Piecewise Spectral Collocation Method for Second Order Volterra Integro-Differential Equations with Nonvanishing Delay

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Abstract. In this paper, the piecewise spectral-collocation method is used to solve the second-order Volterra integral differential equation with nonvanishing delay. In this collocation method, the main discontinuity point of the solution of the equation is used to divide the partitions to overcome the disturbance of the numerical error convergence caused by the main discontinuity of the solution of the equation. Derivative approximation in the sense of integral is constructed in numerical format, and the convergence of the spectral collocation method in the sense of the L^∞ and L^2 norm is proved by the Dirichlet formula. At the same time, the error convergence also meets the effect of spectral accuracy convergence. The numerical experimental results are given at the end also verify the correctness of the theoretically proven results.

1 Introduction

Second-order Volterra integro-differential equations (VIDEs) have long appeared in mathematical models of physical phenomena and biological phenomena, which has led many scholars to develop a theoretical and numerical analysis of these equations. For some early research results, such as the general linear method [7], the linear multistep

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method [8,9], the Runge-Kutta method [10,12]. Over the years, polynomial spline collocation method [13–16,27,28], spectral Galerkin method [20] and Bologna [29] have developed an asymptotic solution for first-order and second-order VIDEs containing arbitrary kernels. In [24,37], the collocation method is used to approximate second-order VIDEs. In the recent papers [42–44], an hp-version of the spectral method has been proposed and analyzed for VIDEs.

The spectral method has been used in applied mathematics and scientific calculations to numerically solve certain partial differential equations (PDEs) [31–34]. In practice, the spectral method has high-precision convergence, the so-called "exponential convergence".

Delay VIDEs have many practical applications, such as competitive ecosystem [1], biology and Species model [2–4], virus transmission [5], and so on. For more information on the application of VIDEs with delay in species models, please refer to reference [6].

Remove the integral term in the delayed VIDEs to get the delayed differential equation(DDEs). The DDEs model appears in many practical problems, such as tumor growth models [38], species dynamics systems [39], hepatitis virus infection models [40], and toxic species presence diffusion models [41]. Reference [39] contains a lot of literature on the application of DDEs.

The literatures using spectral methods for solving delay VIDEs are [17–19,21,22,24–26,35]. The numerical methods used in these articles are used to solve the vanishing delay VIDEs. So far, there are few numerical methods for solving the nonvanishing delay type VIDEs that can make the numerical error converge to the spectral accuracy. When the VIDEs with nonvanishing delay are numerically solved, it is necessary to overcome that the solution of the equation has a main discontinuity point, which is inconsistent with the spectral method's requirement for the global smoothness of the equation solution. In the literature [36], the author gives a method of slicing the spectrum for solving VIDEs with nonvanishing delay first-order VIDEs, and gives proof of convergence. So far, few scholars have studied the spectral approximation of second-order VIDEs for the piecewise spectral-collocation method.

The second-order Volterra integro-differential equation with nonvanishing delay considered here is as follows:

$$
\begin{aligned}
y''(t) = {} & a_1(t)y(t) + a_2(t)y'(t) + b_1(t)y(\theta(t)) + b_2(t)y'(\theta(t)) + g(t) \\
& + \int_0^t K_1(t,s)y(s)ds + \int_0^t K_2(t,s)y'(s)ds \\
& + \int_0^{\theta(t)} R_1(t,s)y(s)ds + \int_0^{\theta(t)} R_2(t,s)y'(s)ds,
\end{aligned}
\tag{1.1a}
$$
$$
y(t) = \phi(t), \quad y'(t) = \varphi(t), \quad t \in [\theta(0),0].
\tag{1.1b}
$$

The functions $y(t)$, $t \in (0,T]$ are unknown functions, and $y \in C^{m+1}([0,T])$. Assume func-

tions (1.1) are $m \geq 1$ order continuous differentiable functions in their domain, i.e.,

$$a_1(t), a_2(t), b_1(t), b_2(t), g(t) \in C^m([0,T]), \quad \phi(t) \in C^m([\theta(0),0]), \tag{1.2a}$$

$$K_1(t,s), K_2(t,s) \in C^m(\Omega_1), \qquad\qquad \Omega_1 := \{(t,s) : 0 \leq s \leq t \leq T\}, \tag{1.2b}$$

$$R_1(t,s), R_2(t,s) \in C^m(\Omega_2), \qquad\qquad \Omega_2 := \{(t,s) : \theta(0) \leq s \leq \theta(t), \, 0 \leq t \leq T\}. \tag{1.2c}$$

And the delay function θ satisfies the following conditions:

$$\theta(t) := t - \tau(t), \quad \tau \in C^m([0,T]), \tag{1.3a}$$

$$\theta \text{ is strictly increasing on } [0,T], \tag{1.3b}$$

$$\tau(t) \geq \tau(0) > 0 \quad \text{for all } t \in [0,T]. \tag{1.3c}$$

In Eq. (1.1), the non-decaying hysteresis function θ causes the solution of the Volterra integro-differential equation to have major discontinuities $\{\xi_\mu\}$. The recurrence relation of these major discontinuities for

$$\theta(\xi_\mu) = \xi_{\mu-1}, \quad \mu \geq 0(\xi_{-1} := \theta(0), \, \xi_0 = 0).$$

And these main discontinuities have the characteristics of consistent interval, that is

$$\xi_\mu - \xi_{\mu-1} = \tau > 0, \quad \mu \geq 0.$$

For the convenience of discussion and elaboration in the following papers, we assume

$$T = \xi_{N+1}, \quad M \geq 1.$$

The rest of the paper is organized as follows: in Section 2, a numerical method of piecewise spectral-collocation is proposed to solve the second-order Volterra integro-differential equation with nonvanishing delay. Some concepts and several useful lemmas are described in detail in Section 3. The convergence proof of the method is presented in Section 4 and the experimental results in Section 5 also verify the effectiveness of the proposed method. some concluding remarks are given in Section 6.

2　Piecewise spectral-collocation method

In this section, our main goal is to obtain the numerical format of the piecewise spectral-collocation method for Eq. (1.1). We use the main discontinuity point of the solution of the equation as the interval node, and use Legendre on each subinterval approximation of Equations by spectral collocation method. By transformation:

$$t(x) = \frac{T}{2}(x+1), \quad s(z) = \frac{T}{2}(z+1). \tag{2.1}$$

Convert Eq. (1.1) into a new equation defined on $[-1,1]$.

$$u''(x) = A_1(x)u(x) + A_2(x)u'(x) + B_1(x)u(\vartheta(x)) + B_2(t)u'(\vartheta(x)) + f(x)$$

$$+ \int_{-1}^{x} S_1(x,z)u(z)dz + \int_{-1}^{x} S_2(x,z)u'(z)dz$$

$$+ \int_{-1}^{\vartheta(x)} Q_1(x,z)u(z)dz + \int_{-1}^{\vartheta(x)} Q_2(x,z)u'(z)dz, \tag{2.2a}$$

$$u(x) = \psi(x), \quad u'(x) = \chi(x), \quad x \in [\vartheta(-1), -1], \tag{2.2b}$$

where

$$u(x) := y(t(x)), \quad A_1(x) := \left(\frac{T}{2}\right)^2 a_1(t(x)), \quad A_2(x) := \frac{T}{2} a_2(t(x)), \tag{2.3a}$$

$$B_1(x) = \left(\frac{T}{2}\right)^2 b_1(t(x)), \quad B_2(x) = \frac{T}{2} b_1(t(x)), \quad f(x) = \left(\frac{T}{2}\right)^2 g(t(x)), \tag{2.3b}$$

$$\vartheta(x) := \frac{2}{T}\theta(t(x)) - 1, \quad \psi(x) := \phi(t(x)), \quad \chi(x) := \varphi(t(x)), \tag{2.3c}$$

$$S_1(x,z) := \left(\frac{T}{2}\right)^3 K_1(t(x),s(z)), \quad S_2(x,z) := \left(\frac{T}{2}\right)^2 K_2(t(x),s(z)), \tag{2.3d}$$

$$Q_1(x,z) := \left(\frac{T}{2}\right)^3 R_1(t(x),s(z)), \quad Q_2(x,z) := \left(\frac{T}{2}\right)^2 R_2(t(x),s(z)). \tag{2.3e}$$

The main breakpoint ξ_μ becomes

$$\eta_\mu := (2\xi_\mu/T) - 1, \quad \mu = -1, 0, 1, \cdots, M.$$

Let

$$\delta_\mu := (\eta_\mu, \eta_{\mu+1}], \quad \mu = -1, 0, 1, \cdots, M.$$

Next, we divide the interval $[-1,1]$ into $M+1$ subintervals,

$$[-1,1] = \bigcup_{\mu=0}^{M} \delta_\mu, \quad \mu = 1, \cdots, M, \tag{2.4}$$

where

$$\delta_0 := [-1, \eta_1], \quad \eta_{M+1} = 1, \quad \delta_\mu := (\eta_\mu, \eta_{\mu+1}].$$

