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Elias N. Houstis
Purdue University, enh@cs.purdue.edu

T. S. Papatheodorou

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AT THE GAUSSIAN POINTS

by

E.N. Houstis
Dept. of Computer Sciences
Purdue University
W. Lafayette, IN 47907

and

T.S. Papatheodorou
Dept. of Mathematics
Clarkson College of Tech.
Potsdam, NY 13676

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ABSTRACT

An interpolation scheme based on piecewise cubic polynomials with the Gaussian points as interpolation points is analyzed. Optimal order a priori estimates are obtained for the interpolation error in the maximum norm.

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Introduction. We consider an interpolation scheme based on piecewise cubic polynomials with continuous first derivatives and the Gaussian points as interpolation points.

This scheme has been applied as a collocation method by DeBoor and Swartz [2] and Houstis [6] for the numerical solution of ordinary differential equations. Also, Douglas and Dupont [3], [4], [5] and Houstis [7], have studied a collocation method for partial differential equations based on the above scheme.

In sections 1 and 2 we present the formulation for one and two dimensions. In section 3 of this report we obtain optimal order asymptotic estimates for the interpolation error in the L_∞-norm.

1. One-dimensional interpolation scheme. Let $\Delta=(x_i)_1^{N+1}$ be a partition of $I\equiv[a,b]$, $h_i\equiv|x_{i+1}-x_i|$, $I_i\equiv[x_i,x_{i+1}]$ and $h=\max h_i$. Throughout this report we denote by P_3 the set of polynomials of degree less than 4, and $P_{3,\Delta}$ the set of functions that reduce to polynomials of degree less than 4 in each subinterval $[x_i,x_{i+1}]$. Also we denote by H_Δ the (2N+2)-dimensional vector space of all continuously differentiable piecewise cubic polynomials with respect to Δ. We take $-1<\rho_1,\rho_2<1$ and $w_j>0, j=1,2$ to be the

Gaussian points and weights respectively, so that

$$\int_{-1}^{+1} p(x)\,dx = \sum_{i=1}^{2} p(\rho_i) w_i, \quad p \epsilon P_3([-1, 1]).$$

The Gaussian points and weights in the subinterval $[x_j, x_{j+1}]$ are

(1.1) $$\xi_{2j+i} \equiv \frac{x_j + x_{j+1}}{2} + \rho_i \frac{h_j}{2} \quad i = 1,2 .$$

We introduce an interpolation operator

$$Q_N : C^1(I) \to H_\Delta$$

such that

(1.2) $$(Q_N f)(\sigma_\ell) = f(\sigma_\ell), \quad \ell=1,\dots,2N+2,$$

where $\sigma_1 = a$, $\sigma_\ell = \xi_{2j+i}$, $j=1,\dots,N, i=1,2$, $\sigma_{2N+2} = b$.

This interpolation scheme is well defined. In fact, if $h(x) \epsilon H_\Delta$ also interpolates f as above, then $e(x) \equiv Q_N f(x) - h(x)$ is a cubic polynomial on $[x_i, x_{i+1}]$, $0 \le i \le N$ and $e(\sigma_i) = 0$, $1 \le i \le 2N+2$. We show that e(x) is identically zero in $[x_i, x_{i+1}]$. If this is not so, then without loss of generality we may assume that $e(x) \ne 0$ for all $x \epsilon [x_1, x_2]$. Rolle's Theorem implies that $e(x_2) D_x e(x_2) > 0$. Similarly, $D_x e$ restricted in $[x_2, x_3]$ has roots in (x_2, σ_4), (σ_4, σ_5). Thus, $e(x_3) D_x e(x_3) > 0$. By induction

$e(x_{N+1}) D_x e(x_{N+1}) > 0$ contradicting the relation $e(x_{N+1})=0$. This proves that $e(x) \equiv 0$ in I.

2. Two-dimensional interpolation scheme. In this section we introduce a two-dimensional analogue of the interpolation scheme of the previous section. Let $\Delta_y = (y_j)_1^{M+1}$ be a partition of $[c,d]$, $J \equiv [c,d]$, $k_j \equiv |y_{j+1} - y_j|$, $J_j \equiv [y_j, y_{j+1}]$ and $k \equiv \max k_j$. Also, we denote by $\rho \equiv \Delta x \Delta y$ a partition of $[a,b] \times [c,d]$ and by H_ρ the vector space of all piecewise bicubic polynomials $p(x,y)$ with respect to ρ, such that $D_x^\ell D_y^\eta p(x,y)$ is continuous on $[a,b] \times [c,d]$ for all $0 \leq \ell, \eta \leq 1$.

