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PILLARS AND TOWERS OF QUADRATIC TRANSFORMATIONS

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Abstract. Infinite pillars of quadratic transformations are used to describe residue fields of subrings of finitely generated ring extensions of the ring of integers. Towers whose underlying quadratic transformations are finite pillars or nonpillars are employed for the construction of basic dicritical divisors.

1. Introduction

In Section 2, I shall define the concepts of pillars and towers of QDTs = Quadratic Transformations, and the concept of dicritical divisors. In Section 4, I shall employ towers whose underlying QDTs are finite pillars, as well as nonpillars, for constructing some basic types of dicritical divisors. I shall say more about dicriticals at a later opportunity. Briefly speaking, dicritical divisors deal with transcendental extensions of residue fields, and finite QDT sequences are ideally suited for constructing such extensions. In a similar manner, infinite QDT sequences do the same job for constructing infinite algebraic extensions of residue fields which may be thought of as different incarnations of transcendental extensions. Thus, infinite pillars are the natural tools for answering the following **very interesting** question raised by Vitezslav Kala.

Question 1.1. Let A be a subring of a domain B which is a finitely generated ring extension of the ring of integers $\mathbb{Z}$. Let P be a maximal ideal in A. Then is the field A/P necessarily finite?

Answer. A resounding NO. Indeed A/P could be any field of finite transcendence degree over its prime subfield. Let $\mathrm{ch}(k)$ denote the characteristic of a ring k. Note that if k is a field with $\mathrm{ch}(k) = 0$, then the prime subfield of k is the rational number field $\mathbb{Q}$, and if k is a field with $\mathrm{ch}(k) =$ a prime number p, then the prime subfield of k is the Galois field $\mathrm{GF}(p)$ consisting of p elements. If k is a field whose transcendence degree over its prime subfield is $n \in \mathbb{N} =$ the set of all nonnegative integers, then we put $\mathrm{kdim}(k) = n$ or $\mathrm{kdim}(k) = n + 1$ according as $\mathrm{ch}(k) \neq 0$ or $\mathrm{ch}(k) = 0$; we call $\mathrm{kdim}(k)$ the **kroneckerian dimension** of k.

Without making a fuss about sets and classes, by Ω_n we denote the set of all fields of kroneckerian dimension n. For any domain A we let $\Omega(A)$ denote the set of all

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residue fields of closed points of spec(A), i.e., all fields A/P with $P \in \mathrm{mspec}(A) =$ the set of all maximal ideals in A. For any domain B we let

$$\widehat{\Omega}(B) = \bigcup \Omega(A),$$

where the union is over all subrings A of B. Now the above answer can be paraphrased as the inclusion

$$\bigcup\nolimits^{*} \Omega_n \subset \bigcup\nolimits' \widehat{\Omega}(B),$$

where the starred union is over all nonnegative integers n, and the primed union is over all domains B which are finitely generated ring extensions of $\mathbb{Z}$. Note that all fields in the LHS or the RHS are countable; recall that a set Γ is countable means that there is a surjective map $\mathbb{N}_+ \to \Gamma$ where $\mathbb{N}_+$ is the set of all positive integers.

This answer is **very surprising** because, letting $\overline{\Omega}$ be the set of all finite fields, it can easily be seen that

$$\bigcup\nolimits' \Omega(B) = \overline{\Omega} \text{ whereas } \Omega_n = \bigcup\nolimits^{\dagger} \Omega_{n,p},$$

where the daggered union is over all primes p augmented by 0 and where

$$\Omega_{n,p} = \begin{cases} \text{set of all fields between } \mathrm{GF}(p)(T_1, \dots, T_n) & \\ \qquad \text{and its algebraic closure} & \text{if } n \geq 0 \neq p, \\ \text{set of all fields between } \mathbb{Q}(T_1, \dots, T_{n-1}) & \\ \qquad \text{and its algebraic closure} & \text{if } n > 0 = p. \end{cases}$$

Actually, in Section 3, we shall show that:

Proposition 1.2. *Given any $n \in \mathbb{N}$, for the $(n+1)$-variable polynomial ring $B = \mathbb{Z}[T_1, \dots, T_{n+1}]$ we have $\Omega_n \subset \widehat{\Omega}(B)$.*

2. Quadratic transformations

The basic references for notation and terminology are my books [Ab3] and [Ab4]. In particular see Sections 1 and 6 of [Ab3] and Quest (Q35) of [Ab4]. For dicriticals see my papers [Ab5] and [Ab6] and my joint papers [AbH] and [AbL] with Heinzer and Luengo respectively.

A quasilocal ring (commutative with 1) is a ring V having exactly one maximal ideal $M(V)$. V dominates a quasilocal ring W means that W is a subring of V with $M(W) = W \cap M(V)$. We let

$$H_V : V \to H(V) = V/M(V)$$

denote the residue class epimorphism. The quasilocal ring V is residually rational (resp: residually algebraic, residually transcendental, etc.) over a subring R means that $H(V) = H_V(R)$ (resp: $H(V)/H_V(R)$ is algebraic, $H(V)/H_V(R)$ is transcendental, etc.). Likewise an element z in an overring of V is residually algebraic (resp: residually transcendental, etc.) over R at V or relative to V means that $z \in V$ and $H_V(z)/H_V(R)$ is algebraic (resp: $H_V(z)/H_V(R)$ is transcendental, etc.).

A **coefficient set** of a quasilocal ring V is a subset κ of V with $\{0, 1\} \subset \kappa$ such that H_V maps κ bijectively onto $H(R)$. A **coefficient ring** of V is a subring S of V such that $H_V(S) = H(V)$; equivalently S is a subring of V such that S contains a coefficient set of V. A **coefficient field** of V is a subfield S of V such that S is a coefficient ring of V; equivalently S is a subfield of V such that S is a coefficient set of V. A local ring is a Noetherian quasilocal ring. As usual $\mathbb{N} =$ the set of all nonnegative integers, $\mathbb{N}_+$ = the set of all positive integers, and spec(B) = the set of all prime ideals in a ring B.

Let A be a domain with quotient field $\mathrm{QF}(A) = L$. The modelic spec $\mathfrak{V}(A)$ of A is the set of all its localizations A_P as P varies over spec(A). If A is quasilocal, then its **local normalization** $A^{\mathfrak{N}}$ is the set of all the members of $\mathfrak{V}(\overline{A})$ which dominate A, where $\overline{A}$ is the integral closure of A in L. If U is a set of quasilocal domains, then we put $U^{\mathfrak{N}} = \bigcup_{B\in U} B^{\mathfrak{N}}$. For any nonzero ideal J in A, the modelic blowup $\mathfrak{W}(A,J)$ of A at J is defined by putting $\mathfrak{W}(A,J) = \bigcup_{0\neq x\in J}\mathfrak{V}(A[Jx^{-1}])$. If A is quasilocal, then the dominating modelic blowup $\mathfrak{W}(A,J)^{\Delta}$ of A at J is the set of all members of $\mathfrak{W}(A,J)$ which dominate A. If U is a set of quasilocal domains and $i \in \mathbb{N}$, then we let U_i = the set of all i-dimensional members of U. In the case of quasilocal A, we are particularly interested in the sets $\mathfrak{W}(A,J)_1^{\Delta}$ and $(\mathfrak{W}(A,J)_1^{\Delta})^{\mathfrak{N}}$. For quasilocal A we put $\mathfrak{D}(A,J) = (\mathfrak{W}(A,J)_1^{\Delta})^{\mathfrak{N}}$ and we call this the **dicritical set** of J in A and we call its members the **dicritical divisors** of J in A. If A is a local domain, then, in view of Krull-Akizuki, $\mathfrak{D}(A,J)$ is a finite set, i.e., $|\mathfrak{D}(A,J)| < \infty$, where $|\ |$ denotes **cardinality**; in the case when J is a pencil (as defined below) in a two dimensional regular local domain $R = A$, this is discussed in the first paragraph of (5.6)($\dagger^*$) of [Ab5]; in the general case, it suffices to note that a nonzero ideal in a Noetherian domain is contained in at most a finite number of height one prime ideals. If A is a positive dimensional local domain, then by a QDT = Quadratic Transform of A we mean a member of $\mathfrak{W}(A, M(A))^{\Delta}$; by a 0-th QDT of A we mean A itself, by a first QDT of A we mean a QDT of A,..., by a j-th QDT of A with $j \in \mathbb{N}_+$ we mean a first QDT of a $(j-1)$-th QDT of A. If A is a positive dimensional regular local domain, then we let $o(A)$ denote the unique DVR with quotient field L such that $\mathrm{ord}_{o(A)}x = \mathrm{ord}_A x$ for all $x \in L$; i.e., $o(A)$ is the unique one dimensional first QDT of A. We call $o(A)$ the **natural** DVR of A.

