Piecewise linear (pre-)wavelets on non-uniform meshes

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PIECEWISE LINEAR (PRE-)WAVELETS ON NON-UNIFORM MESHES

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Abstract. In this paper, an explicit construction of compactly supported prewavelets on linear finite element spaces is introduced on non-uniform meshes on polyhedron domains and on boundaries of such domains. The obtained bases are stable in the Sobolev spaces $\mathcal{H}^r$ for $|r| < \frac{3}{2}$. The only condition we need is that of uniform refinements. Compared to existing prewavelets bases on uniform meshes, with our construction the basis transformation from wavelet- to nodal basis (the wavelet transform) can be implemented more efficiently.

1. Motivation and background

Let us denote by $\mathcal{H}^s$, $s \in \mathbf{R}$ (or $|s| \leq t$) a scale of Sobolev spaces on a d-dimensional bounded polyhedral domain or sufficiently smooth manifold. For $s < 0$, $\mathcal{H}^s$ is to be understood as $(\mathcal{H}^{-s})'$. Consider a variational problem: Given $f \in \mathcal{H}^{-r}$, search $u \in \mathcal{H}^r$ such that

$$a(u,v) = \overline{f}(v) \qquad (v \in \mathcal{H}^r), \tag{1.1}$$

where a is a scalar product satisfying $a(v,v) \eqsim \|v\|^2_{\mathcal{H}^r}$, i.e. the problem (1.1) is bounded and *elliptic of order* $2r$. For completeness, with $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

Model examples for (1.1) are given by variational formulations of the *differential equation* $-\nabla \cdot B\nabla u + cu = f$ on a domain Ω, where $B(x) \eqsim I$ and $0 \leq c(x) \lesssim 1$, with suitable boundary conditions ($r = 1$), and of reformulations of Laplace' equation as an *integral equation* on $\Gamma = \partial\Omega$ as the single layer potential equation ($r = -\frac{1}{2}$), the hypersingular equation ($r = \frac{1}{2}$), or, in case of a smooth Γ, the double layer potential equation ($r = 0$).

We consider Galerkin discretizations of (1.1) on a sequence of nested linear finite element spaces

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \mathcal{M}_J \subset \cdots \subset \mathcal{H}^r,$$

where we assume dyadic refinements and conforming triangulations (no hanging nodes). Let $\{\phi_{J,x} : x \in \Omega_J\}$ be a basis of $\mathcal{M}_J$, where as index set Ω_J we may use the set of nodal

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points in the mesh, excluding possible nodes in which essential boundary conditions are prescribed. With respect to this basis, the stiffness matrix $\mathbf{A}_J \in \mathbb{C}^{(\dim \mathcal{M}_J)^2}$ is defined by

$$<\mathbf{A}_J\mathbf{u}_J, \mathbf{v}_J>= a(\sum_x (\mathbf{u}_J)_x \phi_{J,x}, \sum_x (\mathbf{v}_J)_x \phi_{J,x}) \qquad (\mathbf{u}, \mathbf{v} \in \mathbb{C}^{\dim \mathcal{M}_J}),$$

where $<,>$ denotes the Euclidean scalar product on the vector space. Taking $\mathbf{b} \in \mathbb{C}^{\dim \mathcal{M}_J}$ to be the vector satisfying

$$<\mathbf{b}_J, \mathbf{v}_J>= \overline{f}(\sum_x (\mathbf{v}_J)_x \phi_{J,x}) \qquad (\mathbf{v} \in \mathbb{C}^{\dim \mathcal{M}_J}),$$

the Galerkin scheme leads to the linear system

$$\mathbf{A}_J \mathbf{u}_J = \mathbf{b}_J. \tag{1.2}$$

For solving (1.2), we are interested in its conditioning. Let $\mathbf{D}_J$ be the diagonal of $\mathbf{A}_J$. Then from

$$\frac{<\mathbf{A}_J\mathbf{u}_J, \mathbf{u}_J>}{<\mathbf{D}_J\mathbf{u}_J, \mathbf{u}_J>} = \frac{a(\sum_x (\mathbf{u}_J)_x \phi_{J,x}, \sum_x (\mathbf{u}_J)_x \phi_{J,x})}{\sum_x |(\mathbf{u}_J)_x|^2 a(\phi_{J,x}, \phi_{J,x})},$$

and $a(v,v) \eqsim \|v\|^2_{\mathcal{H}^r}$, we see that the spectral condition number $\kappa(\mathbf{D}_J^{-1}\mathbf{A}_J)$ satisfies

$$\kappa(\mathbf{D}_J^{-1}\mathbf{A}_J) \eqsim \frac{\Lambda_J(r)}{\lambda_J(r)},$$

where $\Lambda_J(r)$ and $\lambda_J(r)$ are the optimum constants in

$$\lambda_J(r) \sum_x |c_{J,x}|^2 \|\phi_{J,x}\|^2_{\mathcal{H}^r} \leq \| \sum_x c_{J,x}\phi_{J,x}\|^2_{\mathcal{H}^r} \leq \Lambda_J(r) \sum_x |c_{J,x}|^2 \|\phi_{J,x}\|^2_{\mathcal{H}^r}.$$

We will call a basis $\mathcal{H}^r$-*stable* when $\frac{\Lambda_J(r)}{\lambda_J(r)}$ is bounded uniformly in J. We conclude that the (diagonally preconditioned) system (1.2) is well-conditioned, i.e. $\kappa(\mathbf{D}_J^{-1}\mathbf{A}_J)$ is bounded uniformly in J, if and only if the basis $\{\phi_{J,x} : x \in \Omega_J\}$ is $\mathcal{H}^r$-stable.

Remark 1.1. In many papers, a basis is called $\mathcal{H}^r$-stable only if $\| \sum_x c_{J,x}\phi_{k,x}\|^2_{\mathcal{H}^r} \eqsim \sum_x |c_{J,x}|^2$, i.e. the scaling of the basis is taken into account. Clearly, if a basis, or a scaled version of it, is $\mathcal{H}^r$-stable according to this definition, then this basis is $\mathcal{H}^r$-stable in our terminology.

It is well-known that the nodal basis is stable only in L_2. Assuming shape regular simplices, for this basis the quotient $\frac{\Lambda_J(r)}{\lambda_J(r)}$ is equivalent to $4^{J|r|}$. So, to solve elliptic problems of order $2r$ for $r \neq 0$, an obvious approach is to search for other bases.

In this paper, we study bases of multiscale type: For $k \geq 0$, let $\mathcal{V}_k \subset \mathcal{M}_k$ be such that

$$\begin{cases} \mathcal{V}_0 = \mathcal{M}_0 \\ \mathcal{V}_k = \mathcal{M}_k \ominus \mathcal{M}_{k-1} & \text{if } k \geq 1, \end{cases}$$

and let $\{\psi_{k,x} : x \in \Omega_k \backslash \Omega_{k-1}\}$ ($\Omega_{-1} := \emptyset$) be a basis of $\mathcal{V}_k$. For reasons of an efficient implementation, we are only interested in $\psi_{k,x}$ that are linear combinations of an (uniformly) bounded number of nodal basis functions of $\mathcal{M}_k$. We have that $\mathcal{M}_J = \oplus_{k=0}^J \mathcal{V}_k$, which type of space decomposition is called a *multiscale decomposition*. As a consequence, $\cup_{k=0}^J \{\psi_{k,x} : x \in \Omega_k \backslash \Omega_{k-1}\}$ is a basis of $\mathcal{M}_J$, called *multiscale*-, *wavelet*- or *hierarchical basis*.

Sufficient conditions for $\mathcal{H}^r$-stability of this wavelet basis are

$$\|\sum_k v_k\|_{\mathcal{H}^r}^2 \eqsim \sum_k 4^{kr}\|v_k\|_{L_2}^2 \qquad (v_k \in \mathcal{V}_k) \tag{1.3}$$

and

$$\{\psi_{k,x} : x \in \Omega_k \backslash \Omega_{k-1}\} \text{ is an } L_2\text{-stable basis of } \mathcal{V}_k. \tag{1.4}$$

Indeed, from (1.3) and (1.4), we obtain that

$$\|\sum_{k,x} c_{k,x}\psi_{k,x}\|_{\mathcal{H}^r}^2 \eqsim \sum_k 4^{kr}\|\sum_x c_{k,x}\psi_{k,x}\|_{L_2}^2 \eqsim \sum_{k,x} |c_{k,x}|^2 4^{kr}\|\psi_{k,x}\|_{L_2}^2 \eqsim \sum_{k,x} |c_{k,x}|^2\|\psi_{k,x}\|_{H^r}^2.$$

Note that (1.3) is equivalent to the condition $\|\sum_k v_k\|_{\mathcal{H}^r}^2 \eqsim \sum_k \|v_k\|_{\mathcal{H}^r}^2$ $(v_k \in \mathcal{V}_k)$, which is clearly a necessary condition for $\mathcal{H}^r$-stability of the wavelet basis, and the additional condition $\|v_k\|_{\mathcal{H}^r} \eqsim 2^{kr}\|v_k\|_{L_2}$, i.e. the energy scalar product restricted to $\mathcal{V}_k$ is well-conditioned.

