# PIECEWISE LINEAR EMBEDDINGS OF MANIFOLDS

BY

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## 0. Introduction

The problem of determining conditions which imply that one piecewise linear manifold embeds in another or that two piecewise linear embeddings are isotopic has been studied through many approaches. One such approach which provides a connection between these two problems is the concept of a concordance between two embeddings. In 1966, J. F. P. Hudson [7] proved that, if the codimension is at least three, the existence of a concordance between two embeddings implies the existence of an isotopy. In other words the "block" arc components of the space of embeddings correspond bijectively to the usual arc components, cf. [16], [17]. Thus it is natural to hope that a similar approach might prove helpful in an attempt to determine the higher homotopy groups of the (semisimplicial) complex of embeddings by studying the block-homotopy groups (or concordance-homotopy groups [1]) of the embeddings and to determine the relationship between these two structures. Indeed C. Morlet [15] has given a method attacking the first part of the problem and a somewhat less successful method for the second part. In [13] the author has provided another approach to this latter problem which leads to more satisfactory results. The purpose of this paper is to describe the implications of this method and its consequences in light of Morlet's block-homotopy results.

In Section 1 the main results are stated and related to previous results. Section 2 contains a discussion of the definitions while Section 3 is devoted to the heart of the development, an analysis of several special cases. The more general results are proved in Section 4. The last section contains a useful calculation of some homotopy groups of $V^{PL}_{n+k,k}$.

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## 1. The main results

By employing a basic result of the study of concordance and isotopy of embeddings of balls in manifolds the following theorem will be proved.

THEOREM 3.2. *Let* $f\colon (D^m, \Sigma^{m-1}) \to (N, \partial N)$ *be a proper piecewise linear embedding. If* $\pi_j(N) = 0$, $j \leq k$, *and* $n - m \geq 3$ *the relative* $i$th *homotopy group comparing the space of embeddings and the block space is zero when* $i \leq n + k - m - 2$.

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This result and an extension of Morlet's sequence for the block homotopy of embedding spaces gives a useful calculation in Theorem 3.12. In the special case of a sphere a calculation of Rourke and Sanderson is recovered [17]. Specifically

$$\pi_i(\tilde{\mathscr{E}}(*, \Sigma^n)) \cong \pi_i(G_{n+1}, G_n), \quad n > 2. \tag{3.15}$$

As a consequence one proves Theorem 3.18 which, in the case of a sphere, reduces to the statement

$$\theta_0 \colon \pi_i(\mathscr{E}(D^m, D^m \times \Sigma^n; j)) \to \pi_{i+m}(G_{n+1}, G_n) \tag{3.19}$$

is an isomorphism for $i < 2n - 3$ and an epimorphism for $i = 2n - 3$. A generalization of this result plays a key role in the general case and in a special case gives an answer to a question of Zeeman [18], cf. Husch [6].

COROLLARY 3.21. *If $n \geq 3$ there are compatible homomorphisms*

$$\theta_k \colon \pi_i(\mathscr{E}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j)) \to \pi_{i+m}(G_{n+k+1}, G_n)$$

*which are isomorphisms if $i < 2n + k - 3$.*

The case $m = 0$ proves the conjecture of Zeeman and provides information well beyond his range. This same result provides the proof of Theorem 5.1 which says that if $n \geq 3$, then

$$\pi_i(V^{PL}_{n+k,k}) \to \pi_i(G_{n+k}, G_n)$$

is an isomorphism for $i < 2n + k - 3$. This is a substantial extension of an old result of Haefliger and Wall [4, Theorem 3].

In Section 4 an extension of Morlet's "Lemme de disjonction" is stated and could be used (as he does) to study embedding spaces. In this paper another method is used to give the basic comparison between the homotopy groups of the space of embeddings, $\mathscr{E}(M, N)$, $n - m \geq 3$, and the block space of embeddings, $\tilde{\mathscr{E}}(M, N)$, in Theorem 4.2. Although the theorem is rather complicated its corollaries, 4.8 and 4.9, state that the respective $i$th homotopy groups are isomorphic if $i < n - m - 2$, for general $N$, and $i < 2(n - m) - 2$ if $N = D^n$. Thus the strength of the theorem is not in its generalities but, rather, in its application to specific cases.

## 2. Definitions

All of the definitions and results concern either the simplicial, block, or piecewise linear categories. For the definitions and properties of the simplicial category see Curtis [2], for the block category see Rourke and Sanderson [16], and for the piecewise linear category see either Zeeman [18] or Hudson [8].

Objects in the simplicial and block categories are distinguished by a tilde ($\tilde{\ }$) appearing over the objects in the block category. Thus $\pi_i(X)$ and $\pi_i(\tilde{X})$ denote the $i$th homotopy groups of an object in the simplicial category and the $i$th block-homotopy groups of a related object in the block category. It will be important to compare these groups by including them in an exact sequence, cf. Morlet [15],

$$\to \pi_i(X) \to \pi_i(\tilde{X}) \to \pi_i^{\mathrm{rel}}(X) \to .$$

This is accomplished by defining $\pi_i^{\mathrm{rel}}(X)$ to be the group $\pi_i(\tilde{X}, X)$.

Let $M$ and $N$ denote compact piecewise linear manifolds of dimensions $m$ and $n$ and boundaries $\partial M$ and $\partial N$, respectively. Let $\Delta^s$ denote the standard $s$ dimensional simplex, $I = [0, 1]$, $rJ = [-r, r]$, $(rJ)^n = rD^n$, and $\Sigma^{n-1} = \partial D^n$. A subspace $(K, K_0)$ of a space $(L, L_0)$ is said to be proper if $K \backslash K_0 \subset L \backslash L_0$. A piecewise linear map $f$: $(K, K_0) \to (L, L_0)$ is proper if $(f(K), f(K_0))$ is a proper subcomplex of $(L, L_0)$.

2.1. Definition. Given a proper piecewise linear embedding of a proper subcomplex of $(M, \partial M)$ in $(N, \partial N)$, $f$: $(K, K_0) \to (N, \partial N)$, let $\mathscr{E}(M, N; f)$ denote the simplicial complex of locally unknotted proper piecewise linear embeddings of $(M, \partial M)$ into $(N, \partial N)$ extending $f$. That is the simplicial complex whose $s$ simplices are proper piecewise linear embeddings

$$\begin{array}{ccc} F\colon (M, \partial M) \times \Delta^s & \longrightarrow & (N, \partial N) \times \Delta^s \\ \downarrow p & & \downarrow p \\ 1\colon \Delta^s & \longrightarrow & \Delta^s \end{array}$$

such that:

(i) the diagram is commutative;
(ii) $F^{-1}(\partial N \times \Delta^s) = \partial M \times \Delta^s$;
(iii) $F \mid (K, K_0) \times \Delta^s = f \times 1$;
(iv) for any simplex $\Delta$ linearly embedded in $\Delta^s$, $(N \times \Delta, F(M \times \Delta))$ is a locally unknotted manifold pair.