Henceforth in this section let R be a two dimensional regular local domain with quotient field L. Recall that $D(R)^{\Delta}$ is the set of all prime divisors of R, i.e., DVRs V with quotient field L such that V dominates R and is residually transcendental over R. For any $z \in L^{\times}$ = the set of all nonzero elements of L, $\mathfrak{D}(R,z)$ denotes the set of all **dicritical divisors** of z in R, i.e., the set of all prime divisors V of R such that z is residually transcendental over R relative to V. We also define the **numerator ideal** $a_R(z)$ of z in R, the **denominator ideal** $b_R(z)$ of z in R, and the **first associated ideal** $J_R(z)$ of z in R by writing $z = a/b$ such that $a \neq 0 \neq b$ in R have no nonunit common factor in R and letting $a_R(z) = aR$, $b_R(z) = bR$, and $J_R(z) = (a,b)R$. Note that now we have

$$\mathfrak{D}(R,z)^{\sharp} \subset \mathfrak{D}(R,z)^{\flat} \subset \mathfrak{D}(R,z) = (\mathfrak{W}(R,J_R(z))_1^{\Delta})^{\mathfrak{N}},$$

where $\mathfrak{D}(R,z)^{\sharp}$ (resp: $\mathfrak{D}(R,z)^{\flat}$) is the set of all **sharp dicritical divisors** (resp: **flat dicritical divisors**) of z in R, i.e., those $V \in D(R)^{\Delta}$ at which z is a residual transcendental generator (resp: residually a polynomial) over R, i.e., $z \in V$ and

$$\begin{cases} H(V) = K'(H_V(z)) \text{ (resp: } H_V(z) \in K'[t] \setminus K' \text{ for some } t \in H(V) \text{ with} \\ H(V) = K'(t) \text{ where } K' \text{ is the relative algebraic closure of } H(R) \text{ in } H(V)). \end{cases}$$

We call $\mathfrak{D}(R,z)$ (resp: $\mathfrak{D}(R,z)^{\sharp}$, $\mathfrak{D}(R,z)^{\flat}$) the **dicritical set** (resp: **sharp dicritical set**, **flat dicritical set**) of z in R.

Geometrically speaking, we may visualize R to be the local ring of a simple point of an algebraic or arithmetical surface, and we think of z as a **rational function** at that simple point which corresponds to the **local pencil** of curves $a = ub$ at that point. We say that z generates a **special pencil at** R to mean that b can be chosen so that $b = x^m$ for some $x \in M(R) \setminus M(R)^2$ and $m \in \mathbb{N}$, i.e., $zx^m \in R$ for some $x \in M(R) \setminus M(R)^2$ and $m \in \mathbb{N}$. We say that z generates a **semispecial pencil at** R to mean that b can be chosen so that $b = x^m y^n$ for some x, y in $M(R)$ and m, n in $\mathbb{N}$ with $M(R) = (x, y)R$, i.e., $zx^m y^n \in R$ for some x, y in $M(R)$ and m, n in $\mathbb{N}$ with $M(R) = (x, y)R$. We say that z generates a **polynomial** or **nonpolynomial** pencil in R according as $\mathfrak{D}(R, z) = \mathfrak{D}(R, z)^\flat$ or $\mathfrak{D}(R, z) \neq \mathfrak{D}(R, z)^\flat$. We say that z generates a **generating** or **nongenerating** pencil in R according as $\mathfrak{D}(R, z) = \mathfrak{D}(R, z)^\sharp$ or $\mathfrak{D}(R, z) \neq \mathfrak{D}(R, z)^\sharp$.

We put

$$Q(R) = \begin{cases} \text{the set of all two dimensional regular local domains} \\ \text{whose quotient field is } L \text{ and which dominate } R. \end{cases}$$

By Section 2 of [Ab1] we see that $S \mapsto o(S)$ gives a **bijection** $o_R : Q(R) \to D(R)^\Delta$. By Section 2 of [Ab1] we also see that, given any V in $D(R)^\Delta$, there exists a unique sequence $(R_j)_{0 \le j \le \nu}$ with $\nu \in \mathbb{N}$ and $R_0 = R$ such that R_{j+1} is a two dimensional first QDT of R_j for $0 \le j < \nu$ and $o(R_\nu) = V$. The sequence $(R_j)_{0 \le j \le \nu}$ is called the **finite QDT sequence** of R along V. Note the disjoint partition

$$Q(R) = \coprod_{j \in \mathbb{N}} Q_j(R), \quad \text{where} \quad Q_j(R) = \begin{cases} \text{the set of all two dimensional} \\ j\text{-th QDTs of } R. \end{cases}$$

Note that $o_R(R_\nu) = V$ and $o_R^{-1}(V) = R_\nu$. Given any $T \in Q(R)$ and any nonzero ideal I in R we define the (R, T)-**transform** of I to be the unique ideal J in T which we shall denote by $(R, T)(I)$ and which is characterized by requiring that

$$IT = J \prod_{M(R) \subset M(W)} (T \cap M(W))^{\mathrm{ord}_W(IT)},$$

where the product is taken over the set $\overline{W}$ of all one dimensional members W of $\mathfrak{V}(T)$ with $M(R) \subset M(W)$. Note that $T = R \Leftrightarrow \overline{W} = \emptyset$, and $T \neq R \Rightarrow$ either $\overline{W} = \{T_{xT}\}$ with $x \in M(T) \setminus M(T)^2$ or $\overline{W} = \{T_{xT}, T_{yT}\}$ with $(x, y)T = M(T)$; see page 367 of [ZaS]. Note that for any $z \in L^\times$ we have

$$(R, T)(J_R(z)) = J_T(z).$$

For $P \subset Q(R)$, define the $Q(R)$-**completion** of P to be $\widehat{P} \subset Q(R)$ obtained by putting

$$\widehat{P} = \{T' \in Q(R) : R \subset T' \subset T \in P\} \text{ or equivalently } \widehat{P} = \bigcup_{T \in P} \{R_0, \ldots, R_\nu\},$$

where $(R_j)_{0 \le j \le \nu}$ is the finite QDT sequence of R along $o(T)$. Note that if P is finite, then so is $\widehat{P}$. Call P **globally unforked** to mean that $S \in P \Rightarrow |Q_1(S) \cap \widehat{P}| \le 1$ and $S \in P \Rightarrow |Q_1(S) \cap P| = 0$. We dissect this definition into local pieces thus. Given points (= elements) S and T of $Q(R)$ we say that T is **contiguous** to S if $T \in Q_1(S)$. We call $S \in Q(R)$ a **terminal point** (resp: **unifurcation point, bifurcation point**) of P if $S \in \widehat{P}$ and $|Q_1(S) \cap \widehat{P}| = 0$ (resp: $|Q_1(S) \cap \widehat{P}| = 1, |Q_1(S) \cap \widehat{P}| > 1$). Clearly S is a terminal point of $P \Rightarrow S \in P$. We say that P is

unforked at $S \in Q(R)$ to mean that S is a nonbifurcation point of P and there is no point of P contiguous to S. Clearly P is globally unforked iff it is unforked at each of its points.

For a moment let J be a nonzero ideal in R. We call J a **pencil** (in R) if $J = yJ_R(z)$ for some $y \in R^\times$ and $z \in L^\times$, and we note that then $\mathfrak{D}(R,J) = \mathfrak{D}(R,z)$. If J is a pencil with $J = yJ_R(z)$, then we let $\mathfrak{D}(R,J)^\sharp = \mathfrak{D}(R,z)^\sharp$ and $\mathfrak{D}(R,J)^\flat = \mathfrak{D}(R,z)^\flat$, and if J is not a pencil, then we let $\mathfrak{D}(R,J)^\sharp = \mathfrak{D}(R,J)^\flat = \emptyset$. Note that now we have

$$\mathfrak{D}(R,J)^\sharp \subset \mathfrak{D}(R,J)^\flat \subset \mathfrak{D}(R,J) = (\mathfrak{W}(R,J)_1^\Delta)^\mathfrak{N}.$$

We say that J is a **polynomial** or **nonpolynomial** ideal in R according as $J = yJ_R(z)$ for some $y \in R^\times$ and $z \in L^\times$ such that z generates a polynomial or nonpolynomial pencil in R. We say that J is a **generating** or **nongenerating** ideal in R according as $J = yJ_R(z)$ for some $y \in R^\times$ and $z \in L^\times$ such that z generates a generating or nongenerating pencil in R. In the last two sentences we may say **pencil** instead of ideal. Regardless whether J is a pencil or not, we say that J is **primary** to mean that the ideal J is $M(R)$-primary. We say that J is **special** (resp: **semispecial**) **at** R if $J = yJ_R(z)$ for some $y \in R^\times$ and $z \in L^\times$ such that z generates a special (resp: semispecial) pencil at R. We put

$$\begin{cases} \mathfrak{B}(R,J)^\sharp = \{o_R^{-1}(V) : V \in \mathfrak{D}(R,J)^\sharp\}, \\ \mathfrak{B}(R,J)^\flat = \{o_R^{-1}(V) : V \in \mathfrak{D}(R,J)^\flat\}, \\ \mathfrak{B}(R,J) = \{o_R^{-1}(V) : V \in \mathfrak{D}(R,J)\}, \\ \mathfrak{Q}(R,J) = \{T \in Q(R) : (R,T)(J) \text{ is not principal}\} \end{cases}$$

and we note that $\mathfrak{Q}(R,J)$ is a finite set (see the proof of (4.1)(v) below and also see Proposition 2 on page 367 of [ZaS]) with

$$\mathfrak{B}(R,J)^\sharp \subset \mathfrak{B}(R,J)^\flat \subset \mathfrak{B}(R,J) \subset \mathfrak{Q}(R,J) \subset Q(R).$$

We say that J **goes through** the members of $\mathfrak{Q}(R,J)$ but not through the members of $Q(R) \setminus \mathfrak{Q}(R,J)$. Here is a pictorial visualization of these sets.

