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# PIECEWISE POLYNOMIAL INTERPOLATIONS IN THE FINITE ELEMENT METHOD

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## 1. INTRODUCTION AND SOME NOTATIONS

At first let us introduce some notations which are used in the paper.

Let $\Omega$ be a plane bounded domain and $\bar{\Omega}$ be its closure, i.e. $\bar{\Omega} = \Omega \cup \partial\Omega$ where $\partial\Omega$ is the boundary of $\Omega$. Then

$W_2^{(k)}(\Omega)$ is the Sobolev space of all functions which together with their generalized derivatives up to the $k$th order inclusive belong to $L_2(\Omega)$. For $u \in W_2^{(k)}(\Omega)$ the norm $\|u\|_{k,\Omega}$ is defined by

$$\|u\|_{k,\Omega}^2 = \sum_{j=0}^{k} |u|_{j,\Omega}^2 , \quad \text{where} \quad |u|_{j,\Omega}^2 = \sum_{|i|=j} \|D^i u\|_{L_2(\Omega)}^2 ,$$

$$D^i u = \frac{\partial^{|i|} u}{\partial x^p \, \partial y^q} , \quad i = (p, q) , \quad |i| = p + q .$$

$C^{(k)}(\bar{\Omega})$ is the set of all functions having in $\bar{\Omega}$ continuous derivatives up to the $k$th order inclusive, $C(\bar{\Omega})$ is the set of all functions which are continuous in $\bar{\Omega}$.

$\mathring{W}_2^{(k)}(\Omega)$ is a subspace of $W_2^{(k)}(\Omega)$ which we get by completing in the norm $\|.\|_{k,\Omega}$ the set of functions from $C^{(k)}(\Omega)$ with compact support in $\Omega$.

By a polygonal domain we understand every plane bounded domain $\Omega$ the boudary of which consists of a finite number of simple closed polygons $\Gamma_j$, $j = 0, 1, \ldots, s$; $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ lie inside $\Gamma_0$ and do not intersect.

By a triangulation $\tau$ of a polygonal domain $\Omega$ we understand a covering of the closure $\bar{\Omega}$ by a finite number of arbitrary closed triangles such that the union of all triangles is $\bar{\Omega}$ and any two triangles are either disjoint or have a common vertex or a common side. When we wish to express that $h$ is the length of the greatest side and $\vartheta$ is the magnitude of the smallest angle of all triangles of the triangulation $\tau$ we write $\tau(h, \vartheta)$.

A polygonal domain $\Omega$ is said to be rectangular polygonal if each of the polygons forming the boundary of $\Omega$ has the sides paralles to the axes of a Cartesian coordinate system.

By a partition $\varrho$ of a rectangular polygonal domain $\Omega$ we understand a covering of the closure $\bar{\Omega}$ by a finite number of arbitrary closed rectangles such that the union of all rectangles is $\bar{\Omega}$ and any two rectangles are either disjoint or have a common vertex or a common side. If we wish to express that $\Delta$ and $\bar{\Delta}$ are, respectively, the lengths of the greatest and of the smallest sides of all rectangles of the partition $\varrho$, we write $\varrho(\Delta, \bar{\Delta})$.

$\delta$ is said to be a partition of a polygonal domain $\Omega$ if $\delta$ defines a partition $\varrho$ of a rectangular polygonal domain $\Omega_1 \subset \Omega$ [1]) and a triangulation $\tau$ of the set $\Omega_2 = \Omega - \bar{\Omega}_1$ [2]) provided that the intersection of an arbitrary rectangle of the partition $\varrho$ with an arbitrary triangle of the triangulation $\tau$ is either void or is their common side or their common vertex. The rectangles of the partition $\varrho$ and the triangles of the triangulation $\tau$ are called the rectangles and the triangles of the partition $\delta$, respectively.

Let $\Delta$ and $\bar{\Delta}$ be, respectively, the lengths of the greatest and of the smallest sides of all rectangles of a partition $\delta$ of the given polygonal domain $\Omega$, and $\vartheta$ be the magnitude of the smallest angle of all triangles of $\delta$. A collection $\mathscr{R}$ of partitions of the domain $\Omega$ is said to be regular if there exist two positive constants $\sigma_0$ and $\vartheta_0$ such that $\bar{\Delta} \geqq \sigma_0 \Delta$, $\vartheta \geqq \vartheta_0$, for all $\delta \in \mathscr{R}$.

In [1] the hierarchy of interpolation polynomials on the triangle is defined in the following way:

*Let $T$ be a triangle. Let $P_j, l_j, \nu_j$, $j = 1, 2, 3$, and $P_0$ be its vertices, sides, normals to the sides and center of gravity, respectively. Further, let the points $Q_j^{(\varrho,r)}$, $\varrho = 1, 2, \ldots, r$, divide the side $l_j$ ($j = 1, 2, 3$) into $r + 1$ equal parts. Finally, let $m$, $\varkappa$ be non-negative integers, $1 \leqq \varkappa \leqq 4$. Under these hypotheses to each $f \in C^{(2m+1)}(\bar{T})$ (for $\varkappa = 1, 2$ it is sufficient if $f \in C^{(2m)}(\bar{T})$) there is assigned a polynomial $p$ of degree at most $n = 4m + \varkappa$ such that*

$$D^i p(P_j) = D^i f(P_j), \quad j = 1, 2, 3, \tag{1}$$

$$D^k p(P_0) = D^k f(P_0), \tag{2}$$

$$\frac{\partial^s p(Q_j^{(\varrho,r)})}{\partial \nu_j^s} = \frac{\partial^s f(Q_j^{(\varrho,r)})}{\partial \nu_j^s}, \quad j = 1, 2, 3, \quad \varrho = 1, 2, \ldots, r, \tag{3}$$

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[1]) For different partitions of the domain $\Omega$ the corresponding rectangular polygonal domains $\Omega_1$ may be different.

