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ABSTRACT

A Pierce-type dispersion relation is derived from field theory which describes the interaction of an intense relativistic electron beam with a cylindrical slow-wave structure of arbitrary corrugation depth. It is shown that near a resonance the Pierce parameter can be expressed in terms of the vacuum dispersion function and the space-charge parameter is proportional to a fill factor. The dispersion relation is valid in both the low-current (Compton) regime and the high-current (Raman) regime. The dispersion characteristics of the interaction, such as the linear instability growth rate and bandwidth, are analyzed in both regimes.

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I. INTRODUCTION

There has been much growth in theoretical and experimental research on the relativistic traveling-wave-tube (TWT) amplifier and the relativistic backward-wave oscillator (BWO) since the work by Nation [1] in 1970. The operation principle of this class of high-power microwave devices is based upon the stimulated interaction of a relativistic electron beam with a slow-wave structure. Extensive studies of these devices have been motivated, to a large extent, by their potential applications in the development of high-gradient, radio-frequency (rf) accelerators and high-power radar. Although the nonlinear regime of the interaction is the primary focus of recent investigations, the linear regime of the interaction is still being explored, particularly in situations where effects associated with time-dependent space charge play an important role in determining the dispersion characteristics of the interaction.

In this paper, we derive from field theory a Pierce-type [2] dispersion relation describing relativistic TWT and backward-wave-tube (BWT) interactions in both the low-current (Compton) regime and the high-current (Raman) regime. In particular, making an expansion to leading order in the coupling constant, we show that near a resonance the Pierce parameter can be expressed in terms of the vacuum dispersion function and the space-charge parameter is proportional to a fill factor. Hence, the present dispersion relation is readily used to determine analytically the linear instability growth rate and bandwidth, provided that the vacuum dispersion characteristics are known from either cold-test measurements or analytical/numerical calculations. The present analysis is carried out in a configuration consisting of a periodically corrugated cylindrical waveguide and thin annular electron beam.

In contrast to the analysis by Kurilkov, *et al.* [3] and Belov, *et al.* [4] which has resulted in an approximate dispersion relation for *small* corrugation depth, the present dispersion relation is applicable for arbitrary corrugation depth. As a generalization of the analysis by Swegle [5] which has led to a Pierce-type dispersion relation in the Compton regime, the present dispersion relation describes in both the Compton regime and the Raman regime. Furthermore, the present analysis provides a clear physical interpretation of the coupling of an relativistic electron beam with a slow-wave structure from the point of view of field theory.

II. BASIC FORMULATION

In this section we review the basic formulation of the problem given by Swegle, Poukey and Leifeste [6]. For present purposes, we consider a thin annular relativistic electron beam propagating at axial velocity $V\vec{e}_z$ through an infinitely long cylindrical waveguide whose radius is given by the periodic function

$$b(z) = b(z+d) \ , \tag{1}$$

where d is the fundamental period. For future references, we express the function $b(z)$ as

$$b(z) = b_0 + b_1(z) \ , \tag{2}$$

where $b_0 = \text{const}$ is the average radius of the waveguide and $\int_0^d b_1(z)dz = 0$. The beam is confined radially by a strong axial magnetic field $B_0\vec{e}_z$. The assumptions in the present analysis are: (1) the beam is infinitely thin and is described by the equilibrium charge density

$$\rho_0(r) = -en_0(r) = -\frac{I_b}{2\pi V r}\delta(r-a) \ , \tag{3}$$

with $-e$ the electron charge and $I_b = 2\pi eV \int_0^b n(r)rdr$ the beam current, (2) the axial magnetic field is infinite ($B_0 \to \infty$), (3) the waveguide is a perfect conductor, and (4) the perturbations are azimuthally symmetric transverse-magnetic (TM) modes.

