : 1 \leqq i \leqq q\}) \cdot T$ and if S_1 is oriented by its complex structure, then as an intersection number formula:

$$\sigma_0 \cdot (\eta_{m,q-1}) = (-1)^{q(q-1)/2}(m^{q-1} \cdot m_c/\pi m_i).$$

Finally, define

$$T_t(J) = \{(y_1, \ldots, y_q) \in S_t : |y_i|^{m_i} = |y_j|^{m_j} \text{ for } i, j \in J\}.$$

From the considerations of Lemma 3.1 it is clear that there exists an $m(J)$-sheeted complex analytic covering space $C_t(J)$ of $A'(J)$, where $m(J) = \text{g.c.d. } \{m_i : i \in J\}$, and the following commutative diagram of fibrations:

$$\text{(3.4)} \qquad \begin{array}{ccc} S_t \cap \mathrm{Cl}\,(Y'(J)) & \subseteq & \mathrm{Cl}\,(Y'(J)) \\ \downarrow \tau(J) & & \\ T_t(J) & & \downarrow \mu(J) \\ \downarrow \pi(J) & & \\ C_t(J) & \xrightarrow{\lambda(J)} & A'(J) \end{array}$$

where π has fibre a $(q'-1)$-real torus and τ has fibre a $(q'-1)$-simplex, $q'=|J|$. Then for $|J|=q-1$, $t=1$, and $j_0 \notin J$, $\lambda(J)^{-1}(P \in A'(J) : |y_{j_0}|^{m(j_0)}=K)$ has m_c components and for $C=$any one of these components:

(3.5) $\pi(J)^{-1}(C) \approx$(one of the $(q-1)$-tori of T_1), with this isomorphism given by the projection $\tau(I)\colon S_1 \cap \mathrm{Cl}\,(Y'(\{1,\ldots,q\})) \to T_1$ where $|I|=q$.

4. **Vanishing bundles and the action of** $F(\theta)$. We now return to the general problem considered in §1 and §2.

THEOREM 4.1. *There are tubular neighborhoods* $Y(i)$ *of* $A(i)$ *in* X *for* $0 \leqq i \leqq h$ *such that for* $Y(I)=\bigcap\{Y(i) : i \in I\}$ *there is a* C^∞*-normal bundle fibration*

$$\mu(I)\colon \mathrm{Cl}\,(Y(I)) \to A(I).$$

Further the following conditions are satisfied:

(1) *For* $W(q)=\bigcup\{Y(I) : |I|=q\}$ *and* $Y'(I)=Y(I)-W(|I|+1)$, $A'(I)=A(I)-W(|I|+1)$, *one has that* $\mu(I)^{-1}(A'(I))=\mathrm{Cl}\,(Y'(I))$.

(2) *For each* $P \in A'(I)$, *there is a* $V \in \mathscr{V}$ *with* $I(V)=I$ *and* $\mu(I)^{-1}(P) \subseteq V$, *where* $\mathscr{V}$ *is the regular covering of Theorem* 2.4.

(3) *For* $J \subseteq I$, $\mu(I) \circ \mu(J)=\mu(I)$ *on* $\mathrm{Cl}\,(Y(I))$.

(4) *For each* $I \subseteq \{0,\ldots,h\}$ *with* $|I|=q$, *there is a constant* $K(I)>0$ *such that, for* $P \in A'(I)$ *and* $V \supseteq \mu^{-1}(P)$, *the mapping* $\mu^{-1}(P) \to \mathbf{C}^q$ *given by* $(y(i, V)(P, Q))_{i \in I}$ *is an isomorphism of* $\mu^{-1}(P)$ *onto the closed polycylinder with radii* $K^{1/m(i)}$ *in* $\mathbf{C}^q$.

(5) $\mu(I)^{-1}(P) \cap A'(J)$ *is a* $(|I|-|J|)$*-real torus for* $P \in A'(I)$, $J \subseteq I$.

(6) $(W(q)-\mathrm{Cl}\,(W(q+1)))$ *is a disjoint union of open sets, one for each* I *with* $|I|=q$.

Proof. To construct $\mu(I)$ use Theorem 2.4, parts 3(b), 4(b), and (5). Then all the properties of the theorem follow from Theorem 2.4 and the behavior of normalized coordinates as seen in §3. ∎

Next let us construct $F(\theta)\colon S_1 \to S_{\exp\theta}$ which is induced by a normal bundle construction as in §1 and which behaves like the $F(\theta)$ constructed in §3 on each $\mu(I)^{-1}(P)$ for $P \in A'(I)$. Let $\delta_i\colon X \to [0, 1]$ be a C^∞-function such that:

(1) $\delta_i \equiv 1$ in a neighborhood of $A(i)$;

(2) $\delta_i \equiv 0$ on $X-Y(i)$;

(3) for $P \in A'(I)$, $i \in I$, and Q, $Q' \in \mu^{-1}(P) \subseteq V$ with

$$|y(i, V)(P, Q)| = |y(i, V)(P, Q')|$$

we have $\delta_i(Q)=\delta_i(Q')$. For t sufficiently small, we have for each $Q \in S_t$ that $\delta_i(Q)=1$ for some $i \in \{1,\ldots,h\}$. We can assume that we have this condition for $|t|<2$. Define, for $Q \in S_1$,

$$\rho_i(Q) = \big(\delta_i(Q)/\textstyle\sum\{\delta_j(Q) : 1 \leqq j \leqq h\}\big)-\delta_0(Q)$$

for $1 \leqq i \leqq h$ and $\rho_0(Q)=\delta_0(Q)$.