We choose the following collocation points on the interval $[-1,1]$:

$$X_N := \bigcup_{\mu=0}^{M} X^\mu, \quad X^\mu := \{x_i^\mu : \eta_\mu = x_0^\mu < x_1^\mu < \cdots < x_N^\mu = \eta_{\mu+1}\}, \tag{2.5}$$

which

$$h_\mu := \frac{\eta_{\mu+1} - \eta_\mu}{2},$$

$$x_i^\mu := h_\mu x_i + \frac{\eta_{\mu+1} + \eta_\mu}{2}.$$

Then Eq. (2.2) is true at x_i^μ, that is,

$$
\begin{aligned}
u''(x_i^\mu) = {} & A_1(x_i^\mu)u(x_i^\mu) + A_2(x_i^\mu)u'(x_i^\mu) + B_1(x_i^\mu)u(\vartheta(x_i^\mu)) + B_2(x_i^\mu)u'(\vartheta(x_i^\mu)) \\
& + \int_{-1}^{x_i^\mu} S_1(x_i^\mu,z)u(z)dz + \int_{-1}^{x_i^\mu} S_2(x_i^\mu,z)u'(z)dz \\
& + \int_{-1}^{\vartheta(x_i^\mu)} Q_1(x_i^\mu,z)u(z)dz + \int_{-1}^{\vartheta(x_i^\mu)} Q_2(x_i^\mu,z)u'(z)dz + f(x_i^\mu), \\
& i = 0,1,\cdots,N, \quad \mu = 0,1,\cdots,M.
\end{aligned}
\tag{2.6}
$$

Next, we use v_i^μ, σ_i^μ and ϱ_i^μ to approximate $u(\vartheta(x_i^\mu))$, $u'(\vartheta(x_i^\mu))$ and $u''(x_i^\mu)$ respectively. We have

$$
\varrho_\mu(x) := \sum_{j=0}^N \varrho_j^\mu F_j^\mu(x), \quad x \in [\eta_\mu, \eta_{\mu+1}].
$$

Therefore, $u''|_{\delta_\mu}(x)$ can be approximated by $\varrho_\mu(x)$. Then $u''(x)$ can be approximated by $\varrho(x)$, that is,

$$
\varrho(x) := \varrho_\mu(x), \quad \text{if } x \in [\eta_\mu, \eta_{\mu+1}].
$$

In order to facilitate the proof of the theoretical part, we make the following definitions:

$$
\rho(x) = \zeta(-1) + \int_{-1}^x \varrho(z)dz,
$$

the derivative relationship is

$$
\frac{d}{dx}\rho(x) = \varrho(x).
$$

Then function $u'(x)$ can be approximated by $\rho(x)$.

Similarly, we define:

$$
\varsigma(x) = \chi(-1) + \int_{-1}^x \rho(z)dz,
$$

the derivative relationship is:

$$
\frac{d}{dx}\varsigma(x) = \rho(x).
$$

Then function $u(x)$ can be approximated by $\varsigma(x)$. Thus (2.6) has the following approximation:

$$
\begin{aligned}
\varrho_i^\mu \approx {} & A_1(x_i^\mu)\varsigma(x_i^\mu) + A_2(x_i^\mu)\rho(x_i^\mu) + B_1(x_i^\mu)v_i^\mu + B_2(x_i^\mu)\sigma_i^\mu + f(x_i^\mu) \\
& + \int_{-1}^{x_i^\mu} S_1(x_i^\mu,z)\varsigma(z)dz + \int_{-1}^{x_i^\mu} S_2(x_i^\mu,z)\rho(z)dz \\
& + \int_{-1}^{\vartheta(x_i^\mu)} Q_1(x_i^\mu,z)\varsigma(z)dz + \int_{-1}^{\vartheta(x_i^\mu)} Q_2(x_i^\mu,z)\rho(z)dz.
\end{aligned}
\tag{2.7}
$$

This can be rewritten as

$$
\begin{aligned}
\varrho_i^\mu \approx & A_1(x_i^\mu)\varsigma(x_i^\mu)+A_2(x_i^\mu)\rho(x_i^\mu)+B_1(x_i^\mu)v_i^\mu+B_2(x_i^\mu)\sigma_i^\mu+f(x_i^\mu) \\
& +\sum_{r=0}^{\mu-1}\int_{\eta_r}^{\eta_{r+1}}S_1(x_i^\mu,z)\varsigma(z)dz+\int_{\eta_\mu}^{x_i^\mu}S_1(x_i^\mu,z)\varsigma(z)dz \\
& +\sum_{r=0}^{\mu-1}\int_{\eta_r}^{\eta_{r+1}}S_2(x_i^\mu,z)\rho(z)dz+\int_{\eta_\mu}^{x_i^\mu}S_2(x_i^\mu,z)\rho(z)dz \\
& +\sum_{r=0}^{\mu-2}\int_{\eta_r}^{\eta_{r+1}}Q_1(x_i^\mu,z)\varsigma(z)dz+\int_{\eta_{\mu-1}}^{\vartheta(x_i^\mu)}Q_1(x_i^\mu,z)\varsigma(z)dz \\
& +\sum_{r=0}^{\mu-2}\int_{\eta_r}^{\eta_{r+1}}Q_2(x_i^\mu,z)\rho(z)dz+\int_{\eta_{\mu-1}}^{\vartheta(x_i^\mu)}Q_2(x_i^\mu,z)\rho(z)dz.
\end{aligned} \tag{2.8}
$$

To approximate the integral term using the Gauss quadrature formula, we replace the integration interval $[a,b]$ to the standard interval $[-1,1]$ with the following variables instead

$$
z(a,b,v):=\frac{b-a}{2}v+\frac{b+a}{2}, \quad v\in[-1,1]. \tag{2.9}
$$

For the convenience of the following derivation, remember

$$
z_r(v):=z(\eta_r,\eta_{r+1},v), \quad v\in[-1,1], \quad r>0. \tag{2.10}
$$

Using the Gauss quadrature formula to approximate the integral term in (2.8), we can get

$$
\begin{aligned}
\varrho_i^\mu = & A_1(x_i^\mu)\varsigma(x_i^\mu)+A_2(x_i^\mu)\rho(x_i^\mu)+B_1(x_i^\mu)v_i^\mu+B_2(x_i^\mu)\sigma_i^\mu+f(x_i^\mu) \\
& +\sum_{r=0}^{\mu-1}h_r\sum_{k=0}^{N}S_1(x_i^\mu,z_r(v_k))\varsigma(z_r(v_k))\omega_k \\
& +h_\mu\frac{x_i+1}{2}\sum_{k=0}^{N}S_1(x_i^\mu,z_\mu(z(-1,x_i,v_k)))\varsigma(z_\mu(z(-1,x_i,v_k)))\omega_k \\
& +\sum_{r=0}^{\mu-1}h_r\sum_{k=0}^{N}S_2(x_i^\mu,z_r(v_k))\rho(z_r(v_k))\omega_k \\
& +h_\mu\frac{x_i+1}{2}\sum_{k=0}^{N}S_2(x_i^\mu,z_\mu(z(-1,x_i,v_k)))\rho(z_\mu(z(-1,x_i,v_k)))\omega_k \\
& +\sum_{r=0}^{\mu-2}h_r\sum_{k=0}^{N}Q_1(x_i^\mu,z_r(v_k))\varsigma(z_r(v_k))\omega_k+h_{\mu-1}\frac{\widetilde{\vartheta}(x_i^\mu)+1}{2} \\
& \times\sum_{k=0}^{N}Q_1(x_i^\mu,z_{\mu-1}(z(-1,\widetilde{\vartheta}(x_i^\mu),v_k)))\varsigma(z_{\mu-1}(z(-1,\widetilde{\vartheta}(x_i^\mu),v_k)))\omega_k
\end{aligned}
$$

$$+\sum_{r=0}^{\mu-2}h_r\sum_{k=0}^{N}Q_2(x_i^\mu,z_r(v_k))\rho(z_r(v_k))\omega_k+h_{\mu-1}\frac{\widetilde{\vartheta}(x_i^\mu)+1}{2}$$

$$\times\sum_{k=0}^{N}Q_2(x_i^\mu,z_{\mu-1}(z(-1,\widetilde{\vartheta}(x_i^\mu),v_k)))\rho(z_{\mu-1}(z(-1,\widetilde{\vartheta}(x_i^\mu),v_k)))\omega_k, \qquad (2.11)$$

where v_k, $k=0,1,\cdots,N$ are $N+1$ Legendre Gauss-Lobatto integral nodes on the interval $[-1,1]$, and the corresponding weight function is ω_k, $k=0,1,\cdots,N$. Additionally,

$$\widetilde{\vartheta}(x_i^\mu):=\frac{1}{h_{\mu-1}}\vartheta(x_i^\mu)-\frac{\eta_\mu+\eta_{\mu-1}}{\eta_\mu-\eta_{\mu-1}}, \quad \mu>0. \qquad (2.12)$$

Simplifying (2.11) gives

$$\varrho_i^\mu=A_1(x_i^\mu)\varsigma(x_i^\mu)+A_2(x_i^\mu)\rho(x_i^\mu)+B_1(x_i^\mu)v_i^\mu+B_2(x_i^\mu)\sigma_i^\mu+f(x_i^\mu)+\beta_1(x_i^\mu)$$
$$+\beta_2(x_i^\mu)+\lambda_1(x_i^\mu)+\lambda_2(x_i^\mu), \quad i=0,1,\cdots,N, \quad \mu=0,1,\cdots,M, \qquad (2.13)$$

where $\beta_1(x_i^\mu)$ and $\beta_2(x_i^\mu)$ can be expressed as:

$$\beta_1(x_i^\mu):=\sum_{r=0}^{\mu-1}h_r\sum_{k=0}^{N}S_1(x_i^\mu,z_r(v_k))\varsigma(z_r(v_k))\omega_k$$
$$+h_\mu\frac{x_i+1}{2}\sum_{k=0}^{N}S_1(x_i^\mu,z_\mu(z(-1,x_i,v_k)))\varsigma(z_\mu(z(-1,x_i,v_k)))\omega_k,$$

$$\beta_2(x_i^\mu):=\sum_{r=0}^{\mu-1}h_r\sum_{k=0}^{N}S_2(x_i^\mu,z_r(v_k))\rho(z_r(v_k))\omega_k$$
$$+h_\mu\frac{x_i+1}{2}\sum_{k=0}^{N}S_2(x_i^\mu,z_\mu(z(-1,x_i,v_k)))\rho(z_\mu(z(-1,x_i,v_k)))\omega_k,$$

