PIECEWISE POLYNOMIAL KERNELS FOR IMAGE INTERPOLATION: A GENERALIZATION OF CUBIC CONVOLUTION

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ABSTRACT

A well-known approach to image interpolation is cubic convolution, in which the ideal sinc function is modelled by a finite extent kernel, which consists of piecewise third order polynomials. In this paper we show that the concept of cubic convolution can be generalized. We derive kernels of up to ninth order and compare them both mutually and to cardinal splines of corresponding orders. From spectral analyses we conclude that the improvements of the higher order schemes over cubic convolution are only marginal. We also conclude that in all cases, cardinal splines are superior.

1. INTRODUCTION

Interpolation of sampled data is required in many digital image processing operations. Examples of these include sample rate conversion (magnification or "minification"), and the application of geometrical transformations (rotations, translations, elastic deformations), which are frequently required for the purpose of image registration or volume visualization. In many situations, it is of paramount importance to limit as much as possible the loss of detail (blurring), or the creation of spurious details (aliasing phenomena), as caused by interpolation operations. From the Whittaker-Shannon sampling theorem [1, 2, 3] it is well known that, to this end, the ideal interpolation kernel is the sinc function, which is of infinite extent and has a low rate of decay. In the attempt to obtain practical and computationally efficient image processing algorithms, a frequently used approach to interpolation is to employ finite extent kernels (FIR filters), obtained by modelling the sinc function, *e.g.* by piecewise polynomials. A well-known example of such a kernel is the so called cubic convolution kernel [4, 5], which consists of piecewise third order polynomials.

In this paper we show that the concept of cubic convolution can be generalized to yield a class of piecewise nth-order polynomial interpolation kernels. We extend the results of a previous paper [6] by deriving kernels of up to ninth order and by comparing them both mutually and to cardinal splines of corresponding orders. From several spectral analyses we conclude that—although the improvement of cubic convolution over linear interpolation is known to be substantial—the improvements of the higher order schemes over cubic convolution are only marginal. We also conclude that in all cases, cardinal splines are superior.

2. PIECEWISE POLYNOMIAL KERNELS

Under the assumption of uniformly sampled data, with inter-sample distance $\xi \in \mathbb{R}$, the general form of a piecewise nth-order polynomial kernel is given by:

$$h_{\mathcal{P}}(x) = \begin{cases} \sum_{i=0}^{n} a_{ij}|x|^i, & j\xi \leqslant |x| < (j+1)\xi \\ 0, & m\xi \leqslant |x| \end{cases} \tag{1}$$

where $j = 0, 1, \ldots, m-1$, and the parameter $m \in \mathbb{N}\backslash\{0\}$ determines the extent of the kernel. In this paper we only consider polynomials of odd degree, *i.e.*, n and m are related by $n = 2m - 1$. The $(n+1)m$ coefficients a_{ij} are to be determined by imposing the following constraints on $h_{\mathcal{P}}$:

1) $h_{\mathcal{P}}(x) = 1$ for $x = 0$, and $h_{\mathcal{P}}(x) = 0$ for $|x| = \xi, \ldots, (m-1)\xi$,

2) $h_{\mathcal{P}}^{(l)}(x)$ must be continuous at $|x| = 0, \xi, \ldots, m\xi$, and for $l = 0, 1, \ldots, k$.

In the second constraint, k must be sufficiently large so as to yield a sufficient number of equations in order to be able to solve for the unknown coefficients a_{ij}. As we have shown recently [6], given the order n of the polynomials constituting the interpolation kernel $h_{\mathcal{P}}$, the proper value of k is 0 for $n = 1$, and $n - 2$ for $n > 1$. From this, it follows that for $n = 1$, the system of equations can be solved uniquely to yield a

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C^0 interpolation kernel, *viz.*, the linear interpolation kernel. For every $n > 1$ (n odd), the constraints result in a C^{n-2} kernel, which is a function of exactly one tunable parameter, denoted by α.

3. PARAMETER TUNING

From the early literature on cubic convolution, several—rather *ad hoc*—approaches are known to fix the free parameter, α. For example, it has been proposed to constrain the slope of the kernel at $|x| = \xi$ to be equal to that of the sinc function [7, 8]. An alternative approach has been to constrain the $(n-1)$th order derivative of the kernel to be continuous at that point [8]. In this section we briefly discuss two alternative approaches which yield the mathematically most precise interpolants.