[2]) The set $\Omega_2$ is either void or is the union of a finite number of disjoit polygonal domains.

*and for n* $= 4m + \varkappa$ *the indices i, k, r, s are determined by* $(4_\varkappa)$:

$(4_1)$ $|i| \leqq 2m$ $|k| \leqq m - 2\,,$ $s = r = 1, 2, \ldots, m\,,$

$(4_2)$ $|i| \leqq 2m\,,$ $|k| \leqq m - 1\,,$ $s = r - 1\,,$ $r = 1, 2, \ldots, m + 1\,,$

$(4_3)$ $|i| \leqq 2m + 1\,,$ $|k| \leqq m\,,$ $s = r = 1, 2, \ldots, m\,,$

$(4_4)$ $|i| \leqq 2m + 1\,,$ $|k| \leqq m + 1\,,$ $s = r - 1\,,$ $r = 1, 2, \ldots, m + 1\,.$

The existence and uniqueness of just introduced interpolation are assured by Theorem 1 of [2] and the error estimate (in Sobolev's norm) is given by Theorem 2 of [2]. If $\Omega$ is a polygonal domain, $\tau$ is any triangulation of $\Omega$ and $f \in C^{(2m+1)}(\bar{\Omega})$, then the piecewise-polynomial function $f_\tau$ coinciding on each triangle of the triangulation $\tau$ with the polynomial determined by the conditions $(1)-(4_\varkappa)$ [3]) belongs to $C^{(m)}(\bar{\Omega})$.

Making use of the above described piecewise-polynomial interpolations for solving linear elliptic boundary value problems of order $2(m + 1)$ by the finite element method (see e.g. [2]), it is necessary to use polynomials of degree not smaller than $4m + 1$. Then, of course, to get an approximate solution we must compute the values and the derivatives of this solution not only at the vertices of the triangles of a triangulation $\tau$ of the considered polygonal domain, but also on the sides and at the centers of gravity of the triangles of $\tau$ (see conditions (2) and (3)). However, the normal derivatives on the sides of the triangles are not necessary in applications and their evaluation prolongs the computation. For that reason Bell proposed in [5] a "reduced" polynomial of the fifth degree. We get it from the polynomial $p(x, y)$ of the fifth degree of the above described hierarchy $(m = \varkappa = 1)$ if we eliminate the values $\partial p(Q_j^{(1,1)})/\partial \nu_j$, $j = 1, 2, 3$, by imposing on $p(x, y)$ the condition that $\partial p/\partial \nu_j$, $j = 1, 2, 3$, be cubic polynomials on the sides of the considered triangle (see Section 2, Theorem 1 for $m = 1$). In this case the highest order of accuracy for the finite element method applied to the fourth order boundary value problems is the third order (see Theorem 15 for $m = 1$), whereas if we used the polynomial of the fifth degree from the above described hierarchy $(m = \varkappa = 1)$ the highest order of accuracy would be the fourth order (see [2], Theorem 4), but this fact is not so essential for practical use. Likewise it is possible to save the computer time by eliminating the parameters prescribed at the center of gravity by imposing some restrictions on the polynomials. That was why Zlámal in [6], for solving second order boundary value problems, proposed a "reduced" cubic polynomial $p^*(x, y)$. If $T$ is a triangle with vertices $P_j$, $j = 1, 2, 3$, and with the center of gravity $P_0$, then $p^*(x, y)$ is on $T$ uniquely determined by nine parameters $D^i p^*(P_j)$, $j = 1, 2, 3$, $|i| \leqq 1$, in such a way that the tenth parameter $p^*(P_0)$ is a certain linear combination of the above nine parameters. We apply the just mentioned devices of Bell and Zlámal to the hierarchy of the polynomials deter-

[3]) We say that the function $f_\tau$ is generated by the interpolation (1) — (4).

mined by the condition $(1)-(4_1)$ and thus we come to a hierarchy of reduced interpolation polynomials which are, moreover, also especially suitable to a combination with the reduced Hermite polynomials of [4]. This combination is carried out in Section 3.

When solving a second order boundary value problem by the finite element method and making use of the polynomials of the fifth degree, either determined by 21 parameters (relations $(1)-(4_1)$ for $m = 1$) or reduced by Bell (18 parameters), the approximate solutions have continuous derivatives of the first order on the closure $\bar{\Omega}$ of the considered domain $\Omega$ even when the exact solution does not belong to $C^{(1)}(\bar{\Omega})$. In this case it would be apparently far better to use a polynomial of the fifth degree defined on a triangle with the vertices $P_j$, $j = 1, 2, 3$, and with the center of gravity $P_0$ by these 21 parameters (see [3]): $D^i p(P_j)$, $|i| \leqq 2, j = 1, 2, 3$, $D^k(P_0)$, $|k| \leqq 1$. Following the just mentioned idea we come to interpolation polynomials with "concentrated" parameters. By the concentration, roughly speaking, we mean such a choice of parameters uniquely determining an interpolation polynomial on a triangle, that as many parameters as possible are prescribed at the vertices and at the center of gravity while only as many parameters (or conditions) as necessary for obtaining the desired smoothness of the piecewise-polynomial interpolation in the domain considered are prescribed on the sides. A combination of the just mentioned polynomials with the polynomials of [4] is again carried out in Section 3. In the last section the piecewise-polynomial interpolations of Section 3 are used for solving $V$-elliptic boundary value problems.

## 2. REDUCTIONS AND CONCENTRATION OF PARAMETERS OF INTERPOLATION POLYNOMIALS ON THE TRIANGLE

At first we are going to deal with a reduction of parameters of the interpolation polynomials determined by the conditions $(1)-(4_1)$. This reduction is also mentioned in [2].