$\lambda_1(x_i^\mu)$ and $\lambda_2(x_i^\mu)$ can be expressed as:

$$\lambda_1(x_i^\mu):=\begin{cases}\dfrac{\vartheta(x_i^0)+1}{2}\displaystyle\sum_{k=0}^{N}Q_1(x_i^0,z(-1,\vartheta(x_i^0),v_k))\psi(z(-1,\vartheta(x_i^0),v_k))\omega_k, & \mu=0,\\[4mm]\displaystyle\sum_{r=0}^{\mu-2}h_r\sum_{k=0}^{N}Q_1(x_i^\mu,z_r(v_k))\varsigma(z_r(v_k))\omega_k+h_{\mu-1}\dfrac{\widetilde{\vartheta}(x_i^\mu)+1}{2}\\[4mm]\quad\times\displaystyle\sum_{k=0}^{N}Q_1(x_i^\mu,z_{\mu-1}(z(-1,\widetilde{\vartheta}(x_i^\mu),v_k)))\varsigma(z_{\mu-1}(z(-1,\widetilde{\vartheta}(x_i^\mu),v_k)))\omega_k, & \mu>0,\end{cases}$$

$$\lambda_2(x_i^\mu):=\begin{cases}\dfrac{\vartheta(x_i^0)+1}{2}\displaystyle\sum_{k=0}^{N}Q_2(x_i^0,z(-1,\vartheta(x_i^0),v_k))\chi(z(-1,\vartheta(x_i^0),v_k))\omega_k, & \mu=0,\\[2ex]\displaystyle\sum_{r=0}^{\mu-2}h_r\sum_{k=0}^{N}Q_2(x_i^\mu,z_r(v_k))\rho(z_r(v_k))\omega_k+h_{\mu-1}\dfrac{\widetilde{\vartheta}(x_i^\mu)+1}{2}\\[2ex]\quad\times\displaystyle\sum_{k=0}^{N}Q_2(x_i^\mu,z_{\mu-1}(z(-1,\widetilde{\vartheta}(x_i^\mu),v_k)))\rho(z_{\mu-1}(z(-1,\widetilde{\vartheta}(x_i^\mu),v_k)))\omega_k, & \mu>0.\end{cases}$$

Relying on the discrete system (2.13) alone is not enough to solve the unknown. We need another four about $\varsigma(x_i^\mu)$, $\rho(x_i^\mu)$, v_i^μ, ϱ_i^μ, σ_i^μ, $i=0,1,\cdots,N$, $\mu=0,1,\cdots,M$ for discrete systems

$$\begin{aligned}\rho(x)&=\chi(-1)+\int_{-1}^{x}\varrho(z)dz\\&=\chi(-1)+\sum_{r=0}^{\mu-1}\int_{\eta_r}^{\eta_{r+1}}\varrho_r(z)dz+\int_{\eta_\mu}^{x}\varrho_\mu(z)dz\\&=\chi(-1)+\sum_{r=0}^{\mu-1}h_r\sum_{k=0}^{N}\varrho_r(z_r(v_k))\omega_k+h_\mu\frac{x+1}{2}\sum_{k=0}^{N}\varrho_\mu(z_\mu(z(-1,\widetilde{x},v_k)))\omega_k\\&=\chi(-1)+\sum_{r=0}^{\mu-1}h_r\sum_{k=0}^{N}\varrho_j^r\sum_{j=0}^{N}F_j^r(z_r(v_k))\omega_k\\&\quad+h_\mu\frac{\widetilde{x}+1}{2}\sum_{k=0}^{N}\varrho_j^\mu\sum_{j=0}^{N}F_j^\mu(z_\mu(z(-1,\widetilde{x},v_k)))\omega_k,\quad x\in[\eta_\mu,\eta_{\mu+1}],\end{aligned}\qquad(2.14)$$

where

$$\widetilde{x}:=\frac{2}{\eta_{\mu+1}-\eta_\mu}x-\frac{\eta_{\mu+1}+\eta_\mu}{\eta_{\mu+1}-\eta_\mu}.$$

According to (2.14), we get

$$u(x_i^\mu)=u(-1)+\int_{-1}^{x_i^\mu}u'(z)dz.\qquad(2.15)$$

Then (2.15) has the following approximation

$$\varsigma(x_i^\mu)=\psi(-1)+\int_{-1}^{x_i^\mu}\rho(z)dz,\quad i=0,1,\cdots,N,\quad\mu=0,1,\cdots,M.\qquad(2.16)$$

Similarly, we can get

$$\begin{aligned}u(\vartheta(x_i^\mu))&=u(-1)+\int_{-1}^{\vartheta(x_i^\mu)}u'(z)dz\\&=\psi(-1)+\sum_{r=0}^{\mu-2}h_r\int_{-1}^{1}u'(z_r(v))dv+h_{\mu-1}\frac{\widetilde{\vartheta}(x_i^\mu)+1}{2}\int_{-1}^{1}u'(z_{\mu-1}(z(-1,\widetilde{\vartheta}(x_i^\mu),v)))dv.\end{aligned}\quad(2.17)$$

Then (2.17) has the following approximation:

$$v_i^\mu = \psi(-1) + \alpha_2(x_i^\mu), \quad i = 0,1,\cdots,N, \quad \mu = 0,1,\cdots,M, \tag{2.18}$$

where

$$\alpha_2(x_i^\mu) = \begin{cases} \psi(\vartheta(x_i^0)) - \psi(-1), & \mu = 0, \\ \displaystyle\sum_{r=0}^{\mu-2} h_r \sum_{k=0}^{N} \rho_k^r \omega_k + h_{\mu-1} \frac{\widetilde{\vartheta}(x_i^\mu)+1}{2} \sum_{j=0}^{N} \rho_j^{\mu-1} \sum_{k=0}^{N} F_j(z(-1,\widetilde{\vartheta}(x_i^\mu),v_k))\omega_k, & \mu > 0. \end{cases}$$

Similarly, we have

$$u'(x_i^\mu) = u'(-1) + \int_{-1}^{x_i^\mu} u''(z)dz$$
$$= \chi(-1) + \sum_{r=0}^{\mu-1} h_r \int_{-1}^{1} u''(z_r(v))dv + h_\mu \frac{x_i+1}{2} \int_{-1}^{1} u''(z_\mu(z(-1,x_i,v)))dv. \tag{2.19}$$

Then (2.19) has the following approximation:

$$\rho(x_i^\mu) = \chi(-1) + \alpha_3(x_i^\mu), \quad i = 0,1,\cdots,N, \quad \mu = 0,1,\cdots,M, \tag{2.20}$$

where

$$\alpha_3(x_i^\mu) = \sum_{r=0}^{\mu-1} h_r \sum_{k=0}^{N} \varrho_k^r \omega_k + h_\mu \frac{x_i+1}{2} \sum_{j=0}^{N} \varrho_j^\mu \sum_{k=0}^{N} F_j(z(-1,x_i,v_k))\omega_k. \tag{2.21}$$

Finally, we have

$$u'(\vartheta(x_i^\mu)) = u'(-1) + \int_{-1}^{\vartheta(x_i^\mu)} u''(z)dz$$
$$= \chi(-1) + \sum_{r=0}^{\mu-2} h_r \int_{-1}^{1} u''(z_r(v))dv + h_{\mu-1} \frac{\widetilde{\vartheta}(x_i^\mu)+1}{2} \int_{-1}^{1} u''(z_{\mu-1}(z(-1,\widetilde{\vartheta}(x_i^\mu),v)))dv. \tag{2.22}$$

Then (2.22) has the following approximation:

$$\sigma_i^\mu = \chi(-1) + \alpha_4(x_i^\mu), \quad i = 0,1,\cdots,N, \quad \mu = 0,1,\cdots,M, \tag{2.23}$$

where

$$\alpha_4(x_i^\mu) = \begin{cases} \chi(\vartheta(x_i^0)) - \chi(-1), & \mu = 0, \\ \displaystyle\sum_{r=0}^{\mu-2} h_r \sum_{k=0}^{N} \varrho_k^r \omega_k + h_{\mu-1} \frac{\widetilde{\vartheta}(x_i^\mu)+1}{2} \sum_{j=0}^{N} \varrho_j^{\mu-1} \sum_{k=0}^{N} F_j(z(-1,\widetilde{\vartheta}(x_i^\mu),v_k))\omega_k, & \mu > 0. \end{cases}$$

The other four discrete systems we are looking for are (2.16), (2.20), (2.18) and (2.23). Our method is to solve for ϱ_i^μ, $\rho(x_i^\mu)$ and $\varsigma(x_i^\mu)$ ($i = 0,1,\cdots,N$, $\mu = 0,1,\cdots,M$) through (2.13), (2.16), (2.20), (2.18) and (2.23).

The approximation of $y(t)$ is $\varsigma(\frac{2}{T}t-1)$ and the approximation of $y'(t)$ is $\frac{2}{T}\rho(\frac{2}{T}t-1)$, the approximation of $y''(t)$ is $(\frac{2}{T})^2\varrho(\frac{2}{T}t-1)$.

3 Some preliminaries and useful lemmas

In this section, we give the lemmas needed in the theoretical proof.

