Pisot Substitutions and Rauzy fractals

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Abstract

We prove that the dynamical system generated by a primitive unimodular substitution of the Pisot type on d letters satisfying a combinatorial condition which is easy to check, is measurably isomorphic to a domain exchange in $\mathbb{R}^{d-1}$, and is a finite extension of a translation on the torus $\mathbb{T}^{d-1}$. In the course of the proof, we introduce some potentially useful notions: the linear maps associated to a substitution and their dual maps, and the σ-structure for a dynamical system with respect to a pair of partitions.

Résumé

Nous prouvons qu'un système dynamique engendré par une substitution sur d lettres, unimodulaire, primitive, de type Pisot, et satisfaisant une condition combinatoire facile à verifier, est mesurablement isomorphe à un échange de domaine dans $\mathbb{R}^{d-1}$, et est une extension finie d'une translation sur le tore $\mathbb{T}^{d-1}$. Au cours de la preuve, nous introduisons des concepts qui devraient se révéler utile ailleurs : les extensions linéaires associées à une substitution et leurs applications duales, et la notion de σ-structure d'un système dynamique par rapport à une paire de partitions.
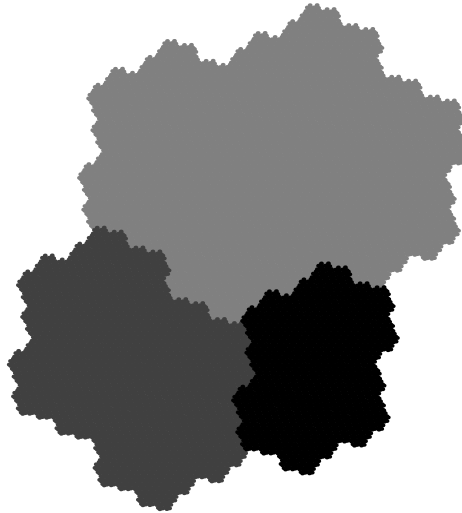

Figure 1: The Rauzy fractal

Introduction

In the paper [Rau82], we find a curious compact domain, called the Rauzy fractal. The domain is constructed in the following manner:

Let σ be the substitution

$$\sigma : \begin{aligned} \mathbf{1} &\to \mathbf{12} \\ \mathbf{2} &\to \mathbf{13} \\ \mathbf{3} &\to \mathbf{1} \end{aligned}$$

Let $w = (w_1, \cdots, w_n, \cdots)$ be the fixed point of this substitution, $E_0(\sigma)$ be the linear map associated to σ by abelianization, $\mathcal{P}$ the contractive invariant plane for the transformation $E_0(\sigma)$, and $\pi : \mathbb{R}^3 \to \mathcal{P}$ be the projection along the eigenvector ${}^t(1, \alpha, \alpha^2)$ with respect to the maximum eigenvalue λ of $E_0(\sigma)$, where $\alpha = 1/\lambda$ is the unique real root of $\alpha + \alpha^2 + \alpha^3 = 1$.

We define X as the closure of the set $\left\{ \pi \sum_{k=1}^n e_{w_k} \mid n = 1, 2, \cdots \right\}$ where e_1, e_2, e_3 is the canonical basis of $\mathbb{R}^3$.

The domain X has a fractal boundary, and admits a natural decomposition into three sets $X^{(\mathbf{i})}, \mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$ defined as the closure of $\left\{ \pi \sum_{k=1}^n e_{w_k} \mid w_n = \mathbf{i}, \ n \in \mathbb{N} \right\}$. This domain is not only interesting from the viewpoint of fractal geometry [IK91], but also in the sense of ergodic theory and number theory. In fact, two dynamical systems, Markov endomorphism with the structure matrix $E_0(\sigma)$ and quasi-periodic motion, act on the domain X (See [Rau82], [IK91], [IO93], [Mes96]). And the projection of the points $\left\{ \sum_{k=1}^{l_n} e_{w_k} \mid n \in \mathbb{N} \right\}$, where l_n is the length of the word $\sigma^n(\mathbf{1})$, gives the best approximations of the point (α, α^2), in other word, the fractal domain is strongly related to the simultaneous approximation of the pair of the cubic number (α, α^2) (See [IH94]). This domain is also related to developments of complex numbers in a base conjugate to α (See [Mes96]).