**Theorem 1.** *Let $T$ be a triangle. Let $P_j$, $l_j$, $v_j$, $j = 1, 2, 3$, and $P_0$ be its vertices, sides, normals to the sides and center of gravity, respectively. Further let $m$ be a non-negative integer. Then to each $f \in C^{(2m)}(\bar{T})$*[4]) *there exists exactly one polynomial $p$ of degree at most $4m + 1$ such that*

(5) $$D^i p(P_j) = D^i f(P_j)\,, \quad j = 1, 2, 3, \quad |i| \leqq 2m\,,$$

(6) $$D^k p(P_0) = D^k f(P_0)\,, \quad |k| \leqq m - 2\,,$$

(7) *on the side $l_j (j = 1, 2, 3)$, the normal derivatives $\partial^r p/\partial v_j^r$, $r = 1, 2, \ldots, m$, are polynomials of degree at most $4m + 1 - 2r$.*[5])

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[4]) Obviously the existence of the derivatives $D^i f$, $|i| \leqq 2m$, on $\bar{T}$ is sufficient.

[5]) We leave out the conditions (6) or (7) if $m = 0, 1$ or $m = 0$, respectively.

Proof. Let $f(x, y) \equiv 0$ and let $p(x, y)$ be a polynomial of degree at most $4m + 1$ satisfying the conditions (5)–(7). If $l_j(x, y) = 0$, $j = 1, 2, 3$, are the equations of the sides of $T$ and $l(x, y) = l_1(x, y)\, l_2(x, y)\, l_3(x, y)$ then, according to (5) and (7),

$$p(x, y) = l^{m+1}(x, y)\, q(x, y)\,,$$

where $q(x, y)$ is a polynomial of degree at most $m - 2$. With respect to (6) $D^k q(P_0) = 0$, $|k| \leqq m - 2$. Then, of course, $q(x, y) \equiv 0$ and hence also $p(x, y) \equiv 0$ which was to be proved.

Theorem 1 may be considered to be a consequence of Theorem 1 of [2] if we take into account that by (7) the parameters $\partial^r p(Q^{(\varrho,r)})/\partial v_j^r$ in the interpolation (1) to ($4_1$) are now given as certain linear combinations of the parameters $D^i p(P_j)$, $j = 1, 2, 3$, $|i| \leqq 2m$. The error estimate for the piecewise-polynomial interpolation generated by the interpolation of Theorem 1 is given by

**Theorem 2.** *Let $\Omega$ be a polygonal domain and let $\tau(h, \vartheta)$ be any triangulation of $\Omega$. Further, let $m$ be a non-negative integer and let $f \in W_2^{(k)}(\Omega)$, $k \geqq 2m + 2$. Finally, let the function $f_\tau$ coincide with the polynomial $p$ described in Theorem 1 on each triangle of the triangulation $\tau$ and let $l = \min(k, 3m + 2)$. Then $f_\tau \in C^{(m)}(\bar{\Omega})$ and for $0 \leqq n \leqq m + 1$*

$$\|f - f_\tau\|_{n,\Omega} \leqq \frac{K}{(\sin \vartheta)^n}\, h^{l-n}\, |f|_{l,\Omega}\,, \tag{8}$$

*where the constant $K$ depends neither on $\Omega$ nor on $f$.*

Proof. The stated smoothness of $f_\tau$ follows from the fact that on each side of any triangle $T$ of $\tau$ the values of $f_\tau$ together with its derivatives up to the order $m$ inclusive are given merely by the values $D^i f$, $|i| \leqq 2m$, at the vertices of $T$ lying on the considered side. The error estimate could be proved in a similar way as the estimate (15) in [2].

Further reduction of parameters will be carried out in such a way that the parameters $D^k p(P_0)$, $|k| \leqq m - 2$, will be given as certain linear combinations of the parameters $D^i p(P_j)$, $j = 1, 2, 3$, $|i| \leqq 2m$. However, a practical way is to retain the parameters $D^k(P_0)$ and use the method of condensation of internal parameters (see [9] or [10]).

If $p(x, y)$ is a polynomial of degree at most $4m + 1$ and $x = x_j(s)$, $y = y_j(s)$ is such a parametric representation of the median $t_j$ ($j = 1, 2, 3$) connecting the vertex $P_j$ with the center of gravity $P_0$ that the values $s = 0, \frac{2}{3}, 1$ correspond to the vertex $P_j$, to the center of gravity $P_0$ and to the center of the side $l_j$ (lying opposite the vertex $P_j$), respectively, then each of the polynomials

$$\bar{\pi}_j^{(\alpha,\beta)}(s) = D^{(\alpha,\beta)} p(x_j(s), y_j(s))\,, \quad \alpha + \beta = r\,, \quad 0 \leqq r \leqq m - 2,$$

is of degree at most $4m + 1 - r$. Let us approximate the polynomial $\bar{\pi}_j^{(\alpha,\beta)}(s)$ by a Hermite polynomial $\pi_j^{(\alpha,\beta)}(s)$ of degree at most $3m + 1 - 2r$ determined in this way:

$$D^i \pi_j^{(\alpha,\beta)}(0) = D^i \bar{\pi}_j^{(\alpha,\beta)}(0)\,, \quad 0 \leqq i \leqq 2m - r\,,$$
$$D^k \pi_j^{(\alpha,\beta)}(1) = D^k \bar{\pi}_j^{(\alpha,\beta)}(1)\,, \quad 0 \leqq k \leqq m - r\,.$$[6]

If the polynomial $p(x, y)$ satisfies the condition (7) then the values $D^i\bar{\pi}_j^{(\alpha,\beta)}(0)$, $0 \leqq i \leqq 2m - r$, $D^k\bar{\pi}_j^{(\alpha,\beta)}(1)$, $0 \leqq k \leqq m - r$, $\alpha + \beta = r$, are linear combinations of the parameters $D^i p(P_n)$, $n = 1, 2, 3$, $|i| \leqq 2m$, hence the same is true for the values

$$\pi_j^{(\alpha,\beta)}(\tfrac{2}{3})\,, \quad j = 1, 2, 3, \quad 0 \leqq \alpha + \beta \leqq m - 2\,.$$

Now, with respect to the just performed consideration, the following theorem is an immediate consequence of Theorem 1.