Visualize J as a source of a 4th of July fireworks display in America or a Diwali Bhuinala fireworks display in India. Visualize members of $\mathfrak{Q}(R,J)$ as stars (or sparks) emanating from the display calling them (quadratic) **stars** of J. Visualize members of $\mathfrak{B}(R,J)$ as **big stars** of J and members of $\mathfrak{Q}(R,J) \setminus \mathfrak{B}(R,J)$ as **small stars** of J. Visualize members of $\mathfrak{B}(R,J)^\sharp$ as **sharp stars** of J and members of $\mathfrak{B}(R,J)^\flat$ as **flat stars** of J.

Visualize $\mathfrak{B}(R,J)^\sharp$ (resp: $\mathfrak{B}(R,J)^\flat, \mathfrak{B}(R,J), \mathfrak{Q}(R,J)$, $\mathfrak{Q}(R,J) \setminus \mathfrak{B}(R,J)$) as the **sharp star** (resp: **flat star, big star, star, small star**) **set** of J. We say that J **terminates** (resp: **unifurcates, bifurcates**) at $S \in Q(R)$ to mean that S is a terminal point (resp: unifurcation point, bifurcation point) of $\mathfrak{B}(R,J)$. We call $S \in Q(R)$ a **terminal** (resp: **unifurcation, bifurcation**) **point** of J to mean that S is a terminal point (resp: unifurcation point, bifurcation point) of $\mathfrak{B}(R,J)$. For reasons of euphony we may say **unifurcated point** (resp: **is unifurcated**) instead of unifurcation point (resp: unifurcates), and so on. We say that J is **globally unforked** (resp: **unforked at** $S \in Q(R)$) to mean that $\mathfrak{B}(R,J)$ is globally unforked (resp: unforked at $S \in Q(R)$).

We call $\mathfrak{D}(R,J)^\sharp$ and $\mathfrak{D}(R,J)^\flat$ the **sharp dicritical set** and **flat dicritical set** of J in R and we call their members **sharp dicritical divisors** and **flat dicritical**

divisors of J in R respectively. For any $S \in Q(R)$ we may visualize $o(S)$ as the "halo" around S. Thus we visualize the dicritical divisors (resp: sharp dicritical divisors, flat dicritical divisors) of J in R as the halos around the big stars (resp: sharp stars, flat stars) of J in R. J is said to be **special** (resp: **semispecial, principal, primary**) **at** $S \in Q(R)$ if the ideal or pencil JS in S is special (resp: semispecial, principal, primary) at S. Note that J goes through $S \in Q(R)$ iff it is nonprincipal at S, i.e., iff the ideal JS in S is nonprincipal. By abuse of language we say that J is **dicritical** or **nondicritical** according as $R \in \mathfrak{B}(R, J)$ or $R \notin \mathfrak{B}(R, J)$.

Now for a moment let (x, y) be generators of $M(R)$, let $K = H(R) = R/M(R)$, and let κ be a coefficient set of R. Referring to Theorem (T158) on page 561 of [Ab4], to get hold of a **concrete set of generators** for the maximal ideal of a QDT of R, given any $R' \in Q_1(R)$ we define generators (x', y') of $M(R')$ and a coefficient set κ' of R' thus. If $x/y \in M(R')$ with $R' \in Q_1(R)$, then $(x', y') = (x/y, y)$ and $\kappa' = \kappa$. If $x/y \notin M(R')$, then $y/x \in R'$ and $(x', y') = (x, \mu(y/x))$, where

$$\mu(Z) = Z^\omega + \sum_{1 \le i \le \omega} \mu_i Z^{\omega - i} \text{ with } \omega \in \mathbb{N}_+ \text{ and } \mu_i \in \kappa$$

is such that

$$Z^\omega + \sum_{1 \le i \le \omega} H_R(\mu_i) Z^{\omega - i}$$

is irreducible in the polynomial ring $K[Z]$, and κ' = the image of κ^ω under the map $\kappa^\omega \to R'$ obtained by sending $(k_0, \dots, k_{\omega-1}) \in \kappa^\omega$ to $\sum_{0 \le i \le \omega - 1} k_i (y/x)^i$. We call (R', x', y', κ') a QDT of (R, x, y, κ). By the **finite** QDT sequence of (R, x, y, κ) along $V \in D(R)^\Delta$ we mean the sequence $(R_j, x_j, y_j, \kappa_j)_{0 \le j \le \nu}$, where $(R_j)_{0 \le j \le \nu}$ is the finite QDT sequence of R along V with $(R_0, x_0, y_0, \kappa_0) = (R, x, y, \kappa)$ and $(R_j, x_j, y_j, \kappa_j)$ is a QDT of $(R_{j-1}, x_{j-1}, y_{j-1}, \kappa_{j-1})$ for $1 \le j \le \nu$. By a **finite** QDT sequence of R we mean the finite QDT sequence of R along some $V \in D(R)^\Delta$; clearly this is equivalent to saying that a finite QDT sequence of R is a sequence of the form $(R_j)_{0 \le j \le \nu}$ with $\nu \in \mathbb{N}$ and $R_0 = R$ such that R_j is a two dimensional QDT of R_{j-1} for $1 \le j \le \nu$. Likewise, by a **finite** QDT sequence of (R, x, y, κ) we mean the finite QDT sequence of (R, x, y, κ) along some $V \in D(R)^\Delta$; clearly this is equivalent to saying that a finite QDT sequence of (R, x, y, κ) is a sequence of the form $(R_j, x_j, y_j, \kappa_j)_{0 \le j \le \nu}$ with $\nu \in \mathbb{N}$ and $(R_0, x_0, y_0, \kappa_0) = (R, x, y, \kappa)$ such that $(R_j, x_j, y_j, \kappa_j)$ is a QDT of $(R_{j-1}, x_{j-1}, y_{j-1}, \kappa_{j-1})$ for $1 \le j \le \nu$. By an **infinite** QDT sequence of R we mean a sequence $(R_j)_{0 \le j < \infty}$ with $R_0 = R$ such that $R_j \in Q_1(R_{j-1})$ for all positive j. By an **infinite** QDT sequence of (R, x, y, κ) we mean a sequence $(R_j, x_j, y_j, \kappa_j)_{0 \le j < \infty}$ with $(R_0, x_0, y_0, \kappa_0) = (R, x, y, \kappa)$ such that $(R_j, x_j, y_j, \kappa_j)$ is a QDT of $(R_{j-1}, x_{j-1}, y_{j-1}, \kappa_{j-1})$ for all positive j. Recall that, given any algebraic field extension k' of a field k and any element ξ of k', the **minimal polynomial** of ξ over k is the unique monic polynomial

$$q(Z) = Z^\omega + \sum_{1 \le i \le \omega} q_i Z^{\omega - i} \text{ with } \omega \in \mathbb{N}_+ \text{ and } q_i \in k$$

such that $q(Z)$ is irreducible in $k[Z]$ and $q(\xi) = 0$.