Concerning Condition (1.3), an attractive choice is to take $\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{L_2}} \mathcal{M}_{k-1}$. With this L_2-orthogonal decomposition, at least for quasi-uniform meshes (1.3) holds for $|r| < \frac{3}{2}$ (e.g. see [Osw94]). Concerning the upperbound $\frac{3}{2}$ for r, note that $\mathcal{M}_J \not\subset \mathcal{H}^{\frac{3}{2}}$.

The lower bound $-\frac{3}{2}$ of the range of stability is not optimal. For the shift-invariant case (uniform, unbounded mesh) in one dimension, for any $t \geq 0$, a (biorthogonal) wavelet basis can be constructed that is $\mathcal{H}^r$-stable for $r \in (-t, \frac{3}{2})$ (see [CDF92]). Moreover, adaptations of these bases to an interval, retaining its nice properties have been constructed in [DKU96]. Yet, these constructions are essentially restricted to uniform meshes. Moreover, we note that the advantages of these biorthogonal bases do not lie so much in having larger ranges of stability, since $r > -\frac{3}{2}$ is sufficient for practical applications, as well as in having more so-called vanishing moments, and the fact that for these bases the basis transformation on $\mathcal{M}_J$ from nodal- to wavelet basis (the *inverse wavelet transform*) can be explicitly computed as a composition of local mappings on all coarser levels, with total costs $\sim \dim\mathcal{M}_J$ operations. As we will see, for preconditioning purposes there is no need for an explicit inverse wavelet transform. We will comment on vanishing moments later on.

In the case of $\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{L_2}} \mathcal{M}_{k-1}$, basis functions of $\mathcal{V}_k$ are often called *prewavelets*; in the general setting of L_2-orthogonal decompositions of a trial space, the name wavelets is usually preserved for basis functions that are also mutually L_2-orthogonal within each complement space.

So far, also for the L_2-orthogonal decomposition, prewavelets have been constructed basically for the uniform mesh case only (cf. [CW92], [KO95, and references cited here]), where moreover the case of a bounded domain is treated only in one dimension. Yet, in [Jun94] the case of linear finite element spaces on the surface of some block-shaped three-dimensional domains is treated, which gives at corners situations different from the uniform mesh case on a domain. In [LO96a] a suggestion is made how non-uniform meshes could be treated.

In this paper, we introduce an explicit construction of prewavelets satisfying (1.4) on *non-uniform meshes* on domains and on surfaces of polyhedral domains in any space dimension. The only condition we need is that of *uniform*, (regular) *refinements*. We keep complete freedom concerning initial meshes and boundary conditions. Since (1.3) is satisfied for $|r| < \frac{3}{2}$, the constructed (pre-)wavelet basis is $\mathcal{H}^r$-stable for this range of r.

For comparison, recall that in two dimensions the standard hierarchical basis ([Yse86]) is $\mathcal{H}^r$-stable only for $1 < r < \frac{3}{2}$, whereas for $r = 1$ the quotient $\frac{\Lambda(r)}{\lambda(r)}$ grows logarithmically as function of J. The situation in more dimensions is worse.

This paper is not restricted to the construction of bases of the L_2-orthogonal complement spaces. In Sect. 2, we give a general framework for constructing (pre-)wavelets. As applications, in Sect. 3 we treat the L_2-orthogonal complement case. In Sect. 4, we show that also the three-point wavelet basis ([Ste95b, Ste95a, Ste96], cf. also [LO96a]) fits into this framework, and in Sect. 6 an alternative three-point wavelet basis is constructed for which we have an explicit inverse wavelet transform.

Finally in this introductory section, to appreciate wavelets from a practical point of view, we recall some facts about implementation. When (1.1) stems from a differential equation, the stiffness matrix with respect to the nodal basis is sparse. Yet, due to interactions between wavelets from different levels, the stiffness matrix corresponding to $\mathcal{M}_J$ with respect to the wavelet basis can be expected to have $\sim J \cdot \dim\mathcal{M}_J$ non-zeros. Concerning efficiency, this means that a naive implementation of the matrix-vector product would partially undo the effect of having a stable basis.

Therefore, let $\mathbf{p}_k$ and $\mathbf{q}_k$ be the representations of the inclusions $\mathcal{M}_{k-1} \to \mathcal{M}_k$ and $\mathcal{V}_k \to \mathcal{M}_k$ respectively with respect to nodal bases on the finite element spaces, and the basis $\{\psi_{k,x} : x \in \Omega_k \backslash \Omega_{k-1}\}$ on $\mathcal{V}_k$. Note that $\mathbf{p}_k$ is the usual multi-grid prolongation, and, by assumption, that the columns of $\mathbf{q}_k$ contain an uniformly bounded number of non-zeros. When we now apply a from top-to-bottom level-wise ordering of the wavelets, the basis transformation $\mathbf{T}_J$ on $\mathcal{M}_J$ from wavelet- to nodal basis (the *wavelet transform*) can be written as $\mathbf{T}_J = \begin{bmatrix} \mathbf{q}_J & \mathbf{p}_J \mathbf{T}_{J-1} \end{bmatrix}$, where, when we equip $\mathcal{V}_0 = \mathcal{M}_0$ with nodal basis, $\mathbf{T}_0 = \mathbf{I}$. So, after having applied $\mathbf{T}_{J-1}$, the application of $\mathbf{T}_J$ costs a number of additional operations $\sim \#(\Omega_J \backslash \Omega_{J-1})$, and so the total costs are $\sim \dim\mathcal{M}_J$.

Let now $\mathbf{A}_J^{(N)} \mathbf{u}_J^{(N)} = \mathbf{b}_J^{(N)}$ and $\mathbf{A}_J^{(W)} \mathbf{u}_J^{(W)} = \mathbf{b}_J^{(W)}$ be the systems (1.2) with respect to nodal- and wavelet basis respectively. Then it is easily verified that $\mathbf{A}_J^{(W)} = \mathbf{T}_J^* \mathbf{A}_J^{(N)} \mathbf{T}_J$, $\mathbf{b}_J^{(W)} = \mathbf{T}_J^* \mathbf{b}_J^{(N)}$ and $\mathbf{u}_J^{(N)} = \mathbf{T}_J \mathbf{u}_J^{(W)}$, and so for the diagonal $\mathbf{D}_J^{(W)}$ of $\mathbf{A}_J^{(W)}$ it holds that $\mathbf{D}_J^{(W)} = \begin{bmatrix} \operatorname{diag}(\mathbf{q}_J^* \mathbf{A}_J^{(N)} \mathbf{q}_J) & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{J-1}^{(W)} \end{bmatrix}$, where $\mathbf{D}_0^{(W)} = \operatorname{diag}\mathbf{A}_0^{(N)}$. We conclude that with $\mathbf{T}_J$, all ingredients for iterating on (1.2) with respect to the wavelet basis, as well as transferring the obtained approximate solution to nodal basis can be implemented in $\sim \dim\mathcal{M}_J$ operations.

The basis transformation on the right-hand side before, as well as the one on the approximate solution after the iteration can even be avoided by operating directly on the system with respect to the nodal basis. From $\mathbf{T}_J {\mathbf{D}_J^{(W)}}^{-1} \mathbf{A}_J^{(W)} \mathbf{T}_J^{-1} = \mathbf{C}_J \mathbf{A}_J^{(N)}$, where $\mathbf{C}_J := \mathbf{T}_J {\mathbf{D}_J^{(W)}}^{-1} \mathbf{T}_J^*$, we conclude that the preconditioned matrices in both systems $\mathbf{C}_J \mathbf{A}_J^{(N)} \mathbf{u}_J^{(N)} = \mathbf{C}_J \mathbf{b}_J^{(N)}$ and ${\mathbf{D}_J^{(W)}}^{-1} \mathbf{A}_J^{(W)} \mathbf{u}_J^{(W)} = {\mathbf{D}_J^{(W)}}^{-1} \mathbf{b}_J^{(W)}$ have equal spectral condition number. Using the definitions of $\mathbf{T}_J$ and $\mathbf{D}_J^{(W)}$, we end up with the following efficient recursive implementation of this multi-level preconditioner $\mathbf{C}_J$ for the system with respect to nodal basis:

$$\begin{cases} \mathbf{C}_J = \mathbf{q}_J(\mathrm{diag}(\mathbf{q}_J^* \mathbf{A}_J^{(N)} \mathbf{q}_J))^{-1}\mathbf{q}_J^* + \mathbf{p}_J \mathbf{C}_{J-1} \mathbf{p}_J^* & J \geq 1 \\ \mathbf{C}_0 = (\mathrm{diag}\mathbf{A}_0^{(N)})^{-1}. \end{cases}$$

In Sect. 5, we will see that for the wavelets introduced in this paper a somewhat modified version of this implementation results in an even more efficient algorithm.