Let (a,b) be a bounded interval on the real number field, and $\omega(x)$ be the Legendre weight function on (a,b). $L_\omega^2(a,b)$ is the measurable function space, and its corresponding norm is

$$\|u\|_{L_\omega^2(a,b)} := \left(\int_a^b |u(x)|^2 \omega(x) dx \right)^{1/2}.$$

$L^\infty(a,b)$ is the measurable function space, and its corresponding norm is

$$\|u\|_{L^\infty(a,b)} := \operatorname{ess}\sup_{x\in(a,b)} |u(x)|.$$

Let Λ_h denote the collection of subintervals δ_μ, $\mu = 0,1,\cdots,M$. We define the broken Sobolev space $H^m(\Lambda_h)$ as

$$H^m(\Lambda_h) := \left\{ u : u|_{\delta_\mu} \in H^m(\delta_\mu), \ \mu = 0,1,\cdots,M \right\}.$$

The associated norm is

$$\|u\|_{H^m(\Lambda_h)} := \left(\sum_{k=0}^m \|u^{(k)}\|_{L^2(\Lambda_h)}^2 \right)^{1/2},$$

where

$$\|u^{(k)}\|_{L^2(\Lambda_h)}^2 := \sum_{\mu=0}^M \|\partial_x^k(u|_{\sigma_\mu})\|_{L^2(\delta_\mu)}^2, \quad k = 0,1,\cdots,m.$$

Lemma 3.1 ([23]). *Let $0 \le M < +\infty$. If the non-negative integrable function $e(x)$ satisfies*

$$e(x) \le v(x) + M \int_1^x e(z)dz,$$

where $v(x)$ is also a non-negative integrable function, then

$$\|e\|_{L^p(-1,1)} \le C \|v\|_{L^p(-1,1)}, \quad p = 2,+\infty.$$

Lemma 3.2 ([36]). *Let $u \in C([-1,1]) \bigcap H^m(\Lambda_h)$. Define I_N as the interpolation operator. Note h as $\max\{h_\mu : \mu = 0,1,\cdots,M\}$. Then when $N \ge m-1$, we have*

$$\|u - I_N u\|_{L^2(-1,1)} \le Ch^m N^{-m} \|u^{(m)}\|_{L^2(\Lambda_h)}, \tag{3.1a}$$

$$\|u - I_N u\|_{L^\infty(-1,1)} \le Ch^{m-1/2} N^{1/2-m} \|u^{(m)}\|_{L^2(\Lambda_h)}, \tag{3.1b}$$

$$\|I_N\|_{L^\infty(-1,1)} \le C\log(N+1), \tag{3.1c}$$

$$\|I_N u\|_{L^2(-1,1)} \le C\|u\|_{L^\infty(-1,1)}. \tag{3.1d}$$

Lemma 3.3 ([11]). *Let $u \in H^m(-1,1), m \geq 1$ and $\varphi \in \mathcal{P}_N$, $\mathcal{P}_N$ be A set of polynomials of all degrees up to N. The existence of a constant C that has nothing to do with N makes it possible to have $N \geq m-1$*

$$\left| \int_{-1}^{1} u(x)\varphi(x) - \sum_{j=0}^{N} u(x_j)\varphi(x_j)\omega_j \right| \leq CN^{-m} \|\partial_x^m u\|_{L^2(-1,1)} \|\varphi\|_{L^2(-1,1)},$$

where x_j is the Legendre Gauss-Lobatto quadrature node point, and the corresponding weight function is ω_j, $j = 0,1,\cdots,N$.

Lemma 3.4 ([36]). *Let $u \in H^m(-1,1), m \geq 1$ and $\varphi|_{\delta_\mu} \in \mathcal{P}_N(\delta_\mu)$, that is, φ is a polynomial whose degree does not exceed N in the interval δ_μ. When $N > m-1$, there is a constant C that is unrelated to N is established as follows*

$$\left| \int_{-1}^{x} u(z)\varphi(z) - S(x) \right| \leq Ch^m N^{-m} \|\partial_z^m u\|_{L^2(-1,x)} \|\varphi\|_{L^2(-1,x)}, \quad x \in \delta_\mu,$$

where

$$S(x) := \sum_{r=0}^{\mu-1} h_r \sum_{j=0}^{N} u|_{\delta_r}(z_r(v_j))\varphi|_{\delta_r}(z_r(v_j))\omega_j$$

$$+ h_\mu \frac{\widetilde{x}+1}{2} \sum_{j=0}^{N} u|_{\delta_\mu}(z_\mu(z(-1,x_i,v_j)))\varphi|_{\delta_\mu}(z_\mu(z(-1,\widetilde{x},v_j)))\omega_j,$$

$$\widetilde{x} := \frac{2}{\eta_{\mu+1}-\eta_\mu} x - \frac{\eta_{\mu+1}+\eta_\mu}{\eta_{\mu+1}-\eta_\mu},$$

v_j is the Legendre Gauss-Lobatto point, the corresponding weight is ω_j, $j = 0,1,\cdots,N$.

4 Convergence analysis

Theorem 4.1. *Let $u(x)$ be the solution of Eq. (2.2). Use the numerical format (2.13), (2.16), (2.20), (2.18) and (2.23) The approximation of $u(x)$ is $\varsigma(x)$, the approximation of its derivative is $\rho(x)$, and the approximation of its second derivative is $\varrho(x)$. Then for a sufficiently large $N \geq m-1$ there is*

$$\|e_i\|_{L^\infty(-1,1)} \leq CN^{1/2-m} R \Big(\|u\|_{L^\infty(-1,1)} + \|u'\|_{L^\infty(-1,1)}$$

$$+ \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1 \Big), \quad i = 0,1,2, \tag{4.1}$$

where

$$e_0(x) := \begin{cases} 0, & x \in [\vartheta(-1), -1], \\ u(x) - \varsigma(x), & x \in (-1, 1], \end{cases}$$

$$e_1(x) := \begin{cases} 0, & x \in [\vartheta(-1), -1], \\ u'(x) - \rho(x), & x \in (-1, 1], \end{cases}$$

$$e_2(x) := \begin{cases} 0, & x \in [\vartheta(-1), -1], \\ u''(x) - \varrho(x), & x \in (-1, 1], \end{cases}$$

R is a constant related only to the m order derivative of $S_j(x,z)$, $Q_j(x,z)$, $\psi(z)$, $\chi(z)$, $(j=1,2)$.

Proof. During the derivation of the numerical format part, we can get

$$u(x_i^\mu) - \varsigma(x_i^\mu) = \int_{-1}^{x_i^\mu} e_1(z)dz, \qquad u'(x_i^\mu) - \rho(x_i^\mu) = \int_{-1}^{x_i^\mu} e_2(z)dz, \qquad (4.2a)$$

$$u(\vartheta(x_i^\mu)) - v_i^\mu = \int_{-1}^{\vartheta(x_i^\mu)} e_1(z)dz, \qquad u'(\vartheta(x_i^\mu)) - \sigma_i^\mu = \int_{-1}^{\vartheta(x_i^\mu)} e_2(z)dz. \qquad (4.2b)$$

Subtracting (2.6) from (2.13) gives

$$u''(x_i^\mu) - \varrho_i^\mu = A_1(x_i^\mu) \int_{-1}^{x_i^\mu} e_1(z)dz + A_2(x_i^\mu) \int_{-1}^{x_i^\mu} e_2(z)dz + B_1(x_i^\mu) \int_{-1}^{\vartheta(x_i^\mu)} e_1(z)dz$$

$$+ B_2(x_i^\mu) \int_{-1}^{\vartheta(x_i^\mu)} e_2(z)dz + \int_{-1}^{x_i^\mu} S_1(x_i^\mu, z) e_0(z)dz + \int_{-1}^{x_i^\mu} S_2(x_i^\mu, z) e_1(z)dz$$

$$+ \int_{-1}^{\vartheta(x_i^\mu)} Q_1(x_i^\mu, z) e_0(z)dz + \int_{-1}^{\vartheta(x_i^\mu)} Q_2(x_i^\mu, z) e_1(z)dz + \sum_{j=0}^{3} E_j(x_i^\mu), \qquad (4.3)$$

where

$$E_0 := \int_{-1}^{\vartheta(x)} Q_1(x, z) \varsigma(z)dz - \lambda_1(x), \qquad x \in [-1, 1],$$

$$E_1 := \int_{-1}^{\vartheta(x)} Q_2(x, z) \rho(z)dz - \lambda_2(x), \qquad x \in [-1, 1],$$

$$E_2 := \int_{-1}^{x} S_1(x, z) \varsigma(z)dz - \beta_1(x), \qquad x \in [-1, 1],$$

$$E_3 := \int_{-1}^{x} S_2(x, z) \rho(z)dz - \beta_2(x), \qquad x \in [-1, 1].$$

Multiply $F_i^\mu(x)$ by the two ends of (4.3) and add $i=0$ to N. Then

$$\sum_{i=0}^{N} u''(x_i^\mu)F_i^\mu(x) - \sum_{i=0}^{N} \varrho_i^\mu F_i^\mu(x)$$

$$=\sum_{i=0}^{N}\left(A_1(x_i^\mu)\int_{-1}^{x_i^\mu}e_1(z)dz\right)F_i^\mu(x)+\sum_{i=0}^{N}\left(A_2(x_i^\mu)\int_{-1}^{x_i^\mu}e_2(z)dz\right)F_i^\mu(x)$$

$$+\sum_{i=0}^{N}\left(B_1(x_i^\mu)\int_{-1}^{\vartheta(x_i^\mu)}e_1(z)dz\right)F_i^\mu(x)+\sum_{i=0}^{N}\left(B_2(x_i^\mu)\int_{-1}^{\vartheta(x_i^\mu)}e_2(z)dz\right)F_i^\mu(x)$$

$$+\sum_{i=0}^{N}\left(\int_{-1}^{x_i^\mu}S_1(x_i^\mu,z)e_0(z)dz\right)F_i^\mu(x)+\sum_{i=0}^{N}\left(\int_{-1}^{x_i^\mu}S_2(x_i^\mu,z)e_1(z)dz\right)F_i^\mu(x)$$

$$+\sum_{i=0}^{N}\left(\int_{-1}^{\vartheta(x_i^\mu)}Q_1(x_i^\mu,z)e_0(z)dz\right)F_i^\mu(x)+\sum_{i=0}^{N}\left(\int_{-1}^{\vartheta(x_i^\mu)}Q_2(x_i^\mu,z)e_1(z)dz\right)F_i^\mu(x)$$

$$+\sum_{j=0}^{3}\sum_{i=0}^{N}E_j(x_i^\mu)F_i^\mu(x),\quad x\in[\eta_\mu,\eta_{\mu+1}]. \tag{4.4}$$