The Gaussian points and weights in the subinterval $[y_i, y_{i+1}]$ are

$$\tau_{2i+j} \equiv \frac{y_i + y_{i+1}}{2} + \rho_j \frac{k_i}{2}, \quad j=1,\ 2 \quad .$$

A two-dimensional interpolation operator is defined as the tensor product

$$Q_\rho \equiv Q_N \otimes Q_M = Q_N Q_M$$

3. Error analysis. In this section, we establish a priori bounds for the interpolation scheme introduced in section 2. For later use, we define the Gramian matrix

$$G_N \equiv \left(B_i(\sigma_j) \ ;\ i,j=1,\ldots,2N+2 \right)$$

of the interpolation operator Q_N. Using the $(2N+2) \times (2N+2)$ matrix

$$H_N \equiv \begin{bmatrix} 1 & & & & & \\ & h & & & 0 & \\ & & \cdot & & & \\ & & & \cdot & & \\ & 0 & & & \cdot & \\ & & & & 1 & \\ & & & & & h \end{bmatrix} .$$

we find that

$$H_N^{-1} G_N \equiv \left[\begin{array}{c|c|c|ccc}
\begin{matrix}1\\0\end{matrix} & A & & & & \\ \hline
& B & A & & & 0 \\ \hline
& & & \ddots & & \\
& 0 & & & \ddots & \\ \hline
& & & B & A & \\ \hline
& & & & B & \begin{matrix}1\\0\end{matrix}
\end{array}\right]$$

where

$$A \equiv \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} , \qquad B \equiv \begin{bmatrix} \beta & \alpha \\ -\delta & -\gamma \end{bmatrix}$$

and

$$\alpha = \frac{9+4\sqrt{3}}{18} , \quad \beta = \frac{9-4\sqrt{3}}{18} , \quad \gamma = \frac{3+\sqrt{3}}{36} , \quad \delta = \frac{3-\sqrt{3}}{36} .$$

We will also use the matrix

$$T \equiv BA^{-1} = \begin{bmatrix} -7 & 48 \\ 1 & -7 \end{bmatrix} .$$

It is easy to see that for all integers n, $(T^{\circ} \equiv I)$,

$$T_n = \begin{bmatrix} a_n & 48c_n \\ c_n & a_n \end{bmatrix}$$

where

$$a_{n+1} = -7a_n + 48c_n \ , \ c_{n+1} = a_n - 7c_n \quad .$$

More generally, from $T^{s+t} = T^s \, T^t$ we get

$$a_{s\pm t} = a_s a_t \pm 48 c_s c_t \ , \ c_{s\pm t} = c_s a_t \pm a_s c_t$$

(3.1)
$$c_s a_t = \frac{1}{2}(c_{s+t} + c_{s-t})$$

$$c_s c_t = \frac{1}{96}(a_{s+t} - a_{s-t})$$

$$a_s a_t = \frac{1}{2}(a_{s+t} + a_{s-t})$$

$$a_{-\ell} = a_\ell \ , \ c_{-\ell} = -c_\ell \quad .$$

Let $\lambda_n \equiv |a_n/c_n| = -a_n/c_n$. Since $\det(T^n) = 1$, we can easily show that λ_n is decreasing with n and for all n

$$\sqrt{48} < \lambda_n \leqq 7, \ \lambda_1 = 7$$

(3.2) $$c_n = (-1)^{n+1}|c_n|, \ a_n = (-1)^n |a_n|$$

$$|a_{n+1}| > |a_n| \ , \ |c_{n+1}| > |c_n|$$

Since

$$|a_n| = \frac{1}{2}(|c_{n+1}|-|c_{n-1}|), |c_n| = \frac{1}{96}(|a_{n+1}|-|a_n|)$$

we also have

$$\sum_{\ell=q}^{p} |a_\ell| = \frac{1}{2}(|c_{p+1}|+|c_p|-|c_q|-|c_{q-1}|)$$

(3.3)

$$\sum_{\ell=q}^{p} |c_\ell| = \frac{1}{96}(|a_{p+1}|+|a_p|-|a_q|-|a_{q-1}|) \quad .$$

