Pinsker inequalities and related Monge-Ampère equations for log concave functions *

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Abstract

In this paper we further develop the theory of f-divergences for log-concave functions and their related inequalities. We establish Pinsker inequalities and new affine invariant entropy inequalities. We obtain new inequalities on functional affine surface area and lower and upper bounds for the Kullback-Leibler divergence in terms of functional affine surface area. The functional inequalities lead to new inequalities for L_p -affine surface areas for convex bodies.

Equality characterizations in these inequalities are related to a Monge Ampère differential equation. We establish uniqueness of the solution of the equation.

1 Introduction

Information theory, probability theory, and statistics have become important in convex geometry and vice versa, and there are many fascinating connections between these areas. Examples are the relation between the entropy power inequality and the Brunn-Minkowski inequality (see, e.g. [47]), the connection, established in [83], between the floating body [14, 100] from convex geometry and data depth from statistics (see also [26], the relation between the L_p -affine surface area and Rényi entropy from information theory and statistics [87, 108, 109] and connections between convex geometry and quantum information theory (e.g., [8, 9, 10, 11, 12, 99]). Further examples can be found in the books by Cover, Dembo, and Thomas [39] and Villani [106] and in [51, 56, 78, 79, 80].

The L_p -affine surface area is a fundamental notion in the theory of convex bodies. It has many remarkable properties. Aside from being linearly invariant, it is a valuation and it satisfies an affine isoperimetric inequality: among all convex bodies with fixed volume, L_p -affine surface area is maximized (minimized, depending on p) by ellipsoids. Not surprisingly, it therefore finds applications in e.g., affine differential geometry [66, 76, 77], geometric flows [103, 105, 107], valuation theory [34, 67, 73, 74, 97, 98] and approximation theory [20, 21, 22, 49, 50, 91, 101]. For extensions to the spherical and hyperbolic setting see [17, 18].

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For a convex body K in \mathbb{R}^n and real $p \neq -n$, it is defined as

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{p+p}}}{\langle x, N_K(x) \rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x), \tag{1}$$

where μ_K is the usual surface measure on ∂K , the boundary of K, $N_K(x)$ is the outer unit normal vector at x to ∂K and $\kappa_K(x)$ is the (generalized) Gauss curvature in $x \in \partial K$. The case p = 1 is the classical affine surface area introduced by Blaschke [19] in dimensions 2 and 3 for sufficiently smooth bodies and extended much later to all convex bodies by Leichtweiss [69], Lutwak [75] and Schütt and Werner [100]. Then, Lutwak, in his ground breaking paper [77], introduced L_p -affine surface area for p > 1. It was finally extended to all $p \in \mathbb{R}, p \neq -n$, and all convex bodies in [102], (see also [53, 81]).

In [108] it was shown that L_p -affine surface areas are Rényi entropies. The latter are specific examples of f-divergences. An f-divergence, which is an important concept from information theory, is a function that measures the difference between (probability) distributions P and Q (see, e.g. [5, 36, 82]). Further examples of f-divergences are Kullback-Leibler divergence [64] $D = D_{KL}(P||Q)$ and total variation distance V = V(P,Q). For various purposes it is of interest to investigate if they can be compared to one another. The most famous such comparison inequality is Pinsker's inequality [89] which states that

$$D \ge \frac{1}{2}V^2. \tag{2}$$

The best constant, $\frac{1}{2}$, is due, independently to Csiszár [37], Kemperman [58] and Kullback [62, 63]. For applications of Pinsker's inequality, see e.g., [16], [38], [104], or Bolley and Villani [24] who showed that Pinsker's inequality is a variant of Talagrand's transportation inequality. Generalizations of Pinsker type inequalities for *f*-divergences were obtained by G. Gilardoni [48], M. Reid and R. Williamson [90].

In recent years much effort has been devoted to extend concepts from convex geometry to a functional setting. Natural analogs of convex bodies are log-concave functions. Much progress has been made in this direction, resulting in functional analogs of the Blaschke Santaló inequality [6, 13, 44, 68], the affine isoperimetric inequality [7, 30], Alexandrov-Fenchel type inequalities [29] and analogs of the John ellipsoid [4] and Löwner ellipsoid [70]. More examples can be found in e.g., [2, 3, 31, 32, 33, 47, 59, 94]). In particular, L_{λ} -affine surface area, the functional analog of L_p -affine surface area for convex bodies, was introduced in [30] for a log concave function $\varphi(x) = e^{-\psi(x)}$ and $\lambda \in \mathbb{R}$,

$$as_{\lambda}(\varphi) = \int \varphi \left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det \left[\nabla^2 \left(-\ln \varphi \right) \right] \right)^{\lambda} dx,$$

where $\nabla \varphi$ is the gradient and $\nabla^2 \varphi$ is the (generalized) Hessian of φ . In that context, entropy inequalities for log-concave functions were established. We only mention a reverse log-Sobolev inequality, proved in [7, 30],

$$\int \ln\left(\det\left(\nabla^{2}\psi\right)\right) e^{-\psi(x)} dx \leq 2\left[\operatorname{Ent}(\varphi) - \operatorname{Ent}(g)\right],\tag{3}$$

where $\varphi : \mathbb{R}^n \to [0, \infty)$ is the density of a log concave probability measure on \mathbb{R}^n , and $\operatorname{Ent}(\varphi)$, resp. $\operatorname{Ent}(g)$ the entropy of φ resp. the Gaussian g. The definitions are given below.

In [28], a theory of f-divergences for log-concave functions was initiated. For instance, Kullback-Leibler divergence and related inequalities for log-concave functions was established as part of the theory. It already yielded entropy inequalities which are stronger than already existing ones. For example, it resulted in the following strengthing of the reverse log-Sobolev inequality (3) of [7],

$$\int \ln\left(\det\left(\nabla^2\psi\right)\right) e^{-\psi(x)} dx \leq 2\left[\operatorname{Ent}(\varphi) - \operatorname{Ent}(g)\right] + \ln\left(\frac{\int e^{-\psi^*}}{(2\pi)^n}\right), \quad (4)$$

where $\varphi : \mathbb{R}^n \to [0, \infty)$ is the density of a log concave probability measure on \mathbb{R}^n , and ψ^* is the Legendre transform of ψ .

In this paper we further develop the theory of f-divergences for log-concave functions and their related inequalities. We establish a Pinsker inequality and new affine invariant entropy inequalities for log-concave functions. We obtain new inequalities on functional affine surface area for log-concave functions and lower and upper bounds for the Kullback-Leibler divergence in terms of functional affine surface area. The inequalities obtained for log-concave functions lead to new inequalities for L_p -affine surface areas for convex bodies.

We start the paper by characterizing the equality case of inequality (4) in Section 2. While equality characterizations of inequality (3) were provided in [30], no such characterizations were available up to date for inequality (4) and for more general f-divergence inequalities. We show first that equality characterization is equivalent to uniqueness of the solution of a Monge Ampère differential equation (also called elliptic Kähler-Einstein equation). Then we show that the Monge-Ampère equation has a unique solution. To do that we use optimal transportation and Cafarelli's regularity theory for optimal transportation.

In Section 3, we show that a Pinsker type inequality for log concave functions follows immediately from a result by G. Gilardoni [48]. Namely, for a convex function $f : (0, \infty) \to \mathbb{R}$, we have

$$D_f(\varphi) \geq \frac{f''(1)}{2} \left(\int \left| \frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det \left[\nabla^2 \left(-\ln \varphi \right) \right]}{\varphi \int e^{-\psi^*}} - \frac{\varphi}{\int \varphi} \right| dx \right)^2,$$

where $D_f(\varphi)$ is the *f*-divergence of the log concave function φ (see Section 2.2 for the definition). A consequence of this Pinsker inequality is an improvement of inequality (4), which takes the form (see Corollary 9 for the precise statement),

$$\int \ln\left(\det\left(\nabla^{2}\psi\right)\right) e^{-\psi(x)} dx \leq 2\left[\operatorname{Ent}(\varphi) - \operatorname{Ent}(g)\right] + \ln\left(\frac{\int e^{-\psi^{*}}}{(2\pi)^{n}}\right) \\ - \frac{1}{2}\left(\int \left|\frac{e^{\psi-\langle\nabla\psi,x\rangle}\det\left(\nabla^{2}\psi\right)}{\int e^{-\psi^{*}}} - e^{-\psi}\right| dx\right)^{2}.$$

In Section 4 we prove difference inequalities for functional affine surface areas and show that λ -affine surface area of a log concave function φ and its polar φ° (see (10) below for the definition) is bounded by a convex combination of 0-affine surface area and 1-affine surface area. We obtain lower and upper bounds for the Kullback-Leibler divergence $D_{KL}(Q_{\varphi}||P_{\varphi})$ in terms of functional affine surface area,

$$as_0(\varphi) - as_1(\varphi) \leq D_{KL}(Q_{\varphi}||P_{\varphi}) \leq as_{-1}(\varphi) - as_0(\varphi)$$

where P_{φ} and Q_{φ} are two distributions with densities $p_{\varphi} = \varphi^{-1} e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det \left[\nabla^2 \left(-\ln \varphi \right) \right]$ and $q_{\varphi} = \varphi$, respectively. Please see (15) for the exact definition of $D_{KL}(Q_{\varphi} || P_{\varphi})$.

We show in Section 5 that on the level of convex bodies these inequalities correspond to inequalities on L_p -affine surface area, e.g., the following ones,

$$as_{\infty}(K) - as_0(K) \leq as_{\frac{p}{n+p}}(K) - as_{\frac{n}{n+p}}(K),$$

for $p \in (-\infty, -n)$, and for p > -n, the inequality is reversed. And

$$as_p(K) \leq \left(\frac{p}{n+p}\right)as_{\infty}(K) + \left(\frac{n}{n+p}\right)as_0(K),$$

in the case when p > 0. For p < 0, those inequalities are reversed. Equality holds trivially if p = 0 or $p = \infty$. Equality also holds for origin symmetric ellipsoids \mathcal{E} whose volume $|\mathcal{E}|$ equals the volume $|B_2^n|$ of the Euclidean unit ball B_2^n .

2 A Monge-Ampère equation and equality in a divergence inequality

2.1 Background on *f*-divergence

Csiszár [36], and independently Morimoto [82] and Ali & Silvery [5] introduced the notion of f-divergence to measure the difference between probability distributions. This notion finds applications in e.g. information theory, statistics, probability theory, signal processing, and pattern recognition [15, 35, 54, 71, 72, 86].

Let (X, μ) be a measure space and let $P = p\mu$ and $Q = q\mu$ be (probability) measures on X that are absolutely continuous with respect to the measure μ . Let $f : (0, \infty) \to \mathbb{R}$ be a convex or a concave function. Then the *f*-divergence $D_f(P, Q)$ of the measures P and Q is defined by

$$D_f(P,Q) = \int_X f\left(\frac{p}{q}\right) q d\mu.$$
(5)

The best known examples of f-divergences are the total variation distance

$$V(P,Q) = \int |p-q| \, d\mu \text{ for } f(t) = |t-1|,$$

and the Kullback-Leibler divergence, or relative entropy

$$D_{KL}(P||Q) = \int p \log\left(\frac{p}{q}\right) d\mu \text{ for } f(t) = t \log t.$$
(6)

We also note that for $f(t) = t^{\alpha}$ we obtain the *Hellinger integrals* (see, e.g., [72])

$$H_{\alpha}(P,Q) = \int_{X} p^{\alpha} q^{1-\alpha} d\mu.$$

Those are related to the *Rényi divergence* of order α , $\alpha \neq 1$, introduced by Rényi [92] (for $\alpha > 0$) as

$$D_{\alpha}(P||Q) = \frac{1}{\alpha - 1} \log\left(\int_{X} p^{\alpha} q^{1 - \alpha} d\mu\right) = \frac{1}{\alpha - 1} \log\left(H_{\alpha}(P, Q)\right).$$
(7)

The case $\alpha = 1$ is the relative entropy $D_{KL}(P||Q)$.