According to the definition of I_N and $\varrho(x)$, we can get

$$I_N u''(x) - \varrho(x) = I_N\left(A_1(x)\int_{-1}^{x}e_1(z)dz\right)+I_N\left(A_2(x)\int_{-1}^{x}e_2(z)dz\right)$$

$$+I_N\left(B_1(x)\int_{-1}^{\vartheta(x)}e_1(z)dz\right)+I_N\left(B_2(x)\int_{-1}^{\vartheta(x)}e_2(z)dz\right)$$

$$+I_N\left(\int_{-1}^{x}S_1(x,z)e_0(z)dz\right)+I_N\left(\int_{-1}^{x}S_2(x,z)e_1(z)dz\right)$$

$$+I_N\left(\int_{-1}^{\vartheta(x)}Q_1(x,z)e_0(z)dz\right)+I_N\left(\int_{-1}^{\vartheta(x)}Q_2(x,z)e_1(z)dz\right)$$

$$+\sum_{j=0}^{3}I_N E_j(x),\quad x\in[-1,1]. \tag{4.5}$$

Through (4.5), we can get:

$$e_2(x)=\sum_{j=0}^{3}I_N E_j(x)+\sum_{j=4}^{12}E_j(x)+A_1(x)\int_{-1}^{x}e_1(z)dz+A_2(x)\int_{-1}^{x}e_2(z)dz$$

$$+B_1(x)\int_{-1}^{\vartheta(x)}e_1(z)dz+B_2(x)\int_{-1}^{\vartheta(x)}e_2(z)dz+\int_{-1}^{x}S_1(x,z)e_0(z)dz$$

$$+\int_{-1}^{x}S_2(x,z)e_1(z)dz+\int_{-1}^{\vartheta(x)}Q_1(x,z)e_0(z)dz+\int_{-1}^{\vartheta(x)}Q_2(x,z)e_1(z)dz, \tag{4.6}$$

where

$$E_4(x) := (I - I_N)u''(x), \qquad\qquad E_5(x) := (I_N - I)A_1(x)\int_{-1}^{x} e_1(z)dz,$$

$$E_6(x) := (I_N - I)A_2(x)\int_{-1}^{x} e_2(z)dz, \qquad E_7(x) := (I_N - I)B_1(x)\int_{-1}^{\vartheta(x)} e_1(z)dz,$$

$$E_8(x) := (I_N - I)B_2(x)\int_{-1}^{\vartheta(x)} e_2(z)dz, \qquad E_9(x) := (I_N - I)\int_{-1}^{x} S_1(x,z)e_0(z)dz,$$

$$E_{10}(x) := (I_N - I)\int_{-1}^{x} S_2(x,z)e_1(z)dz, \qquad E_{11}(x) := (I_N - I)\int_{-1}^{\vartheta(x)} Q_1(x,z)e_0(z)dz,$$

$$E_{12}(x) := (I_N - I)\int_{-1}^{\vartheta(x)} Q_2(x,z)e_1(z)dz.$$

Using Dirichlet formula

$$\int_{-1}^{x}\int_{-1}^{\tau}\Phi(\tau,s))dsd\tau = \int_{-1}^{x}\int_{s}^{x}\Phi(\tau,s))d\tau ds. \tag{4.7}$$

Acting on the right end of (4.6) gives:

$$A_1(x)\int_{-1}^{x} e_1(z)dz = \int_{-1}^{x} A_1(x)(x-\tau)e_2(\tau)d\tau,$$

$$B_1(x)\int_{-1}^{\vartheta(x)} e_1(z)dz = \int_{-1}^{\vartheta(x)} B_1(x)(x-\tau)e_2(\tau)d\tau,$$

$$\int_{-1}^{x} S_1(x,z)e_0(z)dz = \int_{-1}^{x}\left[\int_{\tau}^{x} S_1(x,z)(z-\tau)dz\right]e_2(\tau)d\tau,$$

$$\int_{-1}^{\vartheta(x)} Q_1(x,z)e_0(z)dz = \int_{-1}^{\vartheta(x)}\left[\int_{\tau}^{\vartheta(x)} Q_1(x,z)(z-\tau)dz\right]e_2(\tau)d\tau,$$

$$\int_{-1}^{x} S_2(x,z)e_1(z)dz = \int_{-1}^{x}\left[\int_{\tau}^{x} S_2(x,z)dz\right]e_2(\tau)d\tau,$$

$$\int_{-1}^{\vartheta(x)} Q_2(x,z)e_1(z)dz = \int_{-1}^{\vartheta(x)}\left[\int_{\tau}^{\vartheta(x)} Q_2(x,z)dz\right]e_2(\tau)d\tau.$$

This leads to the existence of constants $C_1, C_2, C > 0$ such that

$$\left| A_1(x)\int_{-1}^{x} e_1(z)dz + A_2(x)\int_{-1}^{x} e_2(z)dz + B_1(x)\int_{-1}^{\vartheta(x)} e_1(z)dz + B_2(x)\int_{-1}^{\vartheta(x)} e_2(z)dz \right.$$

$$\left. + \int_{-1}^{x} S_1(x,z)e_0(z)dz + \int_{-1}^{x} S_2(x,z)e_1(z)dz + \int_{-1}^{\vartheta(x)} Q_1(x,z)e_0(z)dz + \int_{-1}^{\vartheta(x)} Q_2(x,z)e_1(z)dz \right|$$

$$
\leq C_1 \int_{-1}^{x} e_2(z)dz + C_2 \int_{-1}^{\vartheta(x)} e_2(z)dz
$$

$$
\leq C \int_{-1}^{x} e_2(z)dz. \tag{4.8}
$$

From Lemma 3.1, we can estimate $e_2(x)$ in (4.5), as follows

$$
\|e_2\|_{L^\infty(-1,1)} \leq C \left(\sum_{j=0}^{3} \|I_N E_j\|_{L^\infty(-1,1)} + \sum_{j=4}^{12} \|E_j\|_{L^\infty(-1,1)} \right). \tag{4.9}
$$

Next, we estimate the right term of the inequality (4.9).

We first estimate $\|I_N E_0\|_{L^\infty(-1,1)}$. Get it from (3.1c)

$$
\left\| I_N E_0 \right\|_{L^\infty(-1,1)} \leq C\log(N+1) \left\| E_0 \right\|_{L^\infty(-1,1)}. \tag{4.10}
$$

By Lemma 3.3, we can get $(x \in \delta_0)$

$$
\left| E_0(x) \right| \leq CN^{-m} \left\| \partial_v^m \Big(Q_1(z(-1,\vartheta(x),\cdot))\psi(z(-1,\vartheta(x),\cdot)) \Big) \right\|_{L^2(-1,1)}
$$

$$
\leq CN^{-m} \left| \frac{\vartheta(x)+1}{2} \right|^m \left\| \partial_z^m (Q_1(x,z)\psi(z))|_{z=z(-1,\vartheta(x),\cdot)} \right\|_{L^2(-1,1)}
$$

$$
\leq CN^{-m} \left\| \partial_z^m (Q_1(x,\cdot)\psi(\cdot)) \right\|_{L^2(\vartheta(x),-1)}. \tag{4.11}
$$

By Lemma 3.4, we have $(x \in \delta_\mu, \mu > 0)$

$$
\left| E_0(x) \right| \leq CN^{-m} \left\| \partial_z^m (Q_1(x,\cdot)) \right\|_{L^2(-1,\vartheta(x))} \|u^N\|_{L^2(-1,1)}
$$

$$
\leq CN^{-m} \left\| \partial_z^m (Q_1(x,\cdot)) \right\|_{L^2(-1,\vartheta(x))} \Big(\|e_0\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} \Big). \tag{4.12}
$$

Then

$$
\left\| E_0 \right\|_{L^\infty(-1,1)} \leq CN^{-m} \widetilde{Q_1} \Big(\|e_0\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} \Big), \tag{4.13}
$$

where

$$
\widetilde{Q_1} := \max \left\{ \max_{x \in \delta_0} \left\| \partial_z^m (Q_1(x,\cdot)\psi(\cdot)) \right\|_{L^2(\vartheta(x),-1)}, \max_{x \in [\eta_1,1]} \left\| \partial_z^m (Q_1(x,\cdot)) \right\|_{L^2(-1,\vartheta(x))} \right\}.
$$

Combining (4.10) and (4.13) gives

$$
\|I_N E_0\|_{L^\infty(-1,1)} \leq CN^{-m}\log(N+1)\widetilde{Q_1} \Big(\|e_0\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} \Big). \tag{4.14}
$$

