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PINCH INSTABILITIES IN MAGNETIC STARS

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SUMMARY

It is suggested that hydromagnetic 'pinch' instabilities are capable of efficient magnetic flux destruction in the early stages of a star's history. The stability of an axisymmetric magnetic field in the convectively stable region of a star is investigated. It is found that a purely poloidal field is unstable to the kink instability, but that a mixed poloidal–toroidal field may be stable if the toroidal component is sufficiently large. One condition necessary for a star to be magnetic could therefore be the presence of a toroidal field strong enough to prevent instability.

INTRODUCTION

Magnetic stars have surface magnetic fields ranging from 100 Gauss (the limit of detectability) to $3\cdot5\times10^4$ Gauss (Babcock 1960). They form only a small fraction of the main-sequence population (roughly 25 per cent of all A-stars are magnetic) and are generally associated with rotation periods well below the average period for stars of comparable temperature (Slettebak 1955). If one assumes that the magnetic field is approximately uniform throughout the star, the ratio of magnetic to centrifugal and gravitational energies for such an object is typically $1 : 10^6 : 10^9$. On the other hand if we are to explain the existence of stellar magnetism as being due to the squeezing together of field lines while the star condenses out of a primeval gas cloud (Mestel 1965a) then we would expect the field to play an important rôle in star formation, and that in some cases the magnetic and gravitational energies would be comparable even after the star reaches the main sequence. We need some mechanism of flux-destruction, to explain not only why some stars have apparently lost much more flux than others, but also why even the most strongly magnetic have a magnetic energy much below the gravitational energy.

Various means for disposing of unwanted flux have been put forward. Mestel & Spitzer (1956) suggested that the field, plus the low density ionized component of the gas, would slip out of a cloud early in its contraction phase. With newer estimates of ion-neutral coupling, and of the efficiency of galactic ionizing mechanisms, Spitzer (1968) finds that this process is likely to occur in dusty clouds but at a somewhat higher density phase. However we may still ask what happens to gravitationally collapsing magnetic clouds and proto-stars in which the ion density is maintained too high for this mechanism to operate. The Hayashi turbulence will tend to tangle and possibly destroy a weak primeval flux; but it seems probable that in a star whose magnetic energy exceeds the turbulent energy (and certainly in any star with comparable magnetic and gravitational energies) the flux can preserve itself, possibly concentrated into filaments (Moss 1970).

One can, however, approach the problem from the other end. The field structure in many stars may be such that the field intensity increases dramatically in their deep interiors. The outer regions, with relatively low conductivity and high susceptibility to circulation currents, may well have twisted and snapped potential emergent field lines at some time in the star's history and trapped them beneath the surface (Mestel 1965b). In the magnetic stars the field could have reacted back on the circulation in such a way that detailed radiative equilibrium is established with an appreciable surface field. Models constructed on this basis (Davies 1968; Wright 1969) display a strong intensification of field towards the centre of the star, but not sufficient to increase the magnetic energy even to the level of the centrifugal energy (although up by a factor of 10^5 from the uniform field approximation).

The possibility to be explored in this paper is that the natural processes of flux-decay are accelerated by the development of instabilities which can reduce the magnetic energy at least to the levels just mentioned. In particular, we shall consider the astrophysical analogues of the pinch instabilities which arise in laboratory attempts to produce controlled thermonuclear reactions. The equilibrium of a hot plasma confined in a strong magnetic field has been found to be highly unstable for all simple field configurations. The most catastrophic of these instabilities take place on a dynamical time-scale and initially attracted the greatest theoretical attention (e.g. Tayler 1959). It was shown that the most easily excited unstable modes were the 'kink' and 'sausage' instabilities, so-called because of the characteristic deformations of the flux tubes. However, even when experimental devices are used to stabilize these modes, somewhat slower 'resistive' modes set in as well as certain types of microinstabilities.

The possibility of resistive instabilities in astrophysical situations has already received attention (Paris 1971); they may be important, for example, for the coalescing of field lines and the formation of neutral points during the early development of a proto-star. In this paper, however, we shall be concerned with the more rapid hydromagnetic instabilities and look for the analogues of the kink and sausage modes. An overall view of the rôle of hydromagnetic instabilities in astrophysics has been given by Tayler (1971), but here we concentrate on the implications for the survival of magnetic fields inside stars. We shall consider a star with an axisymmetric magnetic field and a neutral line which encircles the axis of symmetry in the radiative envelope. The gas in the region of the neutral line exists inside a magnetic torus similar in structure to those of the terrestrial plasma-containment models, with the important differences that it possesses no sharp boundaries and is acted upon by a strong gravitational field. The gravitational field exerts a considerable stabilizing influence: since we are in a non-convecting region any displacement with a component parallel to the gravitational field will be acted upon by strong restoring forces due to the super-adiabatic temperature gradient. Thus in our search for instabilities we shall investigate only those perturbations which displace the field lines horizontally. In the first section of the paper we consider the stability of purely poloidal field-structures against such perturbations by applying a technique developed by Moss & Tayler (1969). An analogue of the kink mode is still found to be unstable, and two other modes are possibly also unstable. In the second section these three modes are separately examined including the influence of a toroidal field. Using the simplified geometry of a straight circular cylinder, it is found that the kink instability can be prevented by a sufficiently large toroidal field, whilst even a very small toroidal field will stabilize the other two modes. The introduction of a toroidal component results in there being no perturbation which is both horizontal and which does not appreciably distort the field lines; the kinking of the toroidal field produces strong restoring forces resisting the growth of the perturbation. The kink instability in 'terrestrial' theory comes about through the interchange of adjacent field lines with minimum distortion, but such interchanges are now strongly stabilized by gravitational restoring forces.

The tentative conclusion that a substantial toroidal flux is necessary for stability may link up with recent work on the oblique rotator model. Mestel & Takhar (1972) have described a dissipative mechanism which gives rise to a spontaneous widening or contracting of the angle between the magnetic and rotation axes according as the star is dynamically prolate or oblate. A 'girdle' of toroidal flux linking the poloidal flux would tend to compress matter inwards at the equator and, if stronger than the opposing effect of the poloidal field, would result in dynamical prolateness. Such a girdle might arise during the early history of a star from the twisting of poloidal field lines by differential rotation and subsequent Ohmic diffusion. Preston (1971) observed magnetic stars to be either very wide-angle or very small-angle oblique rotators. If we interpret this as a grouping of dynamically prolate and oblate stars respectively, then any stability criterion must permit the existence of stable configurations whose toroidal flux does not *necessarily* produce dynamical prolateness. Computations are under way to determine the critical ratio of toroidal to poloidal flux required to yield a dynamically prolate star (Milsom & Wright, in preparation).

THE STABILITY OF POLOIDAL FIELDS

We consider the stability of a perfectly-conducting gas-cloud with an axially-symmetric poloidal magnetic field. We shall assume that the bulk of the mass is concentrated in a small volume at the centre of the cloud and hence that the effect of self-gravitation may be neglected in the regions of interest. Bernstein *et al.* (1958) have given the following general expression for the change in potential energy of an axially-symmetric static equilibrium due to an arbitrary perturbation $\boldsymbol{\xi}(\mathbf{r}, t)$:

$$\delta W = \tfrac{1}{2}\int d\tau[\mathbf{Q}^2 - \mathbf{j}.\mathbf{Q}\wedge\boldsymbol{\xi} + \gamma P(\nabla.\boldsymbol{\xi})^2 + (\boldsymbol{\xi}.\nabla P)\nabla.\boldsymbol{\xi} - (\boldsymbol{\xi}.\nabla\Phi)\nabla.(\rho\boldsymbol{\xi})] \tag{1}$$

where $\mathbf{B}, \mathbf{j}, \rho, P, \Phi, \gamma$ are the magnetic field, electric current, density, pressure, gravitational potential and ratio of specific heats in the undisturbed equilibrium, and $\mathbf{Q} = \nabla\wedge(\boldsymbol{\xi}\wedge\mathbf{B})$. If there exists a perturbation such that δW is negative then the equilibrium is unstable. Thus to examine stability we must minimize (1) with respect to $\boldsymbol{\xi}$.