By a **tower** at R we mean a sequence $(R_j, J_j)_{0 \le j \le \nu}$ with $\nu \in \mathbb{N}$, where $(R_j)_{0 \le j \le \nu}$ is a finite QDT sequence of R and J_j is a primary pencil in R_j for $0 \le j \le \nu$ such that for $1 \le j \le \nu$ we have that R_{j-1} is a unifurcation point of J_{j-1} and $(R_{j-1}, R_j)(J_{j-1}) = J_j$. The sequence $(J_j)_{0 \le j \le \nu}$ is called the **staircase** of the tower. By a **staircase** of a finite QDT sequence $(R_j)_{0\le j\le\nu}$ of R we mean a sequence $(J_j)_{0\le j\le\nu}$ such that $(R_j,J_j)_{0\le j\le\nu}$ is a tower at R. We call the tower and the staircase **special** (resp: **semispecial**) if J_j is special (resp: semispecial) at R_j for $0\le j\le\nu$; clearly, J_0 is special at $R_0 \Rightarrow J_0$ is semispecial at $R_0 \Rightarrow$ the tower and the staircase are semispecial. We call the tower and the staircase **terminal** if R_ν is a terminal point of J_ν. We call the tower and the staircase **small** if R_j is a small star of J_j for $0\le j<\nu$. We call the tower and the staircase **big** or **dicritical** if R_j is a big star of J_j for some $j\in\{0,1,\dots,\nu-1\}$. By the **big stars** or **dicriticals** of the tower and the staircase we mean those R_j, with $0\le j\le\nu$, which are big stars of J_j. We call $\nu+1$ the **height** of the tower.

By an **infinite pillar** at (R,x), with $x\in M(R)\setminus M(R)^2$, we mean an infinite QDT sequence $(R_j)_{0\le j<\infty}$ of R such that $M(R_{j-1})R_j = xR_j$ for $0\le j<\infty$. By a **pillar** at (R,x), with $x\in M(R)\setminus M(R)^2$, we mean a finite QDT sequence $(R_j)_{0\le j\le\nu}$ of R such that $M(R_{j-1})R_j = xR_j$ for $0\le j\le\nu$. We call these pillars **rational** if R_j is residually rational over R for $0\le j<\infty$ in the first case and for $0\le j\le\nu$ in the second case. In the proof Proposition 3.1 we use infinite pillars which are mostly nonrational. In Section 4 we use finite pillars which are mostly rational.

3. Big residue fields

(Cf. Example 3 on page 102 of [ZaS].) We shall deduce (1.2) from part (IV) of:

Proposition 3.1. *Let R be a two dimensional regular local domain with quotient field L, generators (x,y) of $M(R)$, and residue class epimorphism*

$$H_R : R \to H(R) = R/M(R) = K.$$

Let K^* be an algebraic field extension of K. Let D and B be subrings of L with $D\subset B\cap R$. Then we have the following:

(I) *Assume that $QF(H_R(D)) = K$ with $\{1/x,y\}\subset B$ and let $\xi\in K^*$. Then there exists $R'\in Q_1(R)$ together with $y'\in B\cap R'$ and a homomorphism $\alpha' : R'\to K^*$ with $\ker(\alpha') = M(R')$ such that upon letting $D' = D[y/x]$ we have $M(R)R' = xR'$ with $M(R') = (x,y')R'$, $D'\subset B\cap R'$ with $QF(H_{R'}(D')) = H(R')$, and $\alpha'(y/x) = \xi$ with $\alpha'(z) = H_R(z)$ for all $z\in R$.*

(II) *Assume that $QF(H_R(D)) = K$ with $\{1/x,y\}\subset B$ and let $\xi_j\in K^*$ for all $j\in\mathbb{N}_+$. Then there exists an infinite QDT sequence $(R_j)_{0\le j<\infty}$ of R together with an element $y_j\in B\cap R_j$ and a homomorphism $\alpha_j : R_j\to K^*$ with $\ker(\alpha_j) = M(R_j)$ for all $j\in\mathbb{N}$ such that, upon letting $D_0 = D$ and $D_j = D[y_0/x, y_1/x,\dots,y_{j-1}/x]$ for all $j\in\mathbb{N}_+$, we have the following:*

(1) *$y_0 = y$, and $\alpha_0(z) = H_R(z)$ for all $z\in R$;*
for all $j\in\mathbb{N}$ we have:

(2_j) *$M(R_j) = (x,y_j)R_j$, $D_j\subset B\cap R_j$, $QF(H_{R_j}(D_j)) = H(R_j)$;*
for all $j\in\mathbb{N}_+$ we have:

(3_j) *$M(R_{j-1})R_j = xR_j$, $\alpha_j(y_{j-1}/x) = \xi_j$, and $\alpha_j(z) = \alpha_{j-1}(z)$ for all $z\in R_{j-1}$.*

(III) *In* (II) *assume that $H_R(D)[\xi_1,\xi_2,\dots] = K^*$. Let $R^* = \bigcup_{0\le j<\infty} R_j$ and $D^* = D[y_0/x, y_1/x,\dots]$. Then R^* is a quasilocal domain with quotient field L and D^* is a subring of $B\cap R^*$ with $\alpha^*(D^*) = K^*$, where $\alpha^* : R^*\to K^*$ is the unique epimorphism with $\ker(\alpha^*) = M(R^*)$ such that $\alpha^*(z) = \alpha_j(z)$ for all $j\in\mathbb{N}$ and $z\in R_j$.*

(IV) *Assume that K is countable and that $QF(H_R(D)) = K$ with $\{1/x, y\} \subset B$. Then there exists a subring D^* of B together with an epimorphism $\phi : D^* \to K^*$ such that $D \subset D^*$ and $\phi(z) = H_R(z)$ for all $z \in D$.*

Proof. To prove (I) let $q(Z)$ be the minimal polynomial of ξ over K. Since $\mathrm{QF}(H_R(D)) = K$, there exists $0 \neq a \in K$ such that $aq(Z) \in H_R(D)[Z]$. Now we can find $\mu(Z) \in D[Z]$ such that applying H_R to the coefficients of $\mu(Z)$ we get $aq(Z)$. Let $R' = R[y/x]_P$, where P is the maximal ideal in $R[y/x]$ generated by (x, y') with $y' = \mu(y/x)$. Then $R' \in Q_1(R)$ with $M(R)R' = xR'$ and $M(R') = (x, y')R'$. Moreover, upon letting $D' = D[y/x]$ we clearly have $D' \subset B \cap R'$ and $\mathrm{QF}(H_{R'}(D')) = H(R')$. Also clearly there exists a unique homomorphism $\alpha' : R' \to K^*$ with $\ker(\alpha') = M(R')$ such that $\alpha'(y/x) = \xi$ and $\alpha'(z) = H_R(z)$ for all $z \in R$. This proves (I).

(II) follows from (I) by induction thus. For any $\nu \in \mathbb{N}$, let (II_ν) be obtained from (II) upon replacing the phrases "an infinite QDT sequence $(R_j)_{0 \leq j < \infty}$" and "for all $j \in \mathbb{N}$" and "for all $j \in \mathbb{N}_+$" by the phrases "a finite QDT sequence $(R_j)_{0 \leq j \leq \nu}$" and "for all $j \in \{0, 1, \dots, \nu\}$" and "for all $j \in \{1, 2, \dots, \nu\}$" respectively. Then (II_0) is trivial. Moreover, for any $\nu \in \mathbb{N}_+$, by (I) we see that $(\mathrm{II}_{\nu-1}) \Rightarrow (\mathrm{II}_\nu)$ with the data for $0 \leq j \leq \nu - 1$ being the same as the data in $(\mathrm{II}_{\nu-1})$, and so on.

(III) follows from (II). To prove (IV), by countability we can find elements $\xi_1, \xi_2, \dots$ in K^* such that $H_R(D)[\xi_1, \xi_2, \dots] = K^*$. Now apply (III) and let $\phi : D^* \to K^*$ be the unique homomorphism such that $\phi(z) = \alpha^*(z)$ for all $z \in D^*$. $\square$

Proof of (1.2). Given any $W \in \Omega_n$ let

$$D = \begin{cases} \mathbb{Z}[T_1, \dots, T_{n-1}] & \text{if } \mathrm{ch}(W) = 0, \\ \mathbb{Z}[T_1, \dots, T_n] & \text{if } \mathrm{ch}(W) = p \neq 0 \end{cases}$$

and

$$(S, x, y) = \begin{cases} (\mathbb{Q}(T_1, \dots, T_{n-1})[T_{n+1}^{-1}, T_n], T_{n+1}^{-1}, T_n) & \text{if } \mathrm{ch}(W) = 0, \\ (\mathbb{Z}_{p\mathbb{Z}}(T_1, \dots, T_n)[T_{n+1}^{-1}], T_{n+1}^{-1}, p) & \text{if } \mathrm{ch}(W) = p \neq 0 \end{cases}$$