3.1. Keys' Approach

Let $I : \mathbb{R} \to \mathbb{R}$ be an arbitrary, continuous, one-dimensional image, and $\widehat{I} : \mathbb{R} \to \mathbb{R}$ the interpolated image, that is,

$$\widehat{I}(x) = \sum_{s=-\lfloor n/2 \rfloor}^{\lceil n/2 \rceil} I((r+s)\xi) h_{\mathcal{P}}(x - (r+s)\xi), \qquad (2)$$

where $r = \lfloor x/\xi \rfloor$. If I is at least C^3, then according to Taylor's theorem

$$I((r+s)\xi) = \sum_{i=0}^{2} I^{(i)}(r\xi) \frac{(s\xi)^i}{i!} + \mathcal{O}(\xi^3). \qquad (3)$$

After substitution of Eq. (3) and the polynomials of $h_{\mathcal{P}}$ as a function of the free parameter α into Eq. (2), the optimal image-independent value for α is obtained by minimizing the difference between I and $\widehat{I}$. It turns out that it is possible to choose α such that

$$I(x) - \widehat{I}(x) = \mathcal{O}(\xi^3), \quad \forall x \in \mathbb{R}, \qquad (4)$$

which implies that the interpolation error converges to zero at a rate proportional to ξ^3.

This approach was described first by Keys [4] in the context of cubic convolution, and is equally well applicable to higher order kernels. For the cubic, quintic, septic, and nonic kernels, this results in the values presented in Table 1. It is important to note that Eq. (4) holds regardless of n.

3.2. Park's Approach

An alternative approach, initially proposed by Park & Schowengerdt [5] and recently generalized by us [6], is based entirely on the analysis of the spectra of the kernels as a function the free parameter α.

n	α
3	$-\frac{1}{2}$
5	$\frac{3}{64}$
7	$-\frac{71}{83232}$
9	$\frac{3829}{788235264}$

Table 1: The optimal image-independent values of the free parameter α for the cubic ($n = 3$), quintic ($n = 5$), septic ($n = 7$), and nonic ($n = 9$) piecewise polynomial interpolation kernels.

Since $h_{\mathcal{P}}$ is real-valued and even, the Fourier transform, $\tilde{H}_{\mathcal{P}}$, is also real-valued and even, from which it follows that the Maclaurin series only consist of even terms:

$$\tilde{H}_{\mathcal{P}}(f) = \lambda_0 + \lambda_2 f^2 + \lambda_4 f^4 + \lambda_6 f^6 + \ldots \qquad (5)$$

It can be shown that the coefficients, λ_i, are linear functions of the parameter α. As argued by Park & Schowengerdt [5] in the context of cubic convolution, the optimal image-independent value for α is obtained by solving $\lambda_2 = 0$, which causes $\tilde{H}_{\mathcal{P}}$ to be flat at $f = 0$, thereby preventing unnecessary high frequency emphasis or low frequency suppression.

This approach is also applicable to higher order kernels. Although we have not demonstrated the equivalence of Keys' and Park's approach, the latter yields the exact same values (Table 1).

4. SPECTRAL ANALYSES

Using the theory described in the previous sections, we have derived and analyzed kernels of up to ninth order. An impression of the characteristics of the spectra of these kernels can be obtained from Fig. 1, where we have assumed the sampling frequency, $F_s = 1/\xi$, to be equal to 1 for convenience. Note that although the high frequency suppression capabilities of the higher order kernels are considerably better than those of the cubic convolution kernel, the low-pass characteristics are only slightly better.

Another well-known approach to piecewise polynomial interpolation is to use so called B-splines, originally introduced by Schoenberg [9], which are obtained by auto-convolution of a rectangular pulse (equal to the zeroth order or nearest neighbor interpolation kernel).