So far, we confined the discussion of the implementation to differential equations. The situation is completely different for integral equations. With respect to both the nodal- as the wavelet basis, the stiffness matrix will generally be dense. Yet, assuming some standard smoothness conditions on the Schwarz kernel of the integral operator outside its diagonal, due to the oscillating nature of wavelets the stiffness matrix with respect to the wavelet basis will be close to a sparse one. This "oscillating nature" can be quantified as follows: A wavelet is said to have ℓ vanishing moments when it is L_2-orthogonal to all polynomials of degree less than ℓ. In [Sch95], it is demonstrated that for an problem of order $2r$, the stiffness matrix can be compressed to a sparse one without loosing the order of convergence when the wavelets have more than $2-2r$ vanishing moments. Things can be cleverly organized such that the complete stiffness matrix does not have to be computed, which would undo the reduction of the order of complexity.

Prewavelets have two vanishing moments, which means that they do not give optimal compression rates for problems of non-positive order. Yet, instead of searching to bases that, on *non-uniform* meshes have both a large range of stability as the prewavelet basis, and more than two vanishing moments, an alternative approach might be to use *different* wavelet bases for compression and preconditioning. In that case, to implement the preconditioner, we would need the basis transformation from "preconditioning basis" to "compression basis" (instead of the nodal one). Sufficient is that for the compression basis we have an explicit inverse wavelet transform.

2. A two-step construction of wavelets

For $k \geq 1$, let $(\,,\,)_{\mathcal{M}_k}$ be a Hermitian sesquilinear form on $\mathcal{M}_k$, which on $\mathcal{M}_{k-1}$ is even a scalar product, i.e. $(u,u)_{\mathcal{M}_k} > 0$ when $0 \neq u \in \mathcal{M}_{k-1}$. As main application we have in mind that $(\,,\,)_{\mathcal{M}_k} = (\,,\,)_{L_2}$. It is easily verified that each $u \in \mathcal{M}_k$ has an unique decomposition $u = w + (u-w)$, where $w \in \mathcal{M}_{k-1}$ and $(u-w, \mathcal{M}_{k-1})_{\mathcal{M}_k} = 0$, and so we may define

$$\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{(,)_{\mathcal{M}_k}}} \mathcal{M}_{k-1}. \tag{2.1}$$

On the other hand, corresponding to any decomposition $\mathcal{M}_k = \mathcal{M}_{k-1} \oplus \mathcal{V}_k$, there exists a (non-unique) $(\,,\,)_{\mathcal{M}_k}$ as above such that (2.1) is valid.

For $\ell \in \{k-1, k\}$, let

$$\{\phi_{\ell,x} : x \in \Omega_l\} \tag{2.2}$$

be a basis of $\mathcal{M}_\ell$. In our applications, we will take the nodal basis. Now, as the first step in the construction of a basis of $\mathcal{V}_k$, suppose we have constructed an *adjoint basis*

$$\{\tilde{\phi}_{k-1,y} : y \in \Omega_{k-1}\} \subset \mathcal{M}_k$$

of the basis $\{\frac{\phi_{k-1,y}}{(\phi_{k-1,y},\phi_{k-1,y})_{\mathcal{M}_k}} : y \in \Omega_{k-1}\}$ of $\mathcal{M}_{k-1}$, i.e.,

$$(\tilde{\phi}_{k-1,y}, \phi_{k-1,z})_{\mathcal{M}_k} = \begin{cases} (\phi_{k-1,y}, \phi_{k-1,y})_{\mathcal{M}_k} & \text{if } y = z \\ 0 & \text{if } y \neq z. \end{cases} \tag{2.3}$$

Clearly, since we search an adjoint basis in a space that is larger than $\mathcal{M}_{k-1}$, this basis is not unique. In our applications we will use this freedom to select an adjoint basis that is locally supported.

Then, as the second step, for $x \in \Omega_k \backslash \Omega_{k-1}$ we define

$$\psi_{k,x} = \phi_{k,x} - \sum_{y \in \Omega_{k-1}} \frac{(\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k}}{(\phi_{k-1,y}, \phi_{k-1,y})_{\mathcal{M}_k}} \tilde{\phi}_{k-1,y}. \tag{2.4}$$

Using (2.3), it is easily verified that $\psi_{k,x} \in \mathcal{V}_k$.

Theorem 2.1. *Suppose that*

(a) $\{\phi_{k,x} : x \in \Omega_k \backslash \Omega_{k-1}\} \cup \{\tilde{\phi}_{k-1,y} : y \in \Omega_{k-1}\}$ *is an* L_2*-stable basis of* $\mathcal{M}_k$*,*

(b) *the spectral norm of the matrix* $\left(\frac{(\phi_{k,x},\phi_{k-1,y})_{\mathcal{M}_k} \|\tilde{\phi}_{k-1,y}\|_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{\mathcal{M}_k} \|\phi_{k,x}\|_{L_2}}\right)_{y \in \Omega_{k-1}, x \in \Omega_k \backslash \Omega_{k-1}}$ *is bounded (uniformly in* k*).*

Then

$$\{\psi_{k,x} : x \in \Omega_k \backslash \Omega_{k-1}\} \textit{ is an } L_2\textit{-stable basis of } \mathcal{V}_k \textit{ (cf. Condition (1.4)).}$$

Proof. Since $\#(\Omega_k \backslash \Omega_{k-1}) = \dim \mathcal{V}_k$, it is sufficient to show that $\|\sum_{x \in \Omega_k \backslash \Omega_{k-1}} c_{k,x} \psi_{k,x}\|_{L_2}^2 \eqsim \sum_{x \in \Omega_k \backslash \Omega_{k-1}} |c_{k,x}|^2 \|\psi_{k,x}\|_{L_2}^2$. By Condition (a), there holds

$$\begin{aligned} &\|\sum_{x \in \Omega_k \backslash \Omega_{k-1}} c_{k,x} \psi_{k,x}\|_{L_2}^2 \\ &\quad \eqsim \sum_{x \in \Omega_k \backslash \Omega_{k-1}} |c_{k,x}|^2 \|\phi_{k,x}\|_{L_2}^2 + \sum_{y \in \Omega_{k-1}} |\sum_{x \in \Omega_k \backslash \Omega_{k-1}} \frac{(\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k}}{(\phi_{k-1,y}, \phi_{k-1,y})_{\mathcal{M}_k}} c_{k,x}|^2 \|\tilde{\phi}_{k-1,y}\|_{L_2}^2. \end{aligned}$$

Condition (b) shows that

$$0 \leq \sum_{y \in \Omega_{k-1}} |\sum_{x \in \Omega_k \backslash \Omega_{k-1}} \frac{(\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k}}{(\phi_{k-1,y}, \phi_{k-1,y})_{\mathcal{M}_k}} c_{k,x}|^2 \|\tilde{\phi}_{k-1,y}\|_{L_2}^2 \lesssim \sum_{x \in \Omega_k \backslash \Omega_{k-1}} |c_{k,x}|^2 \|\phi_{k,x}\|_{L_2}^2.$$

The proof is completed by combining both estimates. □

Remark 2.2. In our applications, both $\sup_{x \in \Omega_k \backslash \Omega_{k-1}} \#\{y \in \Omega_{k-1} : (\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k} \neq 0\}$ and $\sup_{y \in \Omega_{k-1}} \#\{x \in \Omega_k \backslash \Omega_{k-1} : (\phi_{k,x}, \phi_{k-1,y})_{\mathcal{M}_k} \neq 0\}$ are bounded (uniformly in k), that is, the number of non-zeros in each row and column of the matrix from Theorem 2.1 Condition (b) is bounded. This means that for (b), we only have to verify whether the elements of this matrix are bounded.

The forms $(\,,\,)_{\mathcal{M}_k}$ we will consider are L_2-bounded, and on $\mathcal{M}_{k-1}$ they will be even equivalent to the L_2-scalar product , i.e. $(u,u)_{\mathcal{M}_k} \eqsim \|u\|^2_{L_2}$ $(u \in \mathcal{M}_{k-1})$. With these properties, a sufficient condition for boundedness of above matrix elements is that $\|\tilde{\phi}_{k-1,y}\|_{L_2} \lesssim \|\phi_{k-1,y}\|_{L_2}$.

In the following sections, for three choices of $(\,,\,)_{\mathcal{M}_k}$, we will construct adjoint bases satisfying the conditions of Theorem 2.1, and so by applying this theorem, we find L_2-stable bases of $\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{(,)_{\mathcal{M}_k}}} \mathcal{M}_{k-1}$. From now on, with $\phi_{\ell,x}$ ((2.2)), we will always mean the *nodal basis* function, i.e.,

$$\phi_{\ell,x}(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \in \Omega_\ell. \end{cases}$$

3. Prewavelets

In this section, we consider the case that $(\,,\,)_{\mathcal{M}_k} = (\,,\,)_{L_2}$, i.e., we are searching for an L_2-stable basis of $\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{L_2}} \mathcal{M}_{k-1}$ consisting of basis functions (prewavelets) that are linear combinations of a bounded number of nodal basis functions of $\mathcal{M}_k$.

The next example shows that for locally refined meshes such a basis generally does not exist.