If $f$ is not specified this complex is denoted by $\mathscr{E}(M, N)$. If

$$f\colon (M, \partial M) \to (N, \partial N)$$

then by $\mathscr{E}(M, N; f)(\mathscr{E}(M, N; f \mid K))$ we understand

$$\mathscr{E}(M, N; f \mid (\partial M, \partial M))(\mathscr{E}(M, N;\ f \mid (\partial(M) \cup K, \partial M)))$$

and $f$ also provides the basepoint of the spaces.

The block complexes, $\tilde{\mathscr{E}}(M, N; f)$ and $\tilde{\mathscr{E}}(M, N)$, are defined as above except that condition (i) is replaced by:

(i) for all faces $\Delta'$ of $\Delta^s$, $F \mid (M, \partial M) \times \Delta'$ is an embedding of $(M, \partial M) \times \Delta'$ into $(N, \partial N) \times \Delta'$.

2.2. Definition. Let $(K, K_0)$ be a piecewise linear space. The complexes of proper piecewise linear homeomorphisms of $(K, K_0)$ onto itself which are fixed on $K_0$ are denoted $\mathscr{H}(K, K_0)$ and $\tilde{\mathscr{H}}(K, K_0)$. If $K$ is a manifold the subcomplex of $\mathscr{H}(K, K_0)$ of homeomorphisms which are fixed on $\partial(K) \cup K_0$ is denoted by $\mathscr{H}_\partial(K, K_0)$. If $K_0$ is empty these spaces are denoted by $\mathscr{H}(K)$ and $\mathscr{H}_\partial(K)$, respectively.

2.3. *Remark.* Inasmuch as all the definitions have analogues in the block category which are apparent, given those in the simplicial category, only the simplicial definitions will be given.

2.4. DEFINITION. The complexes of germs of proper embeddings of $(M, \partial M)$ into $(N, \partial N)$ extending $f$ and of germs of homeomorphisms of $K$ which fix $K_0$ are denoted by $\mathscr{G}\mathscr{E}(M, N; f)$ and $\mathscr{G}\mathscr{H}(K, K_0)$, respectively. The first, for example, consists of the simplices of the complexes of proper embeddings of neighborhoods $(U, U_0)$ of $(K, K_0)$ into $(N, \partial N)$ which extend $f \mid (K, K_0)$.

2.5. DEFINITION. The complex of proper piecewise linear maps of $(L, L_0)$ into $(L', L_0')$ extending $f\colon (K, K_0) \to (L', L_0')$ is denoted by

$$\mathscr{M}((L, L_0), (L_0', L_0'); f).$$

If $(L, L_0) = (L', L_0')$ this complex is denoted by $\mathscr{M}(L, L_0; f)$.

2.6. DEFINITION. Let $i\colon (L_0, L_0) \to (L, L_0)$ denote the inclusion. The subcomplex of $\mathscr{M}(L, L_0; i)$ consisting of homotopy equivalences is denoted by $\hbar(L, L_0)$, that is to say if $F$ is an $s$-simplex of $\hbar(L, L_0)$ there is another $s$-simplex $G$ of $\hbar(L, L_0)$ such that $FG$ and $GF$ are homotopic to the identity through homotopies which are fixed on $L_0$, leave $L \backslash L_0$ setwise invariant, and respect the projection to $\Delta^s \times I$. If $L$ is a manifold and $L_0 = \partial L$ then $\hbar(L, L_0)$ is denoted by $\hbar_\partial(L)$. The subcomplex of those equivalences keeping a subcomplex $K$ fixed is denoted by $\hbar_\partial(L)_K$.

2.7. *Remark.* It is easy to see that $\pi_j(\hbar(L, L_0))$ is isomorphic to

$$\pi_j(\mathscr{M}(L, L_0; i)),$$

via the inclusion, for $j \geq 1$.

2.8. DEFINITION. The complex of proper concordances of (germs of) proper embeddings extending a proper embedding $f\colon (K, K_0) \to (N, \partial N)$ of a proper subcomplex of $(M, \partial M)$ is denoted by $\mathscr{C}(M, N; f)(\mathscr{G}\mathscr{C}(M, N; f))$. It is the subcomplex of

$$\mathscr{E}(I \times M, I \times N; 1 \times f)(\mathscr{G}\mathscr{E}(I \times M, I \times N; 1 \times f))$$

whose $s$-simplices $F$ satisfy

(i) $F^{-1}(\{0\} \times N \times \Delta^s) = \{0\} \times M \times \Delta^s$ and
(ii) $F^{-1}(\{1\} \times N \times \Delta^s) = \{1\} \times M \times \Delta^s$.

2.9. DEFINITION. The complex of (germs of) proper isotopies of proper embeddings extending a proper embedding $f\colon (K, K_0) \to (N, \partial N)$ of a proper subcomplex of $(M, \partial M)$ is denoted by $\mathscr{I}(M, N; f)(\mathscr{G}\mathscr{I}(M, N; f))$. It is the complex of paths in $\mathscr{E}(M, N; f)(\mathscr{G}\mathscr{E}(M, N; f))$ or, equivalently, the subcomplex of

$$\mathscr{E}(I \times M, I \times N; 1 \times f)(\mathscr{G}\mathscr{E}(I \times M, I \times N, 1 \times f))$$

such that, for any $s$-simplex $F$, the diagram

$$\begin{array}{ccc} F: I \times M \times \Delta^s & \longrightarrow & I \times N \times \Delta^s \\ p\downarrow & & p\downarrow \\ 1: I \times \Delta^s & \longrightarrow & I \times \Delta^s \end{array}$$

is commutative.

2.10. Definition. Let $f: (M\ \partial M) \to (N, \partial N)$ be a proper piecewise linear embedding. $\mathscr{C}_\partial(M, N; f \mid K)(\mathscr{C}_{\partial_0}(M, N; f \mid K))$ and $\mathscr{I}_\partial(M, N; f \mid K)(\mathscr{I}_{\partial_0}(M, N; f \mid K))$ are the subcomplexes of $\mathscr{C}(M, N; f \mid \partial M \cup K)$ and $\mathscr{I}(M, N; f \mid \partial M \cup K)$, respectively, whose simplices $F$ satisfy the property that

$$F \mid (\{0, 1\} \times M \times \Delta^s) = 1 \times f \times 1(F \mid (\{0\}) \times M \times \Delta^s) = 1 \times f \times 1).$$

The analogous definitions for concordances and isotopies of germs of embeddings are also assumed.

