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Pinning control of scale-free dynamical networks

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Abstract

Recently, it has been demonstrated that many large complex networks display a scale-free feature, that is, their connectivity distributions have the power-law form. In the present work, control of a scale-free dynamical network by applying local feedback injections to a fraction of network nodes is investigated. The specifically and randomly pinning schemes are considered. The specifically pinning of the most highly connected nodes is shown to require a significantly smaller number of local controllers as compared to the randomly pinning scheme. The method is applied to an array of Chua's oscillators as an example. © 2002 Published by Elsevier Science B.V.

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1. Introduction

Over the past decades, analysis and control of complex behaviors in dynamical networks consisting of a large number of dynamical nodes has become a topic of great interest. However, most of these works have been concentrated on networks with completely regular topological structures, such as chains, grids, lattices, and fully-connected graphs. Two typical cases are the discrete-time coupled map lattice (CML) [1] and the continuous-time cellular neural network (CNN) [2]. The main benefit of these simple architectures is that it allows to focus on the complexity caused by the nonlinear dynamics of the nodes, without considering additional complexity in the network structure itself [3].

Recently, there is increasing interest in trying to understand the generic features that characterize the formation and topology of various complex networks. Typical examples

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include the Internet [4], the World Wide Web (WWW) [5], food web [6], metabolic networks [7], scientific-collaboration networks [8], social networks [9], etc. The apparent ubiquity of such complex networks leads to a fascinating set of common problems concerning how the network structure facilitates and constraints network behaviors. However, due to the large size and the complexity of interactions of such networks, it has become possible only very recently to gather and to analyze the huge amount of data from such intricate systems due to the availability of high computing power.

Traditionally, a network of complex topology is described by a completely random graph, which is at the opposite end of the spectrum from a completely regular network—the so-called ER model [10]. However, many real-world complex networks are neither completely regular nor completely random. In order to describe the transition from a regular network to a random network, Watts and Strogatz (WS) recently introduced the so-called small-world network [11]. A common feature of the ER model and the WS model is that the connectivity distribution of the network peaks at an average value and decays exponentially. Such an exponential network is homogeneous in nature: each node has roughly the same number of connections.

Another significant recent discovery in the field of complex networks is the observation that a number of large-scale and complex networks are scale-free, that is, their connectivity distributions have the power-law form [12,13]. A scale-free network is inhomogeneous in nature: most nodes have very few connections and a few nodes have many connections. It has been argued that the inhomogeneous feature makes the connectivity of a scale-free network error-tolerant but vulnerable to attacks [14–16]. More precisely, the connectivity of such networks is highly robust against random failures (such as random removal of nodes, e.g., random failures of routers in the Internet) but it is also extremely fragile to attacks (e.g., specific removal of the most highly connected nodes).

Recently, we have shown that the synchronizability of a scale-free dynamical network is robust against random removal of nodes, and yet is fragile to specific removal of the most highly connected nodes [17]. In the present work, we investigate the control problem for a scale-free dynamical network by applying local linear feedback injections to a small fraction of network nodes. Feedback pinning has been a common technique for the control of spatiotemporal chaos in regular dynamical networks [18–20]. We investigate the effects of two pinning schemes here. In the specifically pinning scheme, a fraction of the most highly connected nodes are pinned; while in the randomly pinning scheme, a fraction of randomly selected nodes are pinned. We show that, due to the extremely inhomogeneous connectivity distribution of a scale-free network, it is much more effective to pin some most highly connected nodes than pinning the same number of randomly selected nodes.

2. The scale-free dynamical network model

2.1. The scale-free network model

It has been suggested that two ingredients of self-organization of a network in a scale-free structure are “growth” and “preferential attachment” [12,13]. These refer to that networks continuously grow by the addition of new nodes and new nodes are preferentially attached to existing nodes with high numbers of connections (the so-called "*rich get richer*" phenomenon).

Based on the two ingredients, Barabasi and Albert proposed a simple scale-free model. The model starts with m_0 nodes. At every time step, a new node is introduced and is connected to m already-existing nodes. The probability Π_i that the new node is connected to node i depends on the degree k_i of node i, such that $\Pi_i = k_i / \sum_j k_j$. For a large time constant, the probability $P(k)$ that a node in the network is connected to k other nodes decays in a power-law of the form $P(k) = 2m^2/k^3$. More recently, Albert and Barabasi proposed an extended model of network evolution that gives a more realistic description of local processing, taking into account the additions of new nodes and new links, and the rewiring of links [21].