We introduce a (2N+2)x(2N+2) matrix R in partition form

$$\left[\begin{array}{cc|c|cc} r_{11} & r_{12} & \cdots & r_{1,2N+1} & r_{1,2N+2} \\ \hline \multicolumn{2}{c|}{R_{11}} & \cdots & \multicolumn{2}{c}{R_{1,N+1}} \\ \multicolumn{2}{c|}{\vdots} & & \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c|}{R_{N,1}} & \cdots & \multicolumn{2}{c}{R_{N,N+1}} \\ \hline r_{2N+2,1} & r_{2N+2,2} & \cdots & r_{2N+2,2N+1} & r_{2N+2,2N+2} \end{array}\right]$$

where the first and last rows are defined as

$$[r_{1,2j-1},\ r_{1,2j}] \equiv \frac{(-1)^{j+1}}{c_N}\,[c_{N-j+1}\ \ a_{N-j+1}]$$

$$j=1,\ldots,N+1$$

$$[r_{2N+2,2j-1},\ r_{2N+2,2j}] \equiv \frac{(-1)^{N-j}}{c_N}\,[-c_{j-1}\ \ a_{j-1}]$$

while the 2x2 matrices $R_{n,m}$ are defined as

$$R_{n,m} \equiv A^{-1}[(-T)^{n-1}Z_m + \sigma_{n,m}(-T)^{n-m}],$$
$$n=1,\ldots,N \quad , \quad m=1,\ldots,N+1$$

with

$$Z_1 \equiv \begin{bmatrix} 0 & \lambda_N \\ 0 & 1 \end{bmatrix}, \quad Z_m \equiv \frac{(-1)^m}{c_N}\begin{bmatrix} c_{N-m+1} & a_{N-m+1} \\ 0 & 0 \end{bmatrix} \quad m=2,\ldots,N+1$$

and

$$\sigma_{n,m} = \begin{cases} 1 & \text{if } 2 \leq m \leq n \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.1. The matrix $H_N^{-1} G_N$ is invertible and its inverse is the matrix R.

Proof: Let $S \equiv R(H_N^{-1} G_N)$. It is enough to show that S=I. We partition S into blocks:

$$S = \begin{bmatrix} s_{11} & \tau_{11} & \cdots & \tau_{1N} & s_{1,2N+2} \\ \omega_{11} & S_{11} & \cdots & S_{1N} & \omega_{1,2N+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \omega_{N1} & S_{N1} & & S_{NN} & \omega_{N,2N+2} \\ s_{2N+2,1} & \tau_{2N+2,1} & \cdots & \tau_{2N+2,N} & s_{2N+2,2N+2} \end{bmatrix}$$

where each s_{ij} is 1x1, S_{ij} is 2x2, ω_{ij} is 2x1 and τ_{ij} is 1x2.

Performing the multiplication of the matrices R and $H_N^{-1} G_N$ we obtain

$$s_{11} = r_{11} = 1$$

$$\begin{aligned} \tau_{ij} = [s_{1,2j} \;\; s_{1,2j+1}] &= [r_{1,2j-1} \;\; r_{1,2j}]\, A + [r_{1,2j-1} \;\; r_{1,2j+2}]B \\ &= \frac{(-1)^j}{c_N}\{[c_{N-j+1} \;\; a_{N-j+1}] - [c_{N-j} a_{N-j}]T\}A \\ &= \frac{(-1)^j}{c_N}\{[c_{N-j+1} \;\; a_{N-j+1}] - [c_{N-j+1} \;\; a_{N-j+1}]\}A \\ &= [0, \;\; 0] \end{aligned}$$

and

$$s_{1,2N+2} = r_{1,2N+1} = 0 \quad .$$

Similarly

$$\omega_{i,1} = \omega_{i,2N+2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; \tau_{2N+2,j} = [0 \; 0], \; i,j = 1,\ldots,N$$

and

$$s_{2N+2,1} = 0 \quad , \; s_{2N+2,2N+2} = 1.$$

For the square blocks $S_{n,m}$ we find

$$S_{n,m} = R_{n,m} A + R_{n,m+1} B$$

$$= A^{-1}(-T)^{n-1}\{Z_m + Z_{m+1} T + (\sigma_{n,m}-\sigma_{n,m+1})(-T)^{1-m}\} A$$

From the definition of Z_m and T we obtain $Z_m + Z_{m+1} T = \delta_1^m I$.

Then from the definition of $\sigma_{n,m}$ we get

$$S_{n,m} = \delta_n^m I.$$

This concludes the proof of Lemma 3.1.

Lemma 3.2. If G_N is the Grammian of the interpolation operator Q_N then

(3.4) $$||(H_N^{-1} G_N)^{-1}||_\infty < 100$$

for all $N \geq 2$.