(where $\mathbb{Z}_{p\mathbb{Z}}(T_1, \dots, T_n) = A_{pA}$ with $A = \mathbb{Z}_{p\mathbb{Z}}[T_1, \dots, T_n]$) and

$$R = S_{(x,y)S} \quad \text{and} \quad L = \mathrm{QF}(B) = \mathbb{Q}(T_1, \dots, T_{n+1})$$

with the observation that if $\mathrm{ch}(W) = 0$, then $n > 0$, and with the understanding that $\mathbb{Z}_{p\mathbb{Z}}$ and $S_{(x,y)S}$ are the localizations of $\mathbb{Z}$ and S at the prime ideals generated by p and (x, y) respectively. Now R is a two dimensional regular local domain with quotient field L, (x, y) are generators of $M(R)$, $\{1/x, y\} \subset B$, and D is a subring of $B \cap R$. Let

$$H_R : R \to H(R) = R/M(R) = K$$

be the residue class epimorphism. Then clearly $\mathrm{QF}(H_R(D)) = K$. Also clearly there exists an algebraic field extension K^* of K together with an isomorphism $K^* \to W$. By (3.1)(IV) we can find a subring D^* of B together with an epimorphism $\phi : D^* \to K^*$ such that $D \subset D^*$ and $\phi(z) = H_R(z)$ for all $z \in D$. Therefore $W \in \widehat{\Omega}(B)$. $\square$

Remark on DVRs (3.2). Concerning (3.1)(III), using results from [Ab2], we see that R^* is the valuation ring V of a valuation v of L and $H(V)$ is isomorphic to K^* and, moreover, if $[K^* : K] = \infty$, then v is **real**; i.e., the value group G_v

may be assumed to be an additive subgroup of $\mathbb{R}$. By the construction of R^* we have $M(R^*) = xR^*$ and hence if $[K^* : K] = \infty$, then V must be a DVR, and so in particular v is **rational**; i.e., for any η, ζ in $L^\times$ with $v(\eta) < 0 < v(\zeta)$ we have $v(\eta^r\zeta^s) = 0$ for some r, s in $\mathbb{N}_+$. In the very special case when $B = \mathbb{Z}[T_1]$ and $[K^* : K] = \infty$, let us sketch an alternative proof of (1.2) by letting $E = B \cap V$ and showing that $H_V(E) = H(V)$ by using the following Lemma 3.4. The said lemma also inspires a sketch of Krull domains. We thank Bill Heinzer for the two sketches and (the late) Jack Ohm for the lemma. □

Sketch of very special case (3.3). Now $H(V) \approx K^*$ is an algebraic field extension of $\mathrm{GF}(p) \approx K$, and hence $H(V)$ is integral over $\mathrm{GF}(p)$. By the following lemma we get $E_P = V$. Thus $\mathrm{GF}(p) \subset H_V(E) \subset \mathrm{QF}(H_V(E)) = H(V)$, and hence the domain $H_V(E)$ is integral over the field $\mathrm{GF}(p)$, and therefore, say by (T38.1) on page 243 of [Ab4], $H_V(E)$ is a field. Consequently $H_V(E) = H(V)$. □

Lemma 3.4. *Let B be a domain with quotient field L. Let V be the valuation ring of a rational valuation of L such that $B \not\subset V$. Assume that the quotient field of $E = B \cap V$ is L. Then upon letting $P = E \cap M(V)$ we have $E_P = V$.*

Proof. Obviously $E_P \subset V$. To prove $V \subset E_P$ it suffices to show that given any $\mu \in V$, there exists $\nu \in E \setminus P$ with $\mu\nu \in E$. Since $B \not\subset V$, we can find $\zeta \in B$ with $\zeta \notin V$, i.e., $v(\zeta) < 0$. Since $\mathrm{QF}(E) = L$, we have (*) $\mu\eta \in E$ for some $0 \neq \eta \in E$. If $v(\eta) = 0$, then $\eta \notin P$ and we can take $\nu = \eta$. So assume that $v(\eta) \neq 0$. Then, because $\eta \in E \subset V$, we get $v(\eta) > 0$. Therefore by the rationality of v we get $v(\eta^r\zeta^s) = 0$ for some r, s in $\mathbb{N}_+$. Let $\nu = \eta^r\zeta^s$. Then $\nu \in B$ because $\{\eta, \zeta\} \subset B$. Now $\nu \in B$ with $v(\nu) = 0$ tells us that $\nu \in E \setminus P$. We shall show that $\mu\nu \in E$, and this will complete the proof. Clearly $\mu\nu = (\mu\eta)(\eta^{r-1}\zeta^s)$ with $(\mu\eta) \in E \subset B$ by (*), and $(\eta^{r-1}\zeta) \in B$ because $\{\eta, \zeta\} \subset B$. Hence $\mu\nu \in B$ and also $\mu\nu \in V$ because $\mu \in V$ and $v(\nu) = 0$; therefore $\mu\nu \in B \cap V = E$. □

Sketch of Krull domains (3.5). Here is a slightly sharper treatment of Krull domains than the treatment given on pages 82-88 of Zariski and Samuel [ZaS]. Let L be a field and let G be the set of all DVRs with quotient field L. Given any DVR $V \in G$ and a family $F \subset G$ let

$$E = \bigcap_{U \in F} U \quad \text{and} \quad B = \bigcap_{U \in F \setminus \{V\}} U.$$

We call V an **essential valuation of** F, or say that V is **essential for** F, if $E \neq B$. F satisfies the **finiteness condition in** L means that $\mathrm{QF}(E) = L$ and each nonzero element of L is a unit in all except finitely many members of F. If every member of F is essential for F and if F satisfies the finiteness condition in L, then we say that E is a **Krull domain** and F is a family of essential valuations of E; note that then F is uniquely determined by E because ($\bullet$) Krull domain $\Rightarrow F = \mathfrak{V}(E)_1$. To prove ($\bullet$), for every $V \in F$, by Lemma 3.4 we have $V = E_{E \cap M(V)} \in \mathfrak{V}(E)_1$. Conversely, given any $V \in \mathfrak{V}(E)_1$, we want to show that $V \in F$. Since $V \in \mathfrak{V}(E)_1$, there exists $0 \neq x \in E \cap M(V)$. By the finiteness condition on F, x is a unit in all except finitely many members $V_1, \ldots, V_n$ of F. Suppose if possible that $V \notin F$. Then we can find $y \in E$ such that $y \in M(V_i)$ for $1 \leq i \leq n$ but $y \notin M(V)$. Now for some $m \in \mathbb{N}_+$ we get $y^m/x \in E \setminus V$, which is a contradiction. Therefore $V \in F$.

It follows that a domain E is a Krull domain iff upon letting $F = \mathfrak{V}(E)_1$ we have that: (i) every member of F is a DVR, (ii) F satisfies the finiteness condition in QF(E), and (iii) $E = \bigcap_{U\in F} U$. Clearly every Krull domain is normal. Thus we have re-proved Theorem 26 on page 83 of Zariski and Samuel [ZaS] and shown that condition (E4) on page 82 of Zariski and Samuel [ZaS] is redundant. $\square$

4. Basic dicritical divisors

Let R be a two dimensional regular local domain with quotient field L. Let (x, y) be generators of $M(R)$ and let $K = H(R) = R/M(R)$. After fixing some notation in Remark (4.0), in Propositions (4.1) we shall prove some basic results about special and semispecial pencils, and in Propositions 4.2 and 4.3 we shall embellish them by constructing some dicritical towers.

Remark 4.0. In this remark we fix some notation to be used in the rest of this section. Given any $0 \neq F \in R$ we can write

$$F = \sum_{i+j=d} \widetilde{F}_{ij} x^i y^j \quad \text{with} \quad d = \mathrm{ord}_R F \quad \text{and} \quad \widetilde{F}_{ij} \in R$$

and we define the **initial form** $\mathrm{info}(F) = \mathrm{info}_{(R,x,y)}F$ of F (relative to R, x, y) to be the nonzero homogeneous polynomial $\Phi = \Phi(X, Y)$ of degree d in indeterminates X, Y with coefficients in K obtained by putting

$$\Phi = \sum_{i+j=d} \Phi_{ij} X^i Y^j \quad \text{with} \quad \Phi_{ij} = H_R(\widetilde{F}_{ij}).$$

For completeness we put $\mathrm{info}(0) = 0$.

Recall that $R \setminus M(R) = U(R) =$ the set of all **units** in R. For any $\zeta \in R$ or $\zeta \subset R$ we put $\mathrm{ord}_{(R/xR)}\zeta = \mathrm{ord}_{(R/xR)}\zeta^\sharp$ and $\mathrm{ord}_{(R/yR)}\zeta = \mathrm{ord}_{(R/yR)}\zeta^\flat$, where, just in this sentence, we are denoting the images of ζ under the canonical epimorphisms $R \to R/xR$ and $R \to R/yR$ by $\zeta^\sharp$ and $\zeta^\flat$ respectively. Recall that for any subset ζ of a regular local domain S we have $\mathrm{ord}_S\zeta = \min\{\mathrm{ord}_S\xi : \xi \in \zeta\}$, where the min is ∞ if $\zeta \subset \{0\}$.