In this paper, we would like to discuss a generalization of the Rauzy fractal. We will describe several constructions related to substitutions. The aim is, for a certain

class of substitutions, to obtain the same result as for the Rauzy substitution; along the way, we will obtain weaker results for larger classes of substitutions.

Call f the natural map from the free monoid on d letters to $\mathbb{Z}^d$ obtained by abelianization. Consider a substitution σ on d letters, defined by $\sigma(\mathbf{i}) = W^{(\mathbf{i})}$, $\mathbf{i} = \mathbf{1}, \ldots, \mathbf{d}$ where $W^{(\mathbf{i})} = W_1^{(\mathbf{i})} \ldots W_{l_{\mathbf{i}}}^{(\mathbf{i})}$. We will use the notation $W^{(\mathbf{i})} = P_k^{(\mathbf{i})} W_k^{(\mathbf{i})} S_k^{(\mathbf{i})}$, where $P_k^{(\mathbf{i})}$ (resp. $S_k^{(\mathbf{i})}$) is the prefix (resp. suffix) of the word $W^{(\mathbf{i})}$ with respect to the k-th letter of $W^{(\mathbf{i})}$.

We say that the substitution σ has a positive strong coincidence of order 1 for $\mathbf{i}, \mathbf{j}$ if there is an integer k such that $W_k^{(\mathbf{i})} = W_k^{(\mathbf{j})}$ and $f(P_k^{(\mathbf{i})}) = f(P_k^{(\mathbf{j})})$.

We say that the substitution σ has positive strong coincidences for all letters if, for any pair $\mathbf{i}, \mathbf{j}$, there is an integer n such that σ^n has a positive strong coincidence of order 1 for $\mathbf{i}, \mathbf{j}$.

The main result of this paper is the following:

Theorem 1. *Let σ be a primitive unimodular substitution of the Pisot type on d letters that has positive strong coincidences for all letters. The uniquely ergodic symbolic dynamical system generated by σ is measurably conjugate to an exchange of pieces by translation on the plane $\mathbb{R}^{d-1}$. It is also measurably semi-conjugate, by a continuous onto map, to a translation on the torus $\mathbb{T}^{d-1}$.*

We can also define negative strong coincidences, using suffixes instead of prefixes, and obtain the same result.

One interest of our method is that it gives an explicit construction of the domain exchange, with a constructive approximation to the domain.

Remark 1. *In all known examples, the substitution system is in fact isomorphic to a toral translation; we do not know whether this is the general case. We can however prove that it is true if the sequence of approximations we build for the domain exchange contains a fixed open set, and this is possible to check explicitly for a given substitution.*

In that case, it is easy to show that the pieces of the domain exchange are the bases of cylinders of a Markov partition for the toral automorphism associated to the substitution.

The substitution has strong coincidences for all letters if, for example, the images of all letters by σ start or end with the same letter; so the theorem applies for the Rauzy substitution, for which all images start with $\mathbf{1}$, and also for more general systems, giving the following corollary, since in that case we can check that the sequence of approximations contains a fixed open set:

Corollary 1. *We take $d = 3$, and define the substitution $\sigma_{\mathbf{i}}$, $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$, by $\sigma_{\mathbf{i}}(\mathbf{i}) = \mathbf{i}$ and $\sigma_{\mathbf{i}}(\mathbf{j}) = \mathbf{j}\mathbf{i}$ if $\mathbf{j} \neq \mathbf{i}$. Let σ be a product of the substitutions $\sigma_{\mathbf{1}}, \sigma_{\mathbf{2}}, \sigma_{\mathbf{3}}$, where all these substitutions occur at least once. The system generated by σ is measurably isomorphic to a translation on the torus $\mathbb{T}^2$.*

Such substitutions have been considered, for example, in [AR91], where it was proved that all systems generated by sequences of complexity $2n+1$ with one trispecial factor of all lengths could be realized as interval exchanges. It was natural to believe, in the light of the above theorem, that such systems could also be realized

as toral translations. However, a recent paper ([CFZ00]) shows that some of these sequences are not balanced, hence they cannot be coding of translation. In that case, we can, using the techniques given here, build a sequence of partitions, but it fails to converge.