Following Bernstein, we can define in a natural way a coordinate system (ψ, χ, ϕ) based on the field lines, with ψ as $1/2\pi$ times the magnetic flux between the axis of symmetry and the relevant field line, and ϕ as the azimuthal angle about the axis. χ is chosen so that its coordinate vector forms a right-handed triad with the other two coordinate vectors, but its definition is for the time being left incomplete. The volume element is given by

$$d\tau = J\, d\psi\, d\phi\, d\chi$$

where the functional form of J will depend on the exact definition of χ. In these coordinates the two components of the equation of hydrostatic balance in the unperturbed state become

$$\left.\begin{aligned} -\varpi B\frac{\partial P}{\partial\psi}+\rho g_\psi-\frac{\varpi B}{J}\frac{\partial(B^2J)}{\partial\psi} &= 0 \\ -\frac{1}{JB}\frac{\partial P}{\partial\chi}+\rho g_\chi &= 0 \end{aligned}\right\} \qquad (2)$$

where $\mathbf{g} = -\nabla\Phi$ and ϖ is the distance from the axis.

Apart from the perturbation $\boldsymbol{\xi}$, all quantities in the integrand of (1) are independent of ϕ. Consequently we may Fourier-analyse $\boldsymbol{\xi}$ in terms of ϕ, giving the complex perturbation as

$$\xi_\psi = \frac{X(\psi,\chi)}{\varpi B}\exp(ip\phi)$$

$$\xi_\phi = \frac{\varpi}{p}Y(\psi,\chi)\exp(ip\phi)$$

$$\xi_\chi = BZ(\psi,\chi)\exp(ip\phi)$$

where p is a non-zero integer (the case $p = 0$ has to be treated separately and has been shown to be unimportant (Moss & Tayler 1969)). For each p we can consider the sign of δW independently. After integrating with respect to ϕ, δW is given by

$$\begin{aligned} \delta W = \pi\int d\psi\,d\chi\Bigg\{&\frac{1}{p^2}\frac{\varpi^2}{J}\frac{\partial Y}{\partial\chi}\frac{\partial Y^*}{\partial\chi}+\frac{1}{\varpi^2B^2J}\frac{\partial X}{\partial\chi}\frac{\partial X^*}{\partial\chi} \\ &+B^2J\left(\frac{\partial X}{\partial\psi}+iY\right)\left(\frac{\partial X^*}{\partial\psi}-iY^*\right)-\frac{\partial(JB^2)}{\partial\psi} \\ &\times\left[X\frac{\partial X^*}{\partial\psi}+iYX^*+\frac{Z}{J}\frac{\partial X^*}{\partial\chi}\right]+\frac{\gamma P}{J}\left[\frac{\partial(JX)}{\partial\psi}+iJY+\frac{\partial Z}{\partial\chi}\right] \\ &\times\left[\frac{\partial(JX^*)}{\partial\psi}-iJY^*+\frac{\partial Z^*}{\partial\chi}\right]+\left[X\left(\frac{\partial P}{\partial\psi}+\frac{\rho}{\varpi B}g_\psi\right)\right. \\ &\left.+Z\left(\frac{1}{J}\frac{\partial P}{\partial\chi}+\rho Bg_\chi\right)\right]\left[\frac{\partial(JX^*)}{\partial\psi}-iJY^*+\frac{\partial Z^*}{\partial\chi}\right] \\ &+J\left[\frac{Xg_\psi}{\varpi B}+ZBg_\chi\right]\left[X^*\frac{\partial\rho}{\partial\psi}+\frac{Z^*}{J}\frac{\partial\rho}{\partial\chi}\right]+\text{complex conjugate}\Bigg\} \end{aligned}$$

with the only explicit dependence upon p occurring in the first term. This is clearly minimized by taking $p\to\infty$, indicating that in the absence of constraining toroidal flux the most unstable perturbations are those with the greatest amount of ' wriggle ' in the azimuthal direction. This integral was minimized by Moss and Tayler with respect to $iY+\partial X/\partial\psi$ and yields the Euler equation

$$(B^2+\gamma P)\left(\frac{\partial X^*}{\partial\psi}-iY^*\right) = -\left[\frac{\gamma P}{J}\frac{\partial J}{\partial\psi}X^*+\frac{\gamma P}{J}\frac{\partial Z^*}{\partial\chi}+\frac{\partial P}{\partial\psi}X^*+\frac{1}{J}\frac{\partial P}{\partial\chi}Z\right] \qquad (3)$$

giving, after substitution,

$$\delta W \propto \int d\psi \int d\chi \left\{ \frac{1}{\varpi^2 B^2 J} \frac{\partial X}{\partial \chi} \frac{\partial X^*}{\partial \chi} - \frac{1}{J} \frac{\partial (B^2 J)}{\partial \psi} Z \frac{\partial X^*}{\partial \chi} + \frac{\gamma P}{J} \frac{\partial Z}{\partial \chi} \frac{\partial Z^*}{\partial \chi} \right.$$
$$+ \left[\frac{\partial J}{\partial \psi} X^* + \frac{\partial Z^*}{\partial \chi} \right] \left[X \left(\frac{\partial P}{\partial \psi} + \frac{\rho}{\varpi B} g_\psi \right) + Z \left(\frac{1}{J} \frac{\partial P}{\partial \chi} + \rho B g_\chi \right) \right]$$
$$+ \frac{\gamma P}{J} \left(\frac{\partial J}{\partial \psi} \right)^2 X X^* + 2 \frac{\gamma P}{J} \frac{\partial J}{\partial \psi} X \frac{\partial Z^*}{\partial \chi}$$
$$+ J \left[\frac{X g_\psi}{\varpi B} + Z B g_\chi \right] \left[X^* \frac{\partial \rho}{\partial \psi} + \frac{Z^*}{J} \frac{\partial \rho}{\partial \chi} \right] - \frac{J}{B^2 + \gamma P}$$
$$\left. \times \left| \frac{\gamma P}{J} \frac{\partial J}{\partial \psi} X + \frac{\gamma P}{J} \frac{\partial Z}{\partial \chi} + \frac{\partial P}{\partial \psi} X + \frac{1}{J} \frac{\partial P}{\partial \chi} Z \right|^2 + \text{complex conjugate} \right\} \tag{4}$$

which can be written in the form

$$\delta W = \int d\psi \int F \, d\chi$$

where F contains no ψ-derivatives of X and Z. It is now an obvious theorem that a separate criterion for stability can be derived for each magnetic surface ψ = const. If any one of these criteria is unsatisfied then there exists a perturbation, localized near the relevant magnetic surface, which yields a negative δW. This result was first demonstrated by Bernstein *et al.* for the case of zero gravitational field, and was found by Moss & Tayler to be also true when $g \neq 0$. Its importance here is that it enables us to isolate the narrow torus formed by the material surrounding the neutral line, and to consider its stability independently of the rest of the star.