**Theorem 3.** *Let $T$, $P_j$, $l_j$, $v_j$, $P_0$ be the notation of Theorem 1 and let $m \geqq 2$ be an integer. Then to each $f \in C^{(2m)}(\bar{T})$ there exists exactly one polynomial $p$ of degree at most $4m + 1$ such that*

(9) $$D^i p(P_j) = D^i f(P_j)\,, \quad j = 1, 2, 3\,, \quad |i| \leqq 2m\,,$$

(10) $$D^{(\alpha,\beta)} p(P_0) = \tfrac{1}{3} \sum_{n=1}^{3} \pi_n^{(\alpha,\beta)}(\tfrac{2}{3})\,, \quad 0 \leqq \alpha + \beta \leqq m - 2\,,$$

*where the values $\pi_n^{(\alpha,\beta)}(\frac{2}{3})$ are linear combinations of the parameters $D^i p(P_j)$, $j = 1, 2, 3$, $|i| \leqq 2m$, by the above described construction,*

(11) *on the side $l_j$ $(j = 1, 2, 3)$ the derivatives $\partial^r p/\partial v_j^r$, $r = 1, 2, \ldots, m$, are polynomials of degree at most $4m + 1 - 2r$.*

From the construction of the parameters $D^k(P_0)$, $|k| \leqq m - 2$, it is clear that the interpolation polynomial for the function $f$ determined by the conditions (9)–(11) agrees with $f$ if $f$ is any polynomial of degree at most $2m + 3$, while the polynomial of Theorem 1 agrees with $f$ if $f$ is any polynomial of degree at most $3m + 1$. This fact implies, as follows from comparing Theorem 2 with the next theorem, that for $m \geqq 2$ and sufficiently smooth functions the error estimate for the interpolation (9)–(11) is worse than that for the interpolation (5)–(7).

**Theorem 4.** *Let $\Omega$ be a polygonal domain and let $\tau(h, \vartheta)$ be any triangulation of $\Omega$. Further, let $m \geqq 2$ be an integer and let $f \in W_2^{(k)}(\Omega)$, $k \geqq 2m + 2$. Finally, let the function $f_\tau$ coincide with the polynomial $p$ described in Theorem 3 on each triangle of the triangulation $\tau$ and let $l = \min(k, 2m + 4)$. Then $f_\tau \in C^{(m)}(\bar{\Omega})$ and*

[6] $D^i \pi_j^{(\alpha,\beta)}(s) = d^i \pi_j^{(\alpha,\beta)}(s)/ds^i$.

*for* $0 \leqq n \leqq m + 1$

$$\|f - f_\tau\|_{n,\Omega} \leqq \frac{K}{(\sin \vartheta)^n} h^{l-n} |f|_{l,\Omega}, \tag{12}$$

*where the constant K depends neither on Ω nor on f.*

Proof. The stated smoothness of the function $f_\tau$ follows from the definition of $f_\tau$ as in Theorem 2. The error estimate could again be proved in a similar way as the estimate (15) in [2].

Turning to the concentration of parameters determining the interpolation polynomials we begin with an interpolation of [3], which is very important for our further considerations. This interpolation is given by

**Theorem 5.** *Let T be a triangle with the vertices $P_j$, $j = 1, 2, 3$, and with the center of gravity $P_0$, and let m be a natural number. Then to each $f \in C^{(m-1)}(\overline{T})$ there exists exactly one polynomial p of degree at most $2m - 1$ such that*

$$D^i p(P_j) = D^i f(P_j), \quad j = 1, 2, 3, \quad |i| \leqq m - 1, \tag{13}$$

$$D^k p(P_0) = D^k f(P_0), \quad |k| \leqq m - 2. \text{[7]} \tag{14}$$

The error estimate for the piecewise-polynomial interpolation generated by the interpolation of Theorem 5 is given in the following theorem (see [3], Theorem 4).

**Theorem 6.** *Let Ω be a polygonal domain and let $\tau(h, \vartheta)$ be any triangulation of Ω. Further, let m be a natural number and let $f \in W_2^{(k)}(\Omega)$, $k \geqq m + 1$. Finally, let the function $f_\tau$ coincide with the polynomial p described in Theorem 5 on each triangle of the triangulation τ and let $l = \min(k, 2m)$. Then $f_\tau \in C(\overline{\Omega})$ and for* $n = 0, 1$

$$\|f - f_\tau\|_{n,\Omega} \leqq \frac{K}{(\sin \vartheta)^n} h^{l-n} |f|_{l,\Omega}, \tag{15}$$

*where the constant K depends neither on Ω nor on f.*

While the function $f_\tau$ of Theorem 6 is, in general, only continuous, the interpolations introduced by the following theorem generate piecewise-polynomial interpolations of higher smoothness.

**Theorem 7.** *Let T, $P_j$, $l_j$, $v_j$ and $P_0$ be the notation of Theorem 1. Further, let the points $Q_j^{(\varrho,r)}$, $\varrho = 1, 2, \ldots, r$, divide the side $l_j$ into $r + 1$ equal parts. Finally, let m, k be non-negative integers. Then to each $f \in C^{(2m+k)}(\overline{T})$ there exists exactly*

[7]) If $m = 1$ no condition is given at $P_0$.