Define $F(\theta, P)\colon \mu(I)^{-1}(P) \cap S_1 \to \mu(I)^{-1}(P) \cap S_{\exp\theta}$ for $P \in A'(I)$ by:

$$F(\theta, P)((y(i, V)(P, Q))_{i\in I}) = ((\exp(\rho_i(Q)\theta/m(i))y(i, V)(P, Q))_{i\in I})$$

where $V \supseteq \mu(I)^{-1}(P)$. Theorem 2.4, parts (3) and (4), shows that this definition is independent of the choice of V and that the $F(\theta, P)$ piece together to give a C^∞-mapping:

$$F(\theta)\colon S_1 \to S_{\exp\theta}. \tag{4.2}$$

This mapping is easily seen to be induced by a normal bundle construction as described in §1. Hence we can assume that, whenever $\{S_t\}$ acquires an ordinary singularity at $t=0$, then $F(\theta)$ is as constructed in (4.2).

We now proceed to introduce the other constructions used in §3 to the general case with the help of normalized coordinates and the fibrations $\mu(I)$.

On $\mu(I)^{-1}(P)$ with $P \in A'(I)$, we will let $(y_i)_{i\in I}$ be any of the coordinate systems $(y(i, V)(P, \;))_{i\in I}$ for $V \supseteq \mu^{-1}(P)$ and $I(V)=I$. By Theorem 2.4, the choice of V will not matter in what follows.

For $0 \notin I$, define

$$T_t(I) = \{Q \in S_t \cap Y'(I) : |y_i|^{m(i)} = |y_j|^{m(j)} \text{ for } i, j \in I\}.$$

Then for t sufficiently small, hence we assume for $0<|t|<2$, $T_t(I) \cap \mu(I)^{-1}(P)$ is, by Lemma 3.1, $m(I)$ real tori each of real dimension $(|I|-1)$, for $P \in A'(I)$. ($m(I)$ = greatest common divisor of $\{m(i) : i \in I\}$.) Then there exists a complex manifold $C_t(I)$, which is an $m(I)$-sheeted covering space of $A'(I)$ and such that $T_t(I)$ is a $(q-1)$-torus bundle over $C_t(I)$, where $q=|I|$.

Further we can define a fibration $\tau(I)\colon (S_t \cap \mathrm{Cl}\,(Y'(I))) \to T_t(I)$ by

$$\tau(I)^{-1}(Q) = \{Q' \in (S_t \cap \mu(I)^{-1}(P)) : \operatorname{Arg} y_i(Q') = \operatorname{Arg} y_i(Q) \text{ for all } i \in I\}$$

where $P=\mu(I)(Q)$. Then we have a commutative diagram of projections for $0<|t|<2$:

$$\begin{array}{ccc} S_t \cap \mathrm{Cl}\,(Y'(I)) & \subseteq & \mathrm{Cl}\,(Y'(I)) \\ \downarrow \tau(I) & & \\ T_t(I) & & \downarrow \mu(I) \\ \downarrow \pi(I) & & \\ C_t(I) & \xrightarrow{\lambda(I)} & A'(I), \end{array} \tag{4.3}$$

which is a replica of (3.4) on each $\mu(I)^{-1}(P)$, $P \in A'(I)$.

By (3.5) and Theorem 4.1(5), $\pi(I)\colon T_1(I) \to C_1(I)$ can be factored into a series of circle bundles. Hence the composition of the Gysin maps associated to each of the circle bundles gives maps:

$$\begin{array}{ccccc} H_*(C_1(I)) & \longrightarrow & H_{*+(q-1)}(T_1(I)) & \longrightarrow & H_{*+(q-1)}(S_1), \\ \downarrow & & \downarrow & & \downarrow \\ H_*(C_1(I), \partial C_1(I)) & \to & H_{*+(q-1)}(T_1(I), \mathrm{Cl}\,(W(q+1))) & \to & H_{*+(q-1)}(S_1, \mathrm{Cl}\,(W(q+1))), \end{array}$$

where, intuitively, the horizontal maps take a cycle of $C_1(I)$ into the fibre above it in $T_1(I)$. Since it will cause no confusion, we will denote both of the horizontal composition maps by $\psi(I)$.

By §3, $\tau(I)$ has fibre a $(q-1)$-simplex so that we can apply the Thom-Gysin isomorphism and construct:

$$H_*(S_1) \to H_*(S_1, K(I)) \approx H_{*-(q-1)}(T_1(I), \mathrm{Cl}\,(W(q+1)))$$

where $K(I) = \mathrm{Cl}\,(S_1 - Y'(I))$. Call this composition $\chi(I)$.