If we now separate X and Z into their real and imaginary parts by writing them as $X_R + iX_I$ and $Z_R + iZ_I$, then $\int F \, d\chi$ separates into two integrals identical in form but one involving only X_R and Z_R, and the other only X_I and Z_I. Thus we drop the suffices and, for a given magnetic surface, minimize the following integral with respect to real functions X and Z:

$$I = \int d\chi \left\{ \frac{1}{\varpi^2 B^2 J} \left(\frac{\partial X}{\partial \chi} \right)^2 - \frac{1}{J} \frac{\partial (B^2 J)}{\partial \psi} Z \frac{\partial X}{\partial \chi} + \gamma P \left(\frac{\partial Z}{\partial \chi} \right)^2 + \frac{\gamma P}{J} \left(\frac{\partial J}{\partial \psi} \right)^2 X^2 \right.$$
$$+ \frac{2\gamma P}{J} \frac{\partial J}{\partial \psi} X \frac{\partial Z}{\partial \chi} + J \left[\frac{X g_\psi}{\varpi B} + Z B g_\chi \right] \left[X \frac{\partial \rho}{\partial \psi} + \frac{Z}{J} \frac{\partial \rho}{\partial \chi} \right]$$
$$+ \left[\frac{\partial J}{\partial \psi} X + \frac{\partial Z}{\partial \chi} \right] \left[X \left(\frac{\partial P}{\partial \psi} + \frac{\rho}{\varpi B} g_\psi \right) + Z \left(\frac{1}{J} \frac{\partial P}{\partial \chi} + \rho B g_\chi \right) \right]$$
$$\left. - \frac{J}{B^2 + \gamma P} \left[\frac{\gamma P}{J} \frac{\partial J}{\partial \psi} X + \frac{\gamma P}{J} \frac{\partial Z}{\partial \chi} + \frac{\partial P}{\partial \psi} X + \frac{1}{J} \frac{\partial P}{\partial \chi} Z \right]^2 \right\}.$$

The integrand may be re-arranged to give

$$I = \int d\chi \left\{ \frac{1}{\varpi^2 B^2 J} \left(\frac{\partial X}{\partial \chi} \right)^2 + \frac{1}{J} \frac{\partial (B^2 J)}{\partial \psi} \left[\left(\frac{\partial J}{\partial \psi} X + \frac{\partial Z}{\partial \chi} \right) X - Z \frac{\partial X}{\partial \chi} \right] \right.$$
$$- \frac{J}{\gamma P} \left(\frac{\partial P}{\partial \psi} X + \frac{1}{J} \frac{\partial P}{\partial \chi} Z \right)^2 + J \left(\frac{X g_\psi}{\varpi B} + Z B g_\chi \right) \left(X \frac{\partial \rho}{\partial \psi} + \frac{Z}{J} \frac{\partial \rho}{\partial \chi} \right)$$
$$\left. + \frac{J B^2}{B^2 + \gamma P} \left[\frac{\gamma P}{J} \left(\frac{\partial J}{\partial \psi} X + \frac{\partial Z}{\partial \chi} \right) + \frac{\partial P}{\partial \psi} X + \frac{1}{J} \frac{\partial P}{\partial \chi} Z \right]^2 \right\}$$

where use has been made of equations (2). We can now easily gather together the terms which arise through buoyancy effects occurring even in a non-magnetic system; so that

$$I = \int d\chi \left\{ \frac{1}{J} \frac{\partial(B^2J)}{\partial\psi} \left[\frac{JX}{\gamma P} M - Z \frac{\partial X}{\partial\chi} \right] + \frac{JB^2}{\gamma P(B^2+\gamma P)} M^2 + \frac{1}{\varpi^2 B^2 J} \left(\frac{\partial X}{\partial\chi} \right)^2 + J\rho(\xi \cdot \mathbf{g}) (\boldsymbol{\xi} \cdot \nabla \ln (\rho/p^{1/\gamma})) \right\}$$

where

$$M = \frac{\gamma P}{J} \left(\frac{\partial J}{\partial\psi} X + \frac{\partial Z}{\partial\chi} \right) + \boldsymbol{\xi} \cdot \nabla P.$$

$\nabla \ln (\rho/p^{1/\gamma})$ is a vector perpendicular to surfaces of constant entropy. In an unperturbed star it would be radial and therefore parallel to $\mathbf{g}$, hence the sign of the last term in the integral would depend on the direction of this vector. This is, of course, the familiar convective criterion and, since we are concerned with the convectively stable region of a star, would yield a positive integral in the absence of a perturbing magnetic field.

In the vicinity of the neutral line the magnetic forces will be very weak in comparison to the gravitational, so we can expect little perturbation from the basic structure, and $\mathbf{g}$ and $\nabla \ln (\rho/p^{1/\gamma})$ will be almost parallel. This is likely to be the case anyway, since we are dealing with a region very close to the stellar equator, where the vectors will be exactly parallel and radial owing to the symmetry of the magnetic field. Thus any perturbation which is not nearly perpendicular to one of these vectors will give rise to a large positive contribution from this term. When $\boldsymbol{\xi}$ is almost perpendicular to $\mathbf{g}$ it is possible that the term may be small and negative (so-called 'sliding' instabilities (James & Kahn 1970)) but we shall simplify the problem by considering only those perturbations which are exactly perpendicular to the gravitational field. Hence

$$\boldsymbol{\xi} \cdot \mathbf{g} = 0$$

and

$$I = \int d\chi \left\{ \frac{1}{\varpi^2 B^2 J} \left(\frac{\partial X}{\partial\chi} \right)^2 + \frac{1}{J} \frac{\partial(JB^2)}{\partial\psi} \left[\frac{JXM}{\gamma P} - Z \frac{\partial X}{\partial\chi} \right] + \frac{JB^2M^2}{\gamma P(B^2+\gamma P)} \right\}$$

where now

$$M = \frac{\gamma P}{J} \left(\frac{\partial J}{\partial\psi} X + \frac{\partial Z}{\partial\chi} \right) - \frac{X}{J} \frac{\partial(B^2J)}{\partial\psi}.$$

We now define the two quantities f_1 and f_2 by

$$f_1 = \frac{\partial(B^2J)}{\partial\psi}$$

$$f_2 = \frac{\partial J}{\partial\psi} - \frac{1}{\gamma P} \frac{\partial(B^2J)}{\partial\psi}$$

which enables us to write the integral as

$$I = \int d\chi \left\{ \frac{1}{\varpi^2 B^2 J} \left(\frac{\partial X}{\partial\chi} \right)^2 + \left(f_1 + \frac{\gamma P B^2}{B^2+\gamma P} f_2 \right) f_2 \frac{X^2}{J} + \frac{f_1}{J} \left(X \frac{\partial Z}{\partial\chi} - Z \frac{\partial X}{\partial\chi} \right) + \frac{\gamma P B^2}{J(B^2+\gamma P)} \left(2f_2 X \frac{\partial Z}{\partial\chi} + \left(\frac{\partial Z}{\partial\chi} \right)^2 \right) \right\}.$$

In order to simplify further we must now make use of our assumptions about the field structure. Although it will not in general be the case, the field lines will be taken to be approximately circular near the neutral line. Then

$$d\psi = -\varpi B\,dr \tag{5}$$

where r is the radial distance from the neutral line, and if we further define χ to be the natural angular measure round the field line, then

$$J = r/B. \tag{6}$$

Using (5) and (6) we have

$$f_1 + \frac{\gamma P B^2}{B^2+\gamma P} f_2 = -\frac{2\gamma P}{\varpi(B^2+\gamma P)}$$

and

$$f_2 + \frac{2}{\varpi B^2} = \frac{\gamma P + B^2}{\varpi B^2 \gamma P}\left(1 + \frac{\partial \ln B}{\partial \ln r}\right)$$

whence

$$I = \int \frac{d\chi}{J}\left\{\frac{1}{\varpi^2 B^2}\left(\frac{\partial X}{\partial \chi}\right)^2 - \frac{2X^2}{\varpi^2 B^2}\left(1 + \frac{\partial \ln B}{\partial \ln r}\right) + \frac{\gamma P}{B^2+\gamma P}\left(B\frac{\partial Z}{\partial \chi} - 2X\right)^2\right\} + \int d\chi\, ZX \frac{\partial}{\partial \chi}\left(\frac{f_1}{J}\right) \tag{7}$$

where the last integral is a consequence of integrating the term $-(f_1/J)\,Z\,(\partial X/\partial\chi)$ by parts round the full length of the field line. This expression for I is the same as would have resulted from putting $\mathbf{g} = 0$, since we have not yet explicitly related X and Z by $\boldsymbol{\xi}.\mathbf{g} = 0$. Before doing this, however, we note that if we take B and J to be independent of χ, then I should be proportional to δW for the classical case of the straight cylindrical plasma under negligible gravitational forces. To see this we Fourier-analyse with respect to χ, i.e.