For any $a \neq 0 \neq b$ in R, upon letting $\mathrm{info}_{(R,x,y)}a = \overline{a}$ and $\mathrm{info}_{(R,x,y)}b = \overline{b}$, we note that $\overline{a}$ and $\overline{b}$ belong to the graded ring

$$\mathrm{grad}(R) = K[X, Y] = \sum_{i\in\mathbb{N}} M(R)^i/M(R)^{i+1}.$$

We call $\overline{a}$ a **prime power** to mean that $\overline{a} = \delta\alpha^n$ for some $n \in \mathbb{N}_+$, some irreducible $\alpha \in K[X, Y]\setminus K$, and some $\delta \in K^\times$. Write $\mathrm{GCD}(\overline{a}, \overline{b})$ for the unique generator of the smallest principal overideal of $(\overline{a}, \overline{b})K[X, Y]$ which is of the form X^pY^q+(terms of Y-degree $< q$). Note that then $\mathrm{GCD}(\overline{a}, \overline{b}) = 1$ iff $\overline{a}$ and $\overline{b}$ have no nonconstant common factor in $K[X, Y]$. For any $f = f(X, Y)$ in the polynomial ring $S[X, Y]$ over a ring S, the symbols $\deg_X f$, $\deg_Y f$, $\deg_{X,Y} f$, $\mathrm{ord}_X f$, $\mathrm{ord}_Y f$, $\mathrm{ord}_{X,Y} f$, $\mathrm{info}_{X,Y} f$ have obvious meanings, where $\mathrm{ord}_{X,Y} f$ and $\mathrm{info}_{X,Y} f$ are defined by regarding f as a member of $S[[X, Y]]$, while $\mathrm{ord}_X f$ and $\mathrm{ord}_Y f$ are defined by regarding f as a member of $(S[Y])[[X]]$ and $(S[X])[[Y]]$ respectively. Note that if $f = 0$, then the degrees are $-\infty$ and the orders are ∞ and the info is 0; if $f \neq 0$, then the degrees as well as the orders belong to $\mathbb{N}$ and the info is a nonzero homogeneous member of $S[X, Y]$. In particular this applies to elements of $K[X, Y]$. Note that

($\bullet$) if $\mathrm{ord}_R a = \mathrm{ord}_R b = e \in \mathbb{N}_+$ and $bR = x^eR$, then $\overline{b}$ is a prime power and for $J = (a, b)R$ we have $\mathrm{ord}_X \overline{a} < e \Leftrightarrow \overline{a}/\overline{b} \notin K \Leftrightarrow R$ is a big star of J.

Proposition 4.1. *For any pencil $J = (a,b)R$ in R with $a \neq 0 \neq b$ in $M(R)$, let* $\mathrm{ord}_R a = d$ *and* $\mathrm{ord}_R b = e$ *with* $\mathrm{info}_{(R,x,y)} a = \overline{a}$ *and* $\mathrm{info}_{(R,x,y)} b = \overline{b}$*. Then:*

(i) *R is a terminal point of $J \Leftrightarrow d = e$ and $GCD(\overline{a}, \overline{b}) = 1$.*

(ii) *R is a big star of $J \Leftrightarrow d = e$ and $\overline{a}/\overline{b} \notin K$.*

(ii*) *R is a terminal point of $J \Rightarrow R$ is a big star of J.*

(iii) *If J is special at R and R is a big star of J, then J is unforked at R.*

(iv) *If J is semispecial at R, then J is semispecial at every $R^* \in Q(R)$.*

(v) *R is a unifurcation point of $J \Leftrightarrow J$ is nonprincipal at R and $GCD(\widetilde{a}, \widetilde{b})$ is a prime power, where $(\widetilde{a}, \widetilde{b}) = (\overline{a}, \overline{b})$ or $(\widetilde{a}, \widetilde{b}) = (\overline{a}, 0)$ or $(\widetilde{a}, \widetilde{b}) = (0, \overline{b})$ according as $d = e$ or $d < e$ or $d > e$.*

Proof. Clearly $d = e$ and $\mathrm{GCD}(\overline{a}, \overline{b}) = 1 \Leftrightarrow (R, R')(J) = R'$ for all $R' \in Q_1(R)$, where in the $\Leftarrow$ part we assume that J is nonprincipal at R; this proves (i).

Also clearly $d = e$ and $\overline{a}/\overline{b} \notin K \Leftrightarrow a/b$ is residually transcendental over R relative to $o(R)$; this proves (ii). (ii*) follows from (i) and (ii).

In view of (ii), to prove (iii), without loss of generality we may assume that $d = e$ with $b = x^e$ and

$$a = \sum_{0 \le i \le e} a_i x^{e-i} y^i,$$

where the elements $a_i \in R$ are such that $a_i \notin M(R)$ for some $i > 0$. Now clearly we have $(R, R')(J) = R'$ for every $R' \in Q_1(R)$ for which $M(R)R' = xR'$. Moreover for the unique $R'' \in Q_1(R)$ for which $M(R'') = (x'' = x/y, y)R''$, upon letting $J'' = (R, R'')(J)$ we have $J'' = (a'' = a/y^e, b'' = b/y^e)R''$, where

$$a'' = \sum_{0 \le i \le e} a_i x''^{e-i}$$

and $b'' = x''^e$. Clearly $\mathrm{ord}_{R''} a'' < e = \mathrm{ord}_{R''} b''$ and hence R'' is not a big star of J. This proves (iii).

To prove (iv), without loss of generality, we may assume that $bR = x^p y^q R$ with p, q in $\mathbb{N}$. Now by induction on j we see that for all $j \in \mathbb{N}$ and $R_j \in Q_j(R)$ we have $bR_j = x_j^{p_j} y_j^{q_j} R_j$ for some generators (x_j, y_j) of $M(R_j)$ and some p_j, q_j in $\mathbb{N}$; hence J is semispecial at R_j. This proves (iv).

To prove (v) it suffices to note that if J is nonprincipal at R, then the members R' of $Q_1(R)$ for which $(R, R')(J) \neq R'$ are in a bijective correspondence with the "tangent directions" of J, i.e., with the homogeneous principal prime ideals P in $\mathrm{grad}(R)$ such that $\{\mathrm{info}_{(R,x,y)} g : g \in J \text{ with } \mathrm{ord}_R g = \mathrm{ord}_R J\} \subset P$. □

Proposition 4.2. *Now* (4.1)(iii) *says that if J is special at a big star R, then J nonbifurcates at R and has no big star contiguous to R. We shall show that this is best possible by constructing a special pencil at R such that R is a big star of J and there is another big star of J in $Q_2(R)$. So consider the special pencils described by*

$$\begin{cases} J = (a,b)R \text{ and } M(R) = (x,y)R, \\ \widehat{J} = (\widehat{a}, \widehat{b})\widehat{R} \text{ and } M(\widehat{R}) = (\widehat{x}, \widehat{y})\widehat{R} \text{ with } \widehat{R} \in Q_1(R) \text{ and } (\widehat{x}, \widehat{y}) = (x/y, y), \\ \widetilde{J} = (\widetilde{a}, \widetilde{b})\widetilde{R} \text{ and } M(\widetilde{R}) = (\widetilde{x}, \widetilde{y})\widetilde{R} \text{ with } (\widetilde{J}, \widetilde{R}, \widetilde{x}, \widetilde{y}) = (\widehat{J}, \widehat{R}, \widehat{x}, \widehat{y} + \widehat{x}), \\ \overline{J} = (\overline{a}, \overline{b})\overline{R} \text{ and } M(\overline{R}) = (\overline{x}, \overline{y})\overline{R} \text{ with } \overline{R} \in Q_1(\widetilde{R}) \text{ and } (\overline{x}, \overline{y}) = (\widetilde{x}, \widetilde{y}/\widetilde{x}), \end{cases}$$

where

$$\begin{cases}(b,a) = (x^5, [x^4y + x^3y^2] + x^3y^3 + 3x^2y^4 + 3xy^6 + y^8),\\ (\widehat{b},\widehat{a}) = (\widehat{x}^5, (\widehat{x}^4 + \widehat{x}^3\widehat{y}) + \widehat{x}^3 + 3\widehat{x}^2\widehat{y} + 3\widehat{x}\widehat{y}^2 + \widehat{y}^3),\\ (\widetilde{b},\widetilde{a}) = (\widetilde{x}^5, \widetilde{x}^3\widetilde{y} + \widetilde{y}^3),\\ (\overline{b},\overline{a}) = (\overline{x}^2, \overline{x}\ \overline{y} + \overline{y}^3).\end{cases}$$