Using an analysis process similar to estimating $\|I_N E_0\|_{L^\infty(-1,1)}$, we get

$$\|I_N E_1\|_{L^\infty(-1,1)} \leq CN^{-m}\log(N+1)\widetilde{Q_2}\Big(\|e_1\|_{L^\infty(-1,1)} + \|u'\|_{L^\infty(-1,1)}\Big), \tag{4.15a}$$

$$\|I_N E_2\|_{L^\infty(-1,1)} \leq CN^{-m}\log(N+1)\widetilde{S_1}\Big(\|e_0\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)}\Big), \tag{4.15b}$$

$$\|I_N E_3\|_{L^\infty(-1,1)} \leq CN^{-m}\log(N+1)\widetilde{S_2}\Big(\|e_1\|_{L^\infty(-1,1)} + \|u'\|_{L^\infty(-1,1)}\Big), \tag{4.15c}$$

where

$$\widetilde{Q_2} := \max\left\{ \max_{x\in\delta_0}\left\|\partial_z^m(Q_2(x,\cdot)\chi(\cdot))\right\|_{L^2(\vartheta(x),-1)}, \max_{x\in[\eta_1,1]}\left\|\partial_z^m(Q_2(x,\cdot))\right\|_{L^2(-1,\vartheta(x))} \right\},$$

$$\widetilde{S_1} := \max_{x\in[-1,1]}\left\|\partial_z^m(S_1(x,\cdot))\right\|_{L^2(-1,x)},$$

$$\widetilde{S_2} := \max_{x\in[-1,1]}\left\|\partial_z^m(S_2(x,\cdot))\right\|_{L^2(-1,x)}.$$

Next estimate is $\|E_j\|_{L^\infty(-1,1)}$, $j=4,5,\cdots,12$. Note $u|_{\delta_\mu}(x)\in H^{m+2}(\delta_\mu)$, $\mu=0,1,\cdots,M$. Apply (3.1b) to $u(x)$ gets

$$\|E_4\|_{L^\infty(-1,1)} \leq CN^{\frac{1}{2}-m}\|u^{(m+2)}\|_{L^2(\Lambda_h)}. \tag{4.16}$$

Now estimate $\|E_{11}\|_{L^\infty(-1,1)}$. For the sake of brevity, let

$$\gamma(x) := \int_{-1}^{\vartheta(x)} Q_1(x,z)e_0(z)dz. \tag{4.17}$$

When $m=1$, we can get from (3.1b):

$$\|(I_N - I)\gamma\|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}}\|\partial_x^1\gamma\|_{L^2(-1,1)}. \tag{4.18}$$

Obviously, we have

$$\begin{aligned}
|\partial_x^1\gamma(x)| &= \left|Q_1(x,\vartheta(x))e_0(\vartheta(x))\vartheta'(x) + \int_{-1}^{\vartheta(x)}\frac{\partial Q_1}{\partial x}(x,z)e_0(z)dz\right| \\
&\leq \|e_0\|_{L^\infty(-1,1)}\left|\left(Q_1(x,\vartheta(x))\vartheta'(x) + \int_{-1}^{\vartheta(x)}\frac{\partial Q_1}{\partial x}(x,z)dz\right)\right| \\
&\leq C\|e_0\|_{L^\infty(-1,1)}.
\end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19), we can get:

$$\|E_{11}\|_{L^\infty(-1,1)} = \|(I_N - I)\gamma\|_{L^\infty(-1,1)} \leq CN^{-\frac{1}{2}}\|e_0\|_{L^\infty(-1,1)}. \tag{4.20}$$

Similarly, we have

$$\|E_5\|_{L^\infty(-1,1)} \le CN^{-\frac{1}{2}}\|e_1\|_{L^\infty(-1,1)}, \qquad \|E_6\|_{L^\infty(-1,1)} \le CN^{-\frac{1}{2}}\|e_2\|_{L^\infty(-1,1)}, \qquad (4.21a)$$

$$\|E_7\|_{L^\infty(-1,1)} \le CN^{-\frac{1}{2}}\|e_1\|_{L^\infty(-1,1)}, \qquad \|E_8\|_{L^\infty(-1,1)} \le CN^{-\frac{1}{2}}\|e_2\|_{L^\infty(-1,1)}, \qquad (4.21b)$$

$$\|E_9\|_{L^\infty(-1,1)} \le CN^{-\frac{1}{2}}\|e_0\|_{L^\infty(-1,1)}, \qquad \|E_{10}\|_{L^\infty(-1,1)} \le CN^{-\frac{1}{2}}\|e_1\|_{L^\infty(-1,1)}, \qquad (4.21c)$$

$$\|E_{12}\|_{L^\infty(-1,1)} \le CN^{-\frac{1}{2}}\|e_1\|_{L^\infty(-1,1)}. \qquad (4.21d)$$

Combining (4.14), (4.31), (4.16), (4.20) and (4.21), we have

$$\|e_2\|_{L^\infty(-1,1)} \le CN^{-m}\Big(\log(N+1)R(\|u\|_{L^\infty(-1,1)}+\|u'\|_{L^\infty(-1,1)})+N^{1/2}\|u^{(m+2)}\|_{L^2(\Lambda_h)}\Big)$$
$$+CN^{-1/2}(\|e_0\|_{L^\infty(-1,1)}+\|e_1\|_{L^\infty(-1,1)}), \qquad (4.22)$$

where

$$R := \max\Big\{\widetilde{Q_1},\widetilde{Q_2},\widetilde{S_1},\widetilde{S_2}\Big\}.$$

Now we need two $\|e_1\|_{L^\infty(-1,1)}$ and $\|e_0\|_{L^\infty(-1,1)}$, $\|e_2\|_{L^\infty(-1,1)}$ and $\|e_1\|_{L^\infty(-1,1)}$ relation.

Multiply both ends of the first term in (4.2) by $F_i^\mu(x)$ and add from $i=0$ to N to get

$$e_0(x) = \widetilde{E}(x) + (I_N - I)\Big(\int_{-1}^x e_1(s)ds\Big) + \int_{-1}^x e_1(s)ds, \qquad (4.23)$$

where

$$\widetilde{E}(x) := (I - I_N)u(x).$$

We have:

$$\|e_0\|_{L^\infty(-1,1)} \le C\Big(\|\widetilde{E}(x)\|_{L^\infty(-1,1)} + \|(I_N-I)\Big(\int_{-1}^x e_1(s)ds\Big)\|_{L^\infty(-1,1)} + \|e_1\|_{L^\infty(-1,1)}\Big).$$

Applying (3.1b) to $\widetilde{E}(x)$, and applying (3.1b) when $m=1$ to the right middle term of the above inequality, we get

$$\|e_0\|_{L^\infty(-1,1)} \le CN^{-m-\frac{3}{2}}\|u^{(m+2)}\|_{L^2(\Lambda_h)} + C\|e_1\|_{L^\infty(-1,1)}. \qquad (4.24)$$

Similarly, for the second term in (4.2) we can get

$$e_1(x) = \widehat{E}(x) + (I_N - I)\Big(\int_{-1}^x e_2(s)ds\Big) + \int_{-1}^x e_2(s)ds, \qquad (4.25)$$

where

$$\widehat{E}(x) := (I - I_N)u'(x).$$

This will get

$$\|e_1\|_{L^\infty(-1,1)} \le CN^{-m-\frac{1}{2}}\|u^{(m+2)}\|_{L^2(\Lambda_h)} + C\|e_2\|_{L^\infty(-1,1)}. \qquad (4.26)$$

We can then derive (4.1) by combining the results (4.22), (4.24) and (4.26). □

Theorem 4.2. *Let $u(x)$ be the solution of Eq. (2.2). Use the numerical format (2.13), (2.16), (2.20), (2.18), and (2.23) The approximation of $u(x)$ is $\varsigma(x)$, the approximation of its derivative is $\rho(x)$, and the approximation of its second derivative is $\varrho(x)$. Then for a sufficiently large $N \geq m-1$ there is*

$$\|e_i\|_{L^2(-1,1)} \leq CN^{-m}(R+1)^2 \Big(\|u\|_{L^\infty(-1,1)} + \|u'\|_{L^\infty(-1,1)}$$
$$+ \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1 \Big), \quad i=0,1,2. \tag{4.27}$$

Proof. Using Lemma 3.1, we get from (4.6) and (4.8)

$$\|e_2\|_{L^2(-1,1)} \leq C \Big(\sum_{j=0}^{3} \|I_N E_j\|_{L^2(-1,1)} + \sum_{j=4}^{12} \|E_j\|_{L^2(-1,1)} \Big). \tag{4.28}$$

Applying $E_0(x)$ from (3.1d) to (4.28), we get

$$\|I_N E_0\|_{L^2(-1,1)} \leq C\|E_0\|_{L^\infty(-1,1)}. \tag{4.29}$$

By (4.13) and Theorem 4.1, we have

$$\|I_N E_0\|_{L^2(-1,1)} \leq CN^{-m}R(R+1) \Big(\|u\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1 \Big), \tag{4.30}$$

Similarly, we have

$$\|I_N E_1\|_{L^2(-1,1)} \leq CN^{-m}R(R+1) \Big(\|u\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1 \Big), \tag{4.31a}$$

$$\|I_N E_2\|_{L^2(-1,1)} \leq CN^{-m}R(R+1) \Big(\|u\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1 \Big), \tag{4.31b}$$

$$\|I_N E_3\|_{L^2(-1,1)} \leq CN^{-m}R(R+1) \Big(\|u\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1 \Big). \tag{4.31c}$$