Several simplicial fibrations and quasi-fibrations will be required in the next section. These are recognized via standard methods, c.f [11, 13, 15], by employing Hudson's $s$-covering isotopy theorem, Theorem 4.1 of [9].

2.11. Proposition. *Let*

$$(D^k, \Sigma^{k-1}) \xrightarrow{\iota} (M, \partial N) \xrightarrow{f} (N, \partial N)$$

*be embeddings with $n - k \geq 3$ and $n - m \geq 3$. In both categories*

(i) $\mathscr{E}(M, N; f \mid \iota(D^k) \subsetneq \mathscr{E}(M. N; f) \to \mathscr{E}(D^k, N; f\iota)$ *is a simplicial fibration.*
(ii) *If* $D^k = D^i \times D^j$,

$$\mathscr{E}(M, N; f \mid \iota(D^i \times \tfrac{1}{2}D^j)) \subsetneq \mathscr{E}(M, N; f \mid \iota(D^i \times \{0\})) \to \mathscr{G}\mathscr{E}(D^i \times D^j, N; f\iota \mid D^i \times \{0\})$$

*is a simplicial quasi-fibration.*

2.12. *Remark.* Indeed Proposition 2.11(ii) would be a simplicial fibration if the neighborhood of the germ had not been normalized to $D^i \times \frac{1}{2}D^j$. Furthermore these imply the existence of analogous fibrations for concordance and isotopy spaces merely by employing Definitions 2.8 and 2.9, respectively. It is the pairs of these fibrations that are actually required.

Finally a simple consequence of the handle-body theory of piecewise linear manifolds will be required, cf. [8]. Briefly, suppose that $M$ is a piecewise linear manifold and $g: \partial(D^r) \times D^{m-r} \to \partial M$ is a piecewise linear embedding. Then $M' = M \cup_g D^r \times D^{m-r}$ is a piecewise linear manifold obtained from $M$ by attaching an $r$-handle to $M$. A handle-body decomposition for a manifold $M$ is a sequence of manifolds

$$D^m = M_0 \subset M_1 \subset \cdots \subset M_{s+1} = M$$

where $M_{i+1}$ is obtained from $M_i$ by attaching an $r_i$-handle.

2.13. PROPOSITION. *Let $M$ be a connected piecewise linear manifold; then $M$ has a handle-body decomposition such that $r_i \geq r_{i+1}$ for all $i$, and*

(i) *if $\partial M = \emptyset$ there is precisely one 0-handle and one $m$-handle, and*
(ii) *if $\partial M \neq \emptyset$ there is precisely one 0-handle and no $m$-handles.*

## 3. Important special cases

A fundamental result in the study of concordances and isotopies is the following theorem.

3.1. THEOREM [13, 1.11]. *Let $f: (D^m, \Sigma^{m-1}) \to (N, \partial N)$ be a proper piecewise linear embedding. If $\pi_j(N) = 0, j \leq k$ and $n - m \geq 3$, then*

$$\pi_s(\mathscr{C}_\partial(D^m, N), \mathscr{I}_\partial(D^m, N)) = 0 \quad \text{if } s \leq n + k - m - 3.$$

As an immediate consequence one has the following proposition relating the homotopy and block-homotopy of embedding spaces. We note that Morlet [15, p. 29] has this theorem with $k$ replaced by $\tilde{k} = \min(n - m - 3, k)$.

3.2. THEOREM. *Let $f: (D^m, \Sigma^{m-1}) \to (N, \partial N)$ be a proper piecewise linear embedding. If $\pi_j(N) = 0, j \leq k$ and $n - m \geq 3$ then $\pi_i^{\mathrm{rel}}(\mathscr{E}(D^m, N; f)) = 0$ for $i \leq n + k - m - 2$ or, equivalently, the homomorphism*

$$\pi_i(\mathscr{E}(D^m, N; f)) \to \pi_i(\tilde{\mathscr{E}}(D^m, N; f))$$

*is an isomorphism if $i < n + k - m - 2$ and an epimorphism if $i = n + k - m - 2$.*

*Proof.* The proof is by induction on $i$, the case $i = 0$ being a direct consequence of the definition. Consider the following commutative diagram,

$$\begin{array}{ccc}
\pi_{i-1}(\mathscr{I}_\partial(D^m, N; f)) & \xrightarrow{\cong} & \pi_i(\mathscr{E}(D^m, N; f)) \\
\text{Theorem 3.1} \downarrow \cong & & \downarrow \text{Theorem 3.2 } (i) \\
\pi_{i-1}(\mathscr{C}_\partial(D^m, N; f)) & & \pi_i(\tilde{\mathscr{E}}(D^m, N; f)) \\
\downarrow \cong & & \downarrow \cong \\
\pi_{i-1}(\mathscr{E}(D^1 \times D^m, D^1 \times N; 1 \times f)) & \underset{\text{Theorem 3.2 } (i-1)}{\xrightarrow{\cong}} & \pi_{i-1}(\tilde{\mathscr{E}}(D^1 \times D^m, D^1 \times N; 1 \times f))
\end{array}$$

which demonstrates that Theorem 3.1 and Theorem 3.2 for $i - 1$ implies Theorem 3.2 for $i$.

Thus one is led to seek specific information concerning the homotopy groups of these particular block embedding spaces. As observed, there are often special "looping" and "delooping" identifications possible in the block category, For example, if $j\colon D^m \to D^m \times N$ is given by $j(x) = (x, *)$, where $*$ is a generic notation for the basepoint of a space, then $\pi_i(\tilde{\mathscr{E}}(D^m, D^m \times N; j))$ is equal to

$$\pi_{i+m}(\tilde{\mathscr{E}}(*, N)), \quad \pi_0(\tilde{\mathscr{E}}(D^{i+m}, D^{i+m} \times N; 1 \times j))$$

and

$$\pi_0(\mathscr{E}(D^{i+m}, D^{i+m} \times N; 1 \times j)).$$

These identifications have provided interesting calculations, in [13], which are extended here. This is accomplished by studying $\pi_i(\mathscr{E}(*, N))$.