2.2 *f*-divergence for log concave functions

Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a convex function. We define Ω_{ψ} to be the interior of the convex domain of ψ , that is

$$\Omega_{\psi} = \operatorname{int} \left(\{ x \in \mathbb{R}^n, \psi(x) < \infty \} \right) = \operatorname{int} \left(\{ \psi < \infty \} \right).$$

We always consider in this paper convex functions ψ such that $\Omega_{\psi} \neq \emptyset$. Then ψ is in particular ψ is proper, i.e., $\psi(x)$ is finite for at least one x and that the affine dimension of Ω_{ψ} is equal to n. This implies that

$$\int_{\Omega_{\psi}} e^{-\psi(x)} dx > 0.$$
(8)

We will also assume throughout that $e^{-\psi(x)}$ is integrable, i.e., $\int_{\Omega_{\psi}} e^{-\psi(x)} dx < \infty$ and we will also often write in short $\int e^{-\psi(x)} dx$.

In the general case, when ψ is neither smooth nor strictly convex, the gradient of ψ , denoted by $\nabla \psi$, exists almost everywhere by Rademacher's theorem (see, e.g., [23]), and a theorem of Alexandrov [1] and Busemann and Feller [27] guarantees the existence of the (generalized) Hessian, denoted by $\nabla^2 \psi$, almost everywhere in Ω_{ψ} . Let

$$X_{\psi} = \left\{ x \in \mathbb{R}^n : \ \psi(x) < \infty, \text{ and } \nabla^2 \psi(x) \text{ exists and is invertible} \right\}.$$

We recall the Legendre transform $\mathcal{L}\psi$ of ψ ,

$$\mathcal{L}\psi(y) = \psi^*(y) = \sup_{x \in \mathbb{R}^n} \left[\langle x, y \rangle - \psi(x) \right].$$
(9)

When ψ is C^2 , then $X_{\psi} = \Omega_{\psi}$ and $X_{\psi^*} = \Omega_{\psi^*}$. More information about duality transforms of convex functions can be found in [93, 96].

A function $\varphi : \mathbb{R}^n \to [0, \infty)$ is log concave, if it is of the form $\varphi(x) = \exp(-\psi(x))$ where $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is convex. The dual function φ° of a log concave function is defined by [6, 13]

$$\varphi^{\circ}(x) = \inf_{y \in \mathbb{R}^n} \left[\frac{e^{-\langle x, y \rangle}}{\varphi(y)} \right].$$

This definition is connected with the Legendre transform, namely,

$$\varphi^{\circ} = e^{-\mathcal{L}(\psi)} = e^{-\psi^*}.$$
(10)

In [28], f-divergences for s-concave and log concave functions were introduced and their basic properties and entropy inequalities were established. It is explained in detail in [28] that the following definition for f-divergence seems to be the correct one.

Definition 1. Let $f : (0, \infty) \to \mathbb{R}$ be a convex or concave function and let $\varphi : \mathbb{R}^n \to [0, \infty)$ be a log concave function. Then the f-divergence $D_f(\varphi)$ of φ is

$$D_{f}(\varphi) = \int_{X_{\psi}} \varphi f\left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^{2}} \det\left[\nabla^{2}\left(-\ln \varphi\right)\right]\right) dx$$
$$= \int_{X_{\psi}} e^{-\psi} f\left(e^{2\psi - \langle \nabla \psi, x \rangle} \det\left[\nabla^{2}\psi\right]\right) dx.$$
(11)

The special case when $f(t) = t^{\lambda}$, $-\infty < \lambda < \infty$ leads to the L_{λ} -affine surface areas of φ [30],

$$as_{\lambda}(\varphi) = \int_{X_{\psi}} \varphi \left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^2} \det \left[\nabla^2 \left(-\ln \varphi \right) \right] \right)^{\lambda} dx$$
$$= \int_{X_{\psi}} e^{(2\lambda - 1)\psi(x) - \lambda \langle x, \nabla \psi(x) \rangle} \left(\det \nabla^2 \psi(x) \right)^{\lambda} dx.$$
(12)

Those were extensively studied in [30]. In particular,

$$as_0(\varphi) = \int_{X_\psi} \varphi dx \tag{13}$$

and, as observed in [30], since det $[\nabla^2 (-\ln \varphi)] = 0$ outside of X_{ψ} , the integral may be taken over Ω_{ψ} for any $\lambda > 0$. Therefore

$$as_{1}(\varphi) = \int_{X_{\psi}} \varphi^{-1} e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det \left[\nabla^{2} \left(-\ln \varphi \right) \right] dx = \int_{\Omega_{\psi}} \varphi^{-1} e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det \left[\nabla^{2} \left(-\ln \varphi \right) \right] dx$$
$$= \int_{X_{\psi^{*}}} \varphi^{\circ}. \tag{14}$$

In analogy to (56) below, the expressions (11) are the appropriate ones to define f-divergences for log concave functions and because of (13) and (14) these expressions can be viewed as the "volume" of φ and the "volume" of φ° with their corresponding "cone measures". This is explained in detail in [28].

Another special case occurs when $f(t) = -\ln t$. The *f*-divergences then becomes the Kullback-Leibler divergence

$$D_{KL}(Q_{\varphi}||P_{\varphi}) = \int_{X_{\psi}} \varphi \ln \left(\varphi^{2} e^{-\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \left(\det \left[\nabla^{2} \left(-\ln \varphi\right)\right]\right)^{-1}\right) dx$$
$$= \int_{X_{\psi}} e^{-\psi} \left(\langle \nabla \psi, x \rangle - 2\psi\right) \ln \left(\det \left[\nabla^{2} \left(\psi\right)\right]\right)^{-1} dx, \qquad (15)$$

where P_{φ} and Q_{φ} are two distributions with densities $p_{\varphi} = \varphi^{-1} e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det \left[\nabla^2 \left(-\ln \varphi \right) \right]$ and $q_{\varphi} = \varphi$, respectively.

2.3 A Monge-Ampère equation

Now we concentrate on the following divergence inequality, which was also proved in [28]. Recall that we assume throughout the paper that $\int e^{-\psi(x)} dx < \infty$.

Theorem 2. [28] Let $f : (0, \infty) \to \mathbb{R}$ be a convex function. Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be a integrable log-concave function that is C^2 . Then

$$D_f(\varphi) \ge f\left(\frac{\int_{\Omega_{\psi^*}} \varphi^{\circ} dx}{\int_{\Omega_{\psi}} \varphi dx}\right) \left(\int_{\Omega_{\psi}} \varphi dx\right).$$
(16)

If f is concave, the inequality is reversed. If f is linear, equality holds in (16). Equality also holds if $\varphi(x) = Ce^{-\langle Ax,x \rangle}$, where C is a positive constant and A is an $n \times n$ positive definite matrix.

A characterization of the equality case of inequality (16) - and several other inequalities proved in [28] - has not been obtained so far. Here we give such a characterization. We show first that characterization of equality in (16) is equivalent to the unique solution of a Monge Ampère differential equation.

We write $\varphi = e^{-\psi}$. It was shown in [28] that inequality (16) is a consequence of Jensen's inequality and the identity (see, e.g., [28]),

$$\int_{\Omega_{\psi^*}} e^{-\psi^*} dx = \int_{\Omega_{\psi}} e^{\psi - \langle \nabla \psi, x \rangle} \det \left(\nabla^2 \psi \right) dx.$$
(17)

Thus, equality holds in (16) if and only if equality holds in Jensen's inequality which happens if and only if

$$\det(\nabla^2 \psi(x)) = C e^{-2\psi(x) + \langle \nabla \psi(x), x \rangle}, \quad \text{a.e. } x \in \mathbb{R}^n.$$
(18)

To determine C, we integrate (18) to get

$$C\int_{\Omega_{\psi}} e^{-\psi(x)} dx = \int_{\Omega_{\psi}} e^{\psi - \langle \nabla \psi, x \rangle} \det \left(\nabla^2 \psi \right) dx,$$

which together with (17) gives that

$$C = \frac{\int_{\Omega_{\psi^*}} e^{-\psi^*} dx}{\int_{\Omega_{\psi}} e^{-\psi} dx} = \frac{\int_{\Omega_{\psi^*}} \varphi^{\circ}(x) dx}{\int_{X_{\psi}} \varphi(x) dx}.$$

Thus, when f is either strictly convex or strictly concave, equality holds in inequality (16) if and only if ψ satisfies

$$\det(\nabla^2 \psi(x)) = \frac{\int_{\Omega_{\psi^*}} e^{-\psi^*} dx}{\int_{\Omega_{\psi}} e^{-\psi} dx} e^{-2\psi(x) + \langle \nabla \psi(x), x \rangle}, \qquad x \in \mathbb{R}^n.$$
(19)

Recall now that

$$\psi(x) + \psi^*(y) \ge \langle x, y \rangle$$

for every $x, y \in \mathbb{R}^n$, with equality if and only if x is in the domain of ψ and $y \in \partial \psi(x)$, the sub differential of ψ at x. In particular

$$\psi^*(\nabla\psi(x)) = \langle x, \nabla\psi(x) \rangle - \psi(x), \quad \text{a.e. in } \Omega_{\psi}.$$
 (20)

Thus, equation (19) can be written as

$$\frac{e^{-\psi}}{\int_{\Omega_{\psi}} e^{-\psi} dx} = \frac{e^{-\psi^* (\nabla \psi(x))}}{\int_{\Omega_{\psi^*}} e^{-\psi^*} dx} \det(\nabla^2 \psi(x)), \tag{21}$$

which is just a Monge Ampère equation (also called elliptic Kähler-Einstein equation).

Note that if ψ solves (21), then it is not difficult to show that $\psi(x) + c$ solves (21) for any constant c. Thus we seek uniqueness of the solution of (21) up to a constant and this is established the following theorem.

Theorem 3. Let $\psi \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function such that

 $e^{-\psi}dx$ and $e^{-\psi^*}dx$

are finite log concave measures. Assume that the mapping $T(x) = \nabla \psi(x)$ pushes forward

$$d\mu = \frac{e^{-\psi}}{\int e^{-\psi} dx} dx \quad onto \quad d\nu = \frac{e^{-\psi^*}}{\int e^{-\psi^*} dx} dx.$$

In addition, assume that μ has logarithmic derivatives for every x_i , $1 \le i \le n$. Then ψ has the form $\psi = \frac{1}{2} \langle Ax, x \rangle + c$, where c is a constant and A is a $n \times n$ positive definite matrix.

Before we prove this theorem we need to establish some preliminary results concerning integral relations for solutions to the optimal transportation problem. For more detail and background we refer to [106].

Let $\mu = e^{-V} dx$ be a probability measure on \mathbb{R}^n . We say that V_{x_i} is a logarithmic (Sobolev) derivative of μ if $V_{x_i} \in L^1(\mu)$ and for every compactly supported smooth test function ξ the following relation holds,

$$\int \xi_{x_i} d\mu = \int \xi V_{x_i} d\mu.$$

As noted above, in the case of a log concave measure $\mu = e^{-V}$ the function V is almost everywhere differentiable on $\Omega_V = \operatorname{int}(\{V < \infty\})$, but this does not mean that μ has logarithmic derivatives. Indeed, in general the integration by parts formula includes a singular term,

$$\int \xi_{x_i} d\mu = \int \xi V_{x_i} d\mu + \int_{\partial \{V < \infty\}} \langle n, e_i \rangle \xi e^{-V} d\mathcal{H}^{n-1}$$

where n is the outward normal vector to $\partial \{V < \infty\}$ and \mathcal{H}^{n-1} is the (n-1)-dimensional Hausdorff measure. Thus μ does not admit a logarithmic derivative if $\{V = \infty\}$ is not empty and e^{-V} is not vanishing on $\partial \{V < \infty\}$.