*one polynomial $p_{m,k}$ of degree at most $4m + 1 + 2k$ such that*

$$(16)\qquad D^i p_{m,k}(P_j) = D^i f(P_j)\,, \quad j = 1, 2, 3\,, \qquad |i| \leqq 2m + k\,,$$

$$(17)\qquad D^s p_{m,k}(P_0) = D^s f(P_0)\,, \quad |s| \leqq m - 2 \quad \text{for} \quad k = 0\,,$$
$$|s| \leqq m + k - 1 \quad \text{for} \quad k > 0\,,$$

$$(18)\qquad \frac{\partial^r p_{m,k}(Q_j^{(\varrho,r)})}{\partial v_j^r} = \frac{\partial^r f(Q_j^{(\varrho,r)})}{\partial v_j^r}\,, \quad j = 1, 2, 3\,, \quad \varrho = 1, 2, \ldots, r\,,$$
$$r = 1, 2, \ldots, m\,,$$

$$(19)\qquad p_{m,k}(R_j^{(\sigma)}) = f(R_j^{(\sigma)})\,, \quad \sigma = 1, 2, \ldots, \quad k - 1\,, \quad j = 1, 2, \ldots, m,$$

*where $R_j^{(\sigma)} \not\equiv P_0$, $\sigma = 1, 2, \ldots, k - 1$, are distinct points lying inside $T$ on a straight line $d_j$ $(j = 1, 2, \ldots, m)$ which passes through the center of gravity and does not pass through any vertex of $T$.*

It is necessary to add that from (17)–(19) we leave out those condition which have no sense for given $m$ and $k$. Thus, for example, we leave out the condition (19) when $k \leqq 1$ or $m = 0$.

Proof. Suppose $m > 0$ and $k > 1$ since for $k \leqq 1$ or $m = 0$ the interpolations were already considered. Let $f(x, y) \equiv 0$ and let the polynomial $p_{m,k}(x, y)$ of degree at most $4m + 1 + 2k$ satisfy the conditions (16)–(19). Let $l_j(x, y) = 0$ and $t_j(x, y) = 0$, $j = 1, 2, 3$, be the equations of the sides and of the medians of $T$, respectively, and let $l(x, y) = l_1(x, y)\, l_2(x, y)\, l_3(x, y)$, $t(x, y) = t_1(x, y)\, t_2(x, y)\, t_3(x, y)$. Then, by virtue of (16) and (18),

$$p_{m,k}(x, y) = l^{m+1}(x, y)\, q(x, y)\,,$$

where $q(x, y)$ is a polynomial of degree at most $m + 2k - 2$ which, according to (16), (17), (19), satisfies the conditions

$$D^i q(P_j) = 0\,, \quad j = 1, 2, 3\,, \qquad |i| \leqq k - 2\,,$$
$$D^s q(P_0) = 0\,, \quad |s| \leqq m + k - 1\,,$$
$$q(R_j^{(\sigma)}) = 0\,, \quad j = 1, 2, \ldots, m\,, \qquad \sigma = 1, 2, \ldots, k - 1\,.$$

Hence

$$q(x, y) = t(x, y)\, h(x, y)\,,$$

where $h(x, y)$ is a polynomial of degree at most $m + 2k - 5$ satisfying the conditions

$$(16')\qquad D^i h(P_j) = 0\,, \quad j = 1, 2, 3\,, \qquad |i| \leqq k - 3\,,$$

$$(17')\qquad D^s h(P_0) = 0\,, \quad |s| \leqq m + k - 4\,,$$

$$(19')\qquad h(R_j^{(\sigma)}) = 0\,, \quad j = 1, 2, \ldots, m\,, \qquad \sigma = 1, 2, \ldots, k - 1\,.$$

If $k = 2$ and $m = 1$, then $h(x, y)$ is a constant and by (19′) $h(x, y) \equiv 0$. Therefore, assume $k + m > 3$. If $d_j(x, y) = 0$ is the equation of the straight line $d_j$ ($j = 1, 2, \ldots, m$) and $d(x, y) = d_1(x, y)\, d_2(x, y) \ldots d_m(x, y)$ then, with respect to (17′) and (19′), the polynomial $h(x, y)$ is divisible by $d(x, y)$. Hence for $k = 2$ $h(x, y) \equiv 0$ and for $k > 2$

$$h(x, y) = d(x, y)\, z(x, y)\,,$$

where $z(x, y)$ is a polynomial of degree at most $2k - 5$ which satisfies

$$D^i z(P_j) = 0\,, \quad j = 1, 2, 3\,, \quad |i| \leqq k - 3\,,$$
$$D^s z(P_0) = 0\,, \quad |s| \leqq k - 4\,.$$

Then, of course, applying Theorem 5, we have $z(x, y) \equiv 0$ and hence $h(x, y) \equiv 0$ again. Thus in all cases $p_{m,k}(x, y) \equiv 0$ and the proof is complete.

Let us underline that between the parameters determining the polynomials $p_{m,k}$ there are normal derivatives of order at most $m$ only and that the piecewise-polynomial approximation generated in a polygonal domain $\Omega$ by these polynomials belongs to $C^{(m)}(\bar{\Omega})$. Further, let us note that for $k = 0$ or $k = 1$ we obtain the hierarchy of the interpolation polynomials (1)–($4_1$) or (1)–($4_3$), respectively, and for $m = 0$ we have the interpolation of Theorem 5. If $k > 1$ and $m > 0$ then, according to (19), the conditions determining the polynomials $p_{m,k}$ are not symmetric (with respect to the triangle $T$) which is a great disadvantage of these interpolations. For some $k > 1$ and $m > 0$ it is possible to replace the condition (19) by a symmetric condition. For example, to each $f \in C^{(6)}(\bar{T})$ there exists exactly one polynomial $p^*_{1,4}$ of degree at most 13 satisfying (16)–(18) for $m = 1$, $k = 4$ and

$$(19^*) \qquad \frac{\partial p^*_{1,4}(S_j)}{\partial \varepsilon_j} = \frac{\partial f(S_j)}{\partial \varepsilon_j}\,, \quad j = 1, 2, 3\,,$$

where $S_j$ is the center of the line segment $P_jP_0$ and $\varepsilon_j$ is the normal to $P_jP_0$. The parameters of this polynomial may again be reduced in the sense of Theorem 1. Thus we obtain a polynomial $\bar{p}_{1,4}$ of degree at most 13 which is uniquely determined by the conditions

$$(16'') \qquad D^i \bar{p}_{1,4}(P_j) = D^i f(P_j)\,, \quad j = 1, 2, 3\,, \quad |i| \leqq 6\,,$$

$$(17'') \qquad D^s \bar{p}_{1,4}(P_0) = D^s f(P_0)\,, \quad |s| \leqq 4\,,$$

(18″) *on the side* $l_j$ ($j = 1, 2, 3$) *the normal derivative* $\partial \bar{p}_{1,4}/\partial v_j$ *is a polynomial of degree at most* 11,

(19″) *on the median* $P_jP_0$ ($j = 1, 2, 3$) *the normal derivative* $\partial \bar{p}_{1,4}/\partial \varepsilon_j$ *is a polynomial of degree at most* 11.