3.3. Proposition. *Let $i(0) \in N \backslash \partial N = N^\circ$, $N$ a piecewise linear manifold of dimension $n \geq 3$, be a fixed point. Then*

(i) *$\mathscr{E}(D^n, N^\circ; i \mid \{0\}) \to \mathscr{E}(D^n, N^\circ) \to \mathscr{E}(*, N^\circ)$ is a fibration, and*

(ii) $$\mathscr{h}_\partial(N, \tfrac{1}{2}D^n) \to \mathscr{h}_\partial(N, *) \xrightarrow{\gamma} G_n$$

*is a weak quasi-fibration where $\gamma$ is the composition of the germ homomorphism to $\mathscr{h}(D^{\circ n}, 0)$ followed by the inverse of the homotopy equivalence from $\mathscr{h}(\Sigma^{n-1})$ given by taking the germ of the cone.*

3.4. *Remark.* There are corresponding fibrations in the block category. Also the natural inclusion of $\mathscr{E}(*, N^\circ)$ into $\mathscr{E}(*, N)$ is a homotopy equivalence.

The following theorem, a direct extension of a result of Morlet [15, p. 19] to the nonsimply connected case, provides a crucial ingredient at this point.

3.5. Theorem. *Let $f\colon (M, \partial M) \to (N, \partial N)$ be a proper embedding, $(E, E_\partial)$ a regular neighborhood of $f(M, \partial M)$ in $(N, \partial N)$ with inclusion $\imath$. If $n - m \geq 3$ and $n \geq 5$ then there is an exact sequence*

$$\begin{array}{c} \downarrow \\ \pi_i(\tilde{\mathscr{E}}(E, N; \imath \mid E_\partial)) \\ \downarrow \\ \pi_i(\tilde{\mathscr{h}}_\partial(N), \tilde{\mathscr{h}}_\partial(N, E)) \\ \downarrow \eta \\ \pi_i(\mathscr{M}((M, \partial M), (G/PL, *))) \\ \downarrow \\ \vdots \\ \downarrow \\ \pi_1(\mathscr{M}((M, \partial M), (G/PL, *))) \end{array} \tag{3.6}$$

*where $\eta$ is the normal invariant.*

3.7. *Remark.* Because of deformation retracts in the other stages, the tildes are actually only required at the stage $\pi_i(\tilde{\mathscr{E}}(E, N; \iota \mid E_\partial))$. As noted by Morlet this group also appears in the exact homotopy sequence of the weak fibration,

$$(3.8) \qquad \mathscr{E}(E, N; \iota \mid E_\partial \cup f(M)) \to \mathscr{E}(E, N; \iota \mid E_\partial) \to \mathscr{E}(M, N; f \mid \partial M).$$

This fibration provides a useful tool in the attempt to describe the homotopy of block embedding spaces by considering a handle-body decomposition of a general $M$ and working inductively over the handles, cf. Morlet [15]. At this point the case $M = *$ is all that is required. Then (3.6) reduces to Morlet's sequence [15, Theorem 5], i.e., if $\dim N \geq 5$ there is an exact homotopy sequence

$$(3.9) \qquad \begin{array}{c} \downarrow \\ \pi_{i+1}(G/PL) \\ \downarrow \\ \pi_i(\tilde{\mathscr{E}}(D^n, N^\circ)) \\ \downarrow \\ \pi_i(\mathscr{h}_\partial(N), \mathscr{h}_\partial(N, D^n)) \\ \downarrow \eta \\ \pi_i(G/PL) \\ \downarrow \end{array}$$

which terminates with $\pi_1(G/PL)$.

An analogous sequence is

$$(3.10) \qquad \begin{array}{c} \downarrow \\ \pi_{i+1}(G/PL) \\ \downarrow \\ \pi_i(\tilde{\mathscr{E}}(D^n, N^\circ; \iota \mid \{0\})) \\ \downarrow \\ \pi_i(\mathscr{h}_\partial(N, *), \mathscr{h}_\partial(N, D^n)) \\ \downarrow \eta \\ \pi_i(G/PL) \\ \downarrow \end{array}$$

also terminating with $\pi_1(G/PL)$. The basic method to arrive at such a sequence is the framed surgery-transversality techniques which proved so successful [3], [12], [17]. Alternatively, one can consider the inclusion of $\tilde{\mathscr{E}}(D^n, D^{\circ n}; \iota \mid \{0\})$ into $\tilde{\mathscr{E}}(D^n, N^\circ; \iota \mid \{0\})$ and note that it is a homotopy equivalence. By taking germs, this former space is easily recognized to be the homotopy type of $\tilde{P}L(n)$ [11]. Thus the exact sequence of the pair $\tilde{P}L(n) \subset \tilde{G}_n$ and the stability results of Rourke and Sanderson [17, Theorem 0.3, Corollary 5.5, and Theorem 1.10] give the following exact sequence, for $i \geq 1$,

$$
(3.11) \qquad
\begin{array}{c}
\downarrow \\
\pi_{i+1}(G/PL) \\
\downarrow \\
\pi_i(\tilde{P}L(n)) \cong \pi_i(\tilde{\mathscr{E}}(D^n, N^\circ; i \mid \{0\})) \\
\downarrow \\
\pi_i(\tilde{\mathscr{G}}_n) \\
\downarrow \eta \\
\pi_i(G/PL) \\
\downarrow
\end{array}
$$

Then Proposition 3.3 (ii) and (3.11) give (3.10).

3.12. THEOREM. *If* $n \geq 5$, *then, for* $i \geq 1$,

$$\pi_i(\tilde{\mathscr{E}}(*, N)) \cong \pi_i(\hbar_\partial(N), \hbar_\partial(N, *)).$$

3.13. COROLLARY. *If* $n \geq 5$, *then, for* $i \geq 1$,

$$\pi_i^{\mathrm{rel}}(\mathscr{E}(*, N)) \cong \pi_i(\hbar_\partial(N)_*, \hbar_\partial(N, *)).$$

*Proof of Theorem* 3.12 *and Corollary* 3.13. The exact sequences (3.9) and (3.10) are related via the following homotopy lattice of a cube:

$$
\begin{array}{ccccccccc}
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \pi_{i+2}(\boxplus) & \xrightarrow{\partial} & \pi_{i+1}(G/PL) & \xrightarrow{\mathrm{id}} & \pi_{i+1}(G/PL) & \longrightarrow & \pi_{i+1}(\boxplus) & \text{—} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \pi_{i+1}(\tilde{\mathscr{E}}(*, N)) & \xrightarrow{\partial} & \pi_i(\tilde{\mathscr{E}}(D^n, N; i \mid \{0\})) & \longrightarrow & \pi_i(\tilde{\mathscr{E}}(D^n, N^\circ)) & \longrightarrow & \pi_i(\tilde{\mathscr{E}}(*, N)) & \text{—} \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & \pi_{i+1}(\square) & \longrightarrow & \pi_i(\hbar_\partial(N, *), \hbar_\partial(N, D^n)) & \longrightarrow & \pi_i(\hbar_\partial(N), \hbar_\partial(N, D^n)) & \longrightarrow & \pi_i(\square) & \text{—} \\
 & \downarrow & & \downarrow \eta & & \downarrow \eta & & \downarrow & \\
\longrightarrow & \pi_{i+1}(\boxplus) & \longrightarrow & \pi_i(G/PL) & \xrightarrow{\mathrm{id}} & \pi_i(G/PL) & \longrightarrow & \pi_i(\boxplus) & \text{—}
\end{array}
$$

in which all rows and columns are exact sequences and all squares are commutative except those involving four $\partial$ homomorphism which are anticommutative [14].