Under the above assumptions, a normal-mode analysis is readily carried out by expressing all perturbing field components in terms of a Floquet series as

$$\delta\psi(r,z,t) = \sum_{n=-\infty}^{\infty} \delta\psi_n(r)e^{i(k_n z-\omega t)} \ , \tag{4}$$

where $k_n = k + 2\pi n/d$ and $0 \leq k < 2\pi/d$. Details of such an analysis can be found in [6], and the main result is given by the following dispersion relation

$$\det \mathbf{D}(\omega, k) = 0. \tag{5}$$

The elements of the matrix $\mathbf{D}(\omega, k)$ in Eq. (5) are defined by

$$D_{mn}(\omega, k) = D^{(0)}_{mn}(\omega, k)\Big\{1 - \alpha I_0^2(p_n a)\Big[\frac{K_0(p_n a)}{I_0(p_n a)} - \frac{K_{0mn}}{I_{0mn}}\Big]\frac{c^2 p_n^2}{(\omega - k_n V)^2}\Big\} , \tag{6}$$

where $n, m = 0, \pm 1, \pm 2, \cdots$. The vacuum dispersion function is defined by

$$D^{(0)}_{mn}(\omega, k) = \frac{I_{0mn}}{p_n^2}\left(k_m k_n - \frac{\omega^2}{c^2}\right) , \tag{7}$$

with $p_n^2 = k_n^2 - \omega^2/c^2$ and c the speed of light in vacuum. The coupling constant is defined by

$$\alpha = \frac{2}{\gamma^3 \beta}\left(\frac{I_b}{I_A}\right) \tag{8}$$

with $\beta = V/c$, $\gamma = (1-\beta^2)^{-1/2}$, and $I_A \approx 17$ kA the Alfvén current. The Fourier integrals of the first- and second-kind modified Bessel functions of order zero, $I_0(x)$ and $K_0(x)$, are defined by

$$I_{0mn} = \frac{1}{d}\int_0^d dz I_0[p_n b(z)] e^{i2\pi(n-m)z/d} \tag{9}$$

and

$$K_{0mn} = \frac{1}{d}\int_0^d dz K_0[p_n b(z)] e^{i2\pi(n-m)z/d} , \tag{10}$$

respectively.

The earlier analysis [6] of the dispersion relation (5) relies on a proper truncation of the infinite matrix $\mathbf{D}(\omega, k)$ to some finite, numerically manageable size. While the analysis yields accurate dispersion characteristics such as the linear instability growth rate and bandwidth and the linear frequency and wave number shifts, it is difficult to grasp the basic physics of the interaction in such a numerical analysis. The purpose of the remainder of the paper is to show that equation (5) can be approximated by a

Pierce-type [2] dispersion relation, thereby allowing us to gain a better understanding of the interaction process, and to characterize different interaction regimes, namely, the low-current (Compton) regime and the high-current (Raman) regime.

It is instructive to examine several limiting cases before deriving expressions for the Pierce and space-charge parameters. First, for a constant waveguide radius [$b(z) = b_0 =$ const], the integrals in Eqs. (9) and (10) are given by $I_{0mn} = I_0(pb_0)\delta_{mn}$ and $K_{0mn} = K_0(pb_0)\delta_{mn}$, where δ_{mn} is the Kronecker delta. By setting $m = n = 0$, it is readily shown that the dispersion relation (5) reduces to [7]

$$I_0(pb_0) = 0 \quad \text{or} \quad \omega^2 = c^2k^2 + \omega_l^2 \tag{11}$$

for $\alpha = 0$, and that it reduces to [8],[9]

$$D_0^{(sc)}(\omega, k) = 1 - \alpha I_0^2(p_0 a)\left[\frac{K_0(p_0 a)}{I_0(p_0 a)} - \frac{K_0(p_0 b_0)}{I_0(p_0 b_0)}\right]\frac{c^2 p_0^2}{(\omega - k_0 V)^2} = 0 \tag{12}$$

for $\alpha \neq 0$. Note that the dispersion relation (11) describes the vacuum TM_{0l} mode in a constant-radius waveguide with $\omega_l = c\nu_l/b_0$ being the cutoff frequency of the mode and ν_l being the l-th zero of $J_0(x)$, and that the dispersion relation (12) describes fast and slow space-charge waves on the electron beam propagating through a constant-radius waveguide.

Second, the dispersion relation describing electromagnetic (structural) waves propagating through the vacuum corrugated waveguide can be obtained from Eq. (5) by setting $\alpha = 0$ but $b_1 \neq 0$. This yields [6]

$$\det \mathbf{D}^{(0)}(\omega, k) = 0 \ , \tag{13}$$

where the elements of the matrix $\mathbf{D}^{(0)}(\omega, k)$ are defined in Eq. (7). The dispersion characteristics of structural waves can be determined numerically with a proper truncation of $\mathbf{D}^{(0)}(\omega, k)$. They can also be determined experimentally from cold-test measurements.