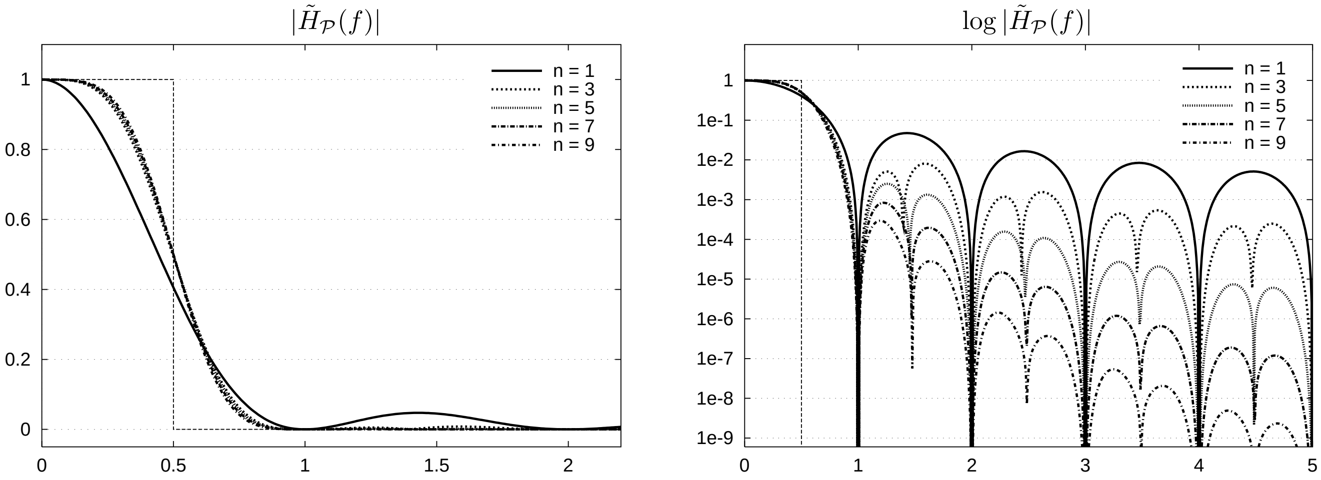

Figure 1: Linear (left) and logarithmic (right) plots of the magnitude of the spectrum $\tilde{H}_{\mathcal{P}}$ of the piecewise polynomial kernel $h_{\mathcal{P}}$ defined in Eq. (1) for $n = 1, 3, 5, 7, 9$. The free parameter α was computed by using the approaches described in Section 3. The spectrum of the sinc function (box filter) is shown for comparison.

The explicit form of an nth-order B-spline reads [10]:

$$\beta^n(x) = \frac{1}{n!}\sum_{i=0}^{n+1}\binom{n+1}{i}(-)^i\left(x - i + \frac{n+1}{2}\right)_+^n \qquad (6)$$

where $(x)_+^n$ denotes the one-sided power function. Since $\beta^n(x) \geqslant 0,\ \forall n \in \mathbb{N},\ \forall x \in \mathbb{R}$, interpolation using B-splines requires preprocessing of the raw image data. This can be done either by matrix manipulations [11], or by means of recursive filtering techniques [12, 13, 14]. As has been stressed by several authors, notably Maeland [15] and Unser *et al.* [12], comparison of interpolation kernels by means of spectral analyses requires for B-spline interpolation to consider, not the Fourier transform of Eq. (6), but that of the so called cardinal splines. The spectrum of a cardinal B-spline of order n, which is the filter obtained by discrete convolution of its corresponding direct and indirect B-spline filters, is given by:

$$\tilde{H}_{\mathcal{S}}(f) = \left(\frac{\sin(\pi f)}{\pi f}\right)^{n+1}\frac{1}{B^n(\mathrm{e}^{i2\pi f})} \qquad (7)$$

where $B^n(z),\ z = \mathrm{e}^{\mathrm{i}2\pi f}$, is the z transform of the sampled version of the B-spline defined in Eq. (6). A comparison of the spectra $\tilde{H}_{\mathcal{P}}$ and $\tilde{H}_{\mathcal{S}}$ is provided in Fig. 2, from which it is concluded that the cardinal splines resemble the sinc function considerably better than the piecewise polynomials described in this paper. This may, in part, be explained from the fact that an nth-order B-spline is an element of C^{n-1} (as can be appreciated from Eq. (6)), while the higher order kernels described in the previous sections are only in C^{n-2}.