$$\boldsymbol{\xi} = \boldsymbol{\xi}_0\, e^{im\chi}$$

and consider the sign of the resulting value of $I(m)$. Since now $\partial/\partial\chi(f_1/J) = 0$, the only term in (7) which contains Z is the positive quantity

$$\frac{\gamma P}{B^2+\gamma P}\left(B\frac{\partial Z}{\partial\chi} - \frac{2X}{\varpi B}\right)^2$$

which is clearly minimized by

$$\frac{\partial Z}{\partial \chi} = \frac{2}{\varpi B^2} X. \tag{8}$$

Together with (3), this can be shown to imply $\nabla.\boldsymbol{\xi} = 0$, and hence that the perturbation is incompressible. The condition that $I(m)$ be positive then yields the criterion

$$\frac{m^2}{2} < 1 + \frac{d \ln B}{d \ln r} \tag{9}$$

for instability. If $m = 0$, $\partial X/\partial \chi = \partial Z/\partial X = 0$, and the instability criterion from (7) is

$$\frac{B^2-\gamma P}{B^2+\gamma P}+\frac{d\ln B}{d\ln r}>0 \tag{10}$$

but now $\nabla . \boldsymbol{\xi} \neq 0$ and we have a compressible mode. (9) and (10) are the fundamental results from the theory of pinched discharges (Tayler 1959; Kadomtsev 1966). The case $m = 0$ is the 'sausage' instability, with field lines alternately being squashed towards and expanded away from the axis. Criterion (9) is most easily satisfied for $m = 1$, which represents the 'kink' or 'wriggle' instability.

We now investigate the analogues of the above criteria when the perturbation is constrained by $\boldsymbol{\xi}.\mathbf{g} = 0$ and the effects of toroidal geometry are included. Let θ be the angle between the direction of $\mathbf{g}$ and the radial distance from the neutral

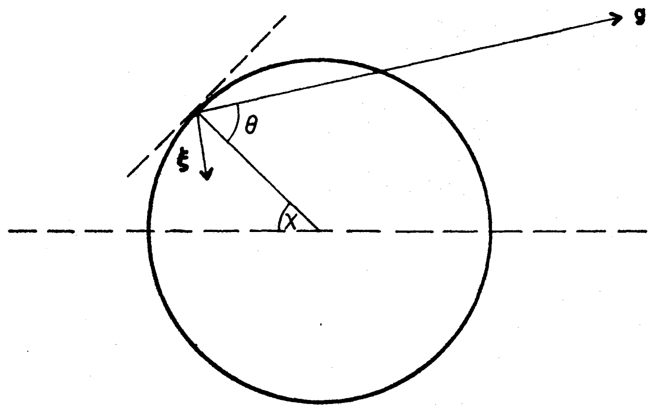

FIG. 1.

line (Fig. 1). Then X and Z are given by

$$X = \alpha(\chi)\sin\theta$$
$$Z = -\frac{\alpha(\chi)}{\varpi B^2}\cos\theta \tag{11}$$

where α is a periodic function of χ. This gives from (7)

$$I = \int_0^{2\pi}\frac{d\chi}{\varpi^2B^2J}\left\{(\alpha'\sin\theta+\theta'\alpha\cos\theta)^2-2\alpha^2(2+\delta)\sin^2\theta\right.$$
$$\left.+\frac{\gamma P}{B^2+\gamma P}\left(\alpha'\cos\theta+(2-\theta')\alpha\sin\theta+\alpha B\left(\frac{1}{B}\right)'\cos\theta\right)^2\right\}$$
$$-\int_0^{2\pi}\frac{d\chi}{\varpi B^2}\alpha^2\sin\theta\cos\theta\left(\frac{f}{J}\right)'$$

where

$$\delta = \frac{\partial\ln B}{\partial\ln r}-1$$

and dashes denote differentiation with respect to χ. If we now set $\kappa = B^2/(B^2+\gamma P)$ and integrate the term in $\alpha\alpha'$ by parts we get

$$I = \int_0^{2\pi}\frac{d\chi}{\varpi^2B^2J}\{(\alpha'^2-\theta'^2\alpha^2)(1-\kappa\cos^2\theta)+G(\chi)\alpha^2\} \tag{12}$$

where

$$G(\chi) = \theta'^2(1+(1-\kappa)\cos^2\theta)-2(2+\delta)\sin^2\theta+\frac{\varpi}{B}\left(\frac{B}{\varpi}(2+\delta)\right)'$$
$$+(1-\kappa)\left((2-\theta')\sin\theta+B\left(\frac{1}{B}\right)'\cos\theta\right)^2-J\left(\frac{F}{J}\right)'$$

and

$$F(\chi) = \theta'\cos\theta\sin\theta+(1-\kappa)\cos\theta\left((2-\theta')\sin\theta+B\left(\frac{1}{B}\right)'\cos\theta\right).$$

In order to proceed further with the minimization of (12) we shall now have to be more explicit about the relation between θ and χ, and also examine the variation of B, J, δ and ϖ with χ. We assume that the star is strongly condensed towards the centre and that the perturbing forces have not disturbed it sufficiently to distort the gravitational field from the radial direction. Thus the configuration is as shown

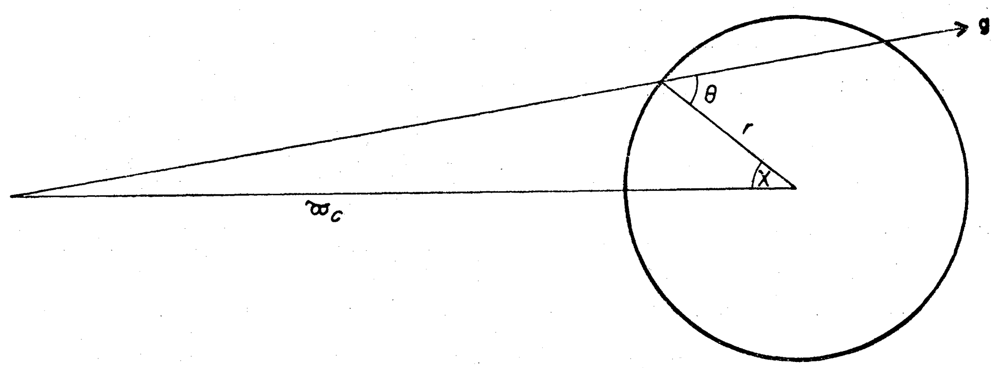

FIG. 2.

in Fig. 2, where ϖ_c is the radial distance of the neutral line from the centre of the star, and $r/\varpi_c \ll 1$. Elementary trigonometry now gives

$$\theta = \chi+\frac{r}{\varpi_c}\sin\chi+\frac{r^2}{\varpi_c^2}\sin\chi\cos\chi+O\left(\frac{r^3}{\varpi_c^3}\right)$$

$$\sin\theta = \sin\chi\left(1+\frac{r}{\varpi_c}\cos\chi+\frac{r^2}{\varpi_c^2}(\cos^2\chi-\tfrac{1}{2}\sin^2\chi)\right)+O\left(\frac{r^3}{\varpi_c^3}\right)$$

$$\cos\theta = \cos\chi-\frac{r}{\varpi_c}\sin^2\chi-\frac{3}{2}\frac{r^2}{\varpi_c^2}\sin^2\chi\cos\chi+O\left(\frac{r^3}{\varpi_c^3}\right)$$

$$\varpi = \varpi_c\left(1-\frac{r}{\varpi_c}\cos\chi\right)$$

and since ϖB is a function of r only,

$$B = B_0(r)\left(1-\frac{r}{\varpi_c}\cos\chi\right)^{-1}$$

and

$$J = \frac{r}{B} = J_0(r)\left(1-\frac{r}{\varpi_c}\cos\chi\right).$$

Finally,

$$\delta = \delta_0 + \frac{r}{\varpi_c}\cos\chi + \frac{r^2}{\varpi_c^2}\cos^2\chi + O\left(\frac{r^3}{\varpi_c^3}\right)$$

where

$$\delta_0 = \frac{d\ln B_0}{d\ln r} - 1.$$

The procedure is now as follows: firstly we let $r/\varpi_c \to 0$ in (12) and minimize the integral with respect to α. This will yield a series of perturbation modes, each giving a stability criterion. Only two of these criteria are likely to be influenced by additional terms of order r/ϖ_c. The relations above will then be used to evaluate the contribution of the geometry in these two cases.