The content of the paper is the following. In the first section, we build the 1-dimensional linear realization of a substitution; this is valid for any substitution. In the second section, we define the dual map associated to a substitution. This definition is only interesting in the case of an unimodular substitution, that is, a substitution whose associated abelianized map has determinant $+1$ or -1. In the third section, we consider primitive unimodular substitutions (a substitution is primitive if there is an n such that, for any $\mathbf{i}$ and $\mathbf{j}$, $\mathbf{i}$ occurs in $\sigma^n(\mathbf{j})$; it is equivalent to say that the matrix of the associated linear map has a strictly positive power) and study some geometric and combinatorial properties of the dual map under this condition. In section 4, we prove that, in that case, we can build a sequence of domains D_n, which tile a hyperplane of $\mathbb{R}^d$, and a sequence of dynamical systems T_n on D_n, that have σ-structure, in the sense defined in that section.

In section 5, we consider Pisot substitutions, that is, substitutions such that the associated matrix has all eigenvalues, except one, of modulus strictly smaller than 1. In that case, we can prove that the sequence of sets D_n, when properly renormalized, converges to a compact set X in the Hausdorff topology. In section 6, we show that, if the substitution has strong coincidences for all letters, the dynamical systems T_n can be used to define a dynamical system on X, which is measurably conjugate to the substitution dynamical system defined by σ. In section 7, we give some examples, and prove the corollary above.

There are different methods to construct the Rauzy fractal: one can use convergent series associated to the substitution, as is done in the original paper [Rau82], or approximate it by a sequence of domains obtained by "exduction", as was already known to Rauzy. We try here to formalize this second approach; however, both methods have common points, and section 5 and 6 of the present paper rely heavily on a (unfortunately unpublished) paper by Bernard Host [Hos92], summarizing the results of a seminar at Villetaneuse (Université Paris 13) on the first approach. The paper [Hos92] contains a deeper result in the 2 letters case: a unimodular substitution σ on two letters is measurably conjugate to a rotation if and only if it has a strong coincidence for the two letters.

We signal that it is unknown whether there exists a unimodular Pisot substitution, on 2 or more letters, that does not have strong coincidences for all letters, and in this regard our work is incomplete.

Another problem is given by the case of unimodular non-Pisot substitutions; we give some elements to study this problem in [AIS01], generalizing the construction of linear extensions of substitutions and their dual maps.

1   The linear maps associated to a substitution

Let $\mathcal{A}$ be an alphabet of d letters $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{d}\}$. We denote by $\mathcal{A}^* = \bigcup_{n=0}^{\infty} \mathcal{A}^n$ the free monoid on $\mathcal{A}$, that is, the set of finite words on the alphabet $\mathcal{A}$, endowed with the concatenation product. A substitution σ on $\mathcal{A}^*$ is a map from $\mathcal{A}$ to $\mathcal{A}^*$, such that,
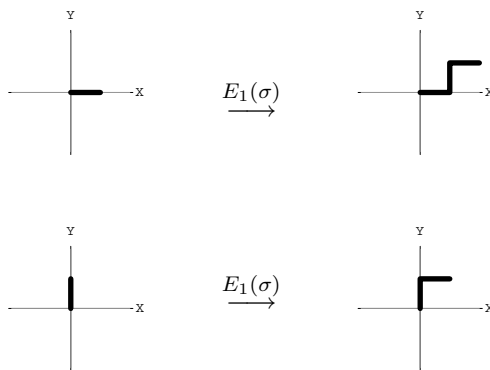
Figure 2: The linear map associated to $\mathbf{1} \mapsto \mathbf{121}$, $\mathbf{2} \mapsto \mathbf{21}$

for any letter $\mathbf{i}$, $\sigma(\mathbf{i})$ is a non-empty word. The substitution σ extends in a natural way to an endomorphism of the monoid $\mathcal{A}^*$, by the rule $\sigma(UV) = \sigma(U)\sigma(V)$.