Since the homomorphisms relating the exact sequences are induced by inclusions the normal invariants of (3.9) are taken identically to those of (3.10), that is the induced homomorphism on $\pi_i(G/PL)$ is the identity. Consequently $\pi_i(\boxplus)$ is trivial and hence $\pi_i(\tilde{\mathscr{E}}(*, N))$ is isomorphic to $\pi_i(\square)$ which is isomorphic to $\pi_i(\hbar_\partial(N), \hbar_\partial(N, *))$.

The exact homotopy sequence of $\mathscr{E}(*, N) \subsetneq \tilde{\mathscr{E}}(*, N)$ splits so that

$$\pi_i(\tilde{\mathscr{E}}(*, N)) \cong \pi_i(\mathscr{E}(*, N)) \oplus \pi_i^{\mathrm{rel}}(\mathscr{E}(*, N)).$$

Thus Theorem 3.9 and the isomorphisms

$$\pi_i(\mathscr{E}(*, N)) \cong \pi_i(N) \cong \pi_i(\mathscr{h}_\partial(N), \mathscr{h}_\partial(N)_*)$$

show that the exact sequence

$$\begin{array}{ccccccc}
\longrightarrow \pi_i(\mathscr{h}_\partial(N)_*, \mathscr{h}_\partial(N, *)) & \longrightarrow & \pi_i(\mathscr{h}_\partial(N), \mathscr{h}_\partial(N, *)) & \longrightarrow & \pi_i(\mathscr{h}_\partial(N), \mathscr{h}_\partial(N)_*) & \longrightarrow \\
 & & \uparrow \cong & & \uparrow \cong & \\
\longleftarrow \quad \pi_i^{\mathrm{rel}}(\mathscr{E}(*, N)) & \longleftarrow & \pi_i(\tilde{\mathscr{E}}(*, N)) & \rightleftarrows & \pi_i(\mathscr{E}(*, N)) & \longleftarrow
\end{array}$$

also splits. These splittings give the isomorphism

$$\pi_i^{\mathrm{rel}}(\mathscr{E}(*, N)) \cong \pi_i(\mathscr{h}_\partial(N)_*, \mathscr{h}_\partial(N, *).$$

3.14. *Remark.* If one takes $N = \Sigma^n$ in Theorem 3.12 the result has a recognizable answer. This follows from the observations that $\mathscr{h}_\partial(\Sigma^n, *)$ is homotopy equivalent to $\mathscr{h}(\Sigma^{n-1}) = G_n$, $\mathscr{h}_\partial(\Sigma^n, D^n)$ is contractible and, employing the standard notation, that $\mathscr{h}_\partial(\Sigma^n)_* = F_n$. Thus one has a version of the calculation of Rourke and Sanderson [17, Corollary 2.18, p. 454] that

(3.15) $$\pi_i(\tilde{\mathscr{E}}(*, \Sigma^n)) \cong \pi_i(G_{n+1}, G_n), \quad n > 2,$$

and

(3.16) $$\pi_i^{\mathrm{rel}}(\mathscr{E}(*, \Sigma^n)) \cong \pi_i(F_n, G_n), \quad n > 2.$$

3.17. *Problem.* The structure of $\pi_i(\mathscr{h}_\partial(N), \mathscr{h}_\partial(N, *))$ for other manifolds has not been determined. This determination appears to be a difficult problem but one which should be attacked so as to gain the most from this approach to embedding spaces.

3.18. THEOREM. *Let $j$: $D^m \to D^m \times N$ be given by $j(x) = (x. *)$. Then if $\pi_j(N) = 0, j \leq k, n \geq 5$ the homomorphism*

$$\pi_i(\mathscr{E}(D^m, D^m \times N; j)) \to \pi_{i+m}(\mathscr{h}_\partial(N), \mathscr{h}_\partial(N, *))$$

*is an isomorphism if $i < n + k - 2$ and an epimorphism if $i = n + k - 2$. (If $N = \Sigma^n$ it is only necessary that $n \geq 3$.)*

*Proof of Theorem* 3.18. This follows easily from Theorem 3.2, the observation that $\pi_i(\tilde{\mathscr{E}}(D^m, D^m \times N; j)$ is equal to $\pi_{i+m}(\tilde{\mathscr{E}}(*, N))$, and Theorem 3.12. If $N = \Sigma^n$, (3.15) is employed in the place of the more general Theorem 3.12.

If $N = \Sigma^n$, $n \geq 3$, this becomes the useful fact [13, Proposition 1.17]:

(3.19) $$\theta_0 \colon \pi_i(\mathscr{E}(D^m, D^m \times \Sigma^n; j)) \to \pi_{i+m}(G_{n+1}, G_n)$$

is an isomorphism if $i < 2n - 3$ and an epimorphism if $i = 2n - 3$.

It is easy to see that, since $\Sigma^n \backslash *$ is contractible,

$$\pi_i(\mathscr{E}(D^m, D^m \times \Sigma^n; j))$$

is isomorphic to

$$\pi_i(\mathscr{E}(D^m \times \Sigma^0, D^m \times \Sigma^n; j)),$$

in both categories, where $j$ also denotes the inclusion $j\colon \Sigma^0 \to \Sigma^n$. This information and the appearance of $\pi_i(\mathscr{E}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j))$ in questions concerning embeddings of arbitrary manifolds as well as its connections with $V^{PL}_{n+k,k}$ make the following theorem and its corollary of some interest.