Another approach to study the spectral behavior of a kernel as a function of frequency is by means of the following error measure:

$$\epsilon(f) = |1 - \tilde{H}(f)|^2 + \sum_{i\in\mathbb{Z}\setminus\{0\}}|\tilde{H}(f - iF_s)|^2 \qquad (8)$$

where $\tilde{H}$ denotes the spectrum of the kernel to be analyzed. This measure—used by Park & Schowengerdt [16] to study the errors introduced by sampling and reconstruction operations —accumulates, for every f, the square deviation of a kernel's spectrum from the spectrum of the sinc function at all repetitions of f. Note that for the sinc function, $\epsilon(f) = 0$ for $|f| \leqslant \frac{1}{2}F_s$ and $\epsilon(f) = 2$ for $|f| > \frac{1}{2}F_s$. The error functions for the spectra of the interpolation kernels $h_{\mathcal{P}}$, as well as those of the cardinal B-splines, are shown in Fig. 3, together with the "error function" of the sinc kernel. From this figure it is clear that the cubic convolution kernel is substantially better than the linear interpolation kernel, but the improvements of higher order kernels are only marginal. It also confirms the statement that cardinal splines of corresponding orders are superior.

5. CONCLUSIONS

In this paper we have shown that the concept of cubic convolution can be generalized to yield a class of piecewise nth-order polynomial interpolation kernels. We have derived kernels of up to ninth order, which were subsequently compared both mutually and to cardinal splines of corresponding orders. The spectral analyses