Example 3.1. For some h small enough with $h^{-1} \in 2\mathbb{N}$, let $\mathcal{M}_{k-1}$, $\mathcal{M}_k \subset H^1_0((0,1))$ be the linear finite element spaces corresponding to the sets of nodal points

$$\Omega_{k-1} = \{ih : i \in \{1, \ldots, h^{-1} - 1\}\}$$

and

$$\Omega_k = \Omega_{k-1} \cup \{(i - \tfrac{1}{2})h : i \in \{1, \ldots, \tfrac{1}{2}h^{-1}\}\}$$

respectively. That is, $\mathcal{M}_k$ is constructed from $\mathcal{M}_{k-1}$ based on an uniform partition, by halving only the elements in $[0, \frac{1}{2}]$.

Let $\tilde{\psi}_k \in \mathcal{V}_k$. By the orthogonality to the coarse-grid nodal basis functions, it is easily seen that $\tilde{\psi}_k(x) \neq 0$ for some $x \geq \frac{1}{2}$ implies that $[\frac{1}{2}, 1] \subset \operatorname{supp}\tilde{\psi}_k$, and in the other direction that

$$\inf \operatorname{supp}\tilde{\psi}_k \in \Omega_{k-1} \cup \{0\} \cup \{h\}. \tag{3.1}$$

Now suppose that $\operatorname{supp}\tilde{\psi}_k \subset [0, \frac{1}{2}]$ and $\tilde{\psi}_k(\frac{h}{2}) = -\frac{1}{6}\tilde{\psi}_k(h)$. It is well-known that for each $x \in I_k := \{x \in \Omega_k \backslash \Omega_{k-1} : \frac{3}{2}h \leq x \leq \frac{1}{2} - \frac{3}{2}h\}$, the function

$$\psi_{k,x} := \phi_{k,x-h} - 6\phi_{k,x-\frac{h}{2}} + 10\phi_{k,x} - 6\phi_{k,x+\frac{h}{2}} + \phi_{k,x+h} \tag{3.2}$$

is in $\mathcal{V}_k$. By adding to $\tilde{\psi}_k$ suitable multiples of $\psi_{k,x}$ for $x \in I_k$, by (3.1) we can create an $\hat{\tilde{\psi}}_k \in \mathcal{V}_k$ with $\operatorname{supp}\hat{\tilde{\psi}}_k \subset [\frac{1}{2} - 2h, \frac{1}{2}]$. It is easily verified that this means that $\hat{\tilde{\psi}}_k = 0$. Since $\#I_k = \dim\mathcal{V}_k - 2$, we conclude that each basis of $\mathcal{V}_k$ contains at least two basis functions $\tilde{\psi}_k$ with $\tilde{\psi}_k(\frac{h}{2}) \neq -\frac{1}{6}\tilde{\psi}_k(h)$ or $[\frac{1}{2}, 1] \subset \operatorname{supp}\tilde{\psi}_k$, and thus at least one basis function $\tilde{\psi}_k$ with $[\frac{1}{2}, 1] \subset \operatorname{supp}\tilde{\psi}_k$.

Above example shows that we have to restrict ourselves to uniform refinements. Considering the meshes used in practice, it is then not a real further restriction to study *uniform, regular and dyadic refinements* only, that is, we assume that when going from level $k-1$ to level k, each d-simplex τ_{k-1} underlying $\mathcal{M}_{k-1}$ is subdivided into $d+1$ congruent d-simplices.

We will search the adjoint basis function $\tilde{\phi}_{k-1,y}$ as a linear combination of $\phi_{k,y}$ and $\phi_{k-1,y}$. Clearly, this $\tilde{\phi}_{k-1,y}$ is L_2-orthogonal to $\phi_{k-1,z}$ for y and z that do not share a common simplex underlying $\mathcal{M}_{k-1}$. Because of the refinement strategy, the values

$$\mu = \frac{\int_{\tau_{k-1}} \phi_{k,y}\phi_{k-1,y}}{\int_{\tau_{k-1}} \phi_{k-1,y}\phi_{k-1,y}} \quad \text{and} \quad \lambda = \frac{\int_{\tau_{k-1}} \phi_{k,y}\phi_{k-1,z}}{\int_{\tau_{k-1}} \phi_{k-1,y}\phi_{k-1,z}}$$

are independent of the simplex τ_{k-1} underlying $\mathcal{M}_{k-1}$, vertices y and z of τ_{k-1} with $z \neq y$, and $k \geq 1$. By writing integrals over the whole domain as sum of integrals over the simplices τ_{k-1}, we see that

$$(\phi_{k,y} - \lambda\phi_{k-1,y}, \phi_{k-1,y})_{L_2} = \begin{cases} (\mu-\lambda)\|\phi_{k-1,y}\|_{L_2}^2 & \text{if } y = z \\ 0 & \text{if } y \neq z \in \Omega_{k-1}. \end{cases}$$

Straightforward computations on a reference d-simplex show that

$$\mu = (\tfrac{1}{2})^{d+2}(d+4) \quad \text{and} \quad \lambda = (\tfrac{1}{2})^{d+1}.$$

Since $\lambda \neq \mu$, we conclude that

$$\{\tilde{\phi}_{k-1,y} := \tfrac{2^{d+2}}{d+2}\phi_{k,y} - \tfrac{2}{d+2}\phi_{k-1,y} : y \in \Omega_{k-1}\} \subset \mathcal{M}_k \tag{3.3}$$

is an L_2-adjoint basis of $\{\frac{\phi_{k-1,y}}{\|\phi_{k-1,y}\|_{L_2}^2} : y \in \Omega_{k-1}\}$.

We will check the conditions of Theorem 2.1. Again using the regular refinement strategy, for y being a vertex of τ_{k-1} one can compute that

$$\gamma := \frac{\int_{\tau_{k-1}} \phi_{k,y}\phi_{k,y}}{\int_{\tau_{k-1}} \phi_{k-1,y}\phi_{k-1,y}} = 2^{-d}, \tag{3.4}$$

and so $\frac{\|\phi_{k-1,y}\|_{L_2}}{\|\phi_{k,y}\|_{L_2}} = \gamma^{-\frac{1}{2}}$. Since $\phi_{k-1,y} - \phi_{k,y} \in \operatorname{span}\{\phi_{k,x} : x \in \Omega_k\backslash\Omega_{k-1}\}$, the basis transformation from $\{\frac{\tilde{\phi}_{k-1,y}}{\|\phi_{k,y}\|_{L_2}} : y \in \Omega_{k-1}\} \cup \{\frac{\phi_{k,x}}{\|\phi_{k,x}\|_{L_2}} : x \in \Omega_k\backslash\Omega_{k-1}\}$ to $\{\frac{\phi_{k,x}}{\|\phi_{k,x}\|_{L_2}} : x \in \Omega_k\}$ can be written in 2×2 block form $\mathbf{S} = \begin{bmatrix} \frac{2^{d+2}-2}{d+2}\mathbf{I} & \mathbf{0} \\ \mathbf{B} & \mathbf{I} \end{bmatrix}$. The well-known L_2-stability of $\{\phi_{k,x} : x \in \Omega_k\}$, and the fact that $\gamma^{-\frac{1}{2}} \lesssim 1$ show that the spectral norm of $\mathbf{B}$ is (uniformly) bounded, and so is the spectral norm of $\mathbf{S}^{-1}$. Again by L_2-stability of the nodal basis, we now conclude that $\{\tilde{\phi}_{k-1,y} : y \in \Omega_{k-1}\} \cup \{\phi_{k,x} : x \in \Omega_k\backslash\Omega_{k-1}\}$ is an L_2-stable basis of $\mathcal{M}_k$ (cf. Remark 1.1), i.e. Condition (a) of Theorem 2.1 is satisfied.

Since $\|\tilde{\phi}_{k-1,y}\|_{L_2} \lesssim \|\phi_{k-1,y}\|_{L_2}$, Remark 2.2 shows that also Condition (b) of Theorem 2.1 is satisfied, and so from this theorem we conclude that

$$\{\psi_{k,x} := \phi_{k,x} - \sum_{y\in\Omega_{k-1}} \frac{(\phi_{k,x}, \phi_{k-1,y})_{L_2}}{(\phi_{k-1,y}, \phi_{k-1,y})_{L_2}}\tilde{\phi}_{k-1,y} : x \in \Omega_k\backslash\Omega_{k-1}\}, \tag{3.5}$$

with $\tilde{\phi}_{k-1,y}$ from (3.3) is an L_2-stable basis of $\mathcal{M}_k \ominus^{\perp_{L_2}} \mathcal{M}_{k-1}$.

We stress that above two-step construction of an L_2-stable basis of $\mathcal{M}_k \ominus^{\perp_{L_2}} \mathcal{M}_{k-1}$ applies *independently of the initial mesh* underlying $\mathcal{M}_0$, so in particular this mesh need not be a regular one, and also *independently of boundary conditions.* The adaptation of the prewavelets to these parameters takes place in the second, explicit step of the construction. Recall that Ω_{k-1} is defined as the set of nodal points in which no essential boundary conditions are prescribed. The factors $\frac{(\phi_{k,x},\phi_{k-1,y})_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{L_2}}$ can be computed using the mass matrix. The first, implicit step of the construction, i.e. solving for α and β in $\tilde{\phi}_{k-1,y} = \alpha\phi_{k,y} + \beta\phi_{k-1,y}$ is independent of the initial mesh and boundary conditions, and so this step does not have to be repeated.