Now let m = a common multiple of $\{m(1), \ldots, m(h)\}$. Define $\eta_m = F(m)_* - $(identity) : $H_*(S_1) \to H_*(S_1)$ where $F(\theta)_*$ is as in §1 and hence can be assumed to be defined by $F(\theta)$ in (4.2). Since there is little danger of confusion, we shall use η_m also to refer to the analogous

$$F(m)_* - (\text{identity}) \colon H_*(S_1) \to H_*(S_1, \mathrm{Cl}\,(W(q+1))).$$

We are now ready to state and prove a generalized Poincaré formula.

Let $\Gamma_p(I)$ denote a basis for $H_p(C_1(I), \partial C_1(I))$, and let $\Gamma_{p'}(I)$ denote its dual basis in $H_{p'}(C_1(I))$ where $p+p' = 2(n-q+1)$. Assume $C_1(I)$ to be oriented from the complex structure of $A'(I)$. For $\gamma \in \Gamma_p(I)$ we shall denote its dual by γ'. Finally let $\mathscr{I}_q = \{I \subseteq \{1, \ldots, h\} : |I| = q\}$.

THEOREM 4.4. *The mapping* $(\eta_m)^{q-1} \colon H_{p+q-1}(S_1) \to H_{p+q-1}(S_1, \mathrm{Cl}\,(W(q+1)))$ *is given by*

$$(\eta_m)^{q-1}(\tilde{\alpha}) = (-1)^r m^{q-1} \sum \{(m(I)/\pi_I m(i))(\tilde{\alpha} \cdot \psi(\gamma'))\psi(\gamma) : \gamma \in \Gamma_p(I), I \in \mathscr{I}_q\}$$

where $r = (q(q-1)/2) + p(q-1)$.

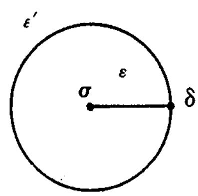
Proof. For $0 \in I$, let $T_1(I) = C_1(I) = A'(I)$ and define $\tau(I) = \mu(I)\,|_{S_1 \cap \mathrm{Cl}(Y'(I))}$ $\pi(I) = \lambda(I) =$ identity map. We will need a series of lemmas.

LEMMA 4.5. *For* $0 \in I$, $P \in A(I)$, $\mu(I)^{-1}(P) \cap S_1$ *is an analytic polydisc of complex dimension* $|I| - 1$.

Proof of lemma. By Definition 2.1, $A(I)$ is a nonsingular subvariety of S_1. Further $\mu(I) \colon Y(I) \to A(I)$ is a normal bundle in X. Hence each $\mu(I)^{-1}(P)$ must intersect S_1 normally. ■

LEMMA 4.6. *There exists a finite cellular decomposition of* $B(1) = S_0 \cup S_\infty$ *such that*:

(1) *The decomposition respects the chain of subspaces* $B(1) \supseteq B(2) \supseteq \cdots B(n+2) = \varnothing$ *as well as each of the subspaces* $A'(I)$, $I \subseteq \{0, \ldots, h\}$;



(2) *for a p-cell* σ *of* $A'(I)$, *there is a* $V \in \mathscr{V}$ *(see Theorem* 2.4) *with* $\mu(I)^{-1}(\sigma) \subseteq V$ *and* $I = I(V)$; *and for* $\varepsilon = \{Q \in (A(J) \cap \mathrm{Cl}\,(Y'(I)) : y_j(Q) \text{ real } \geqq 0\}$ *and*

$$\varepsilon' = \{Q \in (A(J) \cap \mathrm{Cl}\,(Y'(I)) : |y_j(Q)| = K(I)^{1/m(j)}\}$$

and $\delta = \varepsilon \cap \varepsilon'$ *with* $J \cup \{j\} = I$, *we have* $\varepsilon, \varepsilon' \subseteq (p+1)$*-skeleton of* $A(J)$, $\delta \subseteq p$*-skeleton of* $A'(J)$;

(3) *the decomposition lifts through* λ *to a cellular decomposition of each* $C_1(I)$.

Proof of lemma. (1) and (3) are immediate from the elementary theory of C-W complexes. For (2) use Theorem 4.1(5). ■

Define

$$Y_k = \bigcup \{\pi(I)^{-1}(\pi(I)^{-1}(p\text{-skeleton of } C_1(I))) : I \subseteq \{0, \ldots, h\}, p+|I| = k\}.$$

LEMMA 4.7. *The natural mapping* $H_{k-1}(Y_k) \to H_{k-1}(S_1)$ *is onto.*

Proof of lemma. Let $K_g = (S_1 - W(g)) \cup Y_k$ and assume $H_{k-1}(K_{g+1}) \to H_{k-1}(S_1)$ is onto. We have an exact sequence:

$$H_{k-1}(K_g) \to H_{k-1}(K_{g+1}) \to H_{k-1}(K_{g+1}, K_g).$$

However by Lemma 4.6(2), and using the Thom-Gysin isomorphism for $\tau(I)$ (see (4.3) and Lemma 4.5), we have for $p = k-g$ and $T_1(I)_p = \pi(I)^{-1}(p$-skeleton of $C_1(I))$:

$$H_{k-1}(K_{g+1}, K_g) \approx \sum \{H_p(T_1(I), T_1(I)_p : I \subseteq \{0, \ldots, h\}, |I| = g\}.$$