III. PIERCE AND SPACE-CHARGE PARAMETERS

We now make an expansion of Eq. (5) to leading order in the coupling constant α, and derive an approximate Pierce-type dispersion relation describing the coupling of the relativistic electron beam with the slow-wave structure. For present purposes, we rewrite Eq. (6) as

$$D_{mn}(\omega, k) = D_{mn}^{(0)}(\omega, k)\Big[D_n^{(sc)}(\omega, k) - \frac{\alpha F_{mn}(\omega, k)}{(\omega - k_n V)^2}\Big] , \tag{14}$$

where the dielectric function

$$D_n^{(sc)}(\omega, k) = 1 - \alpha I_0^2(p_n a)\Big[\frac{K_0(p_n a)}{I_0(p_n a)} - \frac{K_0(p_n b_0)}{I_0(p_n b_0)}\Big]\frac{c^2 p_n^2}{(\omega - k_n V)^2} \tag{15}$$

is a generalization of the dielectric function $D_0^{(sc)}(\omega, k)$ defined in Eq. (12) and describes fast and slow space-charge waves of the n-th spatial harmonic on the electron beam propagating through the waveguide of the average radius b_0. The function

$$F_{mn}(\omega, k) = c^2 p_n^2 I_0^2(p_n a)\Big[\frac{K_0(p_n b_0)}{I_0(p_n b_0)} - \frac{K_{0mn}}{I_{0mn}}\Big] \tag{16}$$

is related to Pierce's parameter defined later in Eq. (27).

As seen below, the separation of D_{mn} into two terms in Eq. (14) is a useful trick because it allows us to interpret, in a natural way, the coupling of an intense relativistic electron beam with a slow-wave structure as that of the space-charge waves on the beam [described in Eq. (15)] with the structure waves [described in Eq. (13)]. The coupling strength is proportional to αF_{mn}, which vanishes as the corrugation b_1 approaches zero.

Let **A** and **B** be matrices with the elements

$$A_{mn} = D_{mn}^{(0)} D_n^{(sc)} \tag{17}$$

and

$$B_{mn} = \frac{D_{mn}^{(0)} F_{mn}}{(\omega - k_n V)^2} , \tag{18}$$

respectively. Assuming $\det \mathbf{A} \neq 0$, we can express the matrix $\mathbf{D}$ as

$$\mathbf{D} = \mathbf{A}(\mathbf{I} - \alpha \mathbf{A}^{-1}\mathbf{B}) , \tag{19}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Since $\det \mathbf{A} \neq 0$ and $\det \mathbf{D} = 0$ is equivalent to $\det(\mathbf{I} - \alpha \mathbf{A}^{-1}\mathbf{B}) = 0$, we can approximate Eq. (5) to leading order in the coupling constant α by

$$1 = \alpha \mathrm{Tr}(\mathbf{A}^{-1}\mathbf{B}) \,, \tag{20}$$

where $\mathrm{Tr}(\cdots)$ denotes the trace. As shown later in Sec. IV, due to the resonance denominator in B_{mn} defined in Eq. (18), the expansion in Eq. (20) is correct to order α^η, where $\alpha^\eta \ll 1$ and $\frac{1}{3} < \eta < \frac{1}{2}$.

It is readily shown from Eq. (17) that the elements of $\mathbf{A}^{-1}$ are given by

$$(A^{-1})_{mn} = \frac{1}{\det \mathbf{D}^{(0)}} \frac{(\mathrm{adj}\ D^{(0)})_{mn}}{D_m^{(sc)}} \,, \tag{21}$$

where the matrix $(\mathrm{adj}\ \mathbf{D}^{(0)})$ is the adjoint of the matrix $\mathbf{D}^{(0)}$. Substituting Eq. (21) into Eq. (20) yields

$$\det \mathbf{D}^{(0)} = \alpha \sum_{n=-\infty}^{\infty} \frac{1}{D_n^{(sc)}} \sum_{m=-\infty}^{\infty} (\mathrm{adj}\ D^{(0)})_{nm} D_{mn}^{(0)} \frac{F_{mn}}{(\omega - k_n V)^2} \,, \tag{22}$$

which involves an infinite number of resonance denominators associated with the generalized dielectric functions $D_n^{(sc)}$. In the vicinity of the n-th spatial harmonic of the space-charge wave mode, because $D_n^{(sc)} \approx 0$, equation (22) can be further approximated by

$$\det \mathbf{D}^{(0)}(\omega, k) \times D_n^{(sc)}(\omega, k) = \alpha \sum_{m=-\infty}^{\infty} (\mathrm{adj}\ D^{(0)})_{nm} D_{mn}^{(0)} \frac{F_{mn}}{(\omega - k_n V)^2} \,, \tag{23}$$

which shows explicitly the coupling of the structural waves with the space-charge waves.