Notations 1. *We denote by $W^{(\mathbf{i})}$ the word $\sigma(\mathbf{i})$; $W^{(\mathbf{i})} = W_1^{(\mathbf{i})} \ldots W_{l_{\mathbf{i}}}^{(\mathbf{i})}$ is a finite word of length $l_{\mathbf{i}}$. We call $P_k^{(\mathbf{i})}$ the prefix of length $k-1$ of $W^{(\mathbf{i})}$(for $k=1$, this is the empty word), and $S_k^{(\mathbf{i})}$ the suffix of length $l_{\mathbf{i}} - k$, so that $W^{(\mathbf{i})} = P_k^{(\mathbf{i})} W_k^{(\mathbf{i})} S_k^{(\mathbf{i})}$.*

There is a natural homorphism $f : \mathcal{A}^* \to \mathbb{Z}^d$, obtained by abelianization of the free monoid, and one can obtain a linear transformation $E_0(\sigma)$ satisfying the following commutation relation:

$$
\begin{array}{ccc}
\mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* \\
f \downarrow & & \downarrow f \\
\mathbb{Z}^d & \xrightarrow{E_0(\sigma)} & \mathbb{Z}^d.
\end{array}
$$

We will now build another linear map, on an infinite dimensional space, that contains more information on σ. The idea is the following: consider the space $\mathbb{R}^d$, with canonical basis $(e_{\mathbf{1}}, \ldots, e_{\mathbf{d}})$. To a finite word $U = U_1 \ldots U_k$, one can associate a path in $\mathbb{R}^d$, consisting in k segments joining respectively $f(U_1 \ldots U_{j-1})$ and $f(U_1 \ldots U_j)$ (this is just a graphic representation of the abelianization). Each segment is a translate, by an element of $\mathbb{Z}^d$, of one of the basic segments $\{\lambda e_{\mathbf{i}} | 0 \leq \lambda \leq 1\}$ (see Fig. 2, in the case $d = 2$). Formally, we can consider the path as a sum of elements $(x, e_{\mathbf{i}})$, where $x \in \mathbb{Z}^d$. It is then quite natural to consider that the substitution σ operates on the set of paths, replacing each segment parallel to $e_{\mathbf{i}}$ by a broken line corresponding to the word $\sigma(\mathbf{i})$. We want the image of a connected path to be a connected path; this implies, if we suppose that the point 0 is fixed, that the image of a path joining 0 to x is a path joining 0 to $E_0(\sigma)(x)$. This completely defines the image of every segment $(x, e_{\mathbf{i}})$, see Fig. 2.

For a formal definition, it is more convenient to consider linear combinations of paths with coefficients in $\mathbb{R}$, so as to get a vector space.

Definition 1. *Let $\mathcal{F}$ be the vector space of maps from $\mathbb{Z}^d \times \mathcal{A}$ to $\mathbb{R}$ that take value zero except for a finite set. For $x \in \mathbb{Z}^d$ and $\mathbf{i} \in \mathcal{A}$, denote by $(x, e_{\mathbf{i}})$ the element of $\mathcal{F}$ which takes value 1 at $(x, \mathbf{i})$, and 0 elsewhere; the set $\{(x, e_{\mathbf{i}}) | x \in \mathbb{Z}^d, \mathbf{i} \in \mathcal{A}\}$ is a basis of $\mathcal{F}$, and we define the 1-dimensional geometric realization $E_1(\sigma)$ of σ by*

$$E_1(\sigma)(x, e_{\mathbf{i}}) = \sum_{k=1}^{l_{\mathbf{i}}} \left(E_0(\sigma)(x) + f(P_k^{(\mathbf{i})}), e_{\mathbf{W_k^{(i)}}} \right).$$

Definition 2. *The* **support** *of an element of $\mathcal{F}$ is the set of $(x, \mathbf{i})$ on which it is not zero. We say that an element of $\mathcal{F}$ is* **geometric** *if it takes only value 0 and 1. To a geometric element, one can associate a finite union of segments, as explained above. If a and b are two geometric elements, we will say for simplicity that a* **contains** *b if the support of a contains that of b.*

If we apply the map $E_1(\sigma)$ to $(0, e_{\mathbf{i}})$, we get the previous intuitive definition. A geometric application is the following: suppose that $\sigma(\mathbf{i})$ starts with $\mathbf{i}$; then, there is an infinite word u, starting with $\mathbf{i}$, which is fixed by σ. This infinite word begins with $\sigma^n(\mathbf{i})$, for all $n \in \mathbb{N}$. It is possible to prove that $E_1(\sigma)^n(0, e_{\mathbf{i}})$ is a geometric element of $\mathcal{F}$, to which one associates a broken line in $\mathbb{R}^d$ which approximates (if the substitution is primitive) the direction of the eigenvector corresponding to the largest eigenvalue; it is the path associated with $\sigma^n(\mathbf{i})$ we discussed above, and the union of all this paths is an infinite path associated with the fixed point. If the substitution is of the Pisot type, this broken line stays within bounded distance of the line through zero in this direction.