With $r/\varpi_c = 0$, $\chi = \theta$ and (12) becomes

$$I \propto \int_0^{2\pi} d\theta\,\{\alpha'^2(1-\kappa\cos^2\theta) - \alpha^2(1-\kappa+(2\delta_0+3\kappa)\sin^2\theta)\} \tag{13}$$

where we have neglected the variation of κ with χ. This is a valid approximation near the neutral line where $\kappa \ll 1$ anyway. We must now minimize (13) under the condition that α is a periodic function of θ. The Euler equation is

$$\frac{d}{d\theta}\left\{(1-\kappa\cos^2\theta)\frac{d\alpha}{d\theta}\right\} = -\alpha\{1-\kappa+(2\delta_0+3\kappa)\sin^2\theta\}. \tag{14}$$

We wish to know the critical value of δ_0 for which (14) has a periodic solution. Then the system will be stable if and only if δ_0 is less than this critical value, i.e. there will exist no periodic $\alpha(\theta)$ which makes (13) negative. If we make the change of variable

$$v = \arctan\{(1-\kappa)^{1/2}\tan\theta\}$$

then (14) becomes

$$\frac{d^2\alpha}{dv^2} = -\frac{\alpha(1-\kappa)(1+2(\delta_0+\kappa)\sin^2 v)}{(1-\kappa\sin^2 v)^2}.$$

This is, to first order in κ,

$$\frac{d^2\alpha}{dv^2} = -\alpha\{a - 2q\cos 2v + 2\mu\cos 4v\} \tag{15}$$

where

$$a = 1+\kappa+\delta_0(1+\kappa/2)$$

$$q = \kappa + \frac{\delta_0}{2}(1+\kappa)$$

$$\mu = \frac{\kappa\delta_0}{4}. \tag{16}$$

We seek the periodic solutions of (15) since α is periodic in θ if and only if it is periodic in v. Note that by taking $\kappa \ll 1$ we in no way assume that the general stellar magnetic field is weak. Since, to first order, $B = B_1(r/R)$, where R is the scale height of the field near the neutral line, the quantity $B^2/\gamma P = (B_1^2/\gamma P)\times(r^2/R^2)$ will be small even if $B_1^2/\gamma P$, the ratio of ambient magnetic pressure to thermal pressure, is of order unity.

Equation (15) is a transformation of Ince's equation (Ince 1924). Unlike the Mathieu equation, to which (15) reduces when $\mu = 0$, the general conditions for the existence of periodic solutions have not been investigated in detail. Interest has largely centred on the special case of finite sine and cosine series solutions (the Ince polynomials). However, for $\mu \ll 1$ the general features of the solutions of (15) are the same as for the Mathieu functions (Arscott 1964). There exists an infinite series of even periodic solutions $Ce_r(v, q, \mu)$ $(r = 0, 1, \ldots)$ and odd periodic solutions $Se_r(v, q, \mu)$ $(r = 1, 2, \ldots)$, and to each of these solutions there corresponds a ' characteristic ' value of a, denoted by $a_r(q, \mu)$ for the even solutions and $b_r(q, \mu)$ for the odd solutions. The integer r is the number of zeros which the solution has in $0 \leqslant v < \pi$. As $\mu \to 0$, Ce_r and Se_r tend to the Mathieu functions ce_r and se_r, while for $\mu = q = 0$ we retrieve the harmonic functions $\cos rv$ and $\sin rv$.

For small μ we now demonstrate an approximate periodic solution of (15). Set

$$a' = a + X_1\mu$$

$$q' = q + X_2\mu$$

where X_1, X_2 are constants. Then from (15)

$$\frac{d^2\alpha}{dv^2} + \alpha(a' - 2q' \cos 2v) = \mu\alpha(X_1 - 2X_2 \cos 2v - 2 \cos 4v). \tag{17}$$

For $\mu = 0$ the solution is $\alpha = e_r$ where $e_r(v, q')$ is a Mathieu function (either even or odd). To a first approximation we therefore put $\alpha = e_r$ on the right-hand side of (17) and look for a periodic particular integral. Now the function

$$\alpha_1 = \frac{\mu}{3q'}\left(e_r \cos 2v - \frac{de_r}{dv} \sin 2v\right)$$

is periodic in v and is even or odd according as e_r is ce_r or se_r; furthermore

$$\frac{d^2\alpha_1}{dv^2} + \alpha_1(a' - 2q' \cos 2v) = \mu e_r\left(-\frac{2}{3} + \frac{4}{3}\frac{a'-1}{q'} \cos 2v - 2 \cos 4v\right).$$

By comparison with (17), we see that α_1 is the desired particular integral if we take

$$X_1 = -\frac{2}{3}, \qquad X_2 = -\frac{2}{3}\frac{a'-1}{q'}.$$

Thus the even periodic solutions of (15) are given to first order in μ by

$$Ce_r(v, q, \mu) = ce_r(v, q') + \frac{\mu}{3q'}\left(ce_r(v, q') \cos 2v - \frac{d(ce_r(v, q'))}{dv} \sin 2v\right)$$

and the odd periodic solutions are obtained by replacing ce_r with se_r. The characteristic values of a are given by the characteristic values of a' for the Mathieu equation

$$\frac{d^2\alpha}{dv^2} + \alpha(a' - 2q' \cos 2v) = 0.$$

Thus

$$a_r(q, \mu) - \frac{2\mu}{3} = a_r\left(q - \frac{2}{3}\frac{a'-1}{q'}\mu\right)$$

which is, to first order in μ,

$$a_r(q, \mu) = a_r(q) - \frac{2}{3}\left(\frac{a_r(q)-1}{q}\frac{da_r}{dq} - 1\right)\mu \tag{18}$$

with a similar expression for $b_r(q, \mu)$. We have written $a_r(q, 0)$ as $a_r(q)$, the characteristic value for the even Mathieu function of order r. To evaluate the critical values of δ_0 corresponding to the Ce_0, Se_1, and Ce_1, modes we make use of the following expansions (McLachlan 1947)

$$a_0(q) = -\frac{1}{2}q^2 + \frac{7}{128}q^4 + O(q^6)$$

$$b_1(q) = 1 - q - \frac{q^2}{8} + O(q^3)$$

$$a_1(q) = 1 + q - \frac{q^2}{8} + O(q^3).$$

Since $\mu/q \sim \kappa \ll 1$, (18) will remain accurate even for small q and we have

$$a_0(q, \mu) = -\tfrac{1}{2}q^2\left(1 + \frac{3\mu}{8}\right) + O(q^4)$$

$$b_1(q, \mu) = 1 - q\left(1 + \frac{\mu}{4}\right) + O(q^2)$$

$$a_1(q, \mu) = 1 + q\left(1 + \frac{\mu}{4}\right) + O(q^2).$$

Substituting from (16) into the first of these relations gives the critical value of δ_0 to first order in κ as

$$\delta_0 = -1 \cdot 15 - 1 \cdot 8\,\kappa \tag{19}$$

for the Ce_0 mode. Whilst for the Se_1 mode

$$\delta_0 = -\tfrac{4}{3}\kappa \tag{20}$$

and for the Ce_1 mode

$$\delta_0 = 0. \tag{21}$$

The Ce_0 perturbation is a kink-type disturbance constrained to move parallel to the symmetry axis of the star. By comparison, the criterion (9) with $m = 1$ gives a critical value of $-1 \cdot 5$ for δ_0. Now near a neutral line carrying a finite current we can write

$$B_0(r) = B_1\frac{r}{R} + B_2\frac{r^2}{R^2} + \dots \tag{22}$$

giving

$$\delta_0 = \frac{B_2}{B_1}\frac{r}{R} + O\left(\frac{r^2}{R^2}\right). \tag{23}$$

Hence $|\delta_0| \ll 1$ and δ_0 will always exceed the value (19). Furthermore we may expect that any adjustment to (19) taking the toroidal geometry into account will be of order r/ϖ_c at most and therefore insignificant. Thus the constraint $\boldsymbol{\xi}.\mathbf{g} = 0$, although increasing the critical level of δ_0, is not sufficiently strong to stabilize the kink instability.