In what follows, we are given two probability measures $d\mu = e^{-V}dx$ and $d\nu = e^{-W}dx$. Let $\nabla \psi$ be the optimal transportation of $\mu = e^{-V}$ onto $\nu = e^{-W}$. We remark that ψ_{x_i} is always understood in the classical sense, i.e. almost everywhere pointwise. The next proposition was proved in [57]. **Proposition 4.** (Proposition 5.5. [57]) Assume that V, W, ψ are smooth functions on the entire \mathbb{R}^n and ν is a log concave measure. Then for every $q \geq 2$, $0 < \tau < 1$, $i = 1, \dots, n$ there exists $C(q, \tau) > 0$ such that

$$\int_{\mathbb{R}^n} |\psi_{x_i x_i}|^q d\mu \le C(q,\tau) \left(\int_{\mathbb{R}^n} |V_{x_i}|^{\frac{2q}{2-\tau}} d\mu + \int_{\mathbb{R}^n} |x_i|^{\frac{2q}{\tau}} d\nu \right).$$

In the case when μ admits logarithmic derivatives which are integrable in a sufficiently high power, it is natural to understand the second derivatives of ψ in the Sobolev sense. More precisely, we say that ψ admits second partial derivatives $\psi_{x_ix_j}$ in the Sobolev sense if for every smooth compactly supported test function ξ ,

$$\int \psi_{x_i x_j} \xi d\mu = -\int \xi_{x_i} \psi_{x_j} d\mu + \int V_{x_i} \psi_{x_j} \xi d\mu$$

We will use throughout differentiation of the Monge-Ampère equation developed in [57] and [61]. Let us briefly explain this machinery. Differentiate the change of variables formula

$$V = W(\nabla \psi) - \log \det \nabla^2 \psi$$

along unit vector e,

$$V_e = \langle \nabla^2 \psi_e, \nabla W(\nabla \psi) \rangle - \operatorname{Tr}(\nabla^2 \psi)^{-1} \nabla^2 \psi_e.$$
(22)

Introduce the second-order differential operator L,

$$Lf = \operatorname{Tr}(\nabla^2 \psi)^{-1} \nabla^2 f - \langle \nabla f, \nabla W(\nabla \psi) \rangle.$$

One can verify (see [61], Lemma 2.1) that measure μ is invariant with respect to L

$$\int L\xi \cdot \eta d\mu = -\int \langle (\nabla^2 \psi)^{-1} \nabla \xi, \nabla \eta \rangle d\mu,$$

where ξ and η are smooth functions with compact supports

$$K_{\xi}, K_{\eta} \subset \{V < \infty\}.$$

Equation (22) can be rewritten as follows:

$$V_e = -L\psi_e.$$

Differentiating second time one gets

$$V_{ee} = -L\psi_{ee} + \langle \nabla^2 W(\nabla\psi), \nabla\psi_e, \nabla\psi_e \rangle + \operatorname{Tr}\left[(\nabla^2 \psi)^{-1} (\nabla^2 \psi_e) \right]^2.$$
(23)

Integrating with respect to μ and using invariance of μ one gets formally the following integral identity obtained in [60]

$$\int V_e^2 d\mu = \int V_{ee} d\mu = \int \langle \nabla^2 W(\nabla \psi) \nabla \psi_e, \nabla \psi_e \rangle d\mu + \int \operatorname{Tr} \left[(\nabla^2 \psi)^{-1} (\nabla^2 \psi_e) \right]^2 d\mu.$$
(24)

Note that $\int V_e^2 d\mu = \int V_{ee} d\mu$ as μ admits logarithmic derivatives.

We stress that it is indeed a formal relation, because we neglect boundary terms which may appear. Formula (24) holds under additional assumptions on the growth and smoothness of V, W.

Now we will apply the following slight extension of Proposition 5.5. of [57] (see Remark 5.6 in [57]), which can be easily obtained by smooth approximations.

Proposition 5. [57] Assume that ν is a log concave measure and that μ admits logarithmic derivatives V_{x_i} for all $1 \leq i \leq n$ which are integrable in any power. Then ψ admits second Sobolev derivatives with respect to μ satisfying

$$\int |\psi_{x_i x_i}|^q d\mu \le C(q,\tau) \left(\int |V_{x_i}|^{\frac{2q}{2-\tau}} d\mu + \int |x_i|^{\frac{2q}{\tau}} d\nu \right),$$
(25)

where $q \ge 2, 0 < \tau < 1$.

Inequality (24) follows from the integration by parts formula applied to an identity obtained by differentiation of the Monge–Ampére equation. Our next aim is to justify (24) in form of inequality in a sufficient general setting.

Proposition 6. Let $\mu = e^{-V}dx$ and $\nu = e^{-W}dx$ be probability measures, $V \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, W \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Let $\nabla \psi$ be the optimal transportation mapping of μ onto ν .

Assume that the following assumptions hold.

- 1. The sets $\{V < \infty\}$ and $\{W < \infty\}$ are open and V and W are twice differentiable on the sets $\{V < \infty\}$ and $\{W < \infty\}$ respectively, with locally Hölder second derivatives.
- 2. The measure μ admits logarithmic derivatives V_{x_i} which are integrable in every power with respect to μ

$$\int |V_{x_i}|^p d\mu < \infty, \ \forall p > 0.$$

3. ν is log-concave.

Then ψ is at least four times continuously differentiable on $\{V < \infty\}$ and the following integral inequality holds for every unit vector $e \in \mathbb{R}^n$,

$$\int_{\{V<\infty\}} V_{ee} d\mu \ge \int \langle \nabla^2 W(\nabla\psi) \nabla\psi_e, \nabla\psi_e \rangle d\mu + \int \operatorname{Tr} \left[(\nabla^2 \psi)^{-1} (\nabla^2 \psi_e) \right]^2 d\mu.$$
(26)

Proof. Observe first that by assumption on V and W, $\Omega_V = \{V < \infty\}$ and $\Omega_W = \{W < \infty\}$. Next we note that by Proposition 5 $\int |\psi_{x_ix_i}|^p d\mu < \infty$ for every p > 0. Using Sobolev embeddings and the fact that V is locally bounded we get that ψ_{x_i} is locally Sobolev with respect to Lebesgue measure on $\{V < \infty\}$ and continuous.

Let us show that ψ is four times differentiable. Since $\nabla \psi$ is well defined and continuous almost everywhere, we can choose $x_0 \in \{V < \infty\}$ such that $\nabla \psi(x_0)$ exists. Since $\{W < \infty\}$ is open, there exists an open bounded convex neighborhood $U_2 \subset \{W < \infty\}$ of $\nabla \psi(x_0)$. By continuity $U_1 = \nabla \psi^{-1}(U_2) \subset \{V < \infty\}$ is an open set. Moreover, since convex functions are locally Lipschitz, $U_1 = \nabla \psi^*(U_2)$ is bounded. Then consider a mass transportation problem of $\mu|_{U_1}$ onto $\nu|_{U_2}$. Thus U_1 and U_2 are bounded open sets, U_2 is convex and $\nabla \psi$ is the optimal transportation mapping of $\mu|_{U_1}$ onto $\nu|_{U_2}$. These measures have $C^{2,\alpha}$ densities for some α , $0 < \alpha \leq 1$, with respect to the Lebesgue measure. Hence by Caffarelli's regularity theory, (see e.g., Theorem 4.14 and Remark 4.15 from [106]), ψ is locally $C^{4,\alpha}$ on $\{V < \infty\}$. Next we apply the following equation obtained by differentiation of the change of variables formula (see [57], [61] for details),

$$V_{ee} = -L\psi_{ee} + \langle \nabla^2 W(\nabla\psi)\nabla\psi_e, \nabla\psi_e \rangle + \mathrm{Tr} \big[(\nabla^2\psi)^{-1} (\nabla^2\psi_e) \big]^2.$$
(27)

Here L is the second-order differential operator satisfying

$$\int L\xi \cdot \eta d\mu = -\int \langle (\nabla^2 \psi)^{-1} \nabla \xi, \nabla \eta \rangle d\mu,$$

where ξ and η are smooth test functions with compact supports $K_{\xi}, K_{\eta} \subset \{V < \infty\}$.

Step 1. Assume that ν is fully supported, i.e., $W < \infty$ everywhere.

Take a smooth test function η that has compact support in $\{V < \infty\}$, multiply (27) by $\xi = \eta(\nabla \psi)$ and integrate with respect to μ . One gets

$$\int_{\{V<\infty\}} V_{ee} \xi d\mu = \int \langle (\nabla^2 \psi)^{-1} \nabla \psi_{ee}, \nabla (\eta (\nabla \psi)) \rangle d\mu + \int \langle \nabla^2 W (\nabla \psi) \nabla \psi_e, \nabla \psi_e \rangle \xi d\mu + \int \operatorname{Tr} \left[(\nabla^2 \psi)^{-1} (\nabla^2 \psi_e) \right]^2 \xi d\mu.$$
(28)

Note that

$$-\int \langle (\nabla^2 \psi)^{-1} \nabla \psi_{ee}, \nabla (\eta (\nabla \psi)) \rangle d\mu = -\int \langle \nabla \psi_{ee}, \nabla \eta \circ \nabla \psi \rangle d\mu$$
$$= -\int \langle (\nabla^2 \psi)^{-\frac{1}{2}} \nabla^2 \psi_e \cdot e, (\nabla^2 \psi)^{\frac{1}{2}} \nabla \eta \circ \nabla \psi \rangle d\mu$$
$$= \int \langle A (\nabla^2 \psi)^{\frac{1}{2}} \cdot e, (\nabla^2 \psi)^{\frac{1}{2}} \nabla \eta \circ \nabla \psi \rangle d\mu \le \int \|A\| \|\nabla^2 \psi\| |\nabla \eta \circ \nabla \psi| d\mu,$$

where $A = (\nabla^2 \psi)^{-\frac{1}{2}} \nabla^2 \psi_e (\nabla^2 \psi)^{-\frac{1}{2}}$ and $\|\cdot\|$ is the operator norm. Next we note that

$$||A||^{2} \le ||A||_{HS}^{2} = \operatorname{Tr}A^{2} = \operatorname{Tr}\left[(\nabla^{2}\psi)^{-1}(\nabla^{2}\psi_{e})\right]^{2}.$$

Hence for every $\varepsilon > 0$

$$-\int \langle (\nabla^2 \psi)^{-1} \nabla \psi_{ee}, \nabla(\eta(\nabla \psi)) \rangle d\mu$$

$$\leq \varepsilon \int \operatorname{Tr} \left[(\nabla^2 \psi)^{-1} (\nabla^2 \psi_e) \right]^2 \xi d\mu + \frac{1}{4\varepsilon} \int \|\nabla^2 \psi\|^2 \frac{|\nabla(\eta(\nabla \psi)|^2}{\xi} d\mu$$

$$= \varepsilon \int \operatorname{Tr} \left[(\nabla^2 \psi)^{-1} (\nabla^2 \psi_e) \right]^2 \xi d\mu + \frac{1}{4\varepsilon} \int \|\nabla^2 \psi\|^2 \frac{|\nabla \eta|^2}{\eta} \circ (\nabla \psi) d\mu$$

$$\leq \varepsilon \int \operatorname{Tr} \left[(\nabla^2 \psi)^{-1} (\nabla^2 \psi_e) \right]^2 \xi d\mu + \frac{1}{4\varepsilon} \left(\int \|\nabla^2 \psi\|^{2p} d\mu \right)^{\frac{1}{p}} \left(\int \frac{|\nabla \eta|^{2q}}{\eta^q} d\nu \right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Thus, for arbitrary $\varepsilon > 0$

$$\int_{\{V<\infty\}} V_{ee}\xi d\mu + \frac{1}{4\varepsilon} \left(\int \|\nabla^2 \psi\|^{2p} d\mu \right)^{\frac{1}{p}} \left(\int \frac{|\nabla \eta|^{2q}}{\eta^q} d\nu \right)^{\frac{1}{q}} \\
\geq \int \langle \nabla^2 W(\nabla \psi) \nabla \psi_e, \nabla \psi_e \rangle \xi d\mu + (1-\varepsilon) \int \operatorname{Tr} \left[(\nabla^2 \psi)^{-1} (\nabla^2 \psi_e) \right]^2 \xi d\mu. \quad (29)$$

Finally, we want to extract (26) from (29). To this end we find a sequence of compactly supported functions $1 \ge \eta_n \ge 0$ with $\eta_n \to 1$ pointwise such that $\lim_n \int \frac{|\nabla \eta_n|^{2q}}{\eta_n^q} d\nu = 0$ and set $\xi_n = \eta_n (\nabla \Phi)$. We omit the description of the precise construction, since it can be easily done taking into account that $\operatorname{supp}(\nu) = \mathbb{R}^n$. We apply inequality (29), where ξ is replaced by ξ_n and pass to the limit letting n to infinity. Note that $\int ||\nabla^2 \psi||^{2p} d\mu < \infty$ by Proposition 4. Passing to the limit and applying that $\varepsilon > 0$ is arbitrary, one gets (26). Moreover, since the integrals $\int ||\nabla^2 \psi||^{2p} d\mu$, $\int \operatorname{Tr} [(\nabla^2 \psi)^{-1} (\nabla^2 \psi_e)]^2 d\mu$ are finite, it is clear that

$$\int \langle (\nabla^2 \psi)^{-1} \nabla \psi_{ee}, \nabla (\eta_n (\nabla \psi)) \rangle d\mu \to 0$$

and we have in fact equality in (26), because we can pass to the limit in (28). The proof of Step 1 is complete.