2.2. The scale-free dynamical network model

Now, suppose that, at some time, the scale-free network consists of N identical linearly and diffusively coupled nodes, with each node being an n-dimensional dynamical system. The state equations of the network are

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + c\sum_{j=1}^{N} a_{ij}\Gamma\mathbf{x}_j, \quad i = 1,2,\ldots,N , \tag{1}$$

where $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \Re^n$ are the state variables of node i, the constant $c > 0$ represents the coupling strength, and $\Gamma \in \Re^{n\times n}$ is a constant 0–1 matrix linking coupled variables. For simplicity, we assume that $\Gamma = diag(r_1, r_2, \ldots, r_n)$ is a diagonal matrix with $r_i = 1$ for a particular i and $r_j = 0$ for $j \neq i$ with respect to this particular i. This means that two coupled nodes are linked through their ith state variables. The coupling matrix $\mathbf{A} = (a_{ij}) \in \Re^{N\times N}$ represents the scale-free coupling configuration of the network. If there is a connection between node i and node j $(i \neq j)$, then $a_{ij} = a_{ij} = 1$; otherwise, $a_{ij} = a_{ji} = 0$ $(i \neq j)$. If the degree k_i of node i is defined to be the number of connection incidents at node i, then

$$\sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij} = \sum_{\substack{j=1\\ j\neq i}}^{N} a_{ij} = k_i, \quad i = 1,2,\ldots,N . \tag{2}$$

The diagonal elements of the coupling matrix are

$$a_{ii} = -k_i, \quad i = 1,2,\ldots,N . \tag{3}$$

Suppose that the network is connected in the sense that there are no isolate clusters. Then, the coupling matrix $\mathbf{A} = (a_{ij})_{N\times N}$ is a symmetric irreducible matrix. In this case, it can be shown that zero is an eigenvalue of $\mathbf{A}$ with multiplicity 1 and all the other eigenvalues of $\mathbf{A}$ are strictly negative. The ith row $a(i,:)$ and ith column $a(:,i)$ of matrix $\mathbf{A}$ will be called the ith row-column pair of $\mathbf{A}$ below.

3. Pinning control of scale-free dynamical networks

3.1. Stability analysis

Suppose that we want to stabilize network (1) on a homogeneous stationary state defined by

$$\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N = \bar{\mathbf{x}} \tag{4}$$

and that

$$f(\bar{\mathbf{x}}) = 0 \; . \tag{5}$$

We want to achieve the goal of control by applying local linear feedback injections to a small fraction δ $(0 < \delta < 1)$ of the nodes. Suppose that the $i_1, i_2, \ldots, i_l$ nodes were selected, where $l = [\delta N]$ stands for the smaller but nearest integer to the real number δN. The equations of the controlled network are

$$\dot{\mathbf{x}}_{i_k} = f(\mathbf{x}_{i_k}) + c\sum_{j=1}^{N} a_{i_k j}\Gamma\mathbf{x}_j - cd\Gamma(\mathbf{x}_{i_k} - \bar{\mathbf{x}}), \quad k = 1,2,\ldots,l \; , \tag{6a}$$

$$\dot{\mathbf{x}}_{i_k} = f(\mathbf{x}_{i_k}) + c\sum_{j=1}^{N} a_{i_k j}\Gamma\mathbf{x}_j, \quad k = l+1, l+2,\ldots,N \; , \tag{6b}$$

where $d > 0$ is the feedback gain.