## 3. COMBINATIONS OF THE INTERPOLATIONS OF SECTION 2 WITH THE REDUCED HERMITE INTERPOLATIONS OF [4]

In [4] reductions of some parameters of Hermite interpolations of [7] were performed. These reduced interpolation polynomials are treated in the following Theorems 8–11.

**Theorem 8.** *Let $R$ be a rectangle with the vertices $P_j$, $j = 1, 2, 3, 4$, the sides of which are parallel to Cartesian coordinate axes and let $m$ be a natural number. Then to each $f \in C^{(m-1)}(\bar{R})$ there exists exactly one polynomial $p(x, y)$ of degree at most $2m - 1$ in each variable,*

$$p(x, y) = \sum_{i,j=0}^{2m-1} \alpha_{ij} x^i y^j ,$$

*such that*

$$\alpha_{ij} = 0 \quad \text{for} \quad \left[\frac{i}{2}\right] + \left[\frac{j}{2}\right] \geqq m \,,\text{[8]}$$

$$D^i p(P_j) = D^i f(P_j) \,, \quad j = 1, 2, 3, 4 \,, \quad |i| \leqq m - 1 \,.$$

**Theorem 9.** *Let $\Omega$ be a rectangular polygonal domain and let $\varrho(\Delta, \bar{\Delta})$, $\bar{\Delta} = \sigma\Delta$, be any partition of $\Omega$. Further, let $m$ be a natural number and let $f \in W_2^{(k)}(\Omega)$, $k \geqq m + 1$. Finally, let the function $f_\varrho$ coincide with the polynomial $p$ described in Theorem 8 on each rectangle of the partition $\varrho$ and let $l = \min(k, 2m)$. Then $f_\varrho \in C(\bar{\Omega})$ and for $n = 0, 1$*

$$\|f - f_\varrho\|_{n,\Omega} \leqq \frac{K}{\sigma^n} \Delta^{l-n} |f|_{l,\Omega} \,, \tag{20}$$

*where the constant $K$ depends neither on $\Omega$ nor on $f$.*

**Theorem 10.** *Let $R$ be a rectangle with the vertices $P_j$, $j = 1, 2, 3, 4$, the sides of which are parallel to the coordinate axes and let $v_j$ be the normal to the side $l_j$ ($j = 1, 2, 3, 4$). Further, let $m$ be a non-negative integer. Then to each $f \in C^{(2m)}(\bar{R})$ there exists exactly one polynomial $p$ of degree at most $4m + 1$ in each variable,*

$$p(x, y) = \sum_{i,j=0}^{4m+1} \alpha_{ij} x^i y^j ,$$

*such that*

$$\alpha_{ij} = 0 \quad \text{for} \quad i, j \geqq 2(m + 1) \,,$$

$$D^i p(P_j) = D^i f(P_j) \,, \quad j = 1, 2, 3, 4 \,, \quad |i| \leqq 2m \,,$$

---

[8]) $[c]$ is the whole part of the number $c$.

*on the side* $l_j$ $(j = 1, 2, 3, 4)$ *the normal derivatives* $\partial^r p / \partial v_j^r$, $r = 1, 2, \ldots, m$, *are polynomials of degree at most* $4m + 1 - 2r$.[9])

**Theorem 11.** *Let* $\Omega$ *be a rectangular polygonal domain and let* $\varrho(\Delta, \bar{\Delta})$, $\bar{\Delta} = \sigma\Delta$, *be any partition of* $\Omega$. *Further, let* $m$ *be a non-negative integer and* $f \in W_2^{(k)}(\Omega)$, $k \geqq 2m + 2$. *Finally, let the function* $f_\varrho$ *coincide with the polynomial* $p$ *described in Theorem* 10 *on each rectangle of the partition* $\varrho$ *and let* $l = \min(k, 3m + 2)$. *Then* $f_\varrho \in C^{(m)}(\bar{\Omega})$ *and for* $0 \leqq n \leqq m + 1$

$$\|f - f_\varrho\|_{n,\Omega} \leqq \frac{K}{\sigma^n} \Delta^{l-n} |f|_{l,\Omega}, \tag{21}$$

*where the constant* $K$ *depends neither on* $\Omega$ *nor on* $f$.

The combinations of the interpolations of Section 2 with the interpolations of [4] are now given in the following theorems. Thus, the combination of Theorems 6 and 9 gives

**Theorem 12.** *Let* $\Omega$ *be a polygonal domain and let* $\delta$ *be any partition of* $\Omega$ *determining a partition* $\varrho(\Delta, \bar{\Delta})$, $\bar{\Delta} = \sigma\Delta$, *of a rectangular polygonal domain* $\Omega_1 \subset \Omega$ *and a triangulation* $\tau(h, \vartheta)$ *of the set* $\Omega_2 = \Omega - \bar{\Omega}_1$. *Further, let* $m$ *be a natural number and let* $f \in W_2^{(k)}(\Omega)$, $k \geqq m + 1$. *Finally, let the function* $f_\delta$ *coincide with the polynomial of Theorem* 5 *on each triangle of the triangulation* $\tau$ *and with the polynomial of Theorem* 8 *on each rectangle of the partition* $\varrho$, *and let* $l = \min(k, 2m)$. *Then* $f_\delta \in C(\bar{\Omega})$ *and for* $n = 0, 1$

$$\|f - f_\delta\|_{n,\Omega} \leqq \frac{K_1}{\sigma^n} \Delta^{l-n} |f|_{l,\Omega_1} + \frac{K_2}{(\sin \vartheta)^n} h^{l-n} |f|_{l,\Omega_2}, \tag{22}$$

*where the constants* $K_1$ *and* $K_2$ *depend neither on* $\Omega$ *nor on* $f$.

