PIECEWISE LINEAR INVOLUTIONS OF $S^1 \times S^2$

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1. INTRODUCTION

Let S^n denote the triangulated n-sphere. Starting with S^0, we define S^n inductively as the suspension of S^{n-1}. Let k_n denote the simplicial involution of S^n that leaves S^{n-1} pointwise fixed and interchanges the suspension vertices. Define two involutions h_1 and h_2 of $S^1 \times S^2$ by

$$h_1(x, y) = (k_1(x), y) \quad \text{and} \quad h_2(x, y) = (x, k_2(y)).$$

In this paper, we prove the following uniqueness results.

THEOREM 1. *Let* h *be a piecewise linear* (PL) *involution of* $S^1 \times S^2$, *with homogeneously* 2-*dimensional fixed point set* F. *If* F *is not connected, then* h *is* PL-*equivalent to* h_1.

THEOREM 2. *Let* h *be a* PL *involution of* $S^1 \times S^2$ *with* 2-*dimensional connected fixed point set* F *and orientable orbit space. Then* h *is* PL-*equivalent to* h_2.

There is a PL-involution of $S^1 \times S^2$ that has a Klein bottle as fixed point set and a nonorientable 2-disk bundle over S^1 as orbit space. Hence the orientability of the orbit space in Theorem 2 is essential. It is equivalent to the assumption that F separates $S^1 \times S^2$.

The following remark applies to both theorems. Since F is 2-dimensional, h has the property that near each point of F it maps one side of F to the other side. If this were not the case, one could produce a small invariant 2-sphere S near F such that $h \mid S$ has a 2-cell as fixed point set. But this is impossible. Hence, near each point of F (and therefore globally), h reverses the orientation. Throughout, we use the singular (or simplicial) homology, with integer coefficients unless it is otherwise specified.

2. PROOF OF THEOREM 1

PROPOSITION 1. F *separates* $S^1 \times S^2$.

Proof. Suppose $S^1 \times S^2 - F$ is connected. Consider the homology sequence

$$0 \to H_3(S^1 \times S^2) \xrightarrow{j_*} H_3(S^1 \times S^2, F) \xrightarrow{\partial_*} H_2(F) \xrightarrow{i_*} H_2(S^1 \times S^2)$$

of $(S^1 \times S^2, F)$ over Z_2. Since $H_3(S^1 \times S^2, F) \simeq Z_2$, rank $H_2(F) \leq 1$. But rank $H_2(F)$ is the number of components of F. Since F is not connected, the result follows.

The following proposition implies that F is orientable.

PROPOSITION 2. $S^1 \times S^2 - F$ *has exactly two components, and they are interchanged under* h.

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Proof. Suppose a component U of $S^1 \times S^2 - F$ is invariant under h. Since h switches sides near each point of F, $\overline{U}$ is a closed manifold and $\overline{U} = S^1 \times S^2$. This contradicts Proposition 1. Hence h maps each component of $S^1 \times S^2 - F$ onto another. Since the interior of the orbit space is connected, there are exactly two components.

PROPOSITION 3. *F has two components.*

Proof. The exact sequence used in the proof of Proposition 1 shows that

$$\text{rank } H_2(F) \leq \text{rank } H_3(S^1 \times S^2, F);$$

that is, F has no more components than $S^1 \times S^2 - F$. Since F is not connected, the result follows.

PROPOSITION 4. *Each component of F is a 2-sphere.*

Proof. Let M_1 and M_2 be the closures of the complementary domains of F in $S^1 \times S^2$. Then $M_1 \cap M_2 = F$.

Since the M_i are retracts of $S^1 \times S^2$, the $H_1(M_i)$ are isomorphic to a direct summand of $H_1(S^1 \times S^2) \simeq Z$. Hence $H_1(M_i) \simeq Z$ or 0.

Consider the following reduced Mayer-Vietoris sequence of $(S^1 \times S^2, M_1, M_2)$:

$$H_2(S^1 \times S^2) \to H_1(F) \to H_1(M_1) + H_1(M_2) \xrightarrow{\psi} H_1(S^1 \times S^2) \xrightarrow{\Delta} \tilde{H}_0(F) \to 0.$$

Now $H_1(S^1 \times S^2) \simeq Z$ and $\tilde{H}_0(F) \simeq Z$. Since Δ is an epimorphism, it must be an isomorphism in this case. Hence ψ is trivial, and

$$\text{rank } H_1(F) \leq 2 \text{ rank } H_1(M_1) + 1 \leq 3.$$

Since each component of F is an orientable surface, rank $H_1(F)$ is 0 or 2. Hence at least one component F_1 of F is a 2-sphere. Let F_2 denote the other component of F.