In above aspect, the two-step construction of prewavelets differs from the proposals known from the literature. There the prewavelet $\psi_{k,x}$ is sought as linear combination of sufficiently many $\phi_{k,z}$ with z close to x; the coefficients in this linear combination are found by imposing L_2-orthogonality of $\psi_{k,x}$ to $\phi_{k-1,y}$ for all $y \in \Omega_{k-1}$ with $\mathrm{supp}\phi_{k-1,y} \cap \mathrm{supp}\psi_{k,x} \neq \emptyset$. For each geometrically new situation this process have to be repeated; the existence of a solution, and the stability of the obtained basis of $\mathcal{M}_k \ominus^{\perp_{L_2}} \mathcal{M}_{k-1}$ are not known a priori.

Furthermore, we note that the two-step construction also works when we would have used a *weighted L_2-scalar product* $(u,v)_{L_2} = \int \omega u\overline{v}$, where $\omega \gtrsim 1$; as long as ω is piecewise constant on the simplices underlying $\mathcal{M}_0$, the same λ and μ are obtained, thus which are in particular constants. Again, adaptations of the prewavelets to the actual ω take place in the second, explicit step of the construction.

An important application of a weighted scalar product is given by the case of linear finite element spaces on *manifolds*; in this case the role of the weight function ω is played by the Gram determinant of the actual parametrization used to define the finite element spaces with. In case the manifold is the boundary of a polyhedron, this Gram determinant will be piecewise constant, which means that this situation fits into above framework. Note that, compared to the domain case, geometrically new situations that occur at "corner points" (points with less or more neighbours in the mesh) are treated automatically in the second, explicit step. For parametrizations of manifolds for which the Gram determinant can be written as a product of a globally smooth function and a piecewise constant function, the L_2-scalar product may be changed by replacing this smooth function by a constant without the adjoint spaces being changed. So also these situations can be fitted to our framework.

We now discuss the number of vanishing moments of prewavelets. For $x \in \Omega_k \backslash \Omega_{k-1}$, let $V_{k-1,x}$ be the union of simplices underlying $\mathcal{M}_{k-1}$ for which the intersection of the interior with $\mathrm{supp}\psi_{k,x}$ is not empty. Then, when there are no essential boundary conditions prescribed on $V_{k-1,x}$, for each $p \in P_1$ there exists an $u_{k-1} \in \mathcal{M}_{k-1}$ such that $u_{k-1} = p$ on $V_{k-1,x}$, and so $(\psi_{k,x}, p)_{L_2} = (\psi_{k,x}, u_{k-1})_{L_2} = 0$. We conclude that "away from essential boundary conditions" prewavelets have *two vanishing moments.* A dimension argument shows that the number of vanishing moments can not be uniformly larger than two.

Finally in this section, we show how the obtained prewavelets look like in the shift-invariant mesh case; i.e. $\mathcal{M}_k$ is the linear finite element space based on the subdivision of

$\mathbb{R}^d$ into the collection of d-simplices

$2^{-k}\{\alpha+\{x \in [0,1]^d : 0 \leq x_{\pi(1)} \leq \cdots \leq x_{\pi(d)} \leq 1,\ \pi$ a permutation of $\{1,\ldots,d\}\} : \alpha \in \mathbb{Z}^d\}$,

and thus $\Omega_k = 2^{-k}\mathbb{Z}^d$.

For $d = 1$, the coefficient $\frac{(\phi_{k,x},\phi_{k-1,y})_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{L_2}}$ from (3.5) is $\frac{3}{8}$ for both coarse-grid parents $y \in \Omega_{k-1}$ of $x \in \Omega_k \backslash \Omega_{k-1}$. Plugging formula (3.3) into (3.5) now gives the familiar 5-point prewavelet (see Fig. 1) (cf. (3.2)).

	x ↓							
3		**3**		1	**-6**	10	**-6**	1

FIGURE 1. Non-zero coefficients $\frac{(\phi_{k,x},\phi_{k-1,y})_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{L_2}}$ and resulting prewavelet mask both multiplied by 8 in the one-dimensional shift-invariant mesh case. The bold-faced numbers correspond to coarse-grid points.

For $d = 2$, the coefficient $\frac{(\phi_{k,x},\phi_{k-1,y})_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{L_2}}$ is $\frac{5}{24}$ for both coarse-grid parents $y \in \Omega_{k-1}$ of $x \in \Omega_k \backslash \Omega_{k-1}$, and it is $\frac{1}{24}$ for both remaining vertices $y \in \Omega_{k-1}$ of coarse-grid triangles that contain x. In Fig. 2, we show $\frac{(\phi_{k,x},\phi_{k-1,y})_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{L_2}}$ and the resulting 23-point prewavelet mask for $x \in \Omega_{k-1} + 2^{-k}(0,1)$. Since there exists an invertible linear mapping that changes all triangles in the mesh into equilateral ones without the volumes being changed, both remaining cases lead to equivalent masks.

						5	5	1	1
	20	**4**			5	**-70**	6	**-14**	1
	x →			1	6	106	6	1	
4	**20**		1	**-14**	6	**-70**	5		
			1	1	5	5			

FIGURE 2. Non-zero coefficients $\frac{(\phi_{k,x},\phi_{k-1,y})_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{L_2}}$ and 23-point prewavelet mask both multiplied by 96 in the two-dimensional shift-invariant mesh case and $x \in \Omega_{k-1} + 2^{-k}(0,1)$.

For this two-dimensional shift-invariant mesh case, prewavelets with 13-point masks have been constructed in [Jun94, KO95]. Remarkably, as we will see in Sect. 5, although our masks are larger, the wavelet transformation can be implemented using less operations. The prewavelet masks will not enter the computations, and in that respect the pictures of these masks are a bit misleading.

For $d = 3$, we have to distinguish between $x \in \Omega_{k-1} + 2^{-k}\{(1,1,0),(1,0,1),(0,1,1)\}$ and $x \in \Omega_{k-1} + 2^{-k}\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$. In the first case $\frac{(\phi_{k,x},\phi_{k-1,y})_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{L_2}}$ is

$\frac{40}{384}$ for the two coarse-grid parents of x, and it is $\frac{10}{384}$ for the four remaining vertices of coarse-grid tetrahedrons that contain x (see Fig. 3). In the second case $\frac{(\phi_{k,x},\phi_{k-1,y})_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{L_2}}$ is $\frac{45}{384}$ for the two coarse-grid parents of x, and it is $\frac{5}{384}$ for the six remaining vertices of coarse-grid tetrahedrons that contain x (see Fig. 4).

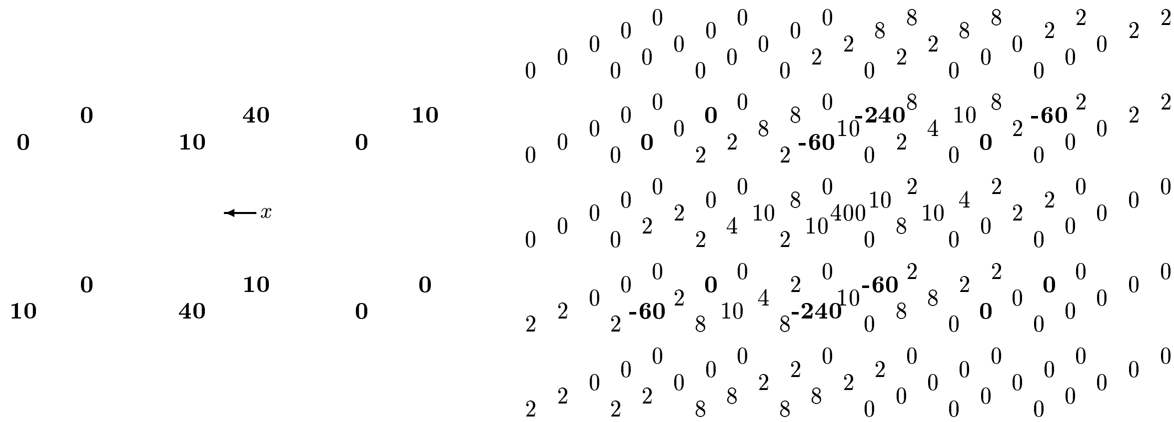

FIGURE 3. Non-zero coefficients $\frac{(\phi_{k,x},\phi_{k-1,y})_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{L_2}}$ and 77-point prewavelet mask both multiplied by 384 in the three-dimensional shift-invariant mesh case and $x \in \Omega_{k-1} + 2^{-k}(0,1,1)$.