Upon combining Theorem 2 with Theorem 11 we find

**Theorem 13.** *Let* $\Omega$, $\delta$, $\Omega_1$, $\Omega_2$, $\varrho(\Delta, \bar{\Delta})$, $\sigma$, $\tau(h, \vartheta)$, $m$ *be the notation of Theorem* 12 *and let* $f \in W_2^{(k)}(\Omega)$, $k \geqq 2m + 2$. *Further, let the function* $f_\delta$ *coincide with the polynomial of Theorem* 1 *on each triangle of* $\tau$ *and with the polynomial of Theorem* 10 *on each rectangle of* $\sigma$, *and let* $l = \min(k, 3m + 2)$. *Then* $f_\delta \in C^{(m)}(\bar{\Omega})$ *and the error estimate* (22) *is valid for* $0 \leqq n \leqq m + 1$.

## 4. APPLICATION TO V-ELLIPTIC BOUNDARY VALUE PROBLEMS

To give some applications of the preceding error bounds, let us consider the Galerkin method for approximating the solutions of $V$-elliptic boundary value problems with homogeneous stable boundary conditions.

[9]) For $m = 0$, p is a bilinear polynomial determined by the values at the vertices of $R$.

Let $\Omega$ be a polygonal domain. Let $V$ be a Hilbert space consisting of all functions from $W_2^{(n)}(\Omega)$ for which all given stable boundary conditions are homogeneous, i.e.

$$\mathring{W}_2^{(n)}(\Omega) \subset V \subset W_2^{(n)}(\Omega), \tag{23}$$

with the norm induced by $W_2^{(n)}(\Omega)$. Let $a(u, v)$ be a complex functional on $W_2^{(n)}(\Omega) \times \times W_2^{(n)}(\Omega)$ which is linear in $u$, antilinear in $v$, bounded and $V$-elliptic, i.e. for some positive constants $M$, $\alpha$

$$|a(u, v)| \leqq M \|u\|_{n,\Omega} \|v\|_{n,\Omega} \quad \forall\, u, v \in W_2^{(n)}(\Omega), \tag{24}$$

$$|a(v, v)| \geqq \alpha \|v\|_{n,\Omega}^2 \quad \forall\, v \in V. \tag{25}$$

Finally, let $F(v)$ be an antilinear bounded functional on $V$. Then a function $u \in V$ is said to be a solution of the boundary value problem (with the homogeneous stable boundary conditions) if

$$a(u, v) = L(v) \quad \forall\, v \in V. \tag{26}$$

Under the above hypotheses, the problem (26) has a unique solution by the Lax-Milgram Lemma (see [8], p. 38).

We consider Galerkin's procedure for obtaining an approximate solution of (26). More precisely, let $S$ be any finite dimensional subspace of $V$. Consider the approximate problem of finding a $w \in S$ such that

$$a(w, v) = L(v) \quad \forall\, v \in S. \tag{27}$$

It is easy to show that the problem (27) has a unique solution. Moreover, if $u \in V$ is the solution of (26), then

$$\|u - w\|_{n,\Omega} \leqq \frac{M}{\alpha} \|u - v\|_{n,\Omega} \quad \forall\, v \in V, \tag{28}$$

where $M$ and $\alpha$ are the constants from (24) and (25), i.e. independent of $S$ (see [7], p. 252).

Let $\Omega$ be a polygonal domain and let $\delta$ be any partition of $\Omega$ determining a partition $\varrho(\Delta, \bar{\Delta})$, $\bar{\Delta} = \sigma\Delta$, of a rectangular polygonal domain $\Omega_1 \subset \Omega$ and a triangulation $\tau(h, \vartheta)$ of the set $\Omega_2 = \Omega - \bar{\Omega}_1$. Let to each function $f$ belonging simultaneously to both $C^{(m-1)}(\bar{\Omega})$ and $V$, $\mathring{W}_2^{(1)}(\Omega) \subset V \subset W_2^{(1)}(\Omega)$, there be assigned a function $f_{m,\delta}$ which coincides with the polynomial of Theorem 8 on each rectangle of the partition $\varrho$ and with the polynomial of Theorem 5 on each triangle of the triangulation $\tau$. The set $G_\delta^m$ of all in this way obtained piecewise-polynomial functions is a finite dimensional subspace of $V$. In a similar way, let to each function $f$ belonging simultaneously to both $C^{(2m)}(\bar{\Omega})$ and $V$, $\mathring{W}_2^{(n)}(\Omega) \subset V \subset W_2^{(n)}(\Omega)$, $n \leqq m + 1$, be assigned a function $f_{m,\delta}$ which coincides with the polynomial of Theorem 10 on each rectangle of the partition $\varrho$ and with the polynomial of Theorem 1 on each triangle of the triangulation $\tau$. The set $H_\delta^m$ of all in this way obtained piecewise-polynomial functions again is a finite dimensional subspace of $V$.