Notice $u''|_{\delta_\mu}(x) \in H^m(\delta_\mu)$, $\mu = 0,1,\cdots,M$. Applying (3.1a) to $u''(x)$, we have

$$\|E_4\|_{L^2(-1,1)} \leq CN^{-m}\|u^{(m+2)}\|_{L^2(\Lambda_h)}. \tag{4.32}$$

Similar to the analysis process of (4.18)-(4.20), applying (3.1a) at $m=1$ to $\gamma(x)$

$$\|E_{11}\|_{L^2(-1,1)} = \|(I-I_N)\gamma(x)\|_{L^2(-1,1)} \leq CN^{-1}\|e_0\|_{L^\infty(-1,1)}. \tag{4.33}$$

Using the estimate of e_0 in Theorem 4.1, we can convert the above inequality to

$$\|E_{11}\|_{L^2(-1,1)} \leq CN^{-m-1/2}R \Big(\|u\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1 \Big). \tag{4.34}$$

Similarly, we have

$$\|E_5\|_{L^2(-1,1)} \leq CN^{-m-1/2}R\left(\|u\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1\right), \quad (4.35a)$$

$$\|E_7\|_{L^2(-1,1)} \leq CN^{-m-1/2}R\left(\|u\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1\right), \quad (4.35b)$$

$$\|E_9\|_{L^2(-1,1)} \leq CN^{-m-1/2}R\left(\|u\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1\right), \quad (4.35c)$$

$$\|E_{10}\|_{L^2(-1,1)} \leq CN^{-m-1/2}R\left(\|u\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1\right), \quad (4.35d)$$

$$\|E_{12}\|_{L^2(-1,1)} \leq CN^{-m-1/2}R\left(\|u\|_{L^\infty(-1,1)} + \|u\|_{L^\infty(-1,1)} + \|u^{(m+2)}\|_{L^2(\Lambda_h)} + 1\right). \quad (4.35e)$$

Similar to the analysis process of (4.18)-(4.20), applying (3.1a) when $m=1$ to get

$$\|E_i\|_{L^2(-1,1)} \leq CN^{-1}\|e_2\|_{L^\infty(-1,1)}, \quad i=6,8. \quad (4.36)$$

Combining (4.28), (4.30), (4.31), (4.32), (4.34), (4.35) and (4.36) gives the estimate of e_2 in (4.27).

Now to estimate e_1. Get from (4.25)

$$\|e_1\|_{L^2(-1,1)} \leq C\left(\|\widehat{E}(x)\|_{L^2(-1,1)} + \left\|(I_N - I)\int_{-1}^x e_2(s)ds\right\|_{L^2(-1,1)} + \|e_2\|_{L^2(-1,1)}\right).$$

Applying (3.1a) to $\widehat{E}(x)$, and applying (3.1a) at $m=1$ to the middle term at the right end of the above inequality

$$\|e_1\|_{L^2(-1,1)} \leq CN^{-m-1}\|u^{(m+2)}\|_{L^2(\Lambda_h)} + C\|e_2\|_{L^2(-1,1)}. \quad (4.37)$$

Substituting the estimate of e_2 in (4.27) into (4.37) gives the estimate of e_1 in (4.27).

Similarly, we can get for (4.23)

$$\|e_0\|_{L^2(-1,1)} \leq CN^{-m-2}\|u^{(m+2)}\|_{L^2(\Lambda_h)} + C\|e_1\|_{L^2(-1,1)}. \quad (4.38)$$

Substituting the estimate of e_1 in (4.27) into (4.38) gives the estimate of e_0 in (4.27). $\qquad\square$

5 Numerical results

Example 5.1. Take in Eq. (1.1)

$$\begin{aligned}
&T=4, \quad a_1(t)=b_1(t)=t, \quad a_2(t)=b_2(t)=t^2, \\
&K_1(t,s)=R_1(t,s)=\sin(t+s), \quad K_2(t,s)=R_2(t,s)=\cos(t+s), \\
&\phi(t)=e^t, \quad \varphi(t)=e^t, \quad \theta(t)=t-1, \\
&g(t)=e^t-te^t-t^2e^t-te^{t-1}-t^2e^{t-1}+2\sin(t)+e^t\sin(2t)+e^{t-1}\cos(2t-1).
\end{aligned}$$

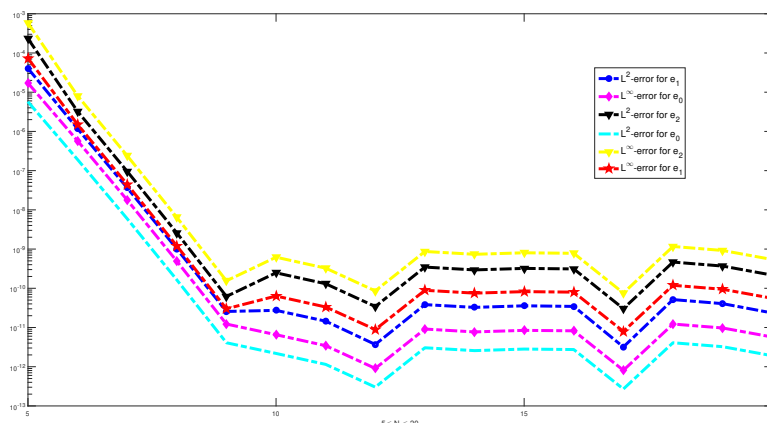Figure 1: Correspondence between error and N for Example 5.1.

Table 1: Correspondence between error and N for Example 5.1.

N	5	8	11	14	20
$\|e_0\|_{L^\infty}$	$4.12e-05$	$1.03e-09$	$1.56e-11$	$3.15e-11$	$2.35e-11$
$\|e_0\|_{L^2}$	$1.73e-05$	$4.87e-10$	$3.46e-12$	$7.42e-12$	$5.74e-12$
$\|e_1\|_{L^\infty}$	$2.36e-04$	$2.54e-09$	$3.39e-10$	$7.35e-10$	$5.79e-10$
$\|e_1\|_{L^2}$	$6.61e-05$	$1.12e-09$	$3.05e-11$	$6.58e-11$	$5.13e-11$
$\|e_2\|_{L^\infty}$	$3.75e-04$	$9.84e-09$	$1.14e-09$	$1.25e-09$	$2.19e-09$
$\|e_2\|_{L^2}$	$6.12e-04$	$5.64e-09$	$8.45e-10$	$9.96e-10$	$8.43e-10$

The exact solution of the equation is $y(t) = e^t, t \in (0, T]$.

Fig. 1 shows the convergence of the error under the L^∞ and L^2 norms at $4 \le N \le 20$. The specific error data is shown in Table 1. Error exponential convergence, which is the same as the convergence proof result.

Example 5.2. Take in Eq. (1.1)

$$T = 3, \quad a_1(t) = a_2(t) = 0, \quad b_1(t) = b_2(t) = g(t) = e^t,$$
$$K_1(t,s) = K_2(t,s) = R_1(t,s) = 0, \quad R_2(t,s) = e^{(t+s)},$$
$$\phi(t) = t, \quad \varphi(t) = 1, \quad \theta(t) = \frac{t-1}{2}.$$

The exact solution of the equation is

$$y(t) = \begin{cases} te^t + \dfrac{2}{27}(3t-7)e^{(3t-1)/2} + \dfrac{5}{9}te^{-1/2} + \dfrac{14}{27}e^{-1/2}, & t \in (0,1], \\ y_1(t), & t \in (1,3], \end{cases}$$
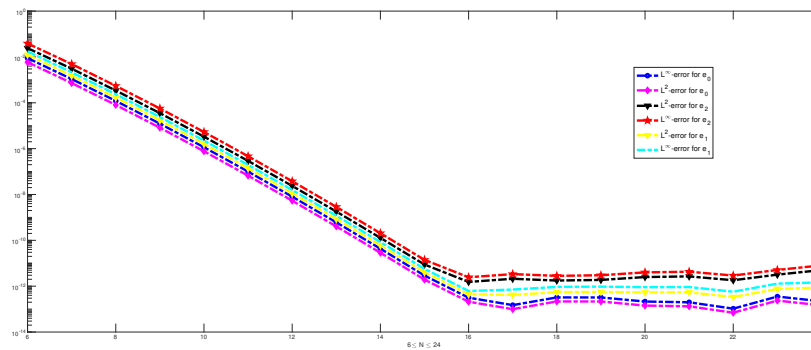
Figure 2: Correspondence between error and N for Example 5.2.

Table 2: Correspondence between error and N for Example 5.2.

N	6	10	14	20	24
$\|e_0\|_{L^\infty}$	$3.87e-02$	$5.32e-06$	$2.04e-10$	$4.00e-12$	$8.09e-12$
$\|e_0\|_{L^2}$	$2.42e-02$	$3.32e-06$	$1.28e-10$	$2.50e-12$	$5.05e-12$
$\|e_1\|_{L^\infty}$	$1.73e-02$	$2.25e-06$	$8.45e-11$	$9.07e-13$	$1.47e-12$
$\|e_1\|_{L^2}$	$1.24e-02$	$1.61e-06$	$6.03e-11$	$5.33e-13$	$8.66e-13$
$\|e_2\|_{L^\infty}$	$8.87e-03$	$1.15e-06$	$4.31e-11$	$2.13e-13$	$2.18e-13$
$\|e_2\|_{L^2}$	$5.91e-03$	$7.67e-07$	$2.87e-11$	$1.42e-13$	$1.45e-13$

where

$$
y_1(t) := \frac{1}{4}(2t-1)e^{2t} + \frac{1}{16}(t-1)e^{2t-1} + \frac{3}{4}e^t + \frac{4}{675}(15t-41)e^{(5t-1)/2}
$$
$$
+ \frac{16}{54675}(45t-271)e^{(9t-7)/4} + \frac{20}{81}e^{(3t-2)/2} + \frac{1}{675}(375t-214)e^{(2t-1)/2}
$$
$$
\left(-\frac{19}{27}e^2 + \frac{1}{36}e - \frac{250}{243}e^{1/2} + \frac{5}{9}e^{-1/2} \right)t + \frac{547}{900}e^2 - \frac{2}{27}e + \frac{764}{2025}e^{1/2} + \frac{31966}{54675}e^{-1/2}.
$$

Fig. 2 depicts the error convergence in the sense of L^∞ and L^2 norms at $6 \le N \le 24$. The specific data errors are shown in Table 2. The error converges exponentially, which is the same as the result of convergence proof.