To investigate the stability of the stationary state $\bar{\mathbf{x}}$, we linearize Eq. (6) about $\bar{\mathbf{x}}$. This leads to

$$\dot{\eta} = \eta[Df(\bar{\mathbf{x}})] + c\mathbf{B}\eta\Gamma \; , \tag{7}$$

where $Df(\bar{\mathbf{x}}) \in \Re^{n\times n}$ is the Jacobian of f on $\bar{\mathbf{x}}$, $\eta = (\eta_1, \eta_2, \ldots, \eta_N)^{\mathrm{T}} \in \Re^{N\times n}$, with

$$\eta_i(t) = \mathbf{x}_i(t) - \bar{\mathbf{x}}, \quad i = 1,2,\ldots,N$$

and

$$\mathbf{B} = \mathbf{A} - \mathbf{D}, \quad \mathbf{D} = diag(d_1, d_2, \ldots, d_N) \tag{8}$$

in which $d_{i_k} = d$ for $1 \leqslant k \leqslant l$ and $d_{i_k} = 0$ for $l+1 \leqslant k \leqslant N$.

Let

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_N$$

be the eigenvalues of matrix $\mathbf{B}$ and $\Phi = [\phi_1, \phi_2, \ldots, \phi_N] \in \Re^{N\times N}$ be the corresponding (generalized) eigenvector basis satisfying

$$\mathbf{B}\phi_k = \lambda_k\phi_k, \quad k = 1,2,\ldots,N \; . \tag{9}$$

By expanding each column η on the basis Φ, we obtain

$$\eta = \Phi\upsilon \; , \tag{10}$$

where the matrix $\upsilon \in \Re^{N\times n}$ satisfies the following equations:

$$\dot{\upsilon} = \upsilon[Df(\bar{\mathbf{x}})] + c\Lambda\upsilon\Gamma , \tag{11}$$

where $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_N)$. Let υ_k be the kth row of υ. We then arrive at

$$\dot{\upsilon}_k^{\mathrm{T}} = [Df(\bar{\mathbf{x}}) + c\lambda_k\Gamma]\upsilon_k^{\mathrm{T}}, \quad k = 1, 2, \ldots, N . \tag{12}$$

We have now transformed the stability problem of the $(n \times N)$-dimensional system (6) to the stability problem of the N numbers of n-dimensional linear systems (12). It follows from the linear system theory that if $[Df(\bar{\mathbf{x}}) + c\lambda_1\Gamma]$ is a Hurwitz matrix in the sense that the real parts of its eigenvalues are all negative, then linear systems (12) are exponentially stable, which implies that the homogeneous stationary state (4) of the controlled network (6) is locally exponentially stable. On the other hand, since $\lambda_1(\mathbf{A}) = 0$, the homogeneous stationary state (4) of the uncontrolled network (1) is unstable if $\bar{\mathbf{x}}$ is not a stable equilibrium point of an isolate node. Thus, in summary, we have the following.

Lemma 1. *Consider the controlled network* (6). *Suppose that there exists a constant* $\rho < 0$ *such that* $[Df(\bar{\mathbf{x}}) + \rho\Gamma]$ *is a Hurwitz matrix. Let* λ_1 *be the largest eigenvalue of matrix* $\mathbf{B} = \mathbf{A} - \mathbf{D}$. *If*

$$c\lambda_1 \leqslant \rho , \tag{13}$$

then the homogeneous stationary state $\bar{\mathbf{x}}$ *of the controlled network* (6) *is locally exponentially stable.*

Since $\lambda_1 < 0$ *and* $\bar{d} < 0$, *inequality* (13) *is equivalent to*

$$c \geqslant \left| \frac{\rho}{\lambda_1} \right| . \tag{14}$$

A small value of λ_1 corresponds to a large value of $|\lambda_1|$, which implies that the controlled network (6) can stabilize onto the stationary state (4) with a small coupling strength c. Therefore, the stability of network (6) with respect to a specific pinning scheme can be characterized by the largest eigenvalue λ_1 of the corresponding matrix $\mathbf{B}$. One should choose the l pinned nodes and the feedback gain d such that λ_1 is as small as possible.