Using that $\tau(I)$ can be factored into circle bundles, each with an ordinary Gysin sequence, it is easily concluded that:

$$H_p(T_1(I), T_1(I)_p) \approx H_p(C_1(I), p\text{-skeleton}) = 0.$$

Hence $H_{k-1}(K_g)$ is onto by the composition of two surjective maps. ■

LEMMA 4.8. *The natural mapping* $H_{k-1}(Y_k) \to H_{k-1}(Y_k, Y_{k-1})$ *is injective.*

Proof of lemma. It suffices to show that $H_{k-1}(Y_{k-1}) = 0$. Define

$$L_g = \bigcup \{\tau(I)^{-1}(\pi(I)^{-1}(p\text{-skeleton of } C_1(I))) : p+|I| = k-1 \text{ and } |I| \geqq g\}.$$

Suppose $H_{k-1}(L_{g+1}) = 0$. L_g has as a strong deformation retract the space

$$L_{g+1} \cup \bigcup \{T_1(I)_p : |I| = g\}$$

for $p = k-1-g$. However each $T_1(I)$ has dimension at most $p+(|I|-1) = k-2$. Hence $H_{k-1}(L_g, L_{g+1}) = 0$ and therefore $H_{k-1}(L_g) = 0$. ■

$F(m)$ as constructed in (4.2) respects all the subspaces constructed above. Hence by Lemmas 4.7 and 4.8, it suffices to characterize

$$\eta_m = F(m)_* - (\text{identity}) : H_{k-1}(Y_k, Y_{k-1}) \to H_{k-1}(Y_k, Y_{k-1}).$$

Let σ be a p-cell of $C_1(I)$ for $p+|I| = k$. Let $M = \tau(I)^{-1}(\pi(I)^{-1}(\sigma))$. Then the boundary of M, which we shall denote by $M^{\cdot}$, is equal to the union of

$N=\tau(I)^{-1}(\pi(I)^{-1}(\sigma^{\cdot}))$ and $\tilde{G}=M-W(|I|)$. Finally let $G=\tilde{G}\cap Y_{k-1}$. Then $H_{k-1}(Y_k, Y_{k-1})$ is the direct sum of groups of the form:

$$H_{k-1}(M, G\cup N); \tag{4.9}$$

hence it suffices to consider $\eta_m: H_{k-1}(M, G\cup N)\to H_{k-1}(M, G\cup N)$, since each subspace of this type is preserved by $F(m)$ as constructed in (4.2). Further we have the exact sequence:

$$H_{k-1}(\tilde{G}, G)\to H_{k-1}(M, G\cup N)\to H_{k-1}(M, \tilde{G}\cup N). \tag{4.10}$$

Using Lemma 4.6(2), the generator for $H_{k-1}(M, \tilde{G}\cup N)$ can be represented by the cell

$$\omega = \{Q\in M : y_i(Q) = r_i\bar{y}_i \text{ for all } i\in I \text{ for some } (r_j)\in R\} \tag{4.11}$$

where R is as in §3 and $\bar{y}_i=y_i(\bar{Q})$ for any fixed $\bar{Q}\in\pi(I)^{-1}(\sigma)$. For the proper choice of $\bar{Q}$, $\omega^{\cdot}\in G\cup N$ by the construction of the cellular decomposition in Lemma 4.6(2).

(4.12) Since $H_{k-1}(\tilde{G}, G)$ is itself a direct sum of groups of the form (4.9), but over p-cells σ' of $C_1(I')$ with $p'+|I|=k$ and $|I'|<|I|$ we can inductively assume that any element of $H_{k-1}(\tilde{G}, G)$ has a representative ω' which is a sum of cells of type (4.11) with real coefficients over p'-cells σ' with $p'>p$.

By (4.10), for any cycle α of $H_{k-1}(M, G\cup N)$, there is a representative chain of the form $a(\sigma)\omega+\omega'$ with ω and ω' as in (4.11) and (4.12) and $a(\sigma)\in \boldsymbol{R}$. Now by (4.2) and the first of Formulae 3.3, $(F(m)-(\text{identity}))^{q-2}(\omega')$ is homologous to a sum of chains of the form $\pi(I')^{-1}(\sigma')$ where σ' is as in (4.12) and $q=|I|$. But $F(m)$ as constructed in (4.2) is homotopic in each fibre to the identity on each $\pi(I')^{-1}(\sigma')$. Hence $(F(m)-(\text{identity}))^{q-1}(\omega')$ is homologous to 0 in $H_{k-1}(M, G\cup N)$. Further by the same formula we have that up to sign:

(4.13) $(F(m)-(\text{identity}))^{q-1}(\omega)\sim(m^{q-1}m(I)/\pi_I m(i))T_\sigma$ if $I\in\mathscr{I}_q$, where

$$T_\sigma = \{Q\in M : |y_i(Q)|^{m(i)} = |y_j(Q)|^{m(j)} \text{ for all } i,j\in I\},$$

and $(F(m)-(\text{identity}))^{q-1}(\omega)\sim 0$ otherwise. ($\sim$ indicates homologous to in $H_{k-1}(M, G\cup N)$.)