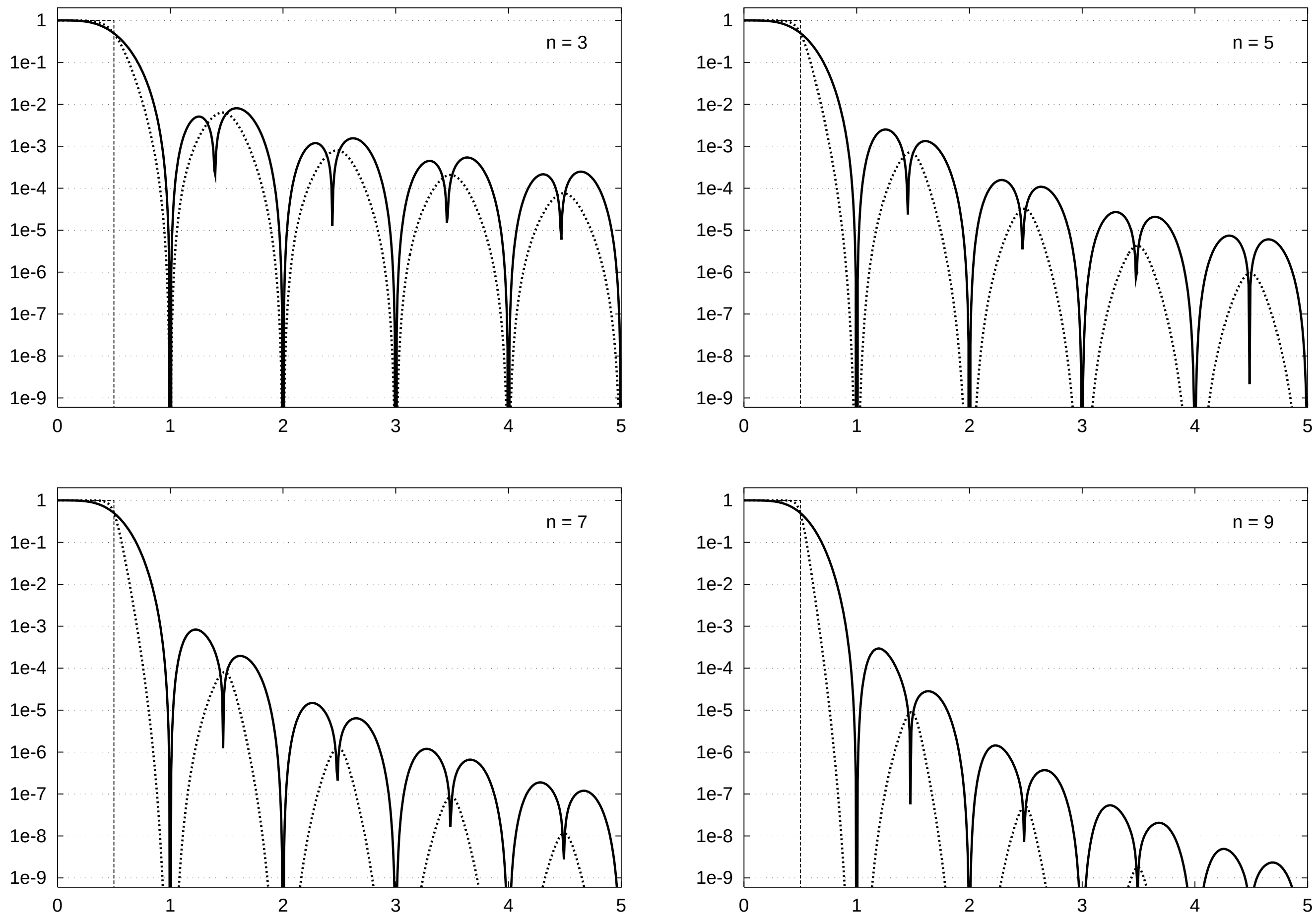

Figure 2: Logarithmic plots of the magnitudes of the spectra $\tilde{H}_{\mathcal{P}}$ (solid curves) of the piecewise polynomial kernel $h_{\mathcal{P}}$ as defined in Eq. (1), and $\tilde{H}_{\mathcal{S}}$ (dotted curves) as defined in Eq. (7), for $n = 3$ (top-left), $n = 5$ (top-right), $n = 7$ (bottom-left), and $n = 9$ (bottom-right). In each of the plots, the spectrum of the sinc function is shown for comparison.

showed consistently that, while the errors made by the cubic convolution scheme are substantially smaller than those made by linear interpolation, the higher order schemes only yield marginal additional improvements over cubic convolution. The analyses also indicated that in all cases, cardinal splines are superior.

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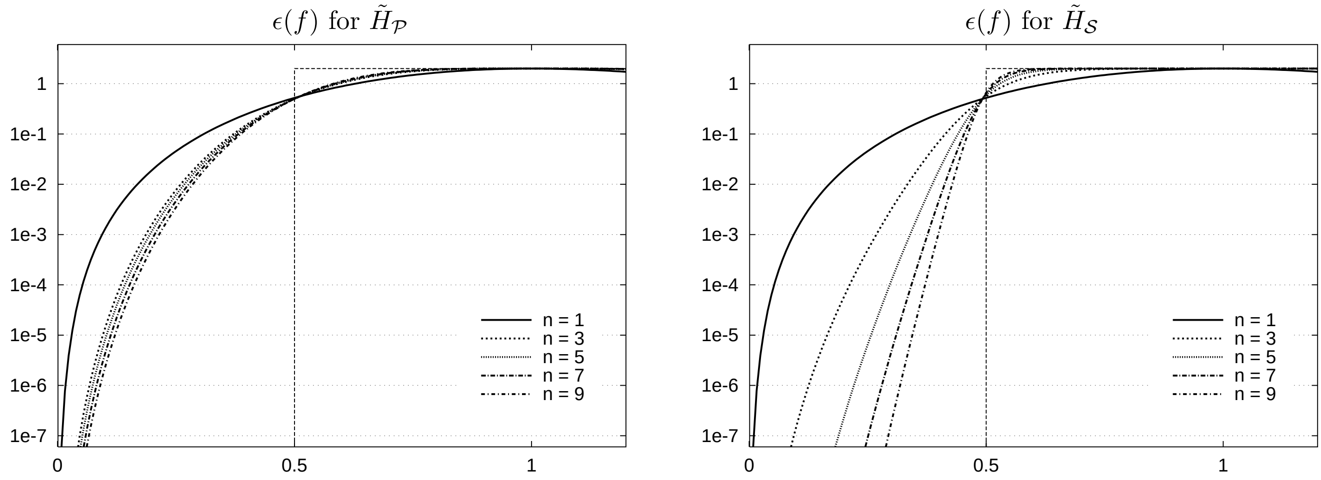

Figure 3: Logarithmic plots of the error function, $\epsilon(f)$, of the spectra $\tilde{H}_{\mathcal{P}}$ (left) and $\tilde{H}_{\mathcal{S}}$ (right) for $n = 1, 3, 5, 7, 9$. In each of the plots, the "error function" of the sinc kernel is shown for comparison.

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