One can check directly that, if we take two distinct segments in this broken line, their images by $E_1(\sigma)$ have disjoint supports; the reason is that, if we take 2 elements $(x, e_{\mathbf{i}})$ and $(y, e_{\mathbf{j}})$ on the line, with y "after" x, the vector $y - x$ has all coordinates positive, and the $\mathbf{i}$-th coordinate strictly positive. Hence we have $E_0(\sigma)(y - x) \geq E_0(\sigma)(e_{\mathbf{i}}) = f(W^{(\mathbf{i})})$, and we can never have $E_0(\sigma)(x) + f(P_k^{(\mathbf{i})}) = E_0(\sigma)(y) + f(P_l^{(\mathbf{j})})$; this is why all the $E_1(\sigma)^n(0, e_{\mathbf{i}})$ are geometric. However, it is false in general: the image of a geometric element is not always geometric. In the next section, we will compute exactly the set of segments that contain a given $(x, e_{\mathbf{i}})$ in their image, and show that it is usually not reduced to one element.

It is possible to generalize this definition to multidimensional extensions of substitutions; this is studied in [AIS01].

2   The dual map associated to a unimodular substitution

In this section, we will only be interested in so-called "unimodular" substitutions.

Definition 3. *A substitution σ is called unimodular if the associated linear map $E_0(\sigma)$ has determinant $+1$ or -1; in other words, its matrix in the canonical basis belongs to the linear group $GL(d, \mathbb{Z})$.*

It is equivalent to say that the square of the matrix belongs to the unimodular linear group $SL(d, \mathbb{Z})$, hence the term unimodular substitution.

We can associate to $E_1(\sigma)$ the dual map $E_1^*(\sigma)$. We will show how to compute explicitly this map. We denote by $\mathcal{F}^*$ the space of linear forms on $\mathcal{F}$ with finite support (that is, linear forms for which there exists a finite set $K \in \mathbb{Z}^d \times \mathcal{A}$, such that the form is 0 on any element of $\mathcal{F}$ whose support does not intersect K) ; this space admits as basis the set $\{(x, e_{\mathbf{i}}^*) | x \in \mathbb{Z}^d, \mathbf{i} \in \mathcal{A}\}$. It is easy to check that $E_1^*(\sigma)$ preserves $\mathcal{F}^*$.
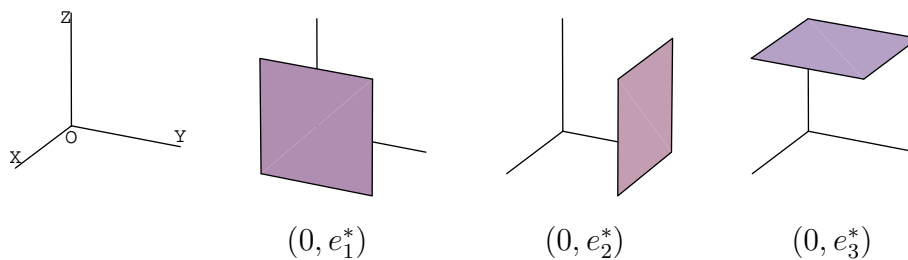
$$(0, e_1^*) \qquad\qquad (0, e_2^*) \qquad\qquad (0, e_3^*)$$

Figure 3: Exemples of geometric models of $(x, e_{\mathbf{i}}^*)$ in dimension 3

Lemma 1. *The map $E_1^*(\sigma)$ is defined on $\mathcal{F}^*$ by:*

$$E_1^*(\sigma)(x, e_{\mathbf{i}}^*) = \sum_{\mathbf{j} \in \mathcal{A}} \sum_{W_k^{(\mathbf{j})} = \mathbf{i}} \left(E_0(\sigma)^{-1} \left(x - f(P_k^{(\mathbf{j})}) \right), e_{\mathbf{j}}^* \right)$$