3.20. THEOREM. *If $n \geq 3$ and $j\colon D^m \times \Sigma^k \subset D^m \times \Sigma^{n+k}$ denotes the inclusion the homomorphism*

$$\pi_i(\mathscr{E}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j)) \to \pi_i(\tilde{\mathscr{E}}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j)$$

*is an isomorphism for $i < 2n + k - 3$ and an epimorphism for $i = 2n + k - 3$.*

3.21. COROLLARY. *If $n \geq 3$ there are compatible homomorphisms*

$$\theta_k\colon \pi_i(\mathscr{E}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j) \to \pi_{i+m}(G_{n+k+1}, G_n)$$

*which are isomorphisms if $i < 2n + k - 3$ and epimorphisms if $i = 2n + k - 3$.*

*Proof of Theorem* 3.20. The proof is by induction on $k$. The case $k = 0$ is simply the observation which precedes the theorem and Theorem 3.2. Suppose the statement has been demonstrated for $0, 1, 2, \ldots, k - 1$. Consider the fibration

$$\text{(3.22)} \quad \mathscr{E}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j \mid D^m \times *) \to \mathscr{E}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j) \to \mathscr{E}(D^m, D^m \times \Sigma^{n+k}; j)$$

in both the categories and the inclusion of one in the other. Theorem 3.2 gives conditions for the inclusion of base spaces to induce an isomorphism on homotopy groups. Thus we must consider the effect upon the homotopy groups of the respective fibers. Information is supplied by considering the following fibrations (in both categories), cf. [11] or [13].

$$\text{(3.23)} \quad \mathscr{E}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j \mid D^m \times D^k) \to \mathscr{E}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j \mid D^m \times *) \to \mathscr{G}\mathscr{E}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j \mid D^m \times *),$$

$$\text{(3.24)} \quad \mathscr{E}(D^m \times D^k, D^m \times D^{n+k}; j \mid D^m \times (\tfrac{1}{2}D^k \cup \Sigma^{k+1})) \to \mathscr{E}(D^m \times D^k, D^m \times D^{n+k}; j \mid D^m \times (* \cup \Sigma^{k-1})) \to \mathscr{G}\mathscr{E}(D^m \times D^k, D^m \times D^{n+k}; j \mid D^m \times (* \cup \Sigma^{k-1})).$$

In the usual fashion (and in both categories) one observes that the base spaces of (3.23) and (3.24) are identical and that the fiber of (3.23) and the total space of

(3.24) are contractible. Finally the fiber of (3.24) has the homotopy type of $\mathscr{E}(D^{m+1} \times \Sigma^{k-1}, D^{m+1} \times \Sigma^{n+k-1}; j)$ so that the induction hypothesis gives conditions relating the homotopy groups of both categories which in turn gives conditions on the homomorphism relating the groups of the fibers (3.22). By the five lemma this implies that the induced homomorphism from

$$\pi_i(\mathscr{E}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j)) \quad \text{to} \quad \pi_i(\tilde{\mathscr{E}}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j))$$

is an isomorphism for $i < 2n + k - 3$ and an epimorphism for $i = 2n + k - 3$.

*Proof of Corollary* 3.21. Define

$$\tilde{\theta}_k\colon \pi_i(\tilde{\mathscr{E}}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}; j)) \to \pi_{i+m}(\tilde{G}_{n+k+1}, \tilde{G}_n)$$

as follows: Let $F\colon D^i \times D^m \times \Sigma^k \to D^i \times D^m \times \Sigma^{n+k}$ represent an element of the homotopy group. Considering $F$ as a concordance,

$$F\colon D^1 \times (D^{m+i+1} \times \Sigma^k) \to D^1 \times (D^{m+i-1} \times \Sigma^{n+k}),$$

apply Hudson's concordance-isotopy theorem [9] to find an ambient isotopy, $H_t$, of

$$D^1 \times (D^{m+i+1} \times \Sigma^{n+k}),$$

fixed on $\{0\} \times (D^{m+i-1} \times \Sigma^{n+k}) \cup D^1 \times \partial(D^{m+i-1} \times \Sigma^{n+k})$, such that

$$H_1 \mid D^1 \times (D^{m+i-1} \times \Sigma^k) = F.$$

Let $\tilde{\theta}_k[F] = [H_1]$. To show that $\tilde{\theta}_k$ is well defined suppose that $H_t$ and $H_t'$ are two such isotopies. Let $H_t'' = H_{1-2t}, 0 \le t \le \frac{1}{2}$ and $H_t'' = H'_{2t-1}, \frac{1}{2} \le t \le 1$. $H'' \mid D^1 \times D^1 \times (D^{m+i-1} \times \Sigma^k)$ defines a concordance, along the second factor, so that Hudson gives an ambient isotopy, $G_t$, fixed on

$$D^1 \times \{0\} \times (D^{m+i-1} \times \Sigma^{n+k}) \cup \Sigma^0 \times D^1 \times (D^{m+i-1} \times \Sigma^{n+k}) \cup D^1 \times D^1 \times \partial(D^{m+i-1} \times \Sigma^{n+k})$$

such that $G_1 \mid D^1 \times D^1 \times (D^{m+-1} \times \Sigma^k) = 1 \times F$. $G_1 H_t''$ defines a homotopy between $H_1$ and $H_1'$. Since $\pi_i(\tilde{G}_p, \tilde{G}_q) \cong \pi_i(G_p, G_q)$ all that remains is to show that the sequences

$$\begin{array}{ccc}
\pi_{i+1}(\tilde{\mathscr{E}}(D^m, D^m \times \Sigma^{n+k})) & \xrightarrow{\tilde{\theta}_0} & \pi_{i+m+1}(G_{n+k+1}, G_{n+k}) \\
\downarrow & & \downarrow \\
\pi_{i-1}(\tilde{\mathscr{E}}(D^{m+1} \times \Sigma^{k-1}, D^{m+1} \times \Sigma^{n+k-1})) & \xrightarrow{\tilde{\theta}_{k-1}} & \pi_{i+m}(G_{n+k}, G_n) \\
\downarrow & & \downarrow \\
\pi_i(\tilde{\mathscr{E}}(D^m \times \Sigma^k, D^m \times \Sigma^{n-k})) & \xrightarrow{\tilde{\theta}_k} & \pi_{i+m}(G_{n+k+1}, G_n) \\
\downarrow & & \downarrow \\
\pi_i(\tilde{\mathscr{E}}(D^m, D^m \times \Sigma^{n+k})) & \xrightarrow{\tilde{\theta}_0} & \pi_{i+m}(G_{n+k+1}, G_{n+k}) \\
\downarrow & & \downarrow
\end{array}$$

are related by homomorphisms so that the diagram is commutative. This is accomplished by checking the construction. One can show that $\tilde{\theta}_k$ is an isomorphism directly via transversality and surgery as in [3], [12], [17] or appeal to the five lemma and use an inductive argument.