Step 2. Proof of the general case: W is twice Hölder differentiable on the open convex domain $\{W < \infty\}$.

Approximate W by everywhere finite and smooth convex functions W_n such that $\nabla W_n \rightarrow \nabla W$ and $\nabla^2 W_n \rightarrow \nabla^2 W$ pointwise on $\{W < \infty\}$. This can be done with the standard convolution technique: set $e^{-W_n} = e^{-W} * \gamma_{\frac{1}{n}}$, where $\gamma_{\frac{1}{n}}$ is the Gaussian measure with zero mean and variance $\frac{1}{n}$. By the Prekopa-Leindler inequality we get that every W_n is convex and, in addition, smooth on the entire \mathbb{R}^n . According to Step 1,

$$\int_{\{V<\infty\}} V_{ee} d\mu = \int \langle \nabla^2 W_n(\nabla \psi_n) \nabla(\psi_n)_e, \nabla(\psi_n)_e \rangle d\mu + \int \operatorname{Tr} \left[(\nabla^2 \psi_n)^{-1} (\nabla^2 (\psi_n)_e) \right]^2 d\mu,$$
(30)

where $\nabla \psi_n$ is the optimal transportation mapping of μ onto $\nu_n = e^{-W_n} dx$. First we observe that it follows from (25) that

$$\sup_{n} \int \left(|\nabla \psi_n|^p + \|\nabla^2 \psi_n\|^p \right) d\mu < \infty, \ \forall p > 0.$$
(31)

We may assume that

$$\int_{\{V<\infty\}} V_{ee} d\mu < \infty$$

as otherwise there is nothing to prove. Then one has by (30)

$$\int \operatorname{Tr}\left[(\nabla^2 \psi_n)^{-1} (\nabla^2 (\psi_n)_e)\right]^2 d\mu = \int_{\{V < \infty\}} V_{ee} d\mu$$
$$-\int \langle \nabla^2 W_n (\nabla \psi_n) \nabla (\psi_n)_e, \nabla (\psi_n)_e \rangle d\mu \le \int_{\{V < \infty\}} V_{ee} d\mu < \infty \qquad (32)$$

and thus

$$\sup_{n} \int \operatorname{Tr} \left[(\nabla^{2} \psi_{n})^{-1} (\nabla^{2} (\psi_{n})_{e}) \right]^{2} d\mu < \infty.$$

Hence

$$\infty > \int_{\{V < \infty\}} V_{ee} d\mu \ge \sup_{n} \int \operatorname{Tr} \left[(\nabla^{2} \psi_{n})^{-1} (\nabla^{2} (\psi_{n})_{e}) \right]^{2} d\mu$$

=
$$\sup_{n} \int \| (\nabla^{2} \psi_{n})^{-1/2} \nabla^{2} (\psi_{n})_{e} (\nabla^{2} \psi_{n})^{-1/2} \|_{HS}^{2} d\mu \ge \sup_{n} \int \frac{\| \nabla^{2} (\psi_{n})_{e} \|_{HS}^{2}}{\| \nabla^{2} \psi_{n} \|_{HS}^{2}} d\mu.$$

By the reverse Hölder inequality we get for all $0 < \varepsilon < 1$,

$$\int \frac{\|\nabla^2(\psi_n)_e\|_{HS}^2}{\|\nabla^2\psi_n\|_{HS}^2} d\mu \ge \left(\int \|\nabla^2(\psi_n)_e\|_{HS}^{2-\varepsilon}\right)^{\frac{2}{2-\varepsilon}} \left(\int \|\nabla^2\psi_n\|_{HS}^{\frac{2(2-\varepsilon)}{\varepsilon}}\right)^{-\frac{\varepsilon}{2-\varepsilon}}.$$

Together with (31) we obtain the following bound on the third derivatives of ψ_n ,

$$\sup_{n} \int \|\nabla^{2}(\psi_{n})_{e}\|^{2-\varepsilon} d\mu < \infty.$$

Since e^{-V} is locally stricktly positive inside of $\{V < \infty\}$, we get, in particular, that for every compact set $K \subset \{V < \infty\}$

$$\sup_{n} \int_{K} \|\nabla^{2}(\psi_{n})_{e}\|^{2-\varepsilon} dx < \infty.$$

Applying the Rellich–Kondrashov embedding theorem and passing to a subsequence (denoted again by $\{\psi_n\}$), one can assume that all the second derivatives $\partial_{x_ix_j}^2 \psi_n$ converge almost everywhere. Applying the same arguments and using the bounds (31) one can assume, in addition, that $\psi_n, \partial_{x_i}\psi_n$ have limits almost everywhere (hence in every $L^p(\mu)$ and $L_{loc}^p(K)$ for all p > 1 and all compact $K \subset \{V < \infty\}$). In addition, one can assume that the third derivatives $(\psi_n)_{x_ix_jx_k}$ converge weakly in $L^{2-\varepsilon}(\mu)$ and $L_{loc}^{2-\varepsilon}(K)$ for every $0 < \varepsilon < 1$. Let us denote the limit of ψ_n by ψ . Clearly, ψ is a convex function. Let us show that

$$\partial_{x_i}\psi_n \to \partial_{x_i}\psi_i$$

Denote the limit of $\partial_{x_i}\psi_n$ by f. Choose a smooth function ξ with compact support $K \subset \{V < \infty\}$. Using convergence in $L^p_{loc}(dx)$ one gets

$$\int f\xi dx = \lim_{n} \int \partial_{x_i} \psi_n \xi dx = -\lim_{n} \int \psi_n \partial_{x_i} \xi dx = -\int \psi \partial_{x_i} \xi dx.$$

Hence f is the Sobolev partial derivative of ψ . Since ψ is convex, it coincides almost everywhere with $\partial_{x_i}\psi$ in the classical sense. In the same way we prove that ψ admits second Sobolev derivatives and

$$\partial_{x_i x_j} \psi_n \to \partial_{x_i x_j} \psi.$$

Finally, using weak convergence of the third derivatives in $L^1_{loc}(K)$, we show that ψ admits third order Sobolev derivatives, which are the weak limits of the corresponding third derivatives of ψ_n .

Let us pass to the limit in (30). By Fatou's lemma,

$$\liminf_{n} \int \langle \nabla^2 W_n(\nabla \psi_n) \nabla (\psi_n)_e, \nabla (\psi_n)_e \rangle d\mu \ge \int \langle \nabla^2 W(\nabla \psi) \nabla \psi_e, \nabla \psi_e \rangle d\mu.$$

Next we note that

$$\mathrm{Tr}[(\nabla^2 \psi_n)^{-1} (\nabla^2 (\psi_n)_e)]^2 = ||A_n||_{HS}^2$$

where $\|\cdot\|_{HS}$ is the Hilbert–Schmidt norm and

$$A_n = (\nabla^2 \psi_n)^{-1/2} \nabla^2 (\psi_n)_e (\nabla^2 \psi_n)^{-1/2}.$$

The space of matrix-valued functions M(x) with the norm $\left(\int \|M\|_{HS}^2 d\mu\right)^{\frac{1}{2}}$ is a Hilbert space. By (32), $\sup_n \int \|A_n\|_{HS}^2 d\mu \leq \int_{\{V < \infty\}} V_{ee} d\mu$, and therefore $\{A_n : n \in \mathbb{N}\}$ is relatively weakly compact in this Hilbert space. Hence there exists a subsequence of $\{A_n\}$, which we denote again by $\{A_n\}$, that converges weakly $A_n \to A$ in the space of matrix-valued functions. Take a matrix valued mapping M(x) such that $\left(\int \|M\|_{HS}^2 d\mu\right)^{\frac{1}{2}} < \infty$. One has

$$\lim_{n} \int \operatorname{Tr}\Big((\nabla^{2}\psi_{n})^{1/2} M(x) (\nabla^{2}\psi_{n})^{1/2} A_{n} \Big) d\mu = \int \operatorname{Tr}\Big((\nabla^{2}\psi)^{1/2} M(x) (\nabla^{2}\psi)^{1/2} A \Big) d\mu.$$

On the other hand

$$\lim_{n} \int \operatorname{Tr} \left((\nabla^{2} \psi_{n})^{1/2} M(x) (\nabla^{2} \psi_{n})^{1/2} A_{n} \right) d\mu = \lim_{n} \int \operatorname{Tr} \left(M(x) \nabla^{2} (\psi_{n})_{e} \right) d\mu$$
$$= \int \operatorname{Tr} \left(M(x) \nabla^{2} \psi_{e} \right) d\mu.$$

Hence

$$\int \operatorname{Tr}\left((\nabla^2 \psi)^{1/2} M(x) (\nabla^2 \psi)^{1/2} A\right) d\mu = \int \operatorname{Tr}\left(M(x) \nabla^2 \psi_e\right) d\mu$$

This implies

$$A = (\nabla^2 \psi)^{-1/2} \nabla^2 \psi_e (\nabla^2 \psi)^{-1/2}.$$

By the properties of the weak convergence and Fatou's lemma,

$$\liminf_{n} \int \|A_n\|_{HS}^2 d\mu \ge \int \|A\|_{HS}^2 d\mu.$$

Passing to the limit in (30), we get (26).

Proof of Theorem 3. Let us show that ψ is a smooth function on $\Omega_{\psi} = \operatorname{int} (\{\psi < \infty\})$. Since

$$\int |\nabla \psi|^p d\mu = \int |x|^p d\nu < \infty$$

for every p > 0, we get with (25) that $\int |\psi_{x_ix_i}|^p d\mu < \infty$ for every p > 0. In particular, by the Sobolev embedding theorem ψ is locally $C^{1,\alpha}$ on Ω_{ψ} . Repeating arguments from Proposition 6, using continuity of $\nabla \psi$ and the fact that ψ and ψ^* are locally Hölder, we obtain that ψ is $C^{2,\alpha}$, $0 < \alpha < 1$, by Theorem 4.14 from [106]. Applying higher order regularity theory (see e.g., Remark 4.15 of [106]), we get by bootstrapping arguments that ψ is C^{∞} on Ω_{ψ} .

Thus we are in position to apply Proposition 6. In our particular case it reads as

$$\int_{\Omega_{\psi}} \psi_{x_i x_i} d\mu \ge \int \langle \nabla^2 \psi^* (\nabla \psi) \nabla \psi_{x_i}, \nabla \psi_{x_i} \rangle d\mu + \int \operatorname{Tr} \left[(\nabla^2 \psi)^{-1} \nabla^2 \psi_{x_i} \right]^2 d\mu$$

Note that

$$\langle \nabla^2 \psi^* (\nabla \psi) \nabla \psi_{x_i}, \nabla \psi_{x_i} \rangle = \psi_{x_i x_i}.$$

Here we use that $\nabla^2 \psi^*(\nabla \psi) = \nabla^2 \psi^{-1}$.

Thus we get $\int \text{Tr}[(\nabla^2 \psi)^{-1} \nabla^2 \psi_{x_i}]^2 d\mu = 0$ and $\nabla^2 \psi_{x_i} = 0$ on Ω_{ψ} . Hence there exists a positive matrix A, a vector b, and an absolute constant c such that

$$\psi(x) = \frac{1}{2} \langle Ax, x \rangle + \langle b, x \rangle + c,$$

for every x satisfying $\psi(x) < \infty$.