Proof. We want to compute $E_1^*(\sigma)(x, e_{\mathbf{i}}^*)$. Using the natural bilinear product on $\mathcal{F}^* \times \mathcal{F}$, we have:

$$\langle E_1^*(\sigma)(x, e_{\mathbf{i}}^*) | (y, e_{\mathbf{j}}) \rangle = \langle (x, e_{\mathbf{i}}^*) | E_1(\sigma)(y, e_{\mathbf{j}}) \rangle$$
$$= \left\langle (x, e_{\mathbf{i}}^*) \left| \sum_{k=1}^{l_{\mathbf{j}}} (E_0(\sigma)(y) + f(P_k^{(\mathbf{j})}), e_{W_k^{(\mathbf{j})}}) \right\rangle \right.$$

The product is not zero if and only if $W_k^{(\mathbf{j})} = \mathbf{i}$ and $E_0(\sigma)(y) + f(P_k^{(\mathbf{j})}) = x$. Since $E_0(\sigma)$, as a map from $\mathbb{Z}^d$ to $\mathbb{Z}^d$, is invertible, this amounts to $y = E_0(\sigma)^{-1} \left(x - f(P_k^{(\mathbf{j})}) \right)$; hence we get the formula given above. ∎

Remark that we can also consider $\mathcal{F}^*$ as the set of maps from $\mathbb{Z}^d \times \mathcal{A}$ to $\mathbb{R}$ with value zero except of a finite set. Hence definition 2 makes sense also here, and we can define the **geometric** elements of $\mathcal{F}^*$ as those who take only value 0 and 1.

We recover in this way, up to a change of coordinates, the formulas already given, with a dynamical motivation, in [IO93]. The signification of this map is not immediately clear; but we can give a geometric model, in the following way:

To $(x, e_{\mathbf{i}}^*)$, we associate the upper hyperface orthogonal to $e_{\mathbf{i}}$ of the unit cube with lower vertex at x. Analytically, this face can be expressed as $\{x + e_{\mathbf{i}} + \sum_{\mathbf{j} \neq \mathbf{i}} \lambda_{\mathbf{j}} e_{\mathbf{j}} | \lambda_{\mathbf{j}} \in [0, 1]\}$ (see Fig. 3). The dual map associates, to a given hyperface, a finite collection of hyperfaces (see Fig. 4 for the dual map associated to the Rauzy substitution).

To a geometric element of $\mathcal{F}^*$, we can associate a finite union of faces; it is however more difficult to represent non-geometric elements. In the general case, the image by $E_1^*(\sigma)$ of a geometric element is no longer geometric, since different faces can have images by $E_1^*(\sigma)$ whose support intersect. We will see in the next section that there are sets of geometric elements invariant by the map; these can be used to approximate, by iteration, the plane defined by the dual eigenvector for the maximal eigenvalue.
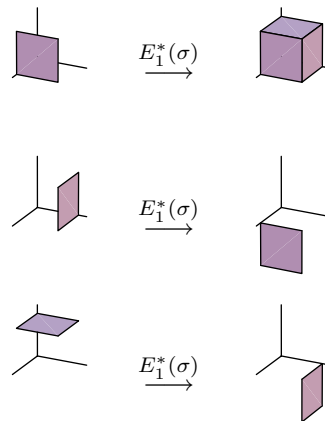
Figure 4: The dual map associated to the Rauzy substitution

3   The case of primitive unimodular substitutions: elementary properties

From now on, we will suppose that the substitution is primitive, that is, there is an n such that, for any pair of letters $(\mathbf{i}, \mathbf{j})$, $\mathbf{i}$ occurs in $\sigma^n(\mathbf{j})$. It is equivalent to say that the matrix of $E_0(\sigma)$ is primitive, and so, by Perron-Frobenius theorem, $E_0(\sigma)$ has an eigenvalue λ that is positive, simple and strictly bigger in modulus than the other eigenvalues. We denote by v_λ a positive eigenvector associated to λ for the transpose of $E_0(\sigma)$, and by $\mathcal{P}$ the plane orthogonal to v_λ. It is clear that $\mathcal{P}$ is invariant by $E_0(\sigma)$. We will denote by $P^<$ the set $\{x \in \mathbb{R}^d | x.v_\lambda < 0\}$, (where $x.v_\lambda$ is the scalar product between vectors of $\mathbb{R}^d$) that is, the open lower half-space associated to $\mathcal{P}$. We can define in the same way $P^{\leq}, P^>, P^{\geq}$.