Now let $\tilde{\alpha}\in H_{p+q-1}(S_1)$. Then $\tilde{\alpha}$ has a representative cycle α in Y_{p+q} which is made up of cells of the form $a(\sigma)\omega$ with ω as in (4.11). Further, for $I\in\mathscr{I}_q$,

$$\sum\{a(\sigma)\sigma : \sigma\ p\text{-cell of } C_1(I)\}$$

is, with appropriate orientation, the cycle of $H_p(C_1(I), \text{Cl}\,(W(q+1)))$ given by $\pi(I)(\chi(I)(\tilde{\alpha}))$. ($\chi(I)$ was defined earlier in §4.) The theorem then follows from (4.13) and the fact that orientation for the mapping $\chi(I)$ can be chosen so that:

$$((\tilde{\alpha}\cdot(\psi(I)(\gamma'))) \text{ in } S_1) = (-1)^{p(q-1)}(((\pi(I)(\chi(I)(\tilde{\alpha})))\cdot\gamma')$$

in $C_1(I)$). The calculation of the sign in the theorem is then essentially given by the second formula of Formulae 3.3. This completes the proof of Theorem 4.4. ∎

Note 4.14. Let $H_0(I)$ be the bundle associated to $\lambda \circ \pi: T_1(I) \to A'(I)$ with fibre $H_0((\lambda_0\pi)^{-1}(P))$ for $P \in A'(I)$, (see [7, p. 151]). Then

$$H_p(C_1(I)) \approx H_p(A'(I); H_0(I)),$$
$$H_p(C_1(I), \partial C_1(I)) \approx H_p(A'(I), \partial A'(I); H_0(I)). \quad \blacksquare$$

COROLLARY 4.15. *Let α be an element of $H_{k-1}(S_1 - W(q+1))$. Then*

$$\eta_m^{q-1}(\alpha) = (-1)^r m^{q-1} \sum \{(m(I)/\pi_I m(i))(\alpha \cdot \psi(\gamma))\psi(\gamma') : \gamma' \in \Gamma_p'(I), I \in \mathscr{I}_q\}$$

in $H_{k-1}(S_1)$ where $r = (q(q-1)/2) + pq$.

Proof. Choose a representative cycle for α in $(Y_k - W(q+1))$ and proceed as in Theorem 4.4. Note that $(\gamma' \cdot \gamma) = (-1)^p$. $\blacksquare$

In the case $q=1$, Theorem 4.4 is true if we define η_m^0 to be the standard homology map

$$H_*(S_1) \to H_*(S_1, \mathrm{Cl}\,(W(2))).$$

However this provides no new information. For the sake of completeness, we shall add here the more appropriate result for the case $q=1$. For $I \in \mathscr{I}_1$ and

$$M(I) = \{\exp(k/m(I)) : k \in \boldsymbol{Z}\}$$

with generator $\xi(I) = \exp(1/m(I))$, we have that $M(I)$ acts on $S_1 \cap \mathrm{Cl}\,(Y'(I))$ as in (4.2). For $\gamma \in H_*(C_1(I), \partial C_1(I))$, $\psi(I)(\gamma) = \gamma$. Let

$$\tilde{\psi}(I)(\gamma) = (\xi(I) \cdot \gamma) - \gamma.$$

As in [5], the group-ring $\boldsymbol{Z}(M(I))$ acts on $H_*(C_1(I), \partial C_1(I))$. Define $\tilde{k} = \exp(k/m(I))$.

THEOREM 4.16. *For $\alpha \in H_p(S_1)$ and $b \geqq 1$, then, in $H_p(S_1, \mathrm{Cl}\,(W(2)))$,*

$$\eta_b(\alpha) = \sum \{(\alpha \cdot \gamma')\tilde{k} \cdot \tilde{\psi}(\gamma) : 0 \leqq k \leqq b-1, \gamma \in \Gamma_p(I), I \in \mathscr{I}_1\}.$$

In particular, if b is a multiple of $m(i)$ for all $1 \leqq i \leqq h$, then $\eta_b(\alpha) = 0$ in $H_p(S_1, \mathrm{Cl}\,(W(2)))$.

Proof. Immediate from the fact that $\eta_b = F(b)_* - $(identity) and from the construction of $F(\theta)$ in (4.2). Note that, for

$$\xi = \sum \{\tilde{k} : 1 \leqq k \leqq m(I)\} \in \boldsymbol{Z}(M(I)),$$

we have $\xi \cdot \psi(\gamma) = 0$ for all $\gamma \in \Gamma_p(I)$. $\blacksquare$