Next we note that the push forward measure of

$$e^{-\psi}dx = e^{-(\frac{1}{2}\langle Ax,x\rangle + \langle b,x\rangle + c)}dx$$

under $y = Ax + b = \nabla \psi(x)$ is

$$Ce^{-(\frac{1}{2}\langle A^{-1}(y-b),(y-b)\rangle+\langle b,A^{-1}(y-b)\rangle+c)}dy$$

Since

$$\psi^*(y) = \frac{1}{2} \langle A^{-1}(y-b), (y-b) \rangle - c,$$

we immediately obtain that the push forward of $e^{-\psi}$ under $\nabla \psi$ coincides with $Ce^{-\psi^*}$ if and only if b = 0. Thus we get that

$$\mu = \frac{1}{Z} I_E e^{-\frac{1}{2} \langle Ax, x \rangle}$$

for some convex set E. We conclude the proof with the observation that $E = \mathbb{R}^n$, as otherwise μ has no logarithmic derivative.

3 Pinsker type inequalities

The original Pinsker inequality compares two important concepts from information theory, the total Variation Distance V = V(P, Q) and Kullback-Leibler divergence $D = D_{KL}(P||Q)$ (see Section 2.1 for the definitions). Comparing these two notions has many advantages, such as transferring results from information theory to probability theory or vice versa (see, e.g. [41, 42]). Pinsker [89] obtained the following inequality

$$D \ge \frac{1}{2}V^2. \tag{33}$$

The best constant, $\frac{1}{2}$, is due, independently to Csiszár [37], Kemperman [58] and Kullback [62, 63]. For applications of Pinsker's inequality, see e.g. [16], [38], [104].

The concept of f-divergence is a generalization of Kullback-Leibler divergence. Thus one wonders whether a Pinsker type inequality holds also for f-divergences. This question was answered by G. Gilardoni [48] and, with different kind of result, by M. Reid and R. Williamson [90].

G. Gilardoni [48] established the following Pinsker type inequality for f-divergences.

Theorem 7. [48] Let $f : (0, \infty) \to \mathbb{R}$ be convex and f(1) = 0. Suppose that the convex function f is differentiable up to order 3 at 1 with f''(1) > 0 and the following inequality holds

$$\left(f(u) - f'(1)(u-1)\right) \left(1 - \frac{f'''(1)}{3f''(1)}(u-1)\right) \ge \frac{f''(1)}{2}(u-1)^2.$$
(34)

Then

$$D_f(P,Q) \ge \frac{f''(1)}{2}V^2.$$

The constant $\frac{f''(1)}{2}$ is best possible. If f is concave, the inequalities are reversed.

In this section, we will consider probability densities. We put

$$q_{\varphi} = \frac{\varphi}{\int_{X_{\psi}} \varphi} \quad \text{and} \quad p_{\varphi} = \frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det\left[\nabla^2 \left(-\ln \varphi\right)\right]}{\varphi \int_{X_{\psi^*}} \varphi^{\circ}}$$
(35)

We use the expressions (35) to define the normalized f-divergences for log concave functions [28].

Definition 8. Let $f : (0, \infty) \to \mathbb{R}$ be a convex or concave function and let $\varphi : \mathbb{R}^n \to [0, \infty)$ be a log concave function. Then the normalized f-divergence $\mathcal{D}_f(P\varphi, Q_\varphi)$ of φ is

$$\mathcal{D}_{f}(\varphi) = \mathcal{D}_{f}(P_{\varphi}, Q_{\varphi}) = \int_{X_{\psi}} \frac{\varphi}{\int_{X_{\psi}} \varphi} f\left(\frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}}}{\varphi^{2}} \frac{\int_{X_{\psi}} \varphi}{\int_{X_{\psi^{*}}} \varphi^{\circ}} det\left[\nabla^{2}\left(-\ln\varphi\right)\right]\right) dx.$$
(36)

From the result of G. Gilardoni [48], we obtain, under additional conditions (see Section 3.1), an information-theoretic inequality for log concave functions.

3.1 Pinsker inequalities for log concave functions

The following entropy inequality for log concave functions follows as an immediate consequence of Theorem 7 with the densities (35) when, for a convex function f, (34) is satisfied:

$$\mathcal{D}_{f}(\varphi) \geq \frac{f''(1)}{2} \left(\int_{X_{\psi}} \left| \frac{e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det \left[\nabla^{2} \left(-\ln \varphi \right) \right]}{\varphi \int_{X_{\psi}^{*}} \varphi^{\circ}} - \frac{\varphi}{\int_{X_{\psi}} \varphi} \right| dx \right)^{2}, \tag{37}$$

with equality if $\varphi(x) = e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix. If f is concave, the inequality is reversed.

This inequality is stronger than the inequality from [28] stated in Theorem 2. Indeed, Theorem 2 says that $D_f(\varphi) \ge f(1)$ and f(1) = 0 for probability densities. Example 10 shows that the right hand side of (37) however is not always 0.

We want to concentrate on the case when $f(t) = -\ln t$. Then the assumptions of Gilardoni 's theorem hold and we get the following corollary.

Recall also that $\operatorname{Ent}(\varphi) = \int_{\Omega_{\psi}} \varphi \ln(\varphi) \, dx - \int_{\Omega_{\psi}} \varphi \ln\left(\int_{\Omega_{\psi}} \varphi\right) dx$. By g, we denote the Gaussian which has entropy $\operatorname{Ent}(g) = -\frac{n}{2} \ln(2\pi e)$. We also use that for functions $\varphi \in C^2$, we have that $X_{\psi} = \Omega_{\psi}$ and that $n \int_{\Omega_{\psi}} \varphi = -\int_{\Omega_{\psi}} \langle \nabla \varphi, x \rangle dx$.

Corollary 9. Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $\varphi = e^{-\psi}$ is a probability density. Then

$$\int_{X_{\psi}} \ln\left(\det\left(\nabla^{2}\psi\right)\right) e^{-\psi(x)} dx \leq 2\left[\operatorname{Ent}(\varphi) - \operatorname{Ent}(g)\right] + \ln\left(\frac{\int_{X_{\psi^{*}}} e^{-\psi^{*}}}{(2\pi)^{n}}\right) \\ - \frac{1}{2}\left(\int_{X_{\psi}} \left|\frac{e^{\psi-\langle\nabla\psi,x\rangle} \det\left(\nabla^{2}\psi\right)}{\int_{X_{\psi^{*}}} e^{-\psi^{*}}} - e^{-\psi}\right| dx\right)^{2}, (38)$$

with equality if $\varphi(x) = e^{-\pi \langle Ax, x \rangle}$ where A is an $n \times n$ positive definite matrix with $\det(A) = 1$.

Remarks.

(i) Note that

$$\int_{X_{\psi}} \left| \frac{e^{\psi - \langle \nabla \psi, x \rangle} \det \left(\nabla^2 \psi \right)}{\int_{X_{\psi^*}} e^{-\psi^*}} - e^{-\psi} \right| dx \ge \left| \int_{X_{\psi}} \frac{e^{\psi - \langle \nabla \psi, x \rangle} \det \left(\nabla^2 \psi \right)}{\int_{X_{\psi^*}} e^{-\psi^*}} - e^{-\psi} dx \right| = 0.$$

The last equality holds as φ is a probability density and by (17). Therefore inequality (38) implies the following inequality, which was proved in [28],

$$\int_{X_{\psi}} \ln\left(\det\left(\nabla^{2}\psi\right)\right) e^{-\psi(x)} dx \leq 2\left[\operatorname{Ent}(\varphi) - \operatorname{Ent}(g)\right] + \ln\left(\frac{\int_{X_{\psi}^{*}} e^{-\psi^{*}}}{(2\pi)^{n}}\right).$$
(39)

(ii) The functional Blaschke Santalo inequality [6, 13, 44, 68] implies that, for a probability density φ , $\left(\frac{\int_{X_{\psi}^{*}} e^{-\psi^{*}}}{(2\pi)^{n}}\right) \leq 1$. Therefore inequality (39) implies $\int_{X_{\psi}} \ln\left(\det\left(\nabla^{2}\psi\right)\right) e^{-\psi(x)} dx \leq 2\left[\operatorname{Ent}(\varphi) - \operatorname{Ent}(g)\right],$ (40)

which was proved in [7]. Thus inequality (38) is the strongest of those entropy inequalities. Indeed, the next example shows that the additional term on the right hand side of (38) is not equal to 0 in general.

Example 10. Let p > 1 and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be given by $\varphi(x) = \frac{1}{A} \cdot e^{-\frac{1}{p}\sum_{i=1}^n |x_i|^p}$, where $A = \int e^{-\frac{1}{p}\sum_{i=1}^n |x_i|^p} = 2^n \left(\Gamma(\frac{1}{p}) p^{\frac{1-p}{p}}\right)^n$. Then $\varphi^\circ = A \cdot e^{-\frac{1}{q}\sum_{i=1}^n |x_i|^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover,

$$\int \varphi^{\circ} = \int e^{-\psi^*} = \int A e^{-\frac{1}{q}\sum_{i=1}^n |x_i|^q} = A \cdot 2^n \left(\Gamma\left(\frac{1}{q}\right) q^{\frac{1-q}{q}}\right)^n,$$

which is just $(2\pi)^n$ if p = q = 2. If p is not 2, then it is not necessarily 2π . And

$$\int \left| \frac{e^{\psi - \langle \nabla \psi, x \rangle} det(\nabla^2 \psi)}{\int e^{-\psi^*}} - e^{-\psi} \right| dx$$
$$= \int \left| \frac{(p-1)^n \prod_{i=1}^n x_i^{p-2}}{2^n \left(\Gamma(\frac{1}{q}) \ q^{\frac{1-q}{q}} \right)^n} \ e^{\frac{1-p}{p} \sum_{i=1}^n |x_i|^p} - A^{-1} e^{-\frac{1}{p} \sum_{i=1}^n |x_i|^p} \right| dx$$

which is only equal to 0, if p = q = 2. If p is not 2, then it is not necessarily 0.

4 Entropy inequalities

The following theorem, which provides bounds for f divergence, was proved by S. Dragomir [40]. Dragomir proved the theorem in the discrete case, that is $p, q \in \mathbb{R}^n_+$, ν is the counting measure on $\{1, \dots, n\}$, $P = p\nu_n$ and $Q = q\nu_n$. Then

$$D_f(P,Q) = \sum_{i=1}^n f\left(\frac{p_i}{q_i}\right) q_i$$

Theorem 11. [40] Suppose that the convex function $f : (0, \infty) \to \mathbb{R}$ is differentiable. Then we have for for all discrete densities $p, q \in \mathbb{R}^n_+$,

$$f'(1)(P_n - Q_n) \leq D_f(P,Q) - f(1)Q_n \leq D_{f'}\left(\frac{P^2}{Q},P\right) - D_{f'}(P,Q),$$
 (41)

where $f': (0, \infty) \to \mathbb{R}$ is the derivative of f and $P_n = \sum_{i=1}^n p_i > 0$, $Q_n = \sum_{i=1}^n q_i > 0$. If f is concave, the inequality is reversed. If f is strictly convex (respectively strictly concave) and $p_i, q_i > 0$ for all $1 \le i \le n$, then equality holds iff p = q.

The result still holds, if the discrete densities are replaced by μ -a.e. positive general densities p and q defined on a measure space (X, μ) . The proof is the same as the one given by by Dragomir in the discrete case [40]. We include it for completeness. We state it when $f: (0, \infty) \to \mathbb{R}$ is convex. If f is concave, the inequalities are reversed. We use the notation $I_P = \int_X p d\mu$ and $I_Q = \int_X q d\mu$.