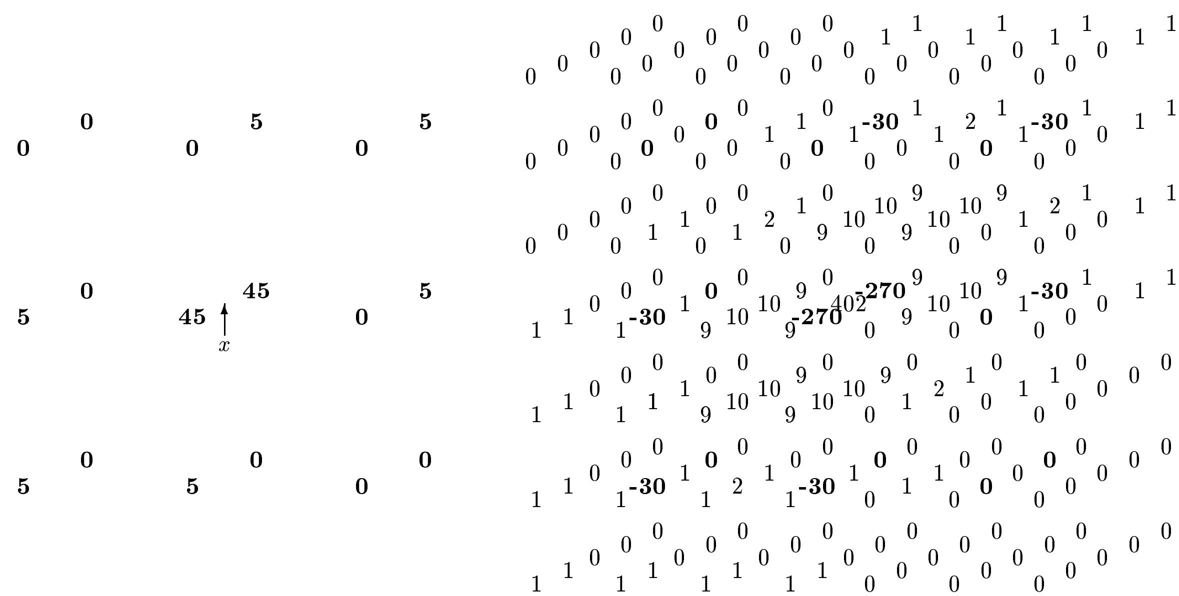

FIGURE 4. Non-zero coefficients $\frac{(\phi_{k,x},\phi_{k-1,y})_{L_2}}{(\phi_{k-1,y},\phi_{k-1,y})_{L_2}}$ and 101-point prewavelet mask both multiplied by 384 in the three-dimensional shift-invariant mesh case and $x \in \Omega_{k-1} + 2^{-k}(0,1,0)$.

4. The three-point wavelet basis

This basis was introduced in [Ste95b, Ste95a], and further analyzed in [LO96a, Ste96]. The basic idea behind the construction originates from [Hac89]. Here we recall some facts about this basis, and show that it fits into the framework of Sect. 2.

The three-point wavelet basis corresponds to the decomposition $\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{(,)_{\mathcal{M}_k}}} \mathcal{M}_{k-1}$, where

$$(u,v)_{\mathcal{M}_k} = (u,v)_{T_k} := \sum_{x\in\Omega_k} w_{k,x} u(x)\overline{v(x)}, \tag{4.1}$$

with $w_{k,x}$ being the sum of the volumes, divided by $d+1$, of the simplices underlying $\mathcal{M}_k$ of which x is a vertex. There holds

$$(u,u)_{L_2} \le (u,u)_{T_k} \le (d+2)(u,u)_{L_2} \qquad (u\in\mathcal{M}_k). \tag{4.2}$$

Also in case of a weighted L_2-scalar product, e.g. the manifold case, with a weight function that is piecewise constant on the simplices, we can construct $(\,,\,)_{T_k}$ satisfying (4.2), simply by multiplying the volumes by the corresponding weights. This is important in order to get stability in the right scale of Sobolev norms. For simplicity, here we consider the standard L_2-scalar product only and refer to [Ste96] for the general case.

For the decomposition $\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{(,)_{T_k}}} \mathcal{M}_{k-1}$, in [Ste96] it has been shown that $\|\sum_k v_k\|^2_{\mathcal{H}^r} \eqsim \sum_k 4^{kr}\|v_k\|^2_{L_2}$ ($v_k\in\mathcal{V}_k$) is valid at least for $\frac{3}{2} > r > \left\{\begin{array}{ll} .29 & \text{if } d=1 \\ .42 & \text{if } d=2 \\ .76 & \text{if } d=3 \end{array}\right\}$ (cf. (1.3)). This result even holds for locally refined meshes, under the assumption that simplices that were not, or irregularly refined when going from level $k-1$ to level k are never refined further on higher levels. This theoretical result concerning the range of stability turns out to be quite pessimistic. In all our experiments in two- and three dimensions reported in [Ste95b, Ste95a], we observed L_2-stability, also in case of locally refined meshes. In ([LO96a]), it was proved that for the shift-invariant case the full range of stability is $r\in(-.990236,\frac{3}{2})$ for $d\in\{1,2,3\}$.

It is immediately clear that an adjoint basis of $\{\frac{\phi_{k-1,y}}{(\phi_{k-1,y},\phi_{k-1,y})_{T_k}} : y\in\Omega_{k-1}\}$ with respect to $(\,,\,)_{T_k}$ is given by

$$\{\tilde{\phi}_{k-1,y} := \frac{(\phi_{k-1,y},\phi_{k-1,y})_{T_k}}{(\phi_{k,y},\phi_{k-1,y})_{T_k}}\phi_{k,y} : y\in\Omega_{k-1}\}.$$

The conditions of Theorem 2.1 are easily verified, and so we conclude that an L_2-stable basis of $\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{(,)_{T_k}}} \mathcal{M}_{k-1}$ (cf. (1.4)) is given by

$$\{\psi_{k,x} := \phi_{k,x} - \sum_{y\in\Omega_{k-1}} \frac{(\phi_{k,x},\phi_{k-1,y})_{T_k}}{(\phi_{k,y},\phi_{k-1,y})_{T_k}}\phi_{k,y} : x\in\Omega_k\backslash\Omega_{k-1}\}. \tag{4.3}$$

Note that the elements in the sum over y in (4.3) are only non-zero when y is a coarse-grid parent of x. This means that $\psi_{k,x}$ is a linear combination of three fine-grid nodal basis functions, independently of the dimension d. For the shift-invariant mesh case in any dimension, $\psi_{k,x}$ can be represented by the mask $[-\frac{1}{2} \quad 1 \quad -\frac{1}{2}]$, where the values $-\frac{1}{2}$ correspond to the coarse-grid parents of x.

Since $(u, \mathbb{1})_{T_k} = (u, \mathbb{1})_{L_2}$ $(u \in \mathcal{M}_k)$, where $\mathbb{1} \in \mathcal{M}_{k-1}$ is defined by $\mathbb{1}(x) = 1$ if $x \in \Omega_{k-1}$, we conclude that "away from essential boundary conditions", the three-point wavelets have one vanishing moment. In areas where locally the meshes on levels $k-1$ and k are uniform, symmetry of $\psi_{k,x}$ yields even two vanishing moments.

Concluding, we note that the three-point wavelet basis is more flexible than the prewavelet basis constructed in Sect. 3, since it can also be constructed in case of non-uniform refinements, e.g. on local refined meshes. As we will discuss in the next section, application of the wavelet transform for the three-point wavelets is also cheaper. Moreover, at least in one dimension, the three-point wavelets give smaller L_2- and $\mathcal{H}^1$ condition numbers. We have to pay for this by having a smaller range of (guaranteed) stability.

5. Implementation

In this section for the wavelet bases discussed in this paper, we compare the number of operations needed for applying the basis transformation from wavelet basis to nodal basis (the *wavelet transform*). From Sect. 1, recall that the wavelet transform $\mathbf{T}_J$ is the only variable part of the multi-level preconditioner corresponding to a wavelet basis.

Let $\mathbf{S}_J$ the basis transformation from $\{\psi_{J,x} : x \in \Omega_J \backslash \Omega_{J-1}\} \cup \{\phi_{J-1,y} : y \in \Omega_{k-1}\}$ to $\{\phi_{J,x} : x \in \Omega_J\}$. Then, by using a splitting of the wavelet basis of $\mathcal{M}_J$ into the sets $\{\psi_{J,x} : x \in \Omega_J \backslash \Omega_{J-1}\}$ and $\cup_{k=0}^{J-1}\{\psi_{k,x} : x \in \Omega_k \backslash \Omega_{k-1}\}$, we have $\mathbf{T}_J = \mathbf{S}_J \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{J-1} \end{bmatrix}$, and so it is sufficient to compare the costs of applying $\mathbf{S}_J$. There holds $\mathbf{S}_J = \begin{bmatrix} \mathbf{q}_J & \mathbf{p}_J \end{bmatrix}$, where $\mathbf{p}_J$ is the standard multi-grid prolongation, and where the columns of $\mathbf{q}_J$ contain the wavelet masks.

In the following, we consider the two-dimensional uniform mesh case, which means that $\mathbf{p}_J$ is represented by the 7-point stencil $\frac{1}{2}\begin{bmatrix} & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & \end{bmatrix}$. Let n be the dimension of $\mathcal{M}_J$. Then a direct implementation of $\mathbf{q}_J$ yields the following operation count for $\mathbf{S}_J$:

- standard hierarchical basis: $1 \cdot \frac{3}{4}n + 7 \cdot \frac{1}{4}n = 2\frac{3}{4}n$
- 3-point wavelet basis: $3 \cdot \frac{3}{4}n + 7 \cdot \frac{1}{4}n = 4n$
- 13-point prewavelet basis: $13 \cdot \frac{3}{4}n + 7 \cdot \frac{1}{4}n = 11\frac{1}{2}n$.