When solving the problem (27) for $S \equiv G_\delta^m$ and estimating the right-hand side of Céa's inequality (28) by Theorem 12, we obtain

**Theorem 14.** *Let $\Omega$ be a polygonal domain and let (23)–(25) be valid for $n = 1$. If the solution $u$ of (26) belongs to $W_2^{(k)}(\Omega)$, $k \geqq m + 1$, $u_\delta^m$ is the solution of (27) for $S \equiv G_\delta^m$ and $l = \min(k, 2m)$, then*

$$\|u - u_\delta^m\|_{n,\Omega} \leqq K\left[\frac{1}{\sigma}\Delta^{l-1}|u|_{l,\Omega_1} + \frac{1}{\sin\vartheta}h^{l-1}|u|_{l,\Omega_2}\right] \tag{29}$$

*where the constant $K$ is independent on the functions $u$ and $u_\delta^m$.*

In a similar fashion, combining the inequality (28) with Theorem 13 we have

**Theorem 15.** *Let $\Omega$ be a polygonal domain and let (23)–(25) be valid. Let $m \geqq n - 1$ and let the solution $u$ of (26) belong to $W_2^{(k)}(\Omega)$, $k \geqq 2m + 2$. If $u_\delta^m$ is the solution of (27) for $S \equiv H_\delta^m$ and $l = \min(k, 3m + 2)$, then*

$$\|u - u_\delta^m\|_{n,\Omega} \leqq K\left[\frac{1}{\sigma^n}\Delta^{l-n}|u|_{l,\Omega_1} + \frac{1}{(\sin\vartheta)^n}h^{l-n}|u|_{l,\Omega_2}\right] \tag{30}$$

*where the constant $K$ does not depend on the functions $u$ and $u_\delta^m$.*

In the case when $\mathscr{R}$ is a regular collection of partitions of $\Omega$ (see Section 1) let us assign to each $\delta \in \mathscr{R}$ determining a partition $\varrho(\Delta, \bar{\Delta})$ of a rectangular polygonal domain $\Omega_1 \subset \Omega$ and a triangulation $\tau(h, \vartheta)$ of the set $\Omega_2 = \Omega - \bar{\Omega}_1$ the parameter $\varkappa = \max(h, \Delta)$. It is clear that if in Theorem 14 $\delta \in \mathscr{R}$, then

$$\|u - u_\delta^m\|_{1,\Omega} \leqq K_1 \varkappa^{l-1}|u|_{l,\Omega},$$

and if in Theorem 15 $\delta \in \mathscr{R}$, then

$$\|u - u_\delta^m\|_{n,\Omega} \leqq K_2 \varkappa^{l-n}|u|_{l,\Omega},$$

where the constants $K_1$, $K_2$ are independent on the functions $u$ and $u_\delta^m$. These inequalities give an asymptotic estimate of the rate of convergence of the approximate solutions $u_\delta^m$ of the problem (26) under the assumptions that the exact solution $u$ of (26) is sufficiently smooth. The following theorems, however, guarantee the convergence of the approximate solutions $u_\delta^m$ to the exact solution $u$ even in the case that $u$ belongs only to $W_2^{(n)}(\Omega)$.

**Theorem 16.** *Let $\Omega$ be a polygonal domain and let $\mathscr{R}$ be a regular collection of partitions of $\Omega$. Let (23)–(25) be valid for $n = 1$ and let $u$ be the solution of (26). Finally, let for each $\delta \in \mathscr{R}$ the function $u_\delta^m$ be the solution of (27), where $S \equiv G_\delta^m$. Then*

$$\|u - u_\delta^m\|_{1,\Omega} \to 0 \quad \text{as} \quad \varkappa \to 0 .$$

Proof. The set of all functions belonging simultaneously to both $V$ and $W_2^{(2m)}(\Omega)$ is dense in $V$. This fact, combined with Theorem 12, implies that to every $\varepsilon > 0$ there exists $v \in G_\delta^m$ such that $\|u - v\|_{1,\Omega} < \varepsilon$ as soon as $\varkappa < \varkappa_\varepsilon$, where $\varkappa_\varepsilon$ depends only on $\varepsilon$. Then, of course, by (28) $\|u - u_\delta^m\|_{1,\Omega} \leqq (M/\alpha)\, \varepsilon$ and the convergence is demonstrated.

**Theorem 17.** *Let $\Omega$ be a polygonal domain and let $\mathscr{R}$ be a regular collection of partitions of $\Omega$. Let (23)–(25) be valid and let $u$ be the solution of (26). Finally, let for each $\delta \in \mathscr{R}$ the function $u_\delta^m$ be the solution of (27), where $S \equiv H_\delta^m$, $m + 1 \geqq n$. Then*

$$\|u - u_\delta^m\|_{n,\Omega} \to 0 \quad \text{as} \quad \varkappa \to 0 .$$

Proof. When noting that the set of all functions from $V$ belonging to $W_2^{(2m+2)}(\Omega)$ is dense in $V$ and using Theorem 13, the proof is exactly analogous to that of the preceding theorem.

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Souhrn

# PO ČÁSTECH POLYNOMICKÉ INTERPOLACE V METODĚ KONEČNÝCH PRVKŮ

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V článku uvádíme některé redukce parametrů, kterými je určen interpolační polynom nad trojúhelníkem v [1]. Dále se zabýváme koncentrací parametrů určujících interpolační polynom nad trojúhelníkem. Přitom koncentrací, zhruba řečeno, rozumíme takovou volbu parametrů určujících interpolační polynom nad trojúhelníkem, že maximální počet těchto parametrů je zadán ve vrcholech a v těžišti trojúhelníka, zatímco na stranách trojúhelníka je zadáno pouze tolik parametrů (nebo podmínek), aby po trojúhelnících polynomická funkce, ztotožňující se nad každým trojúhelníkem triangulace dané polygonální oblasti $\Omega$ s interpolačním polynomem uvažovaného typu, měla v $\bar{\Omega}$ spojitost požadovaného řádu. Získané interpolace, jak s redukovanými tak i s koncentrovanými parametry, jsou kombinovány s redukovanými hermitovskými interpolacemi ze [4] a těchto kombinací je užito k řešení eliptických okrajových úloh metodou konečných prvků.

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