Theorem 12. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and convex. Let p and q be μ -a.e. positive densities on X such that I_p and I_q are finite. Then

$$I_Q f\left(\frac{I_P}{I_Q}\right) \le D_f(P,Q) \le f(1)I_Q + D_{f'}\left(\frac{P^2}{Q},P\right) - D_{f'}(P,Q)$$
 (42)

If f is linear, equality holds on both sides and we get

$$D_f(P,Q) = f(1)I_Q + f'(1)(I_Q - I_P).$$

If f is strictly convex, equality holds on the left inequality if and only if $p = c q \mu$ -a.e where c > 0 is a constant and equality holds on the right inequality if and only if $p = q \mu$ -a.e.

Proof of Theorem 12. By Jensen's inequality,

$$D_f(P,Q) = \int_X f\left(\frac{p}{q}\right) q d\mu = I_Q \int_X f\left(\frac{p}{q}\right) \frac{q}{I_Q} d\mu \ge I_Q f\left(\int_X \frac{p}{I_Q} d\mu\right)$$
$$= I_Q f\left(\frac{I_P}{I_Q}\right),$$

which proves the left inequality of (42).

It is easy to see that equality holds on both sides of (42) if f is linear. If f is strictly convex, equality holds in the left inequality of (42) if and only if equality holds in Jensen's inequality which happens if and only if $p = c q \mu$ -a.e where c > 0 is a constant.

Since f is differentiable and convex, we have for all $s, t \in (0, \infty)$,

$$f'(t)(t-s) \ge f(t) - f(s) \ge f'(s)(t-s).$$
(43)

Let $x \in X$ be such that q(x) > 0 and let $t = \frac{p(x)}{q(x)}$. As q > 0 μ -a.e., it is enough to consider only such x. Let s = 1. By inequality (43),

$$f'\left(\frac{p(x)}{q(x)}\right)\left(\frac{p(x)}{q(x)}-1\right) \ge f\left(\frac{p(x)}{q(x)}\right) - f(1).$$

We multiply both sides by q(x) and integrate

$$\int_X (p(x) - q(x)) f'\left(\frac{p(x)}{q(x)}\right) d\mu(x) \ge D_f(P, Q) - f(1) \int_X q(x) d\mu(x).$$

Since

$$\int_X (p(x) - q(x))f'\left(\frac{p(x)}{q(x)}\right)d\mu(x) = D_{f'}\left(\frac{P^2}{Q}, P\right) - D_{f'}(P, Q),$$

we obtain the desired result.

Equality holds in (43) for a strictly convex function f iff s = t. Therefore, if f is strictly convex, equality holds on the right inequality of (42), iff $p = q \mu$ -a.e.

Remark. Under the assumptions of Theorem 12, we have that

(i) When p and q are probability densities, then

$$f(1) \leq D_f(P,Q) \leq f(1) + D_{f'}\left(\frac{P^2}{Q},P\right) - D_{f'}(P,Q)$$
 (44)

and that $D_f(P,Q) = f(1)$, if f is linear. When f is strictly convex, equality holds in both inequalities if and only if $p = q \mu$ -a.e.

(ii) If we let $t = \frac{I_P}{I_Q}$ and s = 1 in (43), then

$$I_Q f\left(\frac{I_P}{I_Q}\right) \ge f(1)I_Q + f'(1)\left(I_P - I_Q\right),$$

which, together with the upper bound of (42), leads to inequalities corresponding to (41),

$$f'(1)(I_P - I_Q) \leq D_f(P,Q) - f(1)I_Q \leq D_{f'}\left(\frac{P^2}{Q},P\right) - D_{f'}(P,Q).$$
 (45)

When f is linear, equality holds in both inequalities. When f is strictly convex, equality holds in both inequalities if and only if $p = q \mu$ -a.e.

With the identity (17), the following corollary is immediate from Theorem 12.

Corollary 13. Let $\varphi : \mathbb{R}^n \to [0,\infty)$ be a log-concave function and $f : (0,\infty) \to \mathbb{R}$ be differentiable and convex. Then

$$f\left(\frac{\int_{X_{\psi^*}}\varphi^{\circ}}{\int_{X_{\psi}}\varphi}\right)\int_{X_{\psi}}\varphi \leq D_f(P_{\varphi}, Q_{\varphi}) \leq f(1)\int_{X_{\psi}}\varphi + D_{f'}\left(\frac{P_{\varphi}^2}{Q_{\varphi}}, P_{\varphi}\right) - D_{f'}(P_{\varphi}, Q_{\varphi}).$$
(46)

If f is concave, the inequality is reversed. If f is linear, equality holds in both inequalities.

If f is strictly convex or strictly concave, then equality holds on the left hand side if $\varphi(x) = c \ e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix and c > 0 is an absolute constant.

If φ is in addition C^2 , then equality holds on the left hand side iff $\varphi(x) = c \ e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix and c > 0 is an absolute constant.

If f is strictly convex or strictly concave, then equality holds on the right hand side if $\varphi(x) = e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix with $\det(A) = 1$.

If φ is in addition C^2 , then equality holds on the right hand side iff $\varphi(x) = e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix with $\det(A) = 1$.

Remark. Inequality (46) is invariant under self adjoint SL(n) maps. This follows as both, $D_f(P_{\varphi}, Q_{\varphi})$ and $D_{f'}\left(\frac{P_{\varphi}^2}{Q_{\varphi}}, P_{\varphi}\right)$ are invariant under self adjoint SL(n)maps, with possibly different degree of homogeneity. For $D_f(P_{\varphi}, Q_{\varphi})$ this was proved in [28]. For $D_{f'}\left(\frac{P_{\varphi}^2}{Q_{\varphi}}, P_{\varphi}\right)$, it is shown similarly.

Proof of Corollary 13. With the identity (17) and the densities

$$q_{\varphi} = \varphi$$
 and $p_{\varphi} = \varphi^{-1} e^{\frac{\langle \nabla \varphi, x \rangle}{\varphi}} \det \left[\nabla^2 \left(-\ln \varphi \right) \right]$

the inequalities of the corollary follow immediately from Theorem 12.

Let A be a positive definite $n \times n$ matrix, c > 0 a constant and $\varphi(x) = ce^{-\frac{1}{2}\langle Ax, x \rangle}$. Then

$$D_f(P_{\varphi}, Q_{\varphi}) = f\left(\frac{\det(A)}{c^2}\right) \frac{c(2\pi)^{n/2}}{\sqrt{\det(A)}}$$

Therefore it is easy to see that we have equality on the left hand side if $\varphi(x) = c e^{-\frac{1}{2}\langle Ax,x \rangle}$ and on the right hand side if $\varphi(x) = e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix with det(A) = 1.

By Theorem 12, equality holds on the right inequality iff $p_{\varphi} = q_{\varphi}$ a.e. and on the left iff $p_{\varphi} = c \ q_{\varphi}$ a.e. where c > 0 is a constant. For functions $\varphi(x) = c \ e^{-\frac{1}{2}\langle Ax,x \rangle}$, where A is an $n \times n$ positive definite matrix with $\det(A) = 1$, it is easy to check that $p_{\varphi} = q_{\varphi}$. The equation $p_{\varphi} = c \ q_{\varphi}$ a.e., is equivalent to the equation

$$\det(\nabla^2 \psi(x)) = c e^{-2\psi(x) + \langle \nabla \psi(x), x \rangle}, \quad \text{a.e. } x \in \mathbb{R}^n.$$

Then, if φ is C^2 , Theorem 3 and the remarks before it, finish the proof of the corollary.

Now we consider special cases of Corollary 13.

If we let $f(t) = -\ln t$ in the previous corollary, we obtain the following affine invariant entropy inequalities which give upper and lower bounds for the relative entropy in terms of the functional affine surface areas. The proof follows immediately from Corollary 13. **Corollary 14.** Let $\varphi : \mathbb{R}^n \to [0,\infty)$ be a log-concave function. Then

$$\ln\left(\frac{as_0(\varphi)}{as_1(\varphi)}\right)as_0(\varphi) \leq D_{KL}(Q_{\varphi}||P_{\varphi}) \leq as_{-1}(\varphi) - as_0(\varphi).$$
(47)

Equality holds on the left hand side if $\varphi(x) = c \ e^{-\frac{1}{2}\langle Ax,x\rangle}$ where A is an $n \times n$ positive definite matrix and c > 0 is an absolute constant and equality holds on the right hand side if $\varphi(x) = e^{-\frac{1}{2}\langle Ax,x\rangle}$ where A is an $n \times n$ positive definite matrix with $\det(A) = 1$,

If φ is in addition C^2 , then equality holds on the left hand side iff $\varphi(x) = c \ e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix and c > 0 is an absolute constant. And equality holds on the right hand side iff $\varphi(x) = e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix with $\det(A) = 1$.

Remarks. 1. If, in the previous corollary, φ is a probability density, then the inequalities become

$$1 - \ln\left(as_1(\varphi)\right) \leq 1 + D_{KL}(Q_{\varphi}||P_{\varphi}) \leq as_{-1}(\varphi).$$

2. Applying (43), with $f(t) = \ln t$, to the left inequality of the previous corollary, we get

$$as_0(\varphi) - as_1(\varphi) \leq D_{KL}(Q_{\varphi}||P_{\varphi}) \leq as_{-1}(\varphi) - as_0(\varphi).$$

$$(48)$$

If we let $f(t) = t^{\lambda}$ in inequality (46), then we obtain functional affine isoperimetric inequalities.

Corollary 15. Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be a log concave function.

(i) If
$$\lambda \ge 1$$
 or $\lambda \le 0$,
 $as_1^{\lambda}(\varphi) \ as_0^{1-\lambda}(\varphi) \le as_{\lambda}(\varphi) \le as_0(\varphi) + \lambda \left(as_{\lambda}(\varphi) - as_{\lambda-1}(\varphi)\right).$ (49)

(ii) If $\lambda \in (0,1)$,

$$as_1^{\lambda}(\varphi) \ as_0^{1-\lambda}(\varphi) \ \ge \ as_{\lambda}(\varphi) \ \ge \ as_0(\varphi) + \lambda \left(as_{\lambda}(\varphi) - as_{\lambda-1}(\varphi)\right). \tag{50}$$

Equality holds trivially if $\lambda = 1$ or $\lambda = 0$.

Equality holds on the left hand sides if $\varphi(x) = c \ e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix and c > 0 is an absolute constant. Equality holds on the right hand sides if $\varphi(x) = e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix with $\det(A) = 1$. If φ is in addition C^2 , then equality holds on the left hand sides iff $\varphi(x) = c \ e^{-\frac{1}{2}\langle Ax,x \rangle}$ and equality holds on the right hand sides iff $\varphi(x) = e^{-\frac{1}{2}\langle Ax,x \rangle}$ with $\det(A) = 1$.

Remarks. 1. Applying (43) to the function $f(t) = t^{\lambda}$ for $t = \frac{as_1^{\lambda}(\varphi)}{as_0^{\lambda}(\varphi)}$ and s = 1, we get from the left inequality of the previous corollary that

$$\lambda \left(as_1(\varphi) - as_0(\varphi) \right) \leq as_\lambda(\varphi) - as_0(\varphi) \leq \lambda \left(as_\lambda(\varphi) - as_{\lambda-1}(\varphi) \right), \tag{51}$$

when $\lambda \geq 1$ or $\lambda \leq 0$ and that

$$\lambda \left(as_1(\varphi) - as_0(\varphi) \right) \geq as_\lambda(\varphi) - as_0(\varphi) \geq \lambda \left(as_\lambda(\varphi) - as_{\lambda-1}(\varphi) \right), \tag{52}$$

when $\lambda \in (0, 1)$. The equality cases are as in the corollary.

2. The following inequalities for the difference of functional affine surface areas follow immediately from (51). For $\lambda \geq 1$,

$$as_1(\varphi) - as_0(\varphi) \leq as_\lambda(\varphi) - as_{\lambda-1}(\varphi).$$
(53)

For $\lambda \leq 1$, the inequality is reversed.