Clearly, a similar implementation of the 23-point prewavelet basis would cost $23 \cdot \frac{3}{4}n + 7 \cdot \frac{1}{4}n = 19n$ operations. Yet, making use of the two-step construction, we can follow a different approach for implementing $\mathbf{q}_J$: First, for each $x \in \Omega_J \backslash \Omega_{J-1}$, express $\psi_{J,x}$ as a linear combination of $\phi_{J,x}$ and four adjoint basis functions $\tilde{\phi}_{J-1,y}$; then, for each $y \in \Omega_{J-1}$, write $\tilde{\phi}_{J-1,y}$ as linear combination of $\phi_{J,y}$ and $\phi_{J-1,y}$; finally, express each $\phi_{J-1,y}$ as a linear combination of seven fine-grid nodal basis functions $\phi_{J,z}$. Note that we had to perform this last step anyway. We end up with the following operation count for $\mathbf{S}_J$:

– 23-point prewavelet basis (two-step construction): $(1+4)\cdot\frac{3}{4}n+2\cdot\frac{1}{4}n+7\cdot\frac{1}{4}n=6n$,

which is almost two times cheaper than when applying the 13-point prewavelets.

Note that this implementation of $\mathbf{q}_J$ using $\mathbf{p}_J$ does not reduce the operation count for the 3-point wavelet basis, because in that case the adjoint basis function $\tilde{\phi}_{J-1,y}$ is just a multiple of $\phi_{J,y}$.

In the three-dimensional uniform mesh case we obtain the following operation counts for $\mathbf{S}_J$:

– standard hierarchical basis: $1\cdot\frac{7}{8}n+15\cdot\frac{1}{8}n=2\frac{1}{2}n$
– 3-point wavelet basis: $3\cdot\frac{7}{8}n+15\cdot\frac{1}{8}n=4\frac{1}{2}n$
– 101/77-point prewavelet basis (two-step construction):

$$((1+8)\cdot\tfrac{4}{8}n+(1+6)\cdot\tfrac{3}{8}n)+2\cdot\tfrac{1}{8}n+15\cdot\tfrac{1}{8}n=9\tfrac{1}{4}n.$$

Note that we would need $(101\cdot\frac{4}{8}n+77\cdot\frac{3}{8}n)+15\cdot\frac{1}{8}n=81\frac{1}{4}n$ operations with a direct implementation of the 101/77 masks.

So, the two-step construction not only makes it possible to construct stable wavelet bases on non-uniform meshes, it also gives rise to a wavelet transform that can be implemented efficiently in much less operations than one would expect on basis of the sizes of the masks.

6. An alternative three-point wavelet basis

A disadvantage of the prewavelet and three-point wavelet bases introduced in Sects. 3 and 4 is that for these bases there does not exist an explicit inverse wavelet transform as we will see below. There are applications of wavelets for which such an explicit wavelet transform is needed. Examples are given by data compression, which belongs to the "classical" field of applications, and the construction of extension operators for use in domain decomposition algorithms.

Recall that the wavelet transform $\mathbf{T}_J$ can be written as $\mathbf{T}_J=\begin{bmatrix}\mathbf{q}_J & \mathbf{p}_J\mathbf{T}_{J-1}\end{bmatrix}$, where $\mathbf{p}_J$ and $\mathbf{q}_J$ are the representations of the inclusions $\mathcal{M}_{J-1}\to\mathcal{M}_J$ and $\mathcal{V}_J\to\mathcal{M}_J$ respectively, with respect to the nodal bases $\{\phi_{J-1,y}: y\in\Omega_{J-1}\}$ and $\{\phi_{J,x}: x\in\Omega_J\}$ on $\mathcal{M}_{J-1}$ and $\mathcal{M}_J$, and the basis $\{\psi_{J,x}: x\in\Omega_J\backslash\Omega_{J-1}\}$ on $\mathcal{V}_J$.

Alternatively, we could have equipped $\mathcal{M}_J$ with the two-level standard hierarchical basis $\{\phi_{J,x}: x\in\Omega_J\backslash\Omega_{J-1}\}\cup\{\phi_{J-1,y}: y\in\Omega_{J-1}\}$. Denoting corresponding matrices with a tilde, we have $\tilde{\mathbf{p}}_J=\begin{bmatrix}\mathbf{0}\\ \mathbf{I}\end{bmatrix}$, $\tilde{\mathbf{q}}_J=\begin{bmatrix}\tilde{\mathbf{q}}_J^F\\ \tilde{\mathbf{q}}_J^C\end{bmatrix}$ and so $\tilde{\mathbf{T}}_J=\begin{bmatrix}\tilde{\mathbf{q}}_J^F & \mathbf{0}\\ \tilde{\mathbf{q}}_J^C & \mathbf{T}_{J-1}\end{bmatrix}$. Assuming a fine-to-coarse two-level ordering of the nodal basis functions as well, the basis transformation on $\mathcal{M}_J$ from two-level standard hierarchical basis to nodal basis is of the form $\begin{bmatrix}\mathbf{I} & \mathbf{p}_J^C\\ \mathbf{0} & \mathbf{I}\end{bmatrix}$. We end up with the following "UL"-factorization of $\mathbf{T}_J$:

$$\mathbf{T}_J=\begin{bmatrix}\mathbf{I} & \mathbf{p}_J^C\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{q}}_J^F & \mathbf{0}\\ \tilde{\mathbf{q}}_J^C & \mathbf{T}_{J-1}\end{bmatrix}.$$

As always assuming that the columns of $\mathbf{q}_J$ and thus of $\tilde{\mathbf{q}}_J$ contain an uniformly bounded number of non-zeros, from this block factorization we conclude that $\mathbf{T}_J^{-1}$ is a composition of explicitly given, level-wise local operators, which means that the application of $\mathbf{T}_J^{-1}$ can be implemented in $\sim \dim\mathcal{M}_J$ operations, if and only if $(\tilde{\mathbf{q}}_J^F)^{-1}$ is local. For the prewavelet and three-point wavelet bases from Sects. 3 and 4 this $(\tilde{\mathbf{q}}_J^F)^{-1}$ is not local. For example, in the one-dimensional shift-invariant mesh case, one may check that $\tilde{\mathbf{q}}_J^F$ is represented by the stencils $\frac{1}{2}[1 \quad \cdot \quad 4 \quad \cdot \quad 1]$ and $\frac{1}{4}[1 \quad \cdot \quad 6 \quad \cdot \quad 1]$ respectively.

Clearly, a sufficient condition for $(\tilde{\mathbf{q}}_J^F)^{-1}$ being local is that $\tilde{\mathbf{q}}_J^F$ is a diagonal matrix. In the context of Sect. 2, this means that the adjoint basis functions $\tilde{\phi}_{k-1,y}$ are in $\mathcal{M}_{k-1}$, or, $\psi_{k,x}$ is a linear combination of $\phi_{k,x}$ and a number of coarse-grid nodal basis functions. Sufficient for $\tilde{\phi}_{J-1,y} \in \mathcal{M}_{k-1}$ is that $\{\phi_{k-1,y} : y \in \Omega_{k-1}\}$ is an orthogonal set with respect to $(\ ,\ )_{\mathcal{M}_k}$.

In this section, for $(\!(\ ,\ )\!)_{\mathcal{M}_k}$ being some L_2-bounded Hermitian sesquilinear form on $\mathcal{M}_k$, we define

$$(u,v)_{\mathcal{M}_k} = (\!(u,v)\!)_{\mathcal{M}_k} - (\!(I_{k-1}u, I_{k-1}v)\!)_{\mathcal{M}_k} + (I_{k-1}u, I_{k-1}v)_{T_{k-1}},$$

where $I_{k-1} : \mathcal{M}_k \to \mathcal{M}_{k-1}$ is the nodal value interpolant, and $(\ ,\ )_{T_{k-1}}$ is defined in (4.1). In order to obtain locally supported wavelets, we assume that $(\!(\ ,\ )\!)_{\mathcal{M}_k}$ is local in the sense that $(\!(u,v)\!)_{\mathcal{M}_k} \neq 0$ only if $\operatorname{supp} u \cap \operatorname{supp} v \neq \emptyset$.

The following statements are easily verified: $(\ ,\ )_{\mathcal{M}_k}$ is an L_2-bounded Hermitian sesquilinear form on $\mathcal{M}_k$; $\{\phi_{k-1,y} : y \in \Omega_{k-1}\}$ is an $(\ ,\ )_{\mathcal{M}_k}$-orthogonal set, and so an adjoint basis with respect to $(\ ,\ )_{\mathcal{M}_k}$ is given by $\{\tilde{\phi}_{k-1,y} := \phi_{k-1,y} : y \in \Omega_{k-1}\}$. By Remark 2.2, both conditions of Theorem 2.1 are easily verified, and we conclude that

$$\{\psi_{k,x} := \phi_{k,x} - \sum_{y \in \Omega_{k-1}} \frac{(\!(\phi_{k,x}, \phi_{k-1,y})\!)_{\mathcal{M}_k}}{(\phi_{k-1,y}, \phi_{k-1,y})_{T_{k-1}}} \phi_{k-1,y} : x \in \Omega_k \backslash \Omega_{k-1}\}$$

is an L_2-stable basis of $\mathcal{V}_k = \mathcal{M}_k \ominus^{\perp_{(\ ,\ )_{\mathcal{M}_k}}} \mathcal{M}_{k-1}$.