3.25. *Remark.* The corollary is simply the observation that

$$\pi_i(\tilde{\mathscr{E}}(D^m \times \Sigma^k, D^m \times \Sigma^{n+k}))$$

is isomorphic to

$$\pi_{i+m}(G_{n+k+1}, G_n).$$

## 4. Embeddings of manifolds

As mentioned in the introduction the basic approach is the comparison between the homotopy groups of the embedding and block embedding spaces. In doing this a handle-body structure on a manifold is employed in addition to the methods and results of Theorem 3.20.

The theorems which are proved here are slight extensions of the results of Morlet [15]. The major differences are in the method of proof and applications to some special cases where these results give substantial improvements. The fundamental tool of Morlet is the "Lemme de disjonction." The following generalization can be given.

4.1. "LEMME DE DISJONCTION". *Let $N$ be a manifold of dimension $n$, $g$, and $h$ disjoint proper embeddings of $D^p$ and $D^q$ in $N$, respectively. If $n - p \geq 3$ and $\pi_j(N) = 0, j \leq k$, then*

$$\pi_i(\mathscr{C}_{\partial_0}(D^p, N; g), \mathscr{C}_{\partial_0}(D^p, N \backslash h(D^q); g) = 0 \quad \textit{for } i \leq 2n - p - q - 4.$$

Unfortunately this additional dimensional freedom is not sufficient to substantially improve upon his general results via the techniques of [15]. Another approach, however, does give a slight improvement.

As noted in (2.13) every manifold admits a handle-body structure. Under assumptions concerning the dimension of the manifold and its connectivity it is possible to show that a certain dimensional range of handles are not required other than the 0-handle, of course. This fact is reflected in the statement of the main theorem.

4.2. THEOREM. *Let $f\colon (M, \partial M) \to (N, \partial N)$ be a proper embedding and suppose that*

$$\begin{array}{ccccccc} D^m = M_0 & \subset & M_1 & \subset \cdots \subset & M_{s+1} & = & M \\ \cap & & \cap & & \cap & & \\ N_0 & \subset & N_1 & \subset \cdots \subset & N_{s+1} & = & N \end{array}$$

*is a handle-body structure on the pair such that $l < r_i < m - l$ for $0 \leq i \leq s + 1$ if $\partial M \neq \emptyset$ and $0 \leq i \leq s$ if $\partial M = \emptyset$. If $n - m \geq 3$ and $\pi_j(N) = 0$ for $j \leq k$ then the homomorphism*

$$\pi_i^{\mathrm{rel}}(\mathscr{E}(M_0, N_0; f \mid M_0)) \to \pi_i^{\mathrm{rel}}(\mathscr{E}(M, N; f))$$

*is an isomorphism if $i < n + l + \tilde{k} - m - 2$ and an epimorphism if $i = n + l + \tilde{k} - m - 2$, where $\tilde{k} = \min(k, n - m - 1)$.*

*Proof of Theorem* 4.2. The proof is by induction on $s$ and will first consider the case $\partial M \neq \emptyset$. For $s = -1$, i.e., $M = M_0$, the result is trivial. Assume then that the theorem has been proved for $t$ handles. Let

$$M_{t+1} = M_t \cup_g D^{r_t} \times D^{m-r_t}$$

and let $i: D^{m-r_t} \to M_{t+1}$ denote the inclusion of the cell transverse to the handle. Since $n - (m - r_t) \geq 3$, Proposition 2.11 implies the existence of a pair of fibrations:

$$(4.3)\qquad \begin{array}{ccc} \mathscr{E}(M_{t+1}, N_{t+1}; f \mid i(D^{m-r_t})) & \to & \tilde{\mathscr{E}}(M_{t+1}, N_{t+1}; f \mid i(D^{m-r_t})) \\ \downarrow & & \downarrow \\ \mathscr{E}(M_{t+1}, N_{t+1}; f) & \to & \tilde{\mathscr{E}}(M_{t+1}, N_{t+1}; f) \\ \downarrow & & \downarrow \\ \mathscr{E}(D^{m-r_t}, N_{t+1}; f \mid D^{m-r_t}) & \to & \tilde{\mathscr{E}}(D^{m-r_t}, N_{t+1}; f \mid D^{m-r_t}) \end{array}$$

Theorem 3.2 implies the homomorphism relating the $i$th homotopy groups of the base spaces is an isomorphism if $i < n + \tilde{k} + l - m - 1$ and an epimorphism if $i = n + \tilde{k} + l - m - 1$.

The fibers of (4.3) are next studied via the fibrations (4.4) where

$$j: \tfrac{1}{2}D^{r_t} \times D^{m-r_t} \to M_{t+1}.$$

$$(4.4)\qquad \begin{array}{ccc} \mathscr{E}(M_{t+1}, N_{t+1}; f \mid j(\tfrac{1}{2}D^{r_t} \times D^{m-r_t})) & \to & \tilde{\mathscr{E}}(M_{t+1}, N_{t+1}; f \mid j(\tfrac{1}{2}D^{r_t} \times D^{m-r_t})) \\ \downarrow & & \downarrow \\ \mathscr{E}(M_{t+1}, N_{t+1}; f \mid i(D^{m-r_t})) & \to & \tilde{\mathscr{E}}(M_{t+1}, N_{t+1}; f \mid i(D^{m-r_t})) \\ \downarrow & & \downarrow \\ \mathscr{G}\mathscr{E}(D^{r_t} \times D^{m-r_t}, N_{t+1}; fi \mid (0 \times D^{m-r_t})) & \to & \mathscr{G}\tilde{\mathscr{E}}(D^{r_t} \times D^{m-r_t}, N_{t+1}; fi \mid (0 \times D^{m-r_t})) \end{array}$$

To accomplish this first note that the base spaces depend neither upon $N$ nor $f$ and are equivalent to the base space of the following fibrations.

$$(4.5)\qquad \begin{array}{ccc} \mathscr{E}(D^{r_t} \times D^{m-r_t}, D^n; i \mid (\tfrac{1}{2}D^{r_t} \times D^{m-r_t})) & \to & \tilde{\mathscr{E}}(D^{r_t} \times D^{m-r_t}, D^n; i \mid (\tfrac{1}{2}D^{r_t} \times D^{m-r_t})) \\ \downarrow & & \downarrow \\ \mathscr{E}(D^{r_t} \times D^{m-r_t}, D^n; i \mid (\{0\} \times D^{m-r_t})) & \to & \tilde{\mathscr{E}}(D^{r_t} \times D^{m-r_t}, D^n; i \mid (\{0\} \times D^{m-r_t})) \\ \downarrow & & \downarrow \\ \mathscr{G}\mathscr{E}(D^{r_t} \times D^{m-r_t}, D^n; i \mid (\{0\} \times D^{m-r_t})) & \to & \mathscr{G}\tilde{\mathscr{E}}(D^{r_t} \times D^{m-r_t}, D^n; i \mid (\{0\} \times D^{m-r_t})) \end{array}$$