For a given pinning scheme, the largest eigenvalue λ_1 of the matrix $\mathbf{B} = \mathbf{A} - \mathbf{D}$ is a decreasing function of d. Suppose that the $i_1, i_2, \ldots, i_l$ nodes are selected as the pinning nodes, and let $\tilde{\mathbf{A}} \in \Re^{(N-l)\times(N-l)}$ be a minor matrix of $\mathbf{A}$ with respect to this pinning scheme, which is obtained by removing the $i_1, i_2, \ldots, i_l$ row–column pairs of $\mathbf{A}$. We then have

$$\lim_{d\to\infty} \lambda_1(\mathbf{B}) = \lambda_1(\tilde{\mathbf{A}}) . \tag{15}$$

In fact, in the limit $d \to \infty$, the states of the controlled $i_1, i_2, \ldots, i_l$ nodes can be pinned to the target state $\bar{\mathbf{x}}$ for sure. Therefore, we only need to investigate the stability of

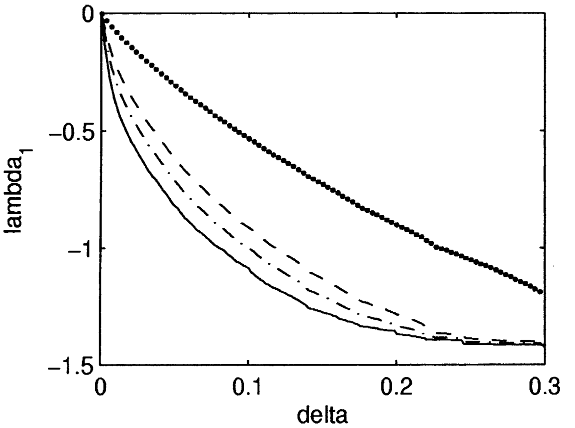

Fig. 1. Value of λ_{1s} as a function of the fraction δ and the feedback gain $d(-, \tilde{\lambda}_{1s}(\delta); \ldots, d = 10; —, d = 50; —, d = 100)$.

the following system:

$$\mathbf{x}_{i_k} = \bar{\mathbf{x}}, \quad k = 1, 2, \ldots, l, \tag{16a}$$

$$\dot{\mathbf{x}}_{i_k} = f(\mathbf{x}_{i_k}) + c \sum_{j=1}^{N} a_{i_k j} \Gamma \mathbf{x}_j, \quad k = l+1, l+2, \ldots, N. \tag{16b}$$

According to Lemma 1, the homogeneous stationary state $\bar{\mathbf{x}}$ of network (16) is locally exponentially stable if

$$c\lambda_1(\tilde{\mathbf{A}}) \leqslant \rho. \tag{17}$$

3.2. Control of a scale-free network with different pinning schemes

Now, we perform a simulation-based analysis on the stability of the controlled scale-free dynamical network (6) with respect to randomly and specifically pinning schemes. In the following simulations, we take $N = 3000$ and $m_0 = m = 3$, namely, the original network contains 3000 nodes with about 9000 connections.

In the randomly pinning scheme, we apply local linear feedback injections to a fraction δ of the randomly selected nodes. In the specifically pinning scheme, we first pin the node with the highest degree, and then continue to select and pin the other nodes in monotonically decreasing order of degrees. For clarity, the feedback gain matrices of the randomly and specifically pinning schemes are denoted as $\mathbf{D}_r$ and $\mathbf{D}_s$, respectively. The largest eigenvalue of the matrices $\mathbf{B}_r = \mathbf{A} - \mathbf{D}_r$ and $\mathbf{B}_s = \mathbf{A} - \mathbf{D}_s$ are denoted as λ_{1r} and λ_{1s}, respectively. Figs. 1 and 2 show the values of λ_{1r} and λ_{1s} as functions of the fraction δ and the feedback gain d. It can be seen that

$$\lim_{d\to\infty} \lambda_{1r}(\delta, d) = \tilde{\lambda}_{1r}(\delta), \qquad \lim_{d\to\infty} \lambda_{1s}(\delta, d) = \tilde{\lambda}_{1s}(\delta), \tag{18}$$

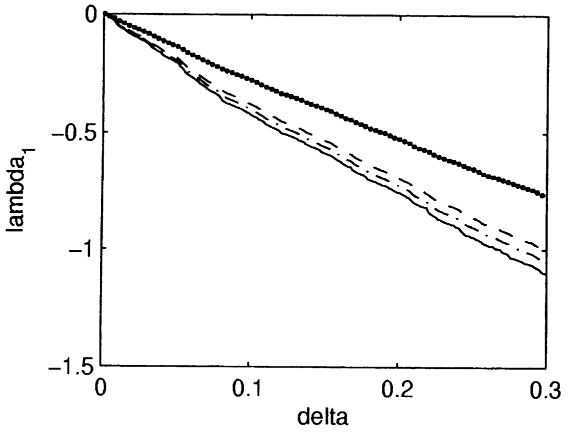

Fig. 2. Value of λ_{1r} as a function of the fraction δ and the feedback gain $d(-, \tilde{\lambda}_{1r}(\delta); \ldots, d = 10; —, d = 50; —, d = 100)$.