To obtain expressions for the Pierce and space-charge parameters in Pierce's TWT theory, we let (ω_c, k_c) denote the intersection of a structural wave, $\det D^{(0)}(\omega, k) = 0$, and the n-th spatial harmonic of the space-charge wave, $D_n^{(sc)}(\omega, k) = 0$, in the dispersion diagram. Expanding $\det D^{(0)}(\omega, k)$ about $\det D^{(0)}(\omega_c, k_c) = 0$, it is readily shown that the dispersion relation (23) can be expressed in the same form as Pierce's TWT dispersion relation, i.e.,

$$[\omega - \omega_c - v_g(k - k_c)][(\omega - k_n V)^2 - (QC)_n] = C_n^3 \,, \tag{24}$$

where $k_n = k + 2\pi n/d$ and

$$v_g = -\left(\frac{\partial \det D^{(0)}}{\partial k} \Big/ \frac{\partial \det D^{(0)}}{\partial \omega}\right)_{\omega=\omega_c, k=k_c} \tag{25}$$

is the group velocity of the structural wave. In Eq. (24), the space-charge parameter is defined by

$$(QC)_n = \alpha c^2 p_n^2 I_0^2(p_n a)\left[\frac{K_0(p_n a)}{I_0(p_n a)} - \frac{K_0(p_n b_0)}{I_0(p_n b_0)}\right] , \tag{26}$$

and the Pierce parameter C_n is defined by

$$C_n^3 = \alpha\left(\frac{\partial \det D^{(0)}}{\partial \omega}\right)^{-1}_{\omega=\omega_c, k=k_c} \sum_{m=-\infty}^{\infty} (\text{adj } D^{(0)})_{nm} D_{mn}^{(0)} F_{mn} . \tag{27}$$

Similar expressions for $(QC)_n$ and C_n^3 can be obtained for the case of a solid beam [10].

It should be stressed that the group velocity v_g, the space-charge parameter $(QC)_n$, and the Pierce parameter C_n can be evaluated using Eqs. (25)-(27) and the dispersion diagram of the structural waves which can be obtained from cold-test measurements or numerical calculations. Therefore, the dispersion relation (24) is readily used to determine the stability properties of the relativistic TWT and BWT interactions.

IV. DISPERSION CHARACTERISTICS IN THE COMPTON AND RAMAN REGIMES

In the Compton regime, the condition $|(QC)_n| \ll |\omega - k_n V|^2$ holds. The dispersion relation (24) can be approximated by

$$\delta\omega^2(\delta\omega - \Delta\Omega_n) = C_n^3 , \tag{28}$$

where $\delta\omega = \omega - k_n V$ and $\Delta\Omega_n = \omega_c + v_g(k - k_c) - k_n V$. The Compton-regime dispersion relation results in instability whenever

$$\frac{3}{4}(\Delta\Omega_n)^4 > \left[C_n^3 + \frac{2}{27}(\Delta\Omega_n)^3\right]^3 . \tag{29}$$

The maximum temporal growth rate,

$$|\mathrm{Im}\delta\omega|_{max} = \frac{\sqrt{3}}{2}|C_n| \propto \alpha^{1/3} , \tag{30}$$

occurs at $\Delta\Omega_n = 0$. Substituting Eqs. (28) and (30) into Eq. (20), we find that the expansion in Eq. (20) is correct to order $\alpha^{1/3}$ in the Compton regime. Indeed, for the Compton-regime approximation to be valid, the inequality $|(QC)_n| \ll |\omega - k_n V|^2$ must be satisfied at maximum growth. This criterion can be expressed as

$$\left(\frac{I_b}{I_A}\right)^{1/3} \ll \left(\frac{\gamma^3\beta}{2}\right)^{1/3} \frac{\left|\left(\frac{\partial \det D^{(0)}}{\partial\omega}\right)^{-1}_{\omega=\omega_c,k=k_c} \sum_{m=-\infty}^{\infty}(\mathrm{adj}\ D^{(0)})_{nm} D^{(0)}_{mn} F_{mn}\right|^{2/3}}{\left|c^2 p_n^2 I_0^2(p_n a)\left[\frac{K_0(p_n a)}{I_0(p_n a)} - \frac{K_0(p_n b_0)}{I_0(p_n b_0)}\right]\right|} , \tag{31}$$

where I_b is the beam current and $I_A \approx 17$ kA is the Alfvén current.