The two total spaces of (4.5) are contractible via the Alexander isotopy [11], [13] while the fibers are easily recognized as

$$\mathscr{E}(D^{m-r_t+1} \times \Sigma^{r_t-1}, D^{m-r_t+1} \times \Sigma^{n-m+r_t-1})$$

and

$$\tilde{\mathscr{E}}(D^{m-r_t+1} \times \Sigma^{r_t-1}, D^{m-r_t+1} \times \Sigma^{n-m+r_t-1}),$$

respectively. Since $n - m + r_t - 1 \geq 3$ the inclusion induces an isomorphism on homotopy for $i < 2(n - m) + l - 3$ and an epimorphism for $i = 2(n - m) + l - 3$. As a consequence the homomorphism between the base spaces of (4.5), and therefore (4.4), is an isomorphism for $i < 2(n - m) + l - 2$ and an epimorphism for $i = 2(n - m) + l - 2$.

This final step in the analysis is the recognition that the fiber of (4.4) is equivalent to

$$\mathscr{E}(M_t, N_t; f \mid M_t) \to \tilde{\mathscr{E}}(M_t, N_t; f \mid M_t).$$

Since $n - m + r_t \geq 3$ and $\pi_j(N) = 0$ for $j \leq k$ standard arguments give $\pi_j(N_t) = 0$ for $j \leq \min(k, n - m + r_t - 2)$. By hypothesis $n - m + r_t - 2 \geq n - l - 1 \geq n - m - 1$ so that the homomorphism between the homotopy groups is an isomorphism if $i < n + l + \tilde{k} - m - 2$ and an epimorphism if $i = n + l + \tilde{k} - m - 2$ by induction. Employing (4.4) and (4.3) this gives the theorem for $\partial M \neq \emptyset$.

If $\partial M = \emptyset$ it is necessary to add the $m$-handle and follow the same argument. Surprisingly, one ends with precisely the same dimensional restrictions.

4.6. COROLLARY. *Under the same assumptions as Theorem* 4.2,

$$\pi_i^{\text{rel}}(\mathscr{E}(M, N; f)) = 0 \quad \textit{if } i \leq n + \tilde{k} - m - 2.$$

The proof is simply to show that $\pi_i^{\text{rel}}(\mathscr{E}(M_0, N_0; f \mid M_0)) = 0$ by employing Theorem 3.2.

4.7. *Remark.* If the handle-body decomposition of $M$ is sufficiently simple one can employ Theorem 3.18 to give precise information. The most striking example of this is Corollary 3.21 where, with $m = 0$, one has an isomorphism, for $n \geq 3$.

$$\theta_k: \pi_i(\mathscr{E}(\Sigma^k, \Sigma^{n+k})) \to \pi_i(G_{n+k+1}, G_n)$$

for $i < 2n + k - 3$. In the general case one recovers Morlet's results. For example:

4.8. COROLLARY. *Let $f$: $(M, \partial M) \to (N, \partial N)$ be a proper embedding with $n - m \geq 3$, then the homomorphism $\pi_i(\mathscr{E}(M, N; f)) \to \pi_i(\tilde{\mathscr{E}}(M, N; f))$ is an isomorphism for $i < n - m - 2$ and an epimorphism for $i = n - m - 2$.*

4.9. COROLLARY. *Let $f$: $(M, \partial M) \to (D^n, \Sigma^{n-1})$ be a proper embedding with $n - m \geq 3$, then the homomorphism $\pi_i(\mathscr{E}(M, D^n; f)) \to \pi_i(\tilde{\mathscr{E}}(M, D^n; f))$ is an isomorphism if $i < 2n - 2m - 2$ and an epimorphism if $i = 2n - 2m - 2$.*

## 5. An application to $V^{PL}_{n+k,k}$

Let $PL(n + k, k) = \mathscr{H}(\Sigma^{n+k}, \Sigma^k)$ with $PL(n) = PL(n, 0)$ and define

$$V^{PL}_{n+k,k} = \frac{PL(n+k)}{PL(n+k,k)} \quad \text{and} \quad \tilde{V}^{PL}_{n+k,k} = \frac{\tilde{P}L(n+k)}{\tilde{P}L(n+k,k)}$$

The spaces appear in fibrations as follows:

$$\begin{array}{ccc}
PL(n+k,k) = \mathscr{H}(\Sigma^{n+k}, \Sigma^k) & \to & \tilde{\mathscr{H}}(\Sigma^{n+k}, \Sigma^k) = \tilde{P}L(n+k,k) \\
\downarrow & & \downarrow \\
PL(n+k) = \mathscr{H}(\Sigma^{n+k}, E^0) & \to & \tilde{\mathscr{H}}(\Sigma^{n+k}, \Sigma^0) = \tilde{P}L(n+k) \\
\downarrow & & \downarrow \\
V^{PL}_{n+k,k} = \mathscr{E}(\Sigma^k, \Sigma^{n+k}; i \mid \Sigma^0) & \to & \tilde{\mathscr{E}}(\Sigma^k, \Sigma^{n+k}; i \mid \Sigma^0) = \tilde{V}^{PL}_{n+k,k}
\end{array}$$

Thus

$$\pi_i^{\mathrm{rel}}(V_{n+k,k}^{PL}) \cong \pi_i^{\mathrm{rel}}(\mathscr{E}(\Sigma^k, \Sigma^{n+k}; i \mid \Sigma^0)).$$

The methods of the previous section show that this is isomorphic to

$$\pi_{i-1}^{\mathrm{rel}}(\mathscr{E}(D^1 \times \Sigma^{k-1}, D^1 \times \Sigma^{n+k-1})).$$

Theorem 3.20 implies that this is zero for $i \leq 2n + k - 3$. Thus we have the following proposition which extends a result of Haefliger and Wall [4, Theorem 3].

5.1. THEOREM. *If $n \geq 3$, then $\pi_i(V_{n+k,k}^{PL}) \to \pi_i(G_{n+k}, G_n)$ is an isomorphism for $i < 2n + k - 3$ and an epimorphism for $i = 2n + k - 3$.*

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