The Se_1 mode is the analogue of the sausage instability. If we let $\kappa \to 0$, then the critical value of $\delta_0 \to 0$ and $Se_1(v) \to \sin\theta$. The resulting $\boldsymbol{\xi}$ is a combination of the $m = 0$ and $m = 2$ modes of the classical pinch (Fig. 3). This is confirmed

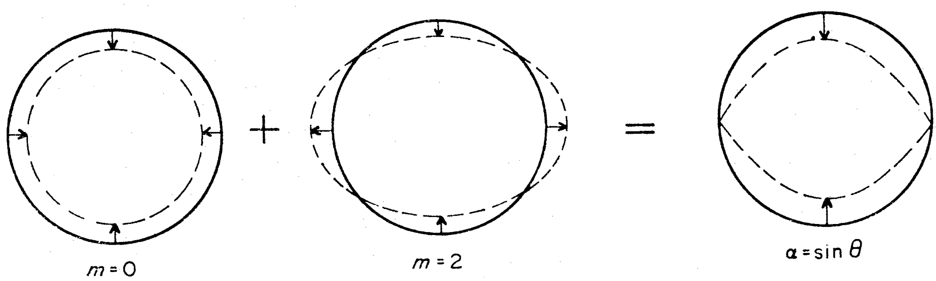

FIG. 3.

by noting that both the criteria (10) for $m = 0$ and (9) for $m = 2$ give instability for $\delta_0 > 0$ if $B^2/\gamma P$ is taken as vanishingly small. In any case $\delta_0 \sim (B_2/B_1)(r/R)$ from (23) and $\kappa \sim (B_1{}^2/\gamma P)(r^2/R^2)$, so the critical value for δ_0 will be zero to first order in r/R: only if $B_2 = 0$ is the quantity $4\kappa/3$ significant. Thus whether a star is unstable to this sausage mode depends on the sign of B_2/B_1. However, for this mode we no longer have any obvious justification for having ignored terms of order r/ϖ_c, since all zero-order terms in (13) have vanished. Therefore we shall perform below a more detailed calculation to determine the influence of the toroidal geometry on the result (20).

The result (21) for the critical δ_0 under the Ce_1 perturbation is identical to that for the $m = 2$ mode in (9). In fact it represents a combination of the $m = 2$ mode and a uniform rotation of the field line (Fig. 4). Since neither of the two component

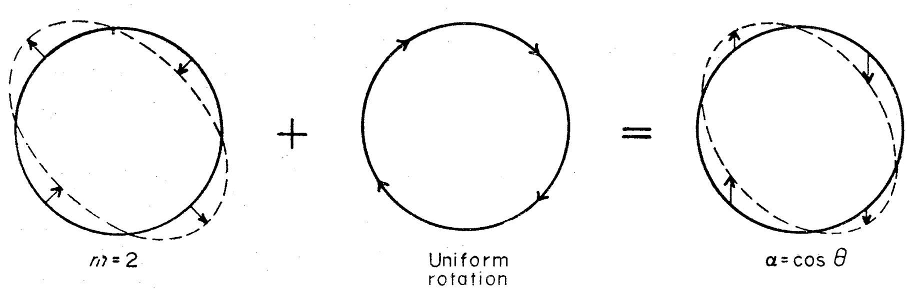

FIG. 4.

displacements is compressible, it is not surprising that κ does not enter (21). As with the Se_1 mode, the neglect of toroidal effects is unjustifiable and the full calculation is given below.

For all other modes Ce_r, Se_r $(r \geqslant 2)$ the resulting critical δ_0's are all positive and at least of order one, which implies from (23) that field lines close to the neutral line are always stable to them.

To evaluate the effects of the toroidal geometry on the criteria for the Se_1 and Ce_1 modes we must return to equation (12). However we first observe that if $\alpha = \sin\theta$ is substituted into (13) the result is

$$I = -\frac{3\pi}{2}\left(\delta_0 + \frac{4\kappa}{3}\right)$$

and with $\alpha = \cos\theta$

$$I = -\frac{\pi\delta_0}{2}.$$

Comparison with (20) and (21) shows that, to first order in κ, $\alpha = \sin\theta$ and $\alpha = \cos\theta$ are also 'critical' perturbations of the system. This suggests that to examine the Se_1 mode in toroidal geometry we can insert in (12)

$$\alpha = \epsilon(\chi)\sin\theta$$

where $\epsilon = 1 + (r/\varpi_c) f(\chi)$ and $f(\chi)$ is a periodic function of χ, and then minimize I with respect to f. We thus have

$$I = \int_0^{2\pi} \frac{d\chi}{\varpi^2 B^2 J}\{\epsilon'^2 \sin^2\theta(1-\kappa\cos^2\theta) + \epsilon^2 H(\chi)\} \tag{24}$$

where

$$H(\chi) = G(\chi)\sin^2\theta + \left(\theta'' + J\left(\frac{1}{J}\right)'\theta'\right)\sin\theta\cos\theta - 2\kappa\theta'^2\cos^2\theta\sin^2\theta.$$

Ignoring terms of order $(r/\varpi_c)\,\kappa$, (24) becomes

$$I = \int_0^{2\pi}\frac{d\chi}{\varpi^2 B^2 J}\left\{\frac{r^2}{\varpi_c^2} f'^2 \sin^2\chi + 2f\frac{r}{\varpi_c}H + H\right\}. \tag{25}$$

Assuming we can neglect the $2fH$ term compared with the last term, the integral is minimized by taking $f' = 0$; and since

$$\frac{H}{J} = -2\delta_0\sin^4\chi\left(1 + \frac{5r}{\varpi_c}\cos\chi\right) - 2\kappa\sin^2\chi$$
$$+\frac{3r}{\varpi_c}\sin^2\chi\cos\chi + \frac{r^2}{\varpi_c^2}\sin^2\chi(16\cos^2\chi - 5\sin^2\chi) + O\left(\frac{r^3}{\varpi_c^3}\right) \tag{26}$$

the minimum I is

$$I = \frac{1}{\varpi_c^2 B_0^2 J}\left(-\frac{3\pi}{4}\delta_0 + O\left(\frac{r^2}{\varpi_c^2}\right)\right). \tag{27}$$

Thus the condition for instability is, to first order in r/ϖ_c,

$$\delta_0 > 0$$

which is also the result from the straight cylinder approximation (taking κ as second order). However this is the valid criterion only when $\delta_0 \sim r/\varpi_c$, i.e. when $B_2/B_1 \sim 1$. If $B_2 = 0$ or $B_2/B_1 \ll 1$, then the neglect of $2fH$ in (25) cannot be justified. The only term of H/J which needs to be included in the product $2fH/J$ is the one of order r/ϖ_c. This is

$$\frac{3r}{\varpi_c}\sin^2\chi\cos\chi = \frac{r}{\varpi_c}\frac{d}{d\chi}\sin^3\chi.$$

Thus the second term in (25) can be integrated by parts and (25) written as

$$I = \frac{r^2}{\varpi_c^2}\int_0^{2\pi}\frac{d\chi}{\varpi_c^2 B_0^2 J_0}[f'-\sin\chi]^2\sin^2\chi+\int_0^{2\pi}\frac{d\chi}{\varpi^2 B^2}\left[\frac{H}{J}-\frac{r^2}{\varpi_c^2}\frac{1}{J_0}\sin^4\chi\right]$$

which is clearly minimized when

$$f' = \sin\chi.$$

Using (26) and performing the integrations over χ the instability criterion becomes

$$\delta_0+\frac{4\kappa}{3}+\frac{1}{3}\frac{r^2}{\varpi_c^2}>0. \tag{28}$$

So the geometry of a torus leads to a second-order de-stabilizing effect.