Here we are given the fourth row $\overline{J} = (\overline{a},\overline{b})\overline{R}$*, which is dicritical by* (4.1)(ii)*. We construct the third row* $\widetilde{J} = (\widetilde{a},\widetilde{b})\widetilde{R}$ *so that its transform* $(\widetilde{R},\overline{R})(\widetilde{J})$ *equals* $\overline{J}$*. Taking new generators for* $M(\widetilde{R})$ *we get the second row* $\widehat{J} = (\widehat{a},\widehat{b})\widehat{R}$*. This gives rise to the first row* $J = (a,b)R$ *whose transform equals* $\widetilde{J}$*. Now* J *is dicritical by* (4.1)(ii) *because the bracketed terms constitute the info of* a*.*

Proposition 4.3. *We shall now generalize* (4.2) *in two pieces. In* (ii) *and* (iii) *we shall deal with the piece* $\overline{J} \to \widetilde{J}$*, whereas in* (iv) *we shall deal with the piece* $\widetilde{J} \to J$*. In* (v)*, which follows from* (ii) *to* (iv)*, by putting the two pieces together, we shall construct a special tower of any height having big stars exactly at any noncontiguous preassigned spots. The first assertion* (i) *is auxiliary and follows from* (4.1)*. Let* R' *and* R'' *be the unique members of* $Q_1(R)$ *such that* $M(R') = (x',y')R'$ *and* $M(R'') = (x'',y'')R''$ *with* $(x',y') = (x,y/x)$ *and* $(x'',y'') = (x/y,y)$*. Let* S *be any coefficient ring of* R*; for instance* $S = R$*. Note that then* S *is a coefficient ring also of* R' *and* R''*.*

Informally speaking, the four pencils $J, \widehat{J}, \widetilde{J}, \overline{J}$ *constructed in* (4.2) *are all of the form* $J = (a,b)R$*, where*

$$(b,a) = (x^e, f(x,y) = y^c + xyw) \ \text{ with } \ w \in R \ \text{ and } \ f = f(X,Y) \in S[X,Y].$$

To define the four types of f *we let* $ord_R a = d$ *and introduce subsets*

$$S(e,d,c), \quad S(e > d = c), \quad S(e < d \le c), \quad S(e = d < c)$$

of $S[X,Y]$*. Here the first subset contains the next three. The second and third subsets correspond to the nondicritical pencils* $\widehat{J}, \widetilde{J}, \overline{J}$*. The last subset corresponds to the dicritical pencil* J*. Formally speaking:*

For any e,d,c *in* $\mathbb{N}_+$*,*

$$\begin{cases}\text{let } S(e,d,c) \text{ be the set of all } f = f(X,Y) \text{ in } S[X,Y]\\ \text{such that } \deg_X f(X,Y) < e \text{ with } \deg_Y f(X,Y) = c\\ \text{and } \mathrm{ord}_{X,Y} f(X,Y) = d \text{ with } \mathrm{ord}_Y f(X,Y) > 0\\ \text{and } f(0,Y) = Y^c.\end{cases}$$

For any $e > d = c$ *in* $\mathbb{N}_+$*, let*

$$S(e > d = c) = \{f \in S(e,d,c) : \mathrm{ord}_{X,Y}[f(X,Y) - f(0,Y)] > c\}.$$

For any $e < d \le c$ *in* $\mathbb{N}_+$*, let*

$$S(e < d \le c) = \{f \in S(e,d,c) : \mathrm{ord}_{X,Y}[f(X,Y) - f(0,Y)] > e\}.$$

For any $e = d < c$ *in* $\mathbb{N}_+$*, let*

$$S(e = d < c) = \{f \in S(e,d,c) : \mathrm{ord}_X \mathrm{info}_{X,Y} f(X,Y) < e\}.$$

For any $f(X,Y) \in S[X,Y]$ *and* $e \in \mathbb{N}$*, upon writing*

$$f(X,Y) = \sum_{0 \le i < \infty} f_i(Y)X^i \quad \text{with} \quad f_i(Y) \in S[Y],$$

we define the (X,e)-truncation $\tau_e f(X,Y)$ of $f(X,Y)$ by putting

$$\tau_e f(X,Y) = \sum_{0\le i<e} f_i(Y)X^i.$$

(i) If $J=(a,b)R$, where $b=x^e$ and $a=f(x,y)$ with

$$f\in S(e>d=c) \quad \text{or} \quad f\in S(e<d\le c) \quad \text{or} \quad f\in S(e=d<c),$$

then J is a special primary pencil at R, and R is respectively a

small star or small star or big star

of J, and we always have that J is unifurcated at R and

$$\mathrm{ord}_R b=e \text{ and } \mathrm{ord}_R a=d \text{ and } \mathrm{ord}_{(R/xR)} a=c \text{ and } \mathrm{ord}_{(R/yR)} a=\infty.$$

(ii) Consider the special primary pencil $J'=(a',b')R'$ at R' given by $b'=x'^e$ and $a'=f'(x',y')$ with $f'\in S(e,d,c)$. Let $f(X,Y)=X^c f'(X,Y/X)$. Then $f\in S(e+c>c=c)$ and for the special primary pencil $J=(a,b)R$ at R given by $b=x^{e+c}$ and $a=f(x,y)$ we have $(R,R')(J)=J'$ and R is a unifurcated small star of J.

Proof. Clearly $f'\in S[X,Y]$ belongs to $S(e,d,c)$ iff

$$f'(X,Y)=Y^c+\sum_{1\le i<e \text{ and } 1\le j\le c} \alpha_{ij}X^iY^j \quad \text{with} \quad \alpha_{ij}\in S$$

and

$$\min(\{c\}\cup\{i+j:\alpha_{ij}\ne 0\})=d.$$

Since $f(X,Y)=X^c f'(X,Y/X)$ we get

$$f(X,Y)=Y^c+\sum_{1\le i<e \text{ and } 1\le j\le c} \alpha_{ij}X^{i+c-j}Y^j$$

and hence $f(X,Y)\in S[X,Y]$ is such that

$$\begin{cases} \deg_X f(X,Y)<e+c \text{ with } \deg_Y f(X,Y)=c \\ \text{and } \mathrm{ord}_{X,Y} f(X,Y)=c \text{ with } \mathrm{ord}_Y f(X,Y)>0 \\ \text{and } f(0,Y)=Y^c \text{ with } \mathrm{ord}_{X,Y}[f(X,Y)-f(0,Y)]>c. \end{cases}$$

It follows that $f\in S(e+c>c=c)$ and for the special primary pencil $J=(a,b)R$ at R given by $b=x^{e+c}$ and $a=f(x,y)$ we have $(R,R')(J)=J'$. By (i) we also see that R is a unifurcated small star of J. $\square$

(iii) Consider the special primary pencil $J''=(a'',b'')R''$ at R'' given by $b''=x''^e$ and $a''=f''(x'',y'')$ with $f''\in S(e,d,c)$. Let $f(X,Y)=Y^e f''(X/Y,Y)$. Then $f(X,Y)\in S[X,Y]$ and letting $d^*=\mathrm{ord}_{X,Y} f(X,Y)$ we have that $e<d^*\le e+c$ and $f\in S(e<d^*\le e+c)$, and for the special primary pencil $J=(a,b)R$ at R given by $b=x^e$ and $a=f(x,y)$ we have $(R,R'')(J)=J''$ and R is a unifurcated small star of J.

Proof. Clearly $f''\in S[X,Y]$ belongs to $S(e,d,c)$ iff

$$f''(X,Y)=Y^c+\sum_{1\le i<e \text{ and } 1\le j\le c} \alpha_{ij}X^iY^j \quad \text{with} \quad \alpha_{ij}\in S$$

and

$$\min(\{c\}\cup\{i+j:\alpha_{ij}\ne 0\})=d.$$

Since $f(X,Y) = Y^e f''(X/Y,Y)$, we get

$$f(X,Y) = Y^{e+c} + \sum_{1\le i<e \text{ and } 1\le j\le c} \alpha_{ij} X^i Y^{j+e-i}$$

and hence $f(X,Y) \in S[X,Y]$ is such that

$$\begin{cases} \deg_X f(X,Y) < e \text{ with } \deg_Y f(X,Y) = e+c \\ \text{and } \operatorname{ord}_{X,Y} f(X,Y) = d^* \text{ with } \operatorname{ord}_Y f(X,Y) > 0 \\ \text{and } f(0,Y) = Y^{e+c} \text{ with } \operatorname{ord}_{X,Y}[f(X,Y) - f(0,Y)] > e \end{cases}$$

with

$$e < d^* \le e + c.$$

It follows that $f \in S(e < d^* \le e+c)$ and for the special primary pencil $J = (a,b)R$ at R given by $b = x^e$ and $a = f(x,y)$ we have $(R,R'')(J) = J''$. By (i) we also see that R is a unifurcated small star of J. □

(iv) Consider the generators (u,v) of $M(R'')$ given by $(u,v) = (x'', y'' + x'')$. Consider the special primary pencil $J'' = (a'', b'')R''$ at R'' given by $b'' = u^e$ and $a'' = f''(u,v)$ with $f'' \in S(e > d = c)$. Then there exists $f \in S(e = e < e+c)$ such that for the special primary pencil $J = (a,b)R$ at R given by $b = x^e$ and $a = f(x,y)$ we have $(R,R'')(J) = J''$ and R is a unifurcated big star of J. Actually we may take f to be given by the explicit formula $f(X,Y) = Y^e \widehat{f}(X/Y,Y)$, where $\widehat{f}(X,Y) = \tau_e f''(X, Y+X)$.