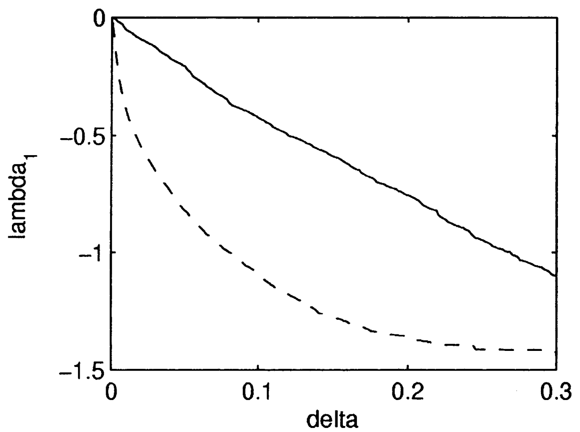

Fig. 3. Values of $\tilde{\lambda}_{1s}(\delta)$ (—) and $\tilde{\lambda}_{1r}(\delta)$ (—) as functions of the fraction δ.

where $\tilde{\lambda}_{1r}(\delta)$ and $\tilde{\lambda}_{1s}(\delta)$ are the largest eigenvalues of the minor matrices of the coupling matrix **A** with respect to the randomly and specifically pinning schemes, respectively.

As shown in Fig. 3, $\tilde{\lambda}_{1s}(\delta)$ is much smaller than $\tilde{\lambda}_{1r}(\delta)$, especially for small values of δ. This implies that, to stabilize a scale-free dynamical network, the specifically pinning scheme requires much less local controllers than the randomly pinning scheme. This feature of scale-free dynamical networks is rooted in their extremely inhomogeneous connectivity distribution. Since most of the nodes in a scale-free network are "small" nodes with very low degrees, $[\delta N]$ "small" nodes will be selected in a randomly pinning scheme with much higher probability if $N \gg 1$ and $\delta \ll 1$. It is not surprising to see that, for the stability of the entire scale-free network, the stability of a few "big" nodes with highest degrees is much more important than the stability of the same number of "small" nodes. This suggests that, to stabilize a scale-free network with a minimum number of local controllers, we should add these local controllers with feedback injections to those "big" nodes with highest degrees.

4. An simulation example

We illustrate the above analysis using a Chua's oscillator as a dynamical node in a scale-free network. In the dimensionless form, a single Chua's oscillator is described by [22]

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \alpha(x_2 - x_1 + f(x_1)) \\ x_1 - x_2 + x_3 \\ -\beta x_2 - \gamma x_3 \end{pmatrix}, \tag{19}$$

where $f(.)$ is a piecewise linear function of the form

$$f(x_1) = \begin{cases} -bx_1 - a + bx_1 > 1 , \\ -ax_1 |x_1| \leqslant 1 , \\ -bx_1 + a - bx_1 < -1 \end{cases} \tag{20}$$

in which $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $a < b < 0$. Suppose that two coupled Chua's oscillators are linked together through the first state variable, i.e., $\Gamma = diag(1, 0, 0)$. The state equations of the entire network are

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} \alpha(x_{i2} - x_{i1} + f(x_{i1})) + c\sum_{j=1}^{N} a_{ij} x_{j1} \\ x_{i1} - x_{i2} + x_{i3} \\ -\beta x_{i2} - \gamma x_{i3} \end{pmatrix}, \quad i = 1, 2, \ldots, N . \tag{21}$$

If the system parameters are chosen to be

$$\alpha = 10.0000, \quad \beta = 15.0000, \quad \gamma = 0.0385, \quad a = -1.2700, \quad b = -0.6800 , \tag{22}$$

then Chua's oscillator (19) has a chaotic attractor, as shown in Fig. 4. In this case, the oscillator (19) has three unstable equilibrium points:

$$\mathbf{x}^{\pm} = [\pm 1.8586 \quad \pm 0.0048 \quad \mp 1.8539]^{\mathrm{T}}, \qquad \mathbf{x}^0 = [0 \quad 0 \quad 0]^{\mathrm{T}} .$$