For the Ce_1 mode we may carry through an exactly analogous calculation using the substitution

$$\alpha = \left(1+\frac{r}{\varpi_c}f\right)\cos\theta$$

in (12). It is found that the criterion is unaltered from the straight cylinder approximation both to first and second order in r/ϖ_c, and remains just

$$\delta_0>0 \tag{29}$$

for instability.

If $B_1>0$ and $B_2\neq 0$ then the first-order criterion for instability against Se_1 and Ce_1 perturbations is

$$B_2>0.$$

Thus the stability of the configuration to these perturbations is determined by the second-order rate of increase of the field away from the neutral line.

When $B_2 = 0$,

$$\delta_0 = \frac{2B_3}{B_1}\frac{r^2}{R^2}+O\left(\frac{r^3}{R^3}\right)$$

and (28) becomes

$$\frac{B_3}{B_1}+\frac{2}{3}\frac{B_1^2}{\gamma P}>-\frac{1}{6}\frac{R^2}{\varpi_c^2}. \tag{30}$$

This concludes the analysis of the unstable modes which can arise in a purely poloidal system, and we now proceed to investigate the stabilizing effect of a toroidal field on these modes.

THE STABILITY OF MIXED POLOIDAL AND TOROIDAL FIELDS

The introduction of a toroidal field component gives rise to a number of mathematical complications. Chief among these is the absence of a theorem permitting us to analyse the stability field line by field line, and in a full treatment one would need to minimize the energy integral over the ψ–χ plane. Euler equations would then exist with respect to each variable. Furthermore, it is not in general the case that the integral is minimized by taking the azimuthal wavenumber, k, as infinitely large, and k must be left arbitrary until the rest of the minimization has been carried out. In order to keep the problem mathematically manageable in face of these difficulties, we shall make some simplifying assumptions dictated largely by the results of the previous section.

Firstly, the effects of the toroidal geometry will not be included and we shall analyse, as before, the stability of a straight circular cylinder in a uniform gravitational field, but with the addition of a uniform field parallel to its axis. This is equivalent to taking $r/\varpi_c \ll 1$, so we are effectively confining the analysis to the volume of gas immediately surrounding the neutral line of the poloidal field. The toroidal geometry of this volume shows itself only through the condition $2\pi\varpi_c k > 1$, which ensures that the azimuthal wavelength is less than the circumference of the torus.

Secondly, the angular variations of the perturbations for the Ce_0-type, Se_1-type, and Ce_1-type modes will be taken to be as in (11) with α = const., $\alpha = \sin\theta$, $\alpha = \cos\theta$ respectively. It has already been pointed out that the last two of these are, to first order in κ, critical perturbations for their respective modes in the poloidal-only case. Taking α = const. in (12) yields

$$I \propto -\left(\delta_0 + 1 + \frac{\kappa}{2}\right) \tag{31}$$

so the stability condition that this be positive is slightly weaker than that of equation (19). For the sake of this small difference it hardly seems worth submitting to the complexities of an exact Ce_0 dependence, especially as this is itself only an approximation to the minimizing angular dependence for the mixed problem.

We take the magnetic field as $(0, B_\theta(r), B_z)$ in cylindrical coordinates (r, θ, z), which from (1) gives the potential energy change as

$$\begin{aligned}\delta W = \int_v r\,dr\,d\theta\,dz \Bigg\{ & \left(\frac{B_\theta}{r}\frac{\partial \xi_r'}{\partial\theta} + B_z\frac{\partial \xi_r'}{\partial z}\right)^2 + \gamma P(\nabla.\boldsymbol{\xi}')^2 \\ & + \left(\frac{B_\theta}{r}\frac{\partial \xi_z'}{\partial\theta} + B_z\frac{\partial \xi_z'}{\partial z} - B_z\nabla.\boldsymbol{\xi}'\right)^2 \\ & + \left(B_\theta\nabla.\boldsymbol{\xi}' - \frac{B_\theta}{r}\frac{\partial \xi_\theta'}{\partial\theta} - B_z\frac{\partial \xi_\theta'}{\partial z} - \frac{2B_\theta}{r}\xi_r'\right)^2 \\ & - \frac{2B_\theta}{r^2}\frac{d(rB_\theta)}{dr}\xi_r'^2\Bigg\}\end{aligned} \tag{32}$$

where now $\boldsymbol{\xi}' = (\xi_r', \xi_\theta', \xi_z')$ is the real perturbation of the system and we have again set $\boldsymbol{\xi}'.\mathbf{g} = 0$. Following our remarks above, we first take a kink-type perturbation of the form

$$\left.\begin{aligned}\xi_r' &= \xi_r(r)\sin\theta\sin kz \\ \xi_\theta' &= \xi_\theta(r)\cos\theta\sin kz \\ \xi_z' &= \xi_z(r)\sin\theta\cos kz\end{aligned}\right\} \tag{33}$$

whence, after integration over θ and z:

$$\begin{aligned}\delta W \propto \int_0^{r_0} r\,dr \Bigg\{ & \xi_r^2\left[\frac{B_\theta^2}{r^2} + k^2B_z^2 - 2B_\theta\frac{d(rB_\theta)}{dr}\right] \\ & + \gamma P\left[\frac{1}{r}\frac{d(r\xi_r)}{dr} - \frac{\xi_\theta}{r} - k\xi_z\right]^2 \\ & + \xi_z^2\frac{B_\theta^2}{r^2} + B_z^2\left(\frac{\xi_\theta}{r} - \frac{1}{r}\frac{d(r\xi_r)}{dr}\right)^2 \\ & + k^2B_z^2\xi_\theta^2 + B_\theta^2\left[\frac{d\xi_r}{dr} - \frac{\xi_r}{r} - k\xi_z\right]^2\Bigg\}\end{aligned} \tag{34}$$

where r_0 is the radius at which the perturbation vanishes and can be chosen arbitrarily as long as the approximation to toroidal geometry remains valid. Since no differentials of ξ_z appear in the integrand of (34) we complete the square for ξ_z and obtain the minimizing relation

$$\xi_z = k\,\frac{\left\{(B_\theta^2+\gamma P)\left(\dfrac{d\xi_r}{dr}+\dfrac{\xi_r}{r}-\dfrac{\xi_\theta}{r}\right)-\xi_\theta\dfrac{B_\theta^2}{r}\right\}}{k^2(B_\theta^2+\gamma P)+B^2{}_\theta/r^2}.$$

Putting $\xi_r = \xi_\theta$ (implied by $\boldsymbol{\xi}'.\mathbf{g} = 0$), (34) becomes

$$\delta W \propto \int_0^{r_0} r\,dr\left\{\frac{\xi_r^2}{r^2}\left[-2(1+\delta_0)\,B_\theta^2-\frac{B_\theta^2k^2\kappa r^2}{k^2r^2+\kappa}+2k^2r^2B_z^2\right]\right.$$
$$\left.-\frac{2\xi_r}{r}\frac{d\xi_r}{dr}\frac{B_\theta^2\kappa}{k^2r^2+\kappa}+\left(\frac{d\xi_r}{dr}\right)^2\left[B_z^2+\frac{B_\theta^2}{k^2r^2+\kappa}\right]\right\}. \quad (35)$$