If we let $\lambda = -1$ in inequality (51) and $\lambda = 1/2$ in Corollary 15, then

$$as_{-1}(\varphi) \leq \frac{as_{0}(\varphi) + as_{-2}(\varphi)}{2}, \quad as_{0}(\varphi) \leq \frac{as_{-1}(\varphi) + as_{1}(\varphi)}{2}$$
$$as_{0}(\varphi) \leq \frac{as_{\frac{1}{2}}(\varphi) + as_{-\frac{1}{2}}(\varphi)}{2}, \quad as_{\frac{1}{2}}(\varphi) \leq \sqrt{as_{1}(\varphi)} as_{0}(\varphi).$$

Similar results hold for dual function φ° by the duality relation $as_{\lambda}(\varphi) = as_{1-\lambda}(\varphi^{\circ})$, which was proved in [30].

We have that $0 < \int_{\mathbb{R}^n} \varphi < \infty$ by (8) and as φ is integrable by assumption. Let $\lambda \in (0, 1)$ and suppose that φ is centered, i.e., $\int_{\mathbb{R}^n} x\varphi(x)dx = 0$. The functional Blaschke Santalò inequality [6, 13, 44, 68] says that for a centered log concave function φ ,

$$\int_{\mathbb{R}^n} \varphi dx \cdot \int_{\mathbb{R}^n} \varphi^{\circ} dx \le (2\pi)^n,$$

with equality iff $\varphi(x) = C \ e^{-\frac{1}{2}\langle Ax,x \rangle}$. For $\lambda \in (0,1)$. We apply this inequality in the left inequality (50). Also using that $as_1(\varphi) = \int_{X_{\psi^*}} \varphi^\circ$, we get

$$as_{\lambda}(\varphi) \leq \left(\int_{X_{\psi^{*}}} \varphi^{\circ}\right)^{\lambda} \left(\int_{X_{\psi}} \varphi\right)^{1-\lambda} \leq \left(\int_{\mathbb{R}^{n}} \varphi^{\circ}\right)^{\lambda} \left(\int_{\mathbb{R}^{n}} \varphi\right)^{\lambda} \left(\int_{X_{\psi}} \varphi\right)^{1-2\lambda}$$
$$\leq (2\pi)^{n\lambda} \left(\int_{X_{\psi}} \varphi\right)^{1-2\lambda},$$

with equality iff $\varphi(x) = c \ e^{-\frac{1}{2}\langle Ax, x \rangle}$.

Similarly, by inequality (49), we get for $\lambda < 0$,

$$as_{\lambda}(\varphi) \ge (2\pi)^{n\lambda} \left(\int_{X_{\psi}} \varphi\right)^{1-2\lambda},$$

with equality iff $\varphi(x) = c \ e^{-\frac{1}{2}\langle Ax, x \rangle}$.

A functional version of the inverse Blaschke Santalò inequality, due to Fradelizi and Meyer [45] says that $\int_{\mathbb{R}^n} \varphi dx \cdot \int_{\mathbb{R}^n} \varphi^{\circ} dx \ge c^n$, where c > 0 is a constant. We use this for $\lambda > 1$ in the left inequality (49) to get that $as_{\lambda}(\varphi) \ge c^{n\lambda} \left(\int_{X_{\psi}} \varphi\right)^{1-2\lambda}$.

Thus we have proved the following corollary which was proved in [30] by different methods.

Corollary 16. Let $\varphi : \mathbb{R}^n \to [0,\infty)$ be a log concave centered function.

(i) If
$$\lambda \in [0,1]$$
, then $as_{\lambda}(\varphi) \leq (2\pi)^{n\lambda} \left(\int_{X_{\psi}} \varphi\right)^{1-2\lambda}$
(ii) If $\lambda \leq 0$, then $as_{\lambda}(\varphi) \geq (2\pi)^{n\lambda} \left(\int_{X_{\psi}} \varphi\right)^{1-2\lambda}$.
(iii) If $\lambda > 1$, then $as_{\lambda}(\varphi) \geq c^{n\lambda} \left(\int_{X_{\psi}} \varphi\right)^{1-2\lambda}$.

Equality holds trivially if $\lambda = 0$.

Equality holds in the first two inequalities iff $\varphi(x) = C e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix and C > 0.

If $\lambda = 1$, the first equality is just the functional Blaschke Santalò inequality.

Another consequence of Corollary 15, along with the duality relation $as_{\lambda}(\varphi) = as_{1-\lambda}(\varphi^{\circ})$ (proved in [30]), is the following (functional) Blaschke Santaló type inequalities, which were originally proved in [30] by different methods.

Corollary 17. Let $\varphi : \mathbb{R}^n \to [0, \infty)$ be a log concave function.

If $\lambda \in [0,1]$ and φ is centered, then $as_{\lambda}(\varphi) as_{\lambda}(\varphi^{\circ}) \leq (2\pi)^n$.

If $\lambda \geq 1$ or $\lambda \leq 0$, then $as_{\lambda}(\varphi) as_{\lambda}(\varphi^{\circ}) \geq \int_{X_{\psi}} \varphi \int_{X_{\psi^{*}}} \varphi^{\circ}$.

Equality holds in the first inequality iff $\varphi(x) = C e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix and C > 0. If φ is in addition C^2 , then equality holds in the second inequality iff $\varphi(x) = C e^{-\frac{1}{2}\langle Ax,x \rangle}$ where A is an $n \times n$ positive definite matrix and C > 0.

Note that if φ is C^2 , then $X_{\psi} = \Omega_{\psi}$ and $X_{\psi^*} = \Omega_{\psi^*}$. Therefore, when $\varphi \in C^2$, we have

$$as_{\lambda}(\varphi) \ as_{\lambda}(\varphi^{\circ}) \ge c^n$$

for $\lambda \geq 1$ or $\lambda \leq 0$, which follows by the inverse functional Blaschke Santaló inequality.

Proof of Corollary 17. For $\lambda \in [0, 1]$, Corollary 15, the duality relation $as_{\lambda}(\varphi) = as_{1-\lambda}(\varphi^{\circ})$, and the functional Blaschke Santaló inequality yield

$$as_{\lambda}(\varphi) \ as_{\lambda}(\varphi^{\circ}) \leq \int_{X_{\psi}} \varphi \int_{X_{\psi^*}} \varphi^{\circ} \leq \int_{\mathbb{R}^n} \varphi \int_{\mathbb{R}^n} \varphi^{\circ} \leq (2\pi)^n.$$

The second inequality is trivial for $\lambda \geq 1$ or $\lambda \leq 0$ by Corollary 15, along with the duality relation $as_{\lambda}(\varphi) = as_{1-\lambda}(\varphi^{\circ})$.

Equality holds in the first inequality iff $\varphi(x) = C e^{-\frac{1}{2}\langle Ax,x\rangle}$, where A is an $n \times n$ positive definite matrix and C > 0. This follows from the equality characterization of the functional Blaschke Santaló inequality. If φ is in addition C^2 , then equality holds in the second inequality iff $\varphi(x) = C e^{-\frac{1}{2}\langle Ax,x\rangle}$, where A is an $n \times n$ positive definite matrix and C > 0.

5 Applications

In this section we will derive applications to convex bodies. We first recall the notion of f-divergence for convex bodies. For general information on convex bodies the books [46, 96] are excellent sources.

In [109], f-divergence and their inequalities were introduced for convex bodies. For details and special cases we refer to [109] and give here only the definition.

Let K be a convex body in \mathbb{R}^n . We assume throughout that K has center of gravity at 0. For $x \in \partial K$, the boundary of a sufficiently smooth convex body K, let $N_K(x)$ denote the outer unit normal to ∂K in x and let $\kappa_K(x)$ be the Gauss curvature in x, and μ_K is the usual surface area measure on ∂K . We put

$$p_K(x) = \frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^n}, \quad q_K(x) = \langle x, N_K(x) \rangle$$
(54)

and

$$P_K = p_K \ \mu_K \quad \text{and} \quad Q_K = q_K \ \mu_K. \tag{55}$$

Then P_K and Q_K are measures on ∂K that are absolutely continuous with respect to μ_K . Note that

$$\int_{\partial K} q_K d\mu = n|K| \quad \text{and} \quad \int_{\partial K} p_K d\mu = n|K^\circ|.$$
(56)

The latter holds, provided K has sufficiently smooth boundary. Thus Q_K and P_K are (up to the factor n) the *cone measures* (e.g., [87]) of K and its polar K° .

Note that $\int_{\partial K} p_K d\mu = n |K^\circ|$ and $\int_{\partial K} q_K d\mu = n |K|$.

Let $f: (0, \infty) \to \mathbb{R}$ be a convex or concave function. The *f*-divergence of *K* with respect to the measures P_K and Q_K was defined in [109] as

$$D_f(P_K, Q_K) = \int_{\partial K} f\left(\frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^{n+1}}\right) \langle x, N_K(x) \rangle d\mu_K.$$
(57)

It is a natural generalization of L_p -affine surface area and measures the difference between the cone measures of K and K°. Those are relevant in many contexts, e.g., [55, 84] as well as e.g., the famous Blaschke Santaló inequality and its still open converse, the Mahler conjecture.

Now we turn to applications to convex bodies of the inequalities we have obtained in the previous sections. There are two equivalent approaches. The first one is to apply the density functions (54) to Theorem 7, Theorem 11 and Theorem 12. Or, we can apply the log-concave function $\varphi_K = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$ to the inequalities of the previous sections for log-concave functions. Here $\|.\|_K$ is the gauge function of K,

$$||x||_{K} = \min\{\lambda \ge 0 : \lambda x \in K\} = \max_{y \in K^{\circ}} \langle x, y \rangle = h_{K^{\circ}}(x).$$

Differentiating with respect to λ at $\lambda = 1$, we get

$$\langle x, \nabla \psi(x) \rangle = 2\psi(x).$$
 (58)

It was already observed in [30] that the L_{λ} -affine surface area for log concave functions is a generalization of L_p -affine surface area for convex bodies. Indeed, it was noted there that if one applies the log concave function $\varphi_K = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$ to Definition (12), then one obtains the L_p -affine surface area for convex bodies,

$$as_{\lambda}(\varphi_K) = \frac{(2\pi)^{n/2}}{n|B_2^n|} as_p(K), \tag{59}$$

where $\lambda = \frac{p}{n+p}$, $p \neq -n$ and B_2^n denotes the *n*-dimensional Euclidean unit ball. Please note also that

$$\int e^{-\frac{\|x\|_K^2}{2}} dx = \frac{(2\pi)^{\frac{n}{2}}|K|}{|B_2^n|} \text{ and } \int e^{-\frac{\|x\|_{K^\circ}^2}{2}} dx = \frac{(2\pi)^{\frac{n}{2}}|K^\circ|}{|B_2^n|}.$$
 (60)

Now we apply the function, $\varphi_K = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$, to Corollary 13 (or apply the densities (54) to Theorem 12) and obtain the following result. P_K , Q_K and $D_f(P_K, Q_K)$ are as above.

Theorem 18. Let K be a convex body in \mathbb{R}^n with 0 in its interior. Let $f : (0, \infty) \to \mathbb{R}$ be convex and differentiable function, then

$$n|K|f\left(\frac{|K^{\circ}|}{|K|}\right) \leq D_{f}(P_{K},Q_{K}) \leq nf(1)|K| + D_{f'}\left(\frac{P_{K}^{2}}{Q_{K}},P_{K}\right) - D_{f'}(P_{K},Q_{K}).$$
(61)

For a concave, differentiable f, the inequalities are reversed.

Equality holds on the left hand side if K is an ellipsoid. If K is C^2_+ , then equality holds on the left hand side iff K is an ellipsoid.

Equality holds on the right hand side if K is an origin symmetric ellipsoid such that $|K| = |B_2^n|$. If K is C_+^2 , then equality holds on the right hand side iff K is an origin symmetric ellipsoid such that $|K| = |B_2^n|$.