COROLLARY 4.17 *Let α be an element of $H_p(S_1 - \mathrm{Cl}\,(W(2)))$. Then*

$$\eta_b(\alpha) = (-1)^p \sum \{(\alpha \cdot \gamma)\tilde{k} \cdot \tilde{\psi}(\gamma') : 0 \leqq k \leqq b-1, \gamma' \in \Gamma_p'(I), I \in \mathscr{I}_1\}$$

in $H_p(S_1)$. $\blacksquare$

5. **Applications.** The proof of Theorem 4.4 actually shows the action of $F(l)_*: H_*(S_1) \to H_*(S_1)$ for any $l \geqq 0$. For if $\alpha \in H_{k-1}(S_1)$, then there is a $\beta \in H_{k-1}(Y_k, Y_{k-1})$ which has a common preimage with α in $H_{k-1}(Y_k)$. Further, β has as representative a sum of cells of the form $a_\omega \cdot \omega$ with $a_\omega \in \mathbf{R}$ and ω as in (4.11). Then $F(l)_*(\beta)$ is represented in $H_{k-1}(Y_k, Y_{k-1})$ by the sum of cells $a_\omega \cdot \omega_l$ where

$$\omega_l = \{Q \in M : y_i(Q) = r_i \bar{y}_i \text{ for all } i \in I \text{ for some } (r_j) \in R_l\}$$

with M as in (4.9) and R_l as in §3. By Lemma 4.8, such an element of $H_{k-1}(Y_k, Y_{k-1})$ comes from a unique element of $H_{k-1}(Y_k)$ which in turn determines $F(l)_*(\alpha) \in H_{k-1}(S_1)$. However, in general this is an unwieldy process.

In this final section we will give a formula which helps characterize $F(m)_*: H_*(S_1) \to H_*(S_1)$ without necessitating the explicit construction outlined above. (m here, as before, is a common multiple of $m(i)$ for $1 \leqq i \leqq h$.) With this formula and Theorem 4.4 we will completely construct

$$\eta_m = F(m)_* - (\text{identity}): H_*(S_1) \to H_*(S_1)$$

for the case $n=2$. (The case $n=0$ is trivial and for $n=1$, which corresponds to the classical Picard-Lefschetz formula, η_m is completely given by Theorem 4.4.)

FORMULA 5.1. Let α be an element of $H_{p'+q-1}(S_1 - W(q+2))$. Then for all $\beta \in H_{p+q-1}(S_1)$ with $p+p' = 2(n-q)$:

$$(\eta_m^{q-1}(\alpha) \cdot \beta) + (-1)^q(\alpha \cdot \eta_m^{q-1}(\beta)) \\ = (-1)^r(q-1)m^q \sum \{(m(I)/\pi_I m(i))(\alpha \cdot \psi(\gamma))(\psi(\gamma') \cdot \beta) : \gamma \in \Gamma_p(I), I \in \mathscr{I}_{q+1}\}$$

where $r = ((q+1)q/2) + p(q+1)$.

Proof. By Corollary 4.15, the right-hand side of the formula is simply $(q-1)\eta_m^q(\alpha) \cdot \beta$. However $\eta_m^q(\alpha)$ is invariant under $F(m)_*$ and $F(m)$ is an orientation-conserving automorphism. Hence $\eta_m^q(\alpha) \cdot \beta = \eta_m^q(\alpha) \cdot F(m)_*^k(\beta)$ for all $k \in \mathbf{Z}$. Thus the right-hand side of the formula is equal to

$$\sum \{\eta_m^q(\alpha) \cdot F(m)_*^k(\beta) : 1 \leqq k \leqq q-1\}$$

which in turn is equal to:

$$\sum \left\{(-1)^{(q-j)} \binom{q}{j} F(m)_*^j(\alpha) \cdot F(m)_*^k(\beta) : 0 \leqq j \leqq q,\ 1 \leqq k \leqq q-1\right\}.$$

This last summation reduces to $\eta_m^{q-1}(\alpha) \cdot \beta + (-1)^q(\alpha \cdot \eta_m^{q-1}(\beta))$ by repeated use of the equality:

$$F(m)_*^j(\alpha) \cdot F(m)_*^k(\beta) = F(m)_*^{j+i}(\alpha) \cdot F(m)_*^{k+i}(\beta) \quad \text{for all } i \in \mathbf{Z}. \qquad \blacksquare$$

Now let us examine $\eta_m: H_j(S_1) \to H_j(S_1)$ for $n=2$. For $j=0$ and 4, $\eta_m \equiv 0$ trivially. For $j=1$, η_m is given by Corollary 4.15. For $j=3$, η_m is given by Theorem 4.4 since the natural map $H_3(S_1) \to H_3(S_1, \text{Cl}(W(3)))$ is an injection. We shall now characterize $\eta_m: H_2(S_1) \to H_2(S_1)$. Note that $W(4) = \varnothing$ and so for any $\alpha, \beta \in H_2(S_1)$ we have that:

$$\eta_m(\alpha)\cdot\beta+\alpha\cdot\eta_m(\beta) = (-1)m^2 \sum \{(m(I)/\pi_I m(i))(\alpha\cdot\psi(\gamma))(\beta\cdot\psi(\gamma)) : \gamma \in \Gamma_0(I), I \in \mathscr{I}_3\}$$

by Formula 5.1. Define the right-hand side of this expression to be equal to $\Phi(\alpha, \beta)$. This formula has as corollaries that:

(5.2) For all $\alpha, \beta \in H_2(S_1)$: $\eta_m(\alpha)\cdot\eta_m(\beta) = -\Phi(\alpha, \beta)$ since $\alpha\cdot\eta_m^2(\beta) = \Phi(\alpha, \beta)$ by Theorem 4.4;

(5.3) $\eta_m^2(\alpha)=0$ for $\alpha \in H_2(S_1)$ if and only if $\eta_m(\alpha)\cdot\eta_m(\alpha)=0$ (since $\eta_m(\alpha)\cdot\eta_m(\alpha)=0$ implies $(\alpha\cdot\psi(\gamma))=0$ for $\gamma \in \Gamma_0(I)$, $I \in \mathscr{I}_3$).