Suppose that we want to stabilize network (21) onto the homogeneous state $\bar{\mathbf{x}} = \mathbf{x}^+$ by applying local linear feedback control to the first state variables of a small fraction δ

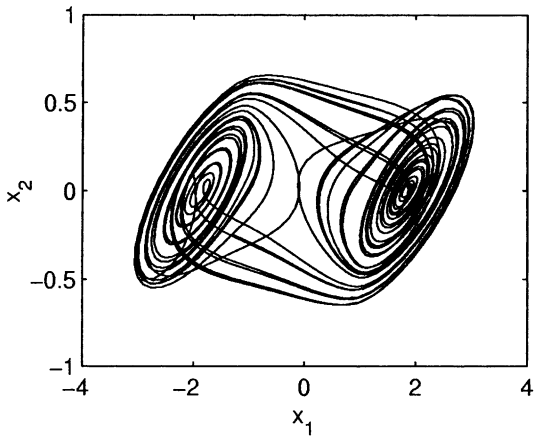

Fig. 4. The chaotic attrator of Chua's oscillator (19).

of nodes. The equations of the controlled network are

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} \alpha(x_{i2} - x_{i1} + f(x_{i1})) + c\sum_{j=1}^{N} a_{ij}x_{j1} + u_i \\ x_{i1} - x_{i2} + x_{i3} \\ -\beta x_{i2} - \gamma x_{i3} \end{pmatrix}, \quad i = 1, 2, \ldots, N ,$$

where

$$u_i = \begin{cases} cd(x_1^+ - x_{i1}), & i = i_1, i_2, \ldots, i_l , \\ 0, & \text{otherwise} . \end{cases}$$

It can be checked that the smallest value of the constant ρ in Lemma 1 is $\rho = -4.71$. For sufficiently large values of feedback gain d, stabilization can be achieved provided that

$$c\lambda_1(\delta) < -4.71 .$$

For a given coupling strength c, let $\hat{\delta}_r$ and $\hat{\delta}_s$ be the smallest value of δ that can achieve stabilization with respect to the randomly and specifically pinning schemes, respectively. Fig. 5 shows the values of $\hat{\delta}_r$ and $\hat{\delta}_s$ as functions of the coupling strength c. It is clear that for a wide range of coupling strengths, $\hat{\delta}_s$ is much smaller than $\hat{\delta}_r$, which verifies that the specifically pinning scheme requires much less local controllers than the randomly pinning scheme.

5. Conclusions

In this paper, we have investigated the stabilization problem for a scale-free dynamical network via local feedback pinning on a small fraction of the network nodes. We

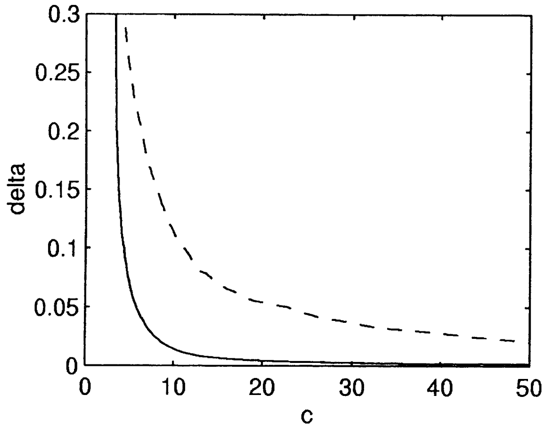

Fig. 5. Values of $\hat{\delta}_r$ (—) and $\hat{\delta}_s$ (—) as functions of coupling strength c.

have shown, both theoretically and numerically, that the most effective arrangement of placing the controllers on the network nodes is to choose those "big" nodes with highest degrees. The main reason is due to the extremely inhomogeneous connectivity distribution of the scale-free network. This may also help explain why complex networks such as the Internet and metabolic networks remain stable despite the frequent instability instances of some local nodes. On the other hand, the instability of a small fraction of the most highly connected nodes may have a severe influence on the stability of the global scale-free network. This indicates once again the so-called "robust yet fragile" feature of the stability of the scale-free networks.

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