Proof. Let $\psi = \frac{\|\cdot\|_K^2}{2}$. In [28], it was proved that

$$D_f(P_{\varphi_K}, Q_{\varphi_K}) = \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} D_f(P_K, Q_K).$$

Clearly,

$$D_{f'}(P_{\varphi_K}, Q_{\varphi_K}) = \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} D_{f'}(P_K, Q_K).$$

We integrate in polar coordinates with respect to the cone measure Q_K (55) of K. Thus, if we write x = rz, with $z \in \partial K$, then $dx = r^{n-1} dr dQ_K(z)$. We also use that the map $x \mapsto \det \nabla^2 \psi(x)$ is 0-homogeneous. With (58),

$$D_{f'}\left(\frac{P_{\varphi_K}^2}{Q_{\varphi_K}}, P_{\varphi_K}\right) = \int_0^{+\infty} r^{n-1} e^{\frac{-r^2}{2}} dr \int_{\partial K} f' \left(\det \nabla^2 \psi(z)\right) \, \det \nabla^2 \psi(z) \, dQ_K(z)$$
$$= \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} \int_{\partial K} f' \left(\det \nabla^2 \psi(z)\right) \, \det \nabla^2 \psi(z) \, dQ_K(z).$$

It is well known (see, e.g., [30]) that for all $z \in \partial K$,

$$\det\left(\nabla^2\psi\right) = \frac{\kappa_K(z)}{\langle z, N_K(z)\rangle^{n+1}}.$$
(62)

Thus,

$$D_{f'}\left(\frac{P_{\varphi_K}^2}{Q_{\varphi_K}}, P_{\varphi_K}\right) = \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} \int_{\partial K} f'\left(\frac{\kappa(x)}{\langle x, N_K(x)\rangle^{n+1}}\right) \frac{\kappa(x)}{\langle x, N_K(x)\rangle^n} d\mu_K(x)$$
$$= \frac{(2\pi)^{\frac{n}{2}}}{n|B_2^n|} D_{f'}\left(\frac{P_K^2}{Q_K}, P_K\right).$$

Therefore, the statement of the theorem follows.

For ellipsoids, $D_f(P_K, Q_K) = n|K|f\left(\frac{|K^\circ|}{|K|}\right)$ and $D_{f'}(P_K, Q_K) = n|K|f'\left(\frac{|K^\circ|}{|K|}\right)$ ([109]). And note that (see [109])

$$D_{f'}\left(\frac{P_K^2}{Q_K}, P_K\right) = \int f'\left(\frac{p_K}{q_K}\right) p_K \ d\mu_K = \int f'\left(\frac{|K^\circ|}{|K|}\right) p_K d\mu_K = f'\left(\frac{|K^\circ|}{|K|}\right) n|K^\circ|.$$

So, equality holds on the left hand side if K is an ellipsoid. If K is C_+^2 , then equality holds on the left hand side iff $p_K = c q_K$. This holds iff K is an ellipsoid by a theorem, due to Petty [88], which says that a C_+^2 convex body K is an ellipsoid iff

$$\frac{\kappa_K(z)}{\langle z, N_K(z) \rangle^{n+1}} = c,$$

where c > 0 a constant.

If $K = \mathcal{E}$ an origin symmetric ellipsoid such that $|\mathcal{E}| = |B_2^n|$, then, by the Blaschke-Santaló inequality, $|K^{\circ}| = |\mathcal{E}^{\circ}| = |B_2^n|$. Therefore, equality holds on the right hand side if K is an origin symmetric ellipsoid such that $|K| = |B_2^n|$. If K is C_+^2 , then equality holds on the right hand side iff $p_K = q_K$, which only holds when K is an origin symmetric ellipsoid such that $|K| = |B_2^n|$.

Now we will consider special cases.

If we let $f(t) = -\ln t$ in inequality (61), then by definitions (1) and (6), we obtain the following affine isoperimetric inequalities.

Corollary 19. Let K be a convex body in \mathbb{R}^n with 0 in its interior. Then

$$n|K|\ln\left(\frac{|K|}{|K^{\circ}|}\right) \leq D_{KL}(Q_K||P_K) \leq as_{-\frac{n}{2}}(K) - as_0(K).$$

Equality holds on the left hand side if K is an ellipsoid. If K is C^2_+ , then equality holds on the left hand side iff K is an ellipsoid.

Equality holds on the right hand side if K is an origin symmetric ellipsoid such that $|K| = |B_2^n|$. If K is C_+^2 , then equality holds on the right hand side iff K is an origin symmetric ellipsoid such that $|K| = |B_2^n|$.

If we let $f(t) = t^{\frac{p}{n+p}}$ in Theorem 18, then by (1), we obtain the following affine isoperimetric inequalities. Recall that $as_0(K) = n|K|$, and if K is C^2_+ , then $as_\infty(K) = n|K^\circ|$.

Corollary 20. Let K be a convex body in \mathbb{R}^n with 0 in its interior. Let $p \leq 0, p \neq -n$. Then

$$n|K^{\circ}|^{\frac{p}{n+p}}|K|^{\frac{n}{n+p}} \leq as_{p}(K) \leq n|K| + \frac{p}{n+p}\left(as_{p}(K) - as_{\frac{-n^{2}}{2n+p}}(K)\right).$$
(63)

If p > 0, then

$$n|K^{\circ}|^{\frac{p}{n+p}}|K|^{\frac{n}{n+p}} \ge as_p(K) \ge n|K| + \frac{p}{n+p}\left(as_p(K) - as_{\frac{-n^2}{2n+p}}(K)\right).$$
(64)

Equality holds trivially for p = 0 or $p = \infty$, if K is C_{+}^{2} .

Equality holds on the left hand side if K is an ellipsoid. If K is C^2_+ , then equality holds on the left hand side iff K is an ellipsoid.

Equality holds on the right hand side if K is an origin symmetric ellipsoid such that $|K| = |B_2^n|$. If K is C_+^2 , then equality holds on the right hand side iff K is an origin symmetric ellipsoid such that $|K| = |B_2^n|$.

Remark. The L_p -affine isoperimetric inequalities state that for $p \ge 0$

$$\frac{as_p(K)}{as_p(B_2^n)} \le \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}$$

and for -n ,

$$\frac{as_p(K)}{as_p(B_2^n)} \ge \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}}.$$

Equality holds trivially if p = 0. In both cases equality holds for $p \neq 0$ if and only if K is an ellipsoid. If p < -n and K is C_{+}^{2} , then

$$c^{\frac{np}{n+p}} \left(\frac{|K|}{|B_2^n|}\right)^{\frac{n-p}{n+p}} \leq \frac{as_p(K)}{as_p(B_2^n)}$$

These inequalities were proved by Lutwak [77] for p > 1 and for all other p by Werner and Ye [110].

In the case of a 0-symmetric convex body K, i.e., K = -K, the left hand sides of (63) and (64), together with the Blaschke Santaló inequality and its equality characterizations, (respectively the inverse Santaló inequality [25, 65, 85]) imply these inequalities and their equality characterizations. Moreover, in the last inequality, we remove the C_{+}^{2} on K of [110].

Another consequence of Corollary 20 are the following Blaschke Santaló type inequalities, which were originally proved in [110] by different methods.

Corollary 21. Let K be a convex body in \mathbb{R}^n with 0 in its interior.

If
$$p \ge 0$$
, then $as_p(K) as_p(K^\circ) \le n^2 |K| |K^\circ|$.
If $p < 0$, $p \ne -n$, then $as_p(K) as_p(K^\circ) \ge n^2 |K| |K^\circ|$.

Equality holds trivially for p = 0 or $p = \infty$, if K is in C_{+}^{2} .

Equality also holds if K is an ellipsoid. If K is in C^2_+ , then equality holds iff K is an ellipsoid.

Remark. Similarly to the Remarks after Corollary 15, we have the following consequences of Corollary 20 for a C_{+}^{2} convex body K:

(i) For p > 0,

$$as_p(K) \leq \left(\frac{p}{n+p}\right)as_{\infty}(K) + \left(\frac{n}{n+p}\right)as_0(K)$$

The duality relation $as_p(K) = as_{\frac{n^2}{2}}(K^{\circ})$ of [110] then yields

$$as_p(K^\circ) \leq \left(\frac{p}{n+p}\right)as_0(K) + \left(\frac{n}{n+p}\right)as_\infty(K).$$

For p < 0, those inequalities are reversed.

(ii) For $p \in (-\infty, -n)$,

$$as_{\infty}(K) - as_0(K) \leq as_p(K) - as_{\frac{-n^2}{2n+p}}(K).$$

For p > -n, the inequality is reversed.

Note also that similar results hold for the dual body K° by the above duality relation. (iii) For p = n, Corollary 20 yields

$$(as_n(K))^2 \leq as_0(K) as_\infty(K).$$

In [28], it was proved that for normalized densities

$$D_f(P_{\varphi_K}, Q_{\varphi_K}) = D_f(P_K, Q_K)$$

Now if we apply $\varphi_K = \exp\left(-\frac{\|\cdot\|_K^2}{2}\right)$ to the inequality (37), we obtain similarly

$$D_f(P_K, Q_K) \ge \frac{f''(1)}{2} \left(\int \left| \frac{|B_2^n| \ e^{-\frac{\|z\|_K^2}{2}}}{(2\pi)^{\frac{n}{2}}} \left(\frac{\kappa_K(z)}{|K^\circ|\langle z, N_K(z)\rangle^{n+1}} - \frac{1}{|K|} \right) \right| dz \right)^2.$$

The relation between the normalized cone measure Q_K and the Hausdorff measure μ_K on ∂K is given by

$$dQ_K(x) = \frac{\langle x, N_K(x) \rangle d\mu_K(x)}{n|K|}$$

We integrate in polar coordinates with respect to the normalized cone measure Q_K of K. Thus, if we write x = rz, with $z \in \partial K$, then $dx = n|K|r^{n-1}dr dQ_K(z)$. We also use

that the map $x \mapsto \det \nabla^2 \psi(x)$ is 0-homogeneous. So, we obtain,

$$\begin{split} D_{f}(P_{K},Q_{K}) &\geq \frac{f''(1)}{2} \frac{n^{2}|K|^{2}|B_{2}^{n}|^{2}}{(2\pi)^{n}} \left(\int_{0}^{+\infty} r^{n-1} e^{\frac{-r^{2}}{2}} dr \!\!\!\!\int_{\partial K} \left| \frac{\kappa_{K}(z)}{|K^{\circ}|\langle z, N_{K}(z)\rangle^{n+1}} - \frac{1}{|K|} \right| dQ_{K}(z) \right)^{2} \\ &= \frac{f''(1)}{2} |K|^{2} \left(\int_{\partial K} \left| \frac{\kappa_{K}(z)}{|K^{\circ}|\langle z, N_{K}(z)\rangle^{n+1}} - \frac{1}{|K|} \right| \frac{dQ_{K}(z)}{n} \right)^{2} \\ &= \frac{f''(1)}{2} \left(\int_{\partial K} \left| \frac{\kappa_{K}(z)}{|K^{\circ}|\langle z, N_{K}(z)\rangle^{n+1}} - \frac{1}{|K|} \right| \frac{\langle z, N_{K}(z)\rangle}{n} d\mu_{K}(z) \right)^{2} \\ &= \frac{f''(1)}{2} \left(\int_{\partial K} \left| \frac{\kappa_{K}(z)}{n|K^{\circ}|\langle z, N_{K}(z)\rangle^{n}} - \frac{\langle z, N_{K}(z)\rangle}{n|K|} \right| d\mu_{K}(z) \right)^{2} \\ &= \frac{f''(1)}{2} V^{2}(P_{K}, Q_{K}). \end{split}$$

Thus we obtained the following Pinsker type inequality for convex bodies.

Corollary 22. Let K be a convex body in \mathbb{R}^n with 0 in its interior. And let $f : (0, \infty) \to \mathbb{R}$ be convex and f(1) = 0. Suppose that the convex function f is differentiable up to order 3 at 1 with f''(1) > 0 and the following inequality holds

$$\left(f(u) - f'(1)(u-1)\right) \left(1 - \frac{f'''(1)}{3f''(1)}(u-1)\right) \geq \frac{f''(1)}{2}(u-1)^2.$$

Then

$$D_f(P_K, Q_K) \ge \frac{f''(1)}{2} V^2(P_K, Q_K).$$

If f is concave, the inequalities are reversed. Equality also holds if K is an ellipsoid.

Similarly, Corollary 9 (the case $f(t) = -\ln t$) becomes

$$D_{KL}(Q_K||P_K) \ge \frac{1}{2} V^2(P_K, Q_K).$$

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