Now F_1 does not separate $S^1 \times S^2$. This is so because (as is well-known) a nice 2-sphere separating $S^1 \times S^2$ bounds a 3-cell. Cut $S^1 \times S^2$ along F_1 to produce a compact 3-manifold X whose boundary is the disjoint union of two 2-spheres A and B. Because h switches sides of F_1, h gives rise to an involution h' of X such that $h'(A) = B$ and the fixed point set is homeomorphic to F_2. Now attach two 3-cells to X along A and B to obtain a closed 3-manifold Y. The involution h' easily extends to an involution h'' of Y with fixed point set homeomorphic to F_2. As in [1], $S^1 \times S^2 \# Y \approx S^1 \times S^2$ and Y is a 3-sphere, and therefore the fixed point set must be a 2-sphere.

PROPOSITION 5. *The orbit space of h is homeomorphic to $[0, 1] \times S^2$.*

Proof. We use the notation of the proof of Proposition 4. The orbit space of h is homeomorphic to the orbit space of h'. Now the orbit space of h'' is homeomorphic to the closure C of the component of Y (the fixed point set) that contains A. Clearly, C is a 3-cell. Therefore the orbit space of h' is homeomorphic to C minus the open 3-cell bounded by A. Hence the orbit space of h is homeomorphic to $[0, 1] \times S^2$.

Proof of Theorem 1. Propositions 2, 3, and 5 show that M_1 and M_2 are homeomorphic to $[0, 1] \times S^2$. Let D_1 and D_2 be arcs in S^1 such that $k_1(D_1) = D_2$ and $D_1 \cap D_2 = S^0$. Define a map $t\colon S^1 \times S^2 \to S^1 \times S^2$ as follows. Let $t \mid M_1$ be a PL homeomorphism u of M_1 onto $D_1 \times S^2$ (such a homeomorphism exists, by Proposition 5). Let $t \mid M_2 = h_1 \cdot u \cdot (h \mid M_2)$. Then t is a PL homeomorphism and $h = t^{-1} h_1 t$.

3. PROOF OF THEOREM 2

PROPOSITION 6. F *separates* $S^1 \times S^2$.

Proof. If the proposition were false, $h \mid S^1 \times S^2 - F$ would be an orientation-reversing involution of a connected set. Hence the orbit space would not be orientable.

Just as in Section 2, it can be shown that F has exactly two complementary domains and that they are interchanged under h. Write $S^1 \times S^2 = M_1 \cup M_2$, with $M_1 \cap M_2 = F$ and with M_i homeomorphic to the orbit space.

PROPOSITION 7. $\pi_1(M_i)$ *is infinite cyclic.*

Proof. Since M_i is a retract of $S^1 \times S^2$, $\pi_1(M_i)$ is infinite cyclic or trivial. If $\pi_1(M_i)$ were trivial, $\pi_1(S^1 \times S^2)$ would also be trivial, since $M_1 \cap M_2$ is path-connected.

The following result may be found in [3].

LEMMA (Tollefson). *Let* M_1 *be a 3-manifold with connected boundary such that the double of* M_1 *is homeomorphic to* $S^1 \times S^2$. *Then* M_1 *is irreducible.*

Recall that the double of a manifold is obtained by attaching two copies of the manifold along the boundary by the identity map. The proof of the lemma is not difficult, and we leave it to the reader.

PROPOSITION 8. M_1 *is homeomorphic to* $S^1 \times D^2$, *where* D^2 *is a 2-cell.*

Proof. By the lemma, Proposition 7, the observation preceding Proposition 7, and [2], M_1 is fibered over the circle with 1-connected fiber. The fiber in this case must be a compact 2-manifold with nonempty boundary. Hence it is a 2-cell. Thus M_1 may be obtained from $[0, 1] \times D^2$ by identifying each $(0, x)$ $(x \in D^2)$ with $(1, g(x))$, where g is a homeomorphism of D^2 onto itself.

Since M_1 is orientable, g must preserve the orientation, and therefore it is isotopic to the identity. Hence M_1 is homeomorphic to $S^1 \times D^2$.

Theorem 2 now follows as in the corresponding part of Section 2.

4. A CORRECTION

I take this opportunity to point out an oversight in the paper *Examples of generalized manifold approaches to topological manifolds* (Vol. 14 (1967), 225-229 of this journal). The argument on page 228 that rules out Case 2 is incorrect. Instead, Case 2 should be ruled out by the following observation:

If X *is a connected manifold with a connected nonorientable boundary* B, *then the inclusion* $B \subset X$ *is essential.*

Proof. Clearly, X is nonorientable. Let $p: \tilde{X} \to X$ be the orientable double covering. Being orientable, $p^{-1}(B)$ must be connected. On the other hand, if $B \subset X$ were inessential, the induced covering $p^{-1}(B) \to B$ would be trivial and therefore disconnected.

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