Proof. Let

$$||R||_\ell \equiv \Sigma_{m=1}^{2N+2} |r_{\ell m}| \quad .$$

From the definition of R and relations (3.1), (3.2), (3.3), we obtain

$$||R||_1 \equiv \Sigma_{j=1}^{N}(|r_{1,2j+1}|+|r_{1,2j}|)$$

$$= \frac{1}{|c_N|} \Sigma_{j=1}^{N}(|c_{N-j+1}|+|a_{N-j+1}|)$$

$$= \frac{1}{|c_N|} \Sigma_{\ell=1}^{N}(|c_\ell|+|a_\ell|)$$

$$\leqq \frac{7}{12}\frac{|a_N|}{|c_N|} + \frac{9}{2} + \frac{5}{15}\frac{|a_1|}{|c_N|} - \frac{7}{2}\frac{|c_1|}{|c_N|}$$

$$\leqq 23/2 \quad .$$

It is easy to see that $||R||_{2N+2} = ||R||_1$. For the remaining rows we use (3.1) (3.2) to get that for $2 \leqq m \leqq n$

$$AR_{n,m} = (-T)^{n-1} z_m + \sigma_{n,m} (-T)^{n-m}$$

$$= \frac{1}{2|c_N|} \begin{bmatrix} -|c_{N-n-m+2}|+|c_{N-n+m}| & |a_{N-n-m+2}|+|a_{N-n+m}| \\ \frac{1}{48}(-|a_{N-n-m+2}|+|a_{N-n+m}|) & |c_{N-n-m+2}|+|c_{N-n+m}| \end{bmatrix}$$

while for $n < m$

$$AR_{n,m} = \frac{1}{2|c_N|} \begin{bmatrix} -|c_{N+n-m}|+|c_{N-n-m+2}| & |a_{N+n-m}|+|a_{N-n-m+2}| \\ \frac{1}{48}(|a_{N+n-m}|-|a_{N-n-m+2}|) & |c_{N+n-m}|-|c_{N-n-m+2}| \end{bmatrix}$$

Finally, for m=1

$$AR_{n,1} = \frac{1}{|c_N|} \begin{bmatrix} 0 & |a_{N-n+1}| \\ 0 & |c_{N-n+1}| \end{bmatrix}$$

Using again the relations (3.1) through (3.3) we now find

$$\sum_{m=1}^{N} ||AR_{n,m}||_1 \leqq$$

$$\frac{1}{2}\left[2\frac{|a_{N-n+1}|}{|c_N|} + \frac{1}{96}\left(\frac{|a_{N-n+1}|}{|c_N|} + \frac{|a_{N-n}|}{|c_N|} + \frac{|a_N|}{|c_N|} + \frac{|a_{N-1}|}{|c_N|}\right)\right.$$

$$\left. + \frac{1}{2}\left(\frac{|c_{N-n}|}{|c_N|} + 9 + \frac{|c_{N-1}|}{|c_N|} + \frac{|a_N|}{|c_N|}\right)\right] \leqq \frac{35}{3}$$

and

$$\sum_{m=1}^{N} ||AR_{n,m}||_2 \leqq$$

$$\frac{1}{2}\left[2\,\frac{|c_{N-n+1}|}{|c_N|} + \frac{1}{96}\left(\frac{|a_{N-n}|}{|c_N|} + 9\,\frac{|a_N|}{|c_N|} + 48 + \frac{|c_{N-n+1}|}{|c_N|}\right.\right.$$

$$\left.\left. + \frac{|c_{N-n}|}{|c_N|} + \frac{|a_{N-1}|}{|c_N|} + \frac{|a_n|}{|c_N|} + \frac{|c_{N-1}|}{|c_N|}\right)\right] \leqq 2 \quad .$$

By definition now, we have for $\ell=1,2$

$$||R||_{2n+\ell} = \sum_{m=1}^{N} ||A^{-1}AR_{n,m}||_\ell \leqq \sum_{m=1}^{N} ||A^{-1}||_\infty ||AR_{n,m}||_\ell$$

while

$$||A^{-1}||_\infty = \frac{7\sqrt{3}+9}{4} \quad .$$

Thus, for the norm $||R||_\infty = \max_i ||R||_i$ the following bound holds

$$||R||_\infty = ||(H^{-1}G_N)^{-1}||_\infty < 100.$$

This concludes the proof of Lemma 3.2.

Remark. As the proof of Lemma 3.2 suggests the bound (3.4) can be improved. Our conjecture is that a more careful analysis will show that the norm $||(H_N^{-1}G_N)^{-1}||_\infty$ is decreasing in N, that

$$\lim_{N\to\infty}||(H_N^{-1}G_N)^{-1}||_\infty = \frac{69-29\sqrt{3}}{2}$$

and that for all $N \geq 2$

$$\frac{69-29\sqrt{3}}{2} \leqq ||(H_N^{-1}G_N)^{-1}||_\infty \leqq ||H_2^{-1}G_2|| = \frac{33\sqrt{3}+9}{7}$$

Numerical experiments confirm this conjecture.