Proof. Clearly $f'' \in S[X,Y]$ belongs to $S(e > d = c)$ iff

$$f''(X,Y) = Y^c + \sum_{1\le i<e \text{ and } 1\le j\le c} \alpha_{ij} X^i Y^j \quad \text{with} \quad \alpha_{ij} \in S$$

and

$$i + j > c \text{ for all } 1 \le i < e \text{ and } 1 \le j \le c \text{ for which } \alpha_{ij} \ne 0.$$

Now upon letting

$$\overline{f}(X,Y) = f''(X, Y+X) - X^c$$

we get $\overline{f}(X,Y) \in S[X,Y]$ with

$$\overline{f}(X,Y) = \sum_{0\le i<\infty} \overline{f}_i(Y) X^i, \quad \text{where} \quad \overline{f}_i(Y) \in S[Y]$$

and

$$\overline{f}_0(Y) = Y^c \text{ and } \deg_Y \overline{f}_i(Y) \le c \text{ with } \overline{f}_i(0) = 0 \text{ for } 0 \le i < \infty.$$

Let

$$g^*(X,Y) = \sum_{0\le i<e} \overline{f}_i(Y) X^i \quad \text{and} \quad g^{**}(X,Y) = \sum_{i\ge e} \overline{f}_i(Y) X^i.$$

Now $g^{**}(X,Y) = Y X^e g(X,Y)$ for some $g(X,Y) \in S[X,Y]$ and hence

$$J'' = (b^*, a^*)R'' \quad \text{with} \quad (b^*, a^*) = (x''^e, x''^c + g^*(x'', y'')).$$

Clearly $g^*(X,Y) \in S(e,d,c)$ and hence, upon letting $f^*(X,Y) = Y^e g^*(X/Y,Y)$, by (iii) we see that $f^* \in S(e < d^* \le e+c)$ for some $d^* \in \mathbb{N}_+$. Let

$$f(X,Y) = X^c Y^{e-c} + f^*(X,Y).$$

Then $f \in S(e = e < e+c)$ and for the special primary pencil $J = (a,b)R$ at R given by $(b,a) = (x^e, f(x,y))$ we have $(R,R'')(J) = (x''^e, x''^c + g^*(x'',y''))R'' = J''$. By

(i) we also see that R is a unifurcated big star of J. It only remains to note that clearly we have $f(X,Y) = Y^e \widehat{f}(X/Y, Y)$, where $\widehat{f}(X,Y) = \tau_e f''(X, Y+X)$. $\square$

(v) Let there be given any finite QDT sequence $(R_j)_{0\le j\le\nu}$ of R and a sequence of integers $0 \le i_1 < \cdots < i_\mu \le \nu$ such that $i_2 - i_1 \ge 2, \dots, i_\mu - i_{\mu-1} \ge 2$. For $0 \le j \le \nu$ let (x_j, y_j) as well as (u_j, v_j) be generators of $M(R_j)$ such that $(u_j, v_j) = (x_j, y_j + \lambda_j x_j)$ with $\lambda_j \in \{0,1\}$. Let $\{0,1,\dots,\nu-1\} = W' \coprod W''$ be a disjoint partition; i.e., W' and W'' are subsets of $\{0,1,\dots,\nu-1\}$ such that $W' \cup W'' = \{0,1,\dots.\nu-1\}$ and $W' \cap W'' = \emptyset$. Assume that for all $j \in W'$ we have $(x_{j+1}, y_{j+1}) = (u_j, v_j/u_j)$, and for all $j \in W''$ we have $(x_{j+1}, y_{j+1}) = (u_j/v_j, v_j)$. Let there be given any e_ν, d_ν, c_ν in $\mathbb{N}_+$ and $f_\nu \in S(e_\nu, d_\nu, e_\nu)$ and a special primary pencil $J_\nu = (a_\nu, b_\nu)R_\nu$ at R_ν such that $(b_\nu, a_\nu) = (u_\nu^{e_\nu}, f_\nu(u_\nu, v_\nu))$; note that by (4.1) we have

$$e_\nu = d_\nu = c_\nu \Leftrightarrow R_\nu \text{ is a terminal point of } J_\nu \Rightarrow R_\nu \text{ is a big star of } J_\nu$$

and

$$R_\nu \text{ is a nonterminal big star of } J_\nu \Rightarrow e_\nu = d_\nu < c_\nu.$$

For $0 \le j < \nu$ let e_j, c_j in $\mathbb{N}_+$ be inductively defined by requiring that

$$(e_j, c_j) = \begin{cases} (e_{j+1} + c_{j+1}, c_{j+1}) & \text{if } j \in W', \\ (e_{j+1}, e_{j+1} + c_{j+1}) & \text{if } j \in W''. \end{cases}$$

Assume that

(1) $i_\mu = \nu$ iff R_ν is a big star of J_ν;
(2) if $i_\mu = \nu$, then $\lambda_\nu = 0$;
(3) if $\lambda_\nu \ne 0$, then $e_\nu > d_\nu = c_\nu$;
(4) if $\lambda_{j+1} \ne 0$ for a nonnegative integer $j < \nu$, then $e_{j+1} > c_{j+1}$.

Let $W^* = \{i_1, i_2, \dots, i_\mu\} \setminus \{\nu\}$ and assume that

$$\{0 \le j < \nu : \lambda_{j+1} \ne 0\} = W^* \subset W''.$$

Then for $0 \le j < \nu$ there exist d_j in $\mathbb{N}_+$ and $f_j \in S(e_j, d_j, c_j)$ and a special primary pencil $J_j = (a_j, b_j)R_j$ at R_j with $(b_j, a_j) = (u_j^{e_j}, f_j(u_j, v_j))$ such that

$$(R_j, R_{j+1})(J_j) = J_{j+1}$$

and

$$\begin{cases} e_j > d_j = c_j & \text{if } j \in W', \\ e_j < d_j \le c_j & \text{if } j \in W'' \setminus W^*, \\ e_j = d_j < c_j & \text{if } j \in W^* \end{cases}$$

and

$$f_j \in \begin{cases} S(e_j > d_j = c_j) & \text{if } j \in W', \\ S(e_j < d_j \le c_j) & \text{if } j \in W'' \setminus W^*, \\ S(e_j = d_j < c_j) & \text{if } j \in W^*. \end{cases}$$

Moreover, for $0 \le j < \nu$ we have that R_j is a unifurcation point of J_j. Finally, for $0 \le j \le \nu$ we have that R_j is a big star of J_j or a small star of J_j according as $j \in \{i_1, \dots, i_\mu\}$ or $j \notin \{i_1, \dots, i_\mu\}$. Actually we may take f_j to be given explicitly for $0 \le j < \nu$ for the inductive formulas

$$\widehat{f}_{j+1}(X,Y) = \begin{cases} f_{j+1}(X,Y) & \text{if } \lambda_{j+1} = 0, \\ (X, e_{j+1})\text{-truncation of } f_{j+1}(X, Y+X) & \text{if } \lambda_{j+1} \ne 0 \end{cases}$$

and

$$f_j(X,Y) = \begin{cases} X_{j+1}^{c_{j+1}} \widehat{f}_{j+1}(X, Y/X) & \text{if } j \in W', \\ Y_{j+1}^{e_{j+1}} \widehat{f}_{j+1}(X/Y, Y) & \text{if } j \in W''. \end{cases}$$

We observe that $(R_j, J_j)_{0 \le j \le \nu}$ **is a special tower at** R **with big stars exactly at** $(R_{i_l})_{1 \le l \le \mu}$, where $0 \le i_1 < \cdots < i_\mu \le \nu$ with any preassigned $\mu \in \mathbb{N}$ and any preassigned integers ≥ 2 as values of $i_2 - i_1, \ldots, i_\mu - i_{\mu-1}$. The tower is terminal iff $e_\nu = d_\nu = c_\nu$.

Note 4.4. The example in (4.2) can be made terminal by inserting two more bottom rows, and then it will correspond to the $\nu = 4$ and $\mu = 3$ case of (4.3)(v).

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