Let

$$E = \bigcup \{A(I) : I \in \mathscr{I}_3\}.$$

(E is called the set of triple points of the hypersurface S_0.) Let $B=\lambda^{-1}(E)$ and for each $P \in C(I)$, $I \in \mathscr{I}_3$, let $T(P)=\pi(I)^{-1}(P)$, $W(P)=\tau(I)^{-1}(\pi(I)^{-1}(P))$. Let $\bar{Q}_P$ be fixed in $T(P)$ and let ω_P be as in (4.11). Let

$$S_1' = (S_1 - W(3)) \cup \bigcup \{\omega_P : P \in B\}.$$

Then by the Thom-Gysin isomorphism and Lemma 4.5, $H_2(S_1') \to H_2(S_1)$ is onto. Let

$$D(J) = \pi(J)(\bigcup \{T_1(J) \cap \omega_P : P \in B\})$$

and let

$$K = \bigcup \{\tau(J)^{-1}(\pi(J)^{-1}(Q) : Q \in D(J)\}.$$

Then, as in §4, we define for $J \in \mathscr{I}_2$:

$$\psi(J): H_1(C_1(J), D(J)) \to H_2(S_1 - W(3), K)$$
$$\tilde{\chi}(J): H_2(S_1') \to H_1(C_1(J), D(J))$$

with orientations chosen as in the final paragraph of the proof of Theorem 4.4. ($\tilde{\chi}(J)$ corresponds to $\pi(J) \circ \chi(J)$ of §4.)

For each $P \in B$, let τ_P be a generator of $H_2(W(P))=H_2(T(P))$. Let $\{\tau_1, \ldots, \tau_s\}$ be a maximal linearly independent set in $\{\tau_P : P \in B\}$ and let $\{\alpha_1, \ldots, \alpha_s\}$ be a dual set under the intersection pairing in $H_2(S_1)$.

LEMMA 5.4. *There are unique elements $\eta_i \in H_2(S_1)$ for $1 \leqq i \leqq s$ such that there exist elements $\xi_P \in H_2(W(P), K)$ for $P \in B$ such that*:

(1) *if $\varphi_i = -m \sum \{(m(J)/\pi_J m(j))\psi(J) \circ \tilde{\chi}(J)(\alpha_1') : J \in \mathscr{I}_2\}$ where α_i' is some pre-image of α_i in $H_2(S_1')$, then the image of η_i in $H_2(S_1, K)$ is given by*

$$\varphi_i + \sum \{(\alpha_i \tau_P)\xi_P : P \in B\};$$

(2) *for $1 \leqq i, j \leqq s$, $\eta_i\alpha_j + \alpha_i\eta_j = \Phi(\alpha_i, \alpha_j)$.*

Proof. There is a representative of α_i' of the form

$$\varepsilon + \sum \{(\alpha_i \cdot \tau_P)a_P\omega_P : P \in B\}$$

where ε is a cycle of $(S_1 - W(3))$ relative to K and the a_P are constants in $\boldsymbol{R}$. Then $F(m)$ fixes $(S_1 - W(3))$, K, and $W(P)$ for each P by (4.2), and so:

(1) $\eta_m\colon H_2(S_1 - W(3), K) \to H_2(S_1 - W(3), K)$ takes $\{\varepsilon\}$ into φ_i by the proof of Theorem 4.4;

(2) $\eta_m\colon H_2(W(P), K) \to H_2(W(P), K)$ takes $\{a_P\omega_P\}$ into an element of $H_2(W(P), K)$ which we will call ξ_P.

Then putting $\eta_i = \eta_m(\alpha_i)$, the existence of a collection $\{\eta_i\}$ satisfying the lemma is proved. For uniqueness, suppose η_i, η_i' for $1 \leqq i \leqq s$ satisfy the conditions of the lemma. Let ξ_P, ξ_P' be the corresponding elements of $H_2(W(P), K)$ for all $P \in B$. Then

$$\eta_i - \eta_i' = \sum \{(\alpha \cdot \tau_P)(\xi_P - \xi_P') : P \in B\}$$

in $H_2(S_1, K)$. However $H_2(K) = 0$ and $\partial_*(\xi_P - \xi_P') = 0$ in $H_1(K)$ and so $(\xi_P - \xi_P')$ comes from an element $b_P\tau_P$ in $H_2(S_1)$. To get the uniqueness of the η_i it suffices to show that for $1 \leqq i \leqq s$:

$$(*) \qquad \sum \{(\alpha_i \cdot \tau_P) b_P \tau_P : P \in B\} = 0.$$

But by condition (2) of the lemma:

$$(\eta_i - \eta_i') \cdot \alpha_j + \alpha_i \cdot (\eta_j - \eta_j') = 0.$$

Hence for all $1 \leqq j \leqq s$, we have that:

$$2 \sum \{b_P(\alpha_i \cdot \tau_P)(\alpha_j \cdot \tau_P) : P \in B\} = 0.$$

Since the α_j form a dual basis to the subspace of $H_2(S_1)$ generated by $\{\tau_P : P \in B\}$, this suffices to prove $(*)$. ∎