6 Conclusions

This paper presents a numerical method of piecewise spectral-collocation for second-order Volterra integro-differential equations with nonvanishing delay based on Legendre polynomials. The convergence of the method is proved and the numerical experiments also verify the correctness of the proof results.

The focus of this work is to construct the derivative approximation in the sense of integration, which makes it possible to use the Dirichlet formula to prove the convergence of the method in the theoretical proof process. In addition, we make the solution at the main discontinuity point of the Volterra integral differential equation solution. The division of intervals in this method greatly overcomes the disturbance of the main discontinuity of the solution of the equation to the convergence of the numerical error.

Acknowledgements

References

[1] W. ZHANG AND M. FAN, *Periodicity in a generalized ecological competition system governed by impulsive differentialequations with delays*, Math. Comput. Model., 39 (2004), pp. 479–493.

[2] K. AL-KHALED, *Numerical approximations for population growth models*, Appl. Math. Comput., 160 (2005), pp. 865–873.

[3] G. A. BOCHAROV AND F. A. RIHAN, *Numerical modelling in biosciences using delay differential equations*, J. Comput. Appl. Math., 125 (2000), pp. 183–199.

[4] D. JIANG, J. WEI AND B. ZHANG, *Positive periodic solutions of functional differential equations and population models*, Electron. J. Differ. Eq., 2002 (2002), pp. 1–13.

[5] F. BRAUER AND P. DRIESSCHE, *Models for transmission of disease with immigration of infectives*, Math. Biosci., 71 (2001), pp. 143–154.

[6] J. M. CUSHING, Integrodifferential Equations and Delay Models in Population Dynamics, Springer Science & Business Media, 2013.

[7] C. ZHANG AND S. VANDEWALLE, *General linear methods for Volterra integro-differential equations with memory*, SIAM J. Sci. Comput., 27 (2006), pp. 2010–2031.

[8] C. T. H. BAKER AND N. J. FORD, *Asymptotic error expansions for linear multistep methods for a class of delay integro-differential equations*, Adv. Comput. Math. Appl., (1993), pp. 59–72.

[9] A. MAKROGLOU, *A bloc-by-block method for the numerical solution of Volterra delay integrodifferential equation*, Computing, 30 (1983), pp. 49–62.

[10] C. T. H. BAKER AND C. A. H. PAUL, *Parallel continuous Runge-Kutta methods and vanishing lag delay differential equations*, Adv. Comput. Math., 1 (1993), pp. 367–394.

[11] C. CANUTO, M. HUSSAINI, A. QUARTERONI AND T. ZANG, Spectral Methods: Fundamentals in Single Domains, Springer Science & Business Media, 2007.

[12] S. LI, *High order contractive Runge–Kutta methods for Volterra functional differential equations*, SIAM J. Numer. Anal., 47 (2010), pp. 4290–4325.

[13] H. BRUNNER, Collocation Methods for Volterra Integral and Related Functional Differential Equations, Cambridge University Press, 2004.

[14] H. BRUNNER, *Recent advances in the numerical analysis of Volterra functional differential equations with variable delays*, J. Comput. Appl. Math., 228 (2009), pp. 524–537.

[15] B. RILEY, *The numerical solution of Volterra integral equations with nonsmooth solutions based on sinc approximation*, Appl. Numer. Math., 9 (1992), pp. 249–257.

[16] M. SHAKOURIFAR AND M. DEHGHAN, *On the numerical solution of nonlinear systems of Volterra integro-differential equations with delay arguments*, Computing, 82 (2008), pp. 241–260.

[17] I. ALI, *Convergence analysis of spectral methods for integro-differential equations with vanishing proportional delays*, J. Comput. Math., 28 (2010), pp. 962–973.

[18] I. ALI, H. BRUNNER AND T. TANG, *A spectral method for pantograph-type delay differential equations and its convergence analysis*, J. Comput. Math., 27 (2009), pp. 254–265.

[19] I. ALI, H. BRUNNER AND T. TANG, *Spectral methods for pantograph-type differential and integral equations with multiple delays*, Front. Math. China, 4 (2009), pp. 49–61.

[20] K. KANT, M. MANDAL AND G. NELAKANTI, *Jacobi spectral Galerkin methods for a class of nonlinear weakly singular Volterra integral equations*, Adv. Appl. Math. Mech., 13 (2021), pp. 1227–1260.

[21] Y. JIANG, *On spectral methods for Volterra-type integro-differential equations*, J. Comput. Appl. Math., 230 (2009), pp. 333–340.

[22] T. TANG AND X. XU, *Accuracy enhancement using spectral postprocessing for differential equations and integral equations*, Commun. Comput. Phys., 5 (2009), pp. 779–792.

[23] T. TANG, X. XU AND J. CHENG, *On spectral methods for Volterra integral equations and the convergence analysis*, J. Comput. Math., 26 (2008), pp. 825–837.

[24] Y. WEI AND Y. CHEN, *Convergence analysis of the Legendre spectral collocation methods for second order Volterra integro-differential equations*, Numer. Math. Theor. Methods Appl., 4 (2011), pp. 419–438.

[25] Y. WEI AND Y. CHEN, *Convergence analysis of the spectral methods for weakly singular Volterra integro-differential equations with smooth solutions*, Adv. Appl. Math. Mech., 4 (2012), pp. 1–20.

[26] Y. WEI AND Y. CHEN, *Legendre spectral collocation methods for pantograph Volterra delay-integro-differential equations*, J. Sci. Comput., 53 (2012), pp. 672–688.

[27] E. RAWASHDEH, D. MCDOWELL AND L. RAKESH, *Polynomial spline collocation methods for secondorder Volterra integro-differential equations*, Int. J. Math. Math. Sci., 2004 (2004), pp. 3011–3022.

[28] M. TARANG, *Stability of the spline collocation method for second order Volterra integrodifferential equations*, Math. Model. Anal., 9 (2004), pp. 79–90.

[29] M. BOLOGNA, *Asymptotic solution for first and second order linear Volterra integro-differential equations with convolution kernels*, J. Phys. A-Math. Theor., 43 (2010), 375203.

[30] M. ZAREBNIA AND Z. NIKPOUR, *Solution of linear Volterra integro-differential equations via Sinc functions*, Int. J. Appl. Math. Comput., 2 (2010), pp. 1–10.

[31] B. GUO AND J. SHEN, *Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval*, Numer. Math., 86 (2000), pp. 635–654.

[32] C. XU AND B. GUO, *Laguerre pseudospectral method for nonlinear partial differential equations*, J. Comput. Math., 20 (2002), pp. 413–428.

[33] G. MASTROIANNI AND G. MONEGATO, *Nystrom interpolants based on zeros of Laguerre polynomials for some Weiner-Hopf equations*, IMA J. Numer. Anal., 17 (1997), pp. 621–642.

[34] J. SHEN, *Stable and efficient spectral methods in unbounded domains using Laguerre functions*, SIAM J. Numer. Anal., 38 (2000), pp. 1113–1133.

[35] Z. GU AND Y. CHEN, *Legendre spectral-collocation method for Volterra integral equations with non-vanishing delay*, Calcolo, 51 (2014), pp. 151–174.

[36] Y. CHEN AND Z. GU, *Legendre spectral-collocation method for Volterra integral differential equations with nonvanishing delay*, Commun. Appl. Math. Comput. Sci., 8 (2013), pp. 67–98.

[37] X. SHI, Y. CHEN, Y. HUANG AND F. HUANG, *Spectral collocation methods for second-order Volterra integro-differential equations with weakly singular kernels*, Adv. Appl. Math. Mech., 12 (2020), pp. 480–502.

[38] M. VILLASANA AND A. RADUNSKAYA, *A delay differential equation model for tumor growth*, J. Math. Biol., 47 (2003), pp. 270–294.

[39] Y. KUANG, Delay Differential Equations with Applications in Population Dynamics, Academic Press, 1993.

[40] S. A. GOURLEY, Y. KUANG AND J. D. NAGY, *Dynamics of a delay differential equation model of hepatitis B virus infection*, J. Biol. Dyn., 2 (2008), pp. 140–153.

[41] J. CHATTOPADHYAY, R. R. SARKAR AND A. ABDLLAOUI, *A delay differential equation model on harmful algal blooms in the presence of toxic substances*, Math. Med. Biol., 19 (2002), pp. 137–161.

[42] L. WANG, L. YI AND H. JIA, *An h-p version of the Chebyshev spectral collocation method for Volterra integro-differential equations with vanishing delays*, J. Integral Equations Appl., 32 (2020), pp. 101–128.

[43] S. LI, L. WANG AND L. YI, *An hp-version of the C^0-continuous Petrov-Galerkin method for second-order Volterra integro-differential equations*, Appl. Numer. Math., 152 (2020), pp. 84–104.

[44] L. WANG AND L. YI, *An h-p version of the continuous Petrov-Galerkin method for Volterra delay-integro-differential equations*, Adv. Comput. Math., 43 (2017), pp. 1437–1467.