We note that the terms in B_z^2 are stabilizing and that for $B_z = 0$ we let $k\to\infty$ to obtain

$$\delta W \propto -2\int_0^{r_0}\frac{B_\theta^2}{r}\,\xi_r^2\left[1+\delta_0+\frac{\kappa}{2}\right]dr$$

giving essentially (31). For $B_z \neq 0$ we examine the dominant terms in (35): we ignore all second-order effects, taking $\kappa \ll 1$ and $B_\theta = B_{1\theta}r/R$. Then $\delta_0 = 0$ to this order and

$$\delta W \propto \int_0^{r_0} r\,drB_{1\theta}^2\left\{-\frac{2\xi_r^2}{R^2}(1-\beta-k^2R^2l^2)+\frac{1}{k^2R^2}\left(\frac{d\xi_r}{dr}\right)^2(1-\beta+k^2R^2l^2)\right\} \quad (36)$$

where the constants l and β are given by

$$l = B_z/B_{1\theta}$$

$$\beta = \frac{B_{1\theta}^2}{B_{1\theta}^2+k^2R^2\gamma P}.$$

The Euler equation minimizing (36) is

$$(r\xi_r')' = -Kr\xi_r \quad (37)$$

where

$$K = 2k^2\frac{(1-\beta-l^2k^2R^2)}{(1-\beta+l^2k^2R^2)} \quad (38)$$

The critical value of l^2 for marginal stability will be given by the value of K for which (37) has a solution satisfying the conditions

$$\xi_r(r_0) = \left(\frac{d\xi_r}{dr}\right)_{r=0} = 0$$

and which does not vanish in $0<r<r_0$ (Greene & Johnson 1965). The appropriate solution is the zero-order Bessel function

$$\xi_r = J_0[K^{1/2}r]$$

giving the condition on K as

$$K^{1/2}r_0 = \rho_0 = 2\cdot4$$

where ρ_0 is the first zero of $J_0[\rho]$. This value of K gives, from (38):

$$l^2 = \frac{1-\beta}{k^2R^2}\frac{(k^2r_0^2-2\cdot 9)}{(k^2r_0^2+2\cdot 9)}.$$

If we now assume that the ambient poloidal field is sufficiently small for β to be neglected (i.e. $B_1^2/\gamma P \ll R^2/\varpi_c^2$) we find that l^2 is maximized for

$$k^2 = \frac{1}{r_0^2}\frac{2\cdot 9}{\sqrt{2}-1}.$$

Note that this value of k is consistent with $2\pi\varpi_c k > 1$. The critical value of l is thus

$$l = \frac{1}{4\cdot 1}\frac{r_0}{R}$$

or

$$B_z = 0\cdot 24\, B_\theta(r_0). \tag{39}$$

To maximize the critical value of B_z we choose r_0 to be the largest radius for which the straight cylinder approximation is valid. If this is such that $r_0 \sim R$, then B_z needs to be roughly a quarter of the ambient poloidal field strength in order to stabilize the configuration. If $r_0 \ll R$ it does not mean, of course, that the configuration is easier to stabilize, but merely that we have only considered the stabilization of the small volume near the neutral line of the poloidal field.

We now perform a similar analysis for the Se_1-type mode and take

$$\xi_r' = \xi_r(r)\sin^2\theta\sin kz$$

$$\xi_\theta' = \xi_\theta(r)\sin\theta\cos\theta\sin kz$$

$$\xi_z' = (\xi_{zs}(r)\sin^2\theta + \xi_{zc}(r)\cos^2\theta)\cos kz.$$

After substitution in (32) and integration over θ and z, the integral can be minimized with respect to ξ_{zs} and ξ_{zc} by completing the square. With $\xi_\theta = \xi_r$ we then have

$$\delta W \propto \int_0^{r_0} r\,dr\left\{\frac{\xi_r{}^2}{r^2}\left[-B_\theta{}^2(6\delta_0+8\kappa)+\frac{4B_\theta{}^2}{k^2r^2+\kappa}+B_z{}^2(3+4k^2r^2)\right.\right.$$
$$\left.\left.+\frac{2\xi_r}{r}\frac{d\xi_r}{dr}\left[B_z{}^2-\frac{8B_\theta{}^2}{k^2r^2+\kappa}\right]+\left(\frac{d\xi_r}{dr}\right)^2\left[3B_z{}^2+\frac{4B_\theta{}^2}{k^2r^2+4\kappa}\right]\right\}.$$

We again notice that B_z is a stabilizing factor, and that we recover the poloidal-only criterion for $B_z = 0$ and $k \to \infty$. If we now take $B_\theta = B_{1\theta}r/R$ then the term involving $(\xi_r/r)(d\xi_r/dr)$ integrates to zero and an obvious sufficient condition for stability is

$$\frac{B_z{}^2}{B_\theta{}^2} > \frac{8B_\theta{}^2}{3(B_\theta{}^2+\gamma P)}$$

or

$$\frac{B_z{}^2}{B_{1\theta}{}^2} > \frac{8\kappa}{3}\frac{r_0{}^2}{R^2}. \tag{40}$$

For $\kappa \ll 1$ this implies that even a very weak toroidal component will stabilize this mode. To find the necessary and sufficient condition we proceed as before, but with the zero-order Coulomb wave function as the solution of the Euler equation. The critical value of $B_z/B_{1\theta}$ behaves as $\kappa^{3/2}$ and the most unstable perturbation has $k \sim 1/\kappa$. There is little point in detailed analysis at this point, since it is clear that an exact criterion would also contain contributions from the toroidal geometry and from second-order terms in the expansion of B_θ which will be at least of order κ. It suffices to point out that any toroidal field which stabilizes the Ce_0 mode will easily stabilize the Se_1 mode also.

Finally we consider the Ce_1-like mode for which

$$\xi_r' = \xi_r(r) \sin\theta \cos\theta \sin kz$$

$$\xi_\theta' = \xi_\theta(r) \cos^2\theta \sin kz$$

$$\xi_z' = \xi_z(r) \sin\theta \cos\theta \cos kz.$$

Minimizing (32) with respect to ξ_z and setting $\xi_r = \xi_\theta$ we obtain

$$\delta W \propto \int_0^{r_0} r\,dr \left\{ \frac{\xi_r^2}{r^2} [4k^2 r^2 B_z^2 - 2B_\theta^2 \delta_0] + \left(\frac{d\xi_r}{dr} - \frac{\xi_r}{r}\right)^2 \left[\frac{B_z^2(k^2(B_\theta^2 + \gamma P) + B_\theta^2/r^2) + B_\theta^2/r^2(B_\theta^2 + \gamma P)}{k^2(B_\theta^2 + \gamma P) + B_\theta^2/r^2}\right]\right\}.$$

When $B_\theta = (r/R)\,B_{1\theta}$ the integral is positive definite for any non-zero B_z. Thus this mode is unlikely to yield instabilities in any field structure other than a purely poloidal one.

CONCLUSIONS

The results above suggest that hydromagnetic instabilities can only be prevented in configurations where the toroidal flux is at least as strong as that given by (39). The ultimate development of these instabilities cannot be described by the linear analysis above and we should also need to include a finite electrical conductivity if magnetic flux-loss were to appear as an explicit consequence of the instability. We can do no more than assume that the distortion of the field lines results in enhanced Ohmic decay and a rapid destruction of flux, although it is conceivable that in a star with a toroidal field just below the stabilizing limit (thus needing r_0 to be as large as possible), the azimuthal wavelength of the perturbation will be so large that insufficient bending of field lines takes place before second-order effects damp growth, and an oscillatory motion is set up.

It should be emphasized, however, that we have only give a necessary condition for stability and that (39) is a lower bound for the stabilizing toroidal field. Future research must tackle the rigorous angular minimization in the mixed field case, and possibly relax the assumption of circular field lines. It would also be desirable not to have to assume that the quantity κ is locally small and thus to consider the stability of those regions where the magnetic pressure of the poloidal field is comparable with the thermal pressure.

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