COROLLARY 5.5. *$\eta_i = \eta_m(\alpha_i) \in H_2(S_1)$ for all $1 \leqq i \leqq s$.* ∎

Now $\Gamma_1(J)$, the basis for $H_1(C_1(J), \partial C_1(J))$ for $J \in \mathscr{J}_2$, can be so chosen that:

$$\Gamma_1(J) = \Delta_1(J) \cup \Omega_1(J), \qquad \Gamma_1'(J) = \Delta_1'(J) \cup \Omega_1'(J)$$

where $\Delta_1(J)$ generates the image of $H_1(C_1(J))$ in $H_1(C_1(J), \partial C_1(J))$ and $\Omega_1'(J)$ generates the image of the mapping $H_1(\partial C_1(J)) \to H_1(C_1(J))$ and $\gamma \in \Delta_1(J)$ implies that $\gamma' \in \Delta_1'(J)$. For any $\alpha \in H_2(S_1)$,

$$\alpha'' = \alpha - \sum \{(\alpha \cdot \tau_i)\alpha_i : 1 \leqq i \leqq s\}$$

comes from an element (which we shall also call α'') of $H_2(S_1 - W(3))$. By Corollary 4.15,

$$\eta_m(\alpha'') = (-1)m \sum \{(m(J)/\pi_J m(j))(\alpha'' \cdot \psi(\gamma))\psi(\gamma') : \gamma' \in \Gamma_1'(J), J \in \mathscr{J}_2\}$$

in $H_2(S_1)$. Now $\Gamma_1'(J) = \Delta_1'(J) \cup \Omega_1'(J)$ and for $\gamma' \in \Omega_1'(J)$, $\psi(\gamma')$ is contained in the subspace of $H_2(S_1)$ generated by $\{\tau_P : P \in B\}$ by (3.5). Hence

$$\eta_m(\alpha'') = (-1)m \sum \{(m(J)/\pi_J m(j))(\alpha'' \cdot \psi(\gamma))\psi(\gamma') : \gamma \in \Delta_1(J), J \in \mathscr{J}_2\}$$
$$+ \sum \{b_i(\alpha'')\tau_i : 1 \leqq i \leqq s\}.$$

However by Formula 5.1 and Corollary 5.5, $(\alpha''\cdot\eta_i)+(\eta_m(\alpha'')\cdot\alpha_i)=\Phi(\alpha'',\alpha_i)=0$ for all $1\leqq i\leqq s$. Thus intersecting this last formula for $\eta_m(\alpha'')$ with α_i we have that:

$$b_i(\alpha'') = m\sum\{(m(J)/\pi_J m(j))(\alpha''\cdot\psi(\gamma))(\alpha_i\cdot\psi(\gamma')) : \gamma\in\Delta_1(J), J\in\mathscr{J}_2\}-(\alpha''\cdot\eta_i).$$

Define

$$\tilde{\psi}(\gamma) = \psi(\gamma)-\sum\{(\alpha_i\cdot\psi(\gamma))\tau_i : 1\leqq i\leqq s\}\in H_2(S_1)$$

for $\gamma\in\Delta_1(J), J\in\mathscr{J}_2$. Then $\alpha\cdot\tilde{\psi}(\gamma)=\alpha''\cdot\psi(\gamma)$. Define

$$\tilde{\eta}_i = \eta_i-m\sum\{(m(J)/\pi_J m(j))(\alpha_i\cdot\psi(\gamma'))\tilde{\psi}(\gamma) : \gamma\in\Delta_1(J), J\in\mathscr{J}_2\}$$
$$-\sum\{(\alpha_k\cdot\eta_i)\tau_k : 1\leqq k\leqq s\}$$

THEOREM 5.6 *For* $\alpha\in H_2(S_1)$, *then*

$$\eta_m(\alpha) = \sum\{(\alpha\cdot\tau_i)\eta_i : 1\leqq i\leqq s\}$$
$$-m\sum\{(m(J)/\pi_J m(j))(\alpha\cdot\tilde{\psi}(\gamma))\psi(\gamma') : \gamma\in\Delta_1(J), J\in\mathscr{J}_2\}$$
$$-\sum\{(\alpha\cdot\tilde{\eta}_i)\tau_i : 1\leqq i\leqq s\}$$

in $H_2(S_1)$, *where m is a common multiple of* $m(j)$ *for all* $1\leqq j\leqq h$.

Proof. The formula is an immediate consequence of the formulae and definitions just preceding. ■

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([2]) (In the above reference, the author's treatment of the triple point case is incorrect.)