In the Raman regime, on the other hand, the condition $|(QC)_n|^{1/2} \gg |\omega - k_n V + (QC)_n^{1/2}|$ holds. The dispersion relation (24) can be approximated by

$$\delta\omega(\delta\omega - \Delta\hat{\Omega}_n) = -\frac{C_n^3}{2(QC)_n^{1/2}} , \tag{32}$$

where $\delta\omega = \omega - k_n V + (QC)_n^{1/2}$ and $\Delta\hat{\Omega}_n = \omega_c + v_g(k - k_c) - k_n V + (QC)_n^{1/2}$. The Raman-regime dispersion relation results in instability whenever

$$(\Delta\hat{\Omega}_n)^2 < \frac{2C_n^3}{(QC)_n^{1/2}} . \tag{33}$$

The maximum temporal growth rate,

$$|\mathrm{Im}\delta\omega|_{max} = \frac{1}{\sqrt{2}}\frac{C_n^{3/2}}{(QC)_n^{1/4}} \propto \alpha^{1/4} , \tag{34}$$

occurs at $\Delta\hat{\Omega}_n = 0$. Substituting Eqs. (32) and (34) into Eq. (20), we find that the expansion in Eq. (20) is correct to order $\alpha^{1/2}$ in the Raman regime. The criteria

$$\alpha^{1/2} \ll 1 \tag{35}$$

and

$$\left(\frac{I_b}{I_A}\right)^{1/4} \gg \left(\frac{\gamma^3\beta}{8}\right)^{1/4} \frac{\left|\left(\frac{\partial \det D^{(0)}}{\partial\omega}\right)^{-1}_{\omega=\omega_c,k=k_c} \sum_{m=-\infty}^{\infty}(\mathrm{adj}\ D^{(0)})_{nm} D^{(0)}_{mn} F_{mn}\right|^{1/2}}{\left|c^2 p_n^2 I_0^2(p_n a)\left[\frac{K_0(p_n a)}{I_0(p_n a)} - \frac{K_0(p_n b_0)}{I_0(p_n b_0)}\right]\right|^{3/4}} \tag{36}$$

must be satisfied to assure the validity of the Raman-regime approximation.

Finally, we estimate the linear gain bandwidth of the interaction in the Raman regime to illustrate the applicability of our theory. Substituting $\Delta\hat{\Omega}_n \approx (v_g - V)(\omega - \omega_c)/v_g$ into Eq. (33), we find that the linear gain bandwidth is given by

$$\mathrm{Re}\delta\omega = 2\epsilon^{1/2}|1 - V/v_g|^{-1}|\mathrm{Im}\delta\omega|_{max} \ , \tag{37}$$

where use has been made of Eq. (34) and $\epsilon \ll 1$ is an arbitrary parameter. For example, the estimated gain bandwidth is $\mathrm{Re}\delta\omega = 0.01\ \omega$, for the choice of parameters corresponding to a BWO: $|\mathrm{Im}\delta\omega|_{max} = 0.04\ \omega$, $\epsilon = 0.05$, and $V/v_g = -0.8$; the estimated gain bandwidth is $\mathrm{Re}\delta\omega = 0.1\ \omega$, for the choice of parameters corresponding to a TWT amplifier: $|\mathrm{Im}\delta\omega|_{max} = 0.04\ \omega$, $\epsilon = 0.05$, and $V/v_g = 0.82$. These estimates indicate that the bandwidth of a TWT amplifier can be broad, whereas the bandwidth of a backward-wave oscillator is intrinsically narrow due to the negative group velocity of the wave.

V. SUMMARY

A Pierce-type dispersion relation has been derived from field theory which describes small-amplitude, relativistic traveling-wave-tube and backward-wave-tube interactions involving a relativistic electron beam and infinitely long, cylindrical slow-wave structure with arbitrary corrugation. In particular, the present Raman-regime dispersion relation has *not* been reported in earlier studies. The dispersion characteristics of the interactions, such as the linear instability growth rate and bandwidth, have been analyzed in both the Compton regime and the Raman regime.

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