So far, we only investigated the case that $(\!(\ ,\ )\!)_{\mathcal{M}_k} = (\ ,\ )_{T_k}$. Then, the coefficients $\frac{(\phi_{k,x}, \phi_{k-1,y})_{T_k}}{(\phi_{k-1,y}, \phi_{k-1,y})_{T_{k-1}}}$ are only non-zero for both coarse-grid parents y of x, independently of the space dimension. For the shift-invariant mesh case, these coefficients are equal to $(\frac{1}{2})^{d+1}$. So, although the resulting wavelet masks are larger than in case of the three-point wavelet basis from Sect. 4, in view of the results about implementation from Sect. 5, we may call this wavelet basis also a three-point wavelet basis. For the operation count of the wavelet transform it makes no difference whether $\tilde{\phi}_{k-1,y}$ is a multiple of $\phi_{k,y}$ or $\phi_{k-1,y}$.

Since $(u, \mathbb{1})_{\mathcal{M}_k} = ((I - I_{k-1})u, \mathbb{1})_{T_k} + (I_{k-1}u, \mathbb{1})_{T_{k-1}} = (u, \mathbb{1})_{L_2}$, we conclude that "away from essential boundary conditions" also these three-point wavelets have at least one vanishing moment. As with the other three-point wavelets, in areas where locally the meshes on levels $k-1$ and k are uniform, symmetry of $\psi_{k,x}$ yields two vanishing moments. So compared to the prewavelets, this basis does not give better compression rates for integral operators (cf. last paragraph of Sect. 1). Further research in this direction is necessary.

The three-point wavelet basis from this section generalizes bases known from the literature to the case of non-uniform meshes. In the one-dimensional shift-invariant mesh case, it equals the (2,2) biorthogonal wavelet basis from [CDF92]. For the two-dimensional shift-invariant mesh case it is equal to the basis proposed in [LO96a, §4.3 Example 2]. Related proposals can be found in [CDP96, VW95, VW96, Swe95].

We will now discuss for which r stability between the levels $\|\sum_k v_k\|^2_{\mathcal{H}^r} \eqsim \sum_k 4^{kr}\|v_k\|^2_{L_2}$ $(v_k \in \mathcal{V}_k)$ is valid (cf. (1.3)). In [LO96a], for the two-dimensional shift-invariant mesh case, stability has been shown for $.02282 < r < \frac{3}{2}$. Numerical experiments reported in [LO96b] show that this wavelet basis is not L_2-stable.

To investigate stability for the non-uniform mesh case, we will follow the analysis from [Ste96]. The crucial thing is to find a $t \in [0, \frac{3}{2})$, such that the L_2-norm of the projection $\Pi^{(J)}_\ell : \mathcal{M}_J \to \mathcal{M}_\ell : \sum_{k=0}^J v_k \mapsto \sum_{k=0}^\ell v_k$ $(v_k \in \mathcal{V}_k)$ is $\lesssim 2^{t(J-\ell)}$. Then $\mathcal{H}^r$-stability for $r \in (t, \frac{3}{2})$ is guaranteed.

Note that $\Pi^{(J)}_\ell = \Pi^{(\ell+1)}_\ell \cdots \Pi^{(J)}_{J-1}$ and that $\Pi^{(k)}_{k-1} : \mathcal{M}_k \to \mathcal{M}_{k-1}$ is the projection onto $\mathcal{M}_{k-1}$ which is orthogonal with respect to $(\,,\,)_{\mathcal{M}_k}$. From $\|u\|_{\mathcal{M}_k} := (u,u)^{\frac{1}{2}}_{\mathcal{M}_k} \eqsim \|u\|_{L_2}$ for $u \in \mathcal{M}_{k-1}$, we have $\|\Pi^{(J)}_\ell\|_{L_2 \leftarrow L_2} \eqsim \|\Pi^{(\ell+1)}_\ell \cdots \Pi^{(J)}_{J-1}\|_{\mathcal{M}_{\ell+1} \leftarrow \mathcal{M}_{J+1}} \leq \prod_{k=\ell}^{J-1} \|\Pi^{(k+1)}_k\|_{\mathcal{M}_{k+1} \leftarrow \mathcal{M}_{k+2}}$.

Following the analysis from [Ste96, §4.2], the factors $\|\Pi^{(k+1)}_k\|_{\mathcal{M}_{k+1} \leftarrow \mathcal{M}_{k+2}}$ can be estimated using local projections. We obtain $\mathcal{H}^r$-stability for $\frac{3}{2} > r > \begin{cases} \frac{1}{2} & \text{if } d = 1 \\ \frac{3}{4} & \text{if } d = 2 \end{cases}$. Compared to Sect. 4, we need the additional assumption that irregular refined simplices do not contain parent nodes for subsequent refinements. Unfortunately, application of this analysis to the three-dimensional case yields the lowerbound $r > {}^2\log\sqrt{4\frac{3}{4}}$ which is larger than 1.

References

[CDF92] A. Cohen, I. Daubechies, and J.C. Feauveau. Biorthogonal bases of compactly supported wavelets. *Comm. Pur. Appl. Math.*, 45:485–560, 1992.

[CDP96] J.M. Carnicer, W. Dahmen, and J.M. Peña. Local decomposition of refinable spaces and wavelets. *Appl. and Comp. Harm. Anal.*, 3:127–153, 1996.

[CW92] C.K. Chui and J.Z. Wang. On compactly supported spline wavelets and a duality principle. *Trans. Amer. Math. Soc.*, 330:903–916, 1992.

[DKS95] W. Dahmen, A. Kunoth, and R. Schneider. Operator equations, multiscale concepts and complexity. Technical report, WIAS-Berlin, December 1995.

[DKU96] W. Dahmen, A. Kunoth, and K. Urban. Biorthogonal spline-wavelets on the interval - stability and moment conditions. Technical report, RWTH Aachen, August 1996.

[Hac89] W. Hackbusch. The frequency decomposition multi-grid method I. Application to anisotropic equations. *Numer. Math.*, 56:229–245, 1989.

[Jun94] J. Junkherr. Multigrid methods for weakly singular integral equations of the first kind. Christian-Albrechts-Universität Kiel, Germany, 1994. PhD. thesis (in German).

[KO95] U. Kotyczka and P. Oswald. Piecewise linear prewavelets of small support. In C.K. Chui and L.L. Schumaker, editors, *Approximation Theory VIII*. World Scientific Publishing Co. Inc., 1995.

[LO96a] R. Lorentz and P. Oswald. Constructing economical Riesz bases for Sobolev spaces. GMD-bericht 993, GMD, Sankt Augustin, May 1996.

[LO96b] R. Lorentz and P. Oswald. Multilevel finite element Riesz bases in Sobolev spaces. In P. Bjørstad, M. Espedal, and Keyes D., editors, *Proc. 9th. Symp. on Domain Decomposition Methods*. John Wiley & Sons Ltd., 1996.

[Osw94] P. Oswald. *Multilevel finite element approximation: Theory and applications.* B.G. Teubner, Stuttgart, 1994.

[Sch95] R. Schneider. Multiskalen- und Wavelet-Matrixkompression: Analysisbasierte Methoden zur Lösung großer volbesetzter Gleigungssysteme. TH Darmstadt, Germany, 1995. Habilitationschrift (in German).

[Ste95a] R.P. Stevenson. Experiments in 3D with a three-point hierarchical basis preconditioner. Technical Report 9558, University of Nijmegen, December 1995. Accepted for publication in *Appl. Numer. Math.*

[Ste95b] R.P. Stevenson. A robust hierarchical basis preconditioner on general meshes. Technical Report 9533, University of Nijmegen, September 1995. Accepted for publication in *Numer. Math.*

[Ste96] R.P. Stevenson. Stable three-point wavelet bases on general meshes. Technical Report 9627, University of Nijmegen, October 1996. Submitted to Numer. Math.

[Swe95] W. Sweldens. The lifting scheme: A construction of second generation wavelets. Technical Report 1995:6, Industrial Mathematics Initiative, Department of Mathematics, University of South Carolina, 1995.

[VW95] P.S. Vassilevski and J. Wang. Stabilizing the hierarchical basis by approximate wavelets, II: Implementation and numerical experiments. CAM report 95-48, Department of Mathematics, UCLA, Los Angeles, 1995. to appear in SIAM J. Sci. Comput.

[VW96] P.S. Vassilevski and J. Wang. Stabilizing the hierarchical basis by approximate wavelets, I: Theory. *Num. Lin. Alg. with Appl.*, 1996. To appear.

[Yse86] H. Yserentant. On the multi-level splitting of finite element spaces. *Numer. Math.*, 49:379–412, 1986.

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