Lemma 3.3. The Gramian matrix G_N of the interpolation operator Q_N is nonsingular and

$$||G_N^{-1}||_\infty \leqq 100\ N \tag{3.5}$$

for all $N \geq 2$.

Proof. (3.5) follows easily from Lemmata 3.1 and 3.2.

Lemma 3.4. Let Q_N be the interpolation operator defined by (2.2). Then, (i) Q_N is a linear projector on $C^1(I)$ with range H_Δ and (ii) there exists a constant c such that $||Q_N|| \leqq c\ N$.

Proof. Conclusion (i) follows easily from Lemma 3.3. To prove (ii) let Λ be the dual space of H_Δ and $\{B_i\}_{i=1}^{2N+2}$ and $\{\delta_{\sigma_i}\}_{i=1}^{2N+2}$ be bases for H_Δ and Λ, where δ_{σ_i} are the point evaluation functionals. Using [1, Prop. 3] one may easily show that

$$||Q_\Delta|| \leqq \max_{a\epsilon R^n} \frac{||\Sigma_i a_i B_i||_\infty}{||a||} ||(\delta_{\sigma_i} B_j)^{-1}||_\infty \max_i ||\delta_{\sigma_i}||$$

$$\leqq 2\ ||G_N^{-1}||_\infty \leqq cN$$

where, $G_N = (\delta_{\sigma_i} B_j)$ and, by (3.5), c = 200. This concludes the proof of Lemma 3.4.

Theorem 3.1. If $f\epsilon W^{4,\infty}(I)$, then

(i) $Q_N f \rightarrow f$, as $N \rightarrow\infty$

and

(ii) for the interpolation error we have

$$||Q_N f-f||_\infty \leqq ch^4$$

where c is independent of h.

Proof. Let $\partial_H f$ be the Hermite interpolant of f, defined by interpolation of f and its first derivative at the nodes of the partition Δ. From the triangle inequality we find

$$(3.6)\quad ||f-Q_N f||_\infty \leqq (1+||Q_N||)||f-\partial_H f||_\infty \quad .$$

Moreover, for the Hermite interpolation error, it is known [10, Thrm 3.6]

$$(3.7) \quad ||f - \partial_H f||_\infty \leqq \frac{1}{384} h^4 ||D^4 f||_\infty \quad .$$

From (3.6), (3.7) and Lemma 3.4, we now get

$$||f - Q_N f||_\infty = O(h^3).$$

This proves conclusion (i).

Also, Theorem 2 [9, p. 251] and conclusion (i) imply that there is a constant K, independent of N, such that

$$(3.8) \quad ||Q_N|| \leqq K \qquad N=2,3,...$$

From (3.6) and (3.8) conclusion (ii) follows.

Theorem 3.2. If $f \in W^{4,\infty}(I \times J)$ then for the interpolation error we have

$$||Q_\rho f - f||_\infty \leqq c|\rho|^4$$

where $|\rho| = \max(h,k)$ and c is a constant independent of h and k.

Proof. From (3.6)-(3.8) and the triangle inequality we have

$$\begin{aligned} ||f-Q_\rho f||_\infty &\leqq ||f-Q_N f||_\infty + ||Q_N f - Q_N Q_M f||_\infty \\ &\leqq ||f-Q_N f||_\infty + ||Q_N|| \; ||f-Q_M f||_\infty \\ &\leqq c\,(h^4+k^4) \leqq c|\rho|^4 \end{aligned}$$

which concludes the proof of the theorem.

4. Numerical results. In this section we present some numerical results concerning the approximation of the functions e^x, x^4 by interpolation at Gaussian points with the space H_Δ. These results indicate that the interpolation scheme introduced at Section 2 is fourth-order accurate in the L_∞-norm. The partition Δ used is uniform with mesh length $h = 1/N$. The rate of convergence estimate

$$-\log \left(\frac{\text{error for } h}{\text{error for } h/2}\right)/\log 2$$

is also given.

N	$\|\|e^x - Q_N e^x\|\|_\infty$	Convergence Rate
3	3.106×10^{-5}	
6	2.325×10^{-6}	3.74
12	1.646×10^{-7}	3.8
24	1.096×10^{-8}	3.9
48	7.070×10^{-10}	3.95

N	$\|\|x^4 - Q_N x^4\|\|_\infty$	Convergence Rate
3	4.155×10^{-4}	
6	2.678×10^{-5}	3.96
12	1.674×10^{-6}	3.99
24	1.047×10^{-7}	4.00
48	6.541×10^{-9}	4.00

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