Remark 2. *To avoid cumbersome terminology, we will always use the 3-dimensional vocabulary instead of d-dimensional, and speak of planes, surface, cube,..., instead of hyperplane, hypersurface, hypercube, ..., since no confusion will ensue, and all our examples will be on 2 or 3 letters.*

Definition 4. *We say that $(x, e_{\mathbf{i}}^*)$, where $x \in \mathbb{Z}^d$, is a unit tip of the stepped surface of $\mathcal{P}$ if $x \in P^<$ and $x + e_{\mathbf{i}} \in P^{\geq}$. We denote by $\mathcal{G}$ the set of finite sums of distinct unit tips.*

The unit tips can be easily visualised, using the geometric model of the preceding section. The element $(x, e_{\mathbf{i}}^*)$ is a unit tip if the interior of the unit cube with lower corner at x intersects the plane $\mathcal{P}$, and its upper face orthogonal to $e_{\mathbf{i}}$ does not intersect $\mathcal{P}$, except maybe on the boundary. The union of all unit tips constitute a stepped surface that is the best approximation of $\mathcal{P}$ by faces contained in the closed upper half-space $P^{\geq}$. The elements of $\mathcal{G}$ can be represented as finite parts of this stepped surface. They are all geometric elements of $\mathcal{F}^*$, so that $E_1^*(\sigma)$ acts on $\mathcal{G}$. The main result of this section is the following:

Proposition 1. *The set $\mathcal{G}$ is invariant by $E_1^*(\sigma)$*

Proof. it is enough to prove that the image of any unit tip is contained in $\mathcal{G}$, and that two different unit tips have disjoint images; this is accomplished in the following two lemmas. ∎

Lemma 2. *If $(x, e_{\mathbf{i}}^*)$ is a unit tip, its image $E_1^*(\sigma)(x, e_{\mathbf{i}}^*)$ belongs to $\mathcal{G}$.*

Proof. From the definition of $E_1^*(\sigma)$, we must prove that, if $W_k^{(\mathbf{j})} = \mathbf{i}$, $(E_0(\sigma)^{-1}(x - f(P_k^{(\mathbf{j})})), e_{\mathbf{j}}^*)$ is a unit tip.

But $f(P_k^{(\mathbf{j})})$ is a positive vector, so its opposite is in the negative cone, included in $P^{\leq}$; x is in $P^{<}$ by definition, so the sum is in $P^{<}$. This set is invariant by $E_0(\sigma)^{-1}$, so that $E_0(\sigma)^{-1}(x - f(P_k^{(\mathbf{j})}))$ is in $P^{<}$.

On the other hand, remark that, from the definition of $E_0(\sigma)$ and f, we have $E_0(\sigma)^{-1}(f(\sigma(\mathbf{j}))) = e_{\mathbf{j}}$. Hence, $E_0(\sigma)^{-1}(x - f(P_k^{(\mathbf{j})})) + e_{\mathbf{j}} = E_0(\sigma)^{-1}(x - f(P_k^{(\mathbf{j})}) + f(\sigma(\mathbf{j}))) = E_0(\sigma)^{-1}(x + e_{\mathbf{i}} + f(S_k^{(\mathbf{j})}))$; since $f(S_k^{(\mathbf{j})})$ is in the positive cone, and $x + e_{\mathbf{i}}$ is in $P^{\geq}$, their sum is in $P^{\geq}$, which is invariant by $E_0(\sigma)^{-1}$, so that $E_0(\sigma)^{-1}(x - f(P_k^{(\mathbf{j})})) + e_{\mathbf{j}} \in P^{\geq}$. This is exactly what we need.

It is easy to check that all the unit tips in the sum that defines $E_1^*(\sigma)(x, e_{\mathbf{i}}^*)$ are distinct. Indeed, take two such unit tips $(E_0(\sigma)^{-1}(x - f(P_k^{(\mathbf{j})})), e_{\mathbf{j}}^*)$ and $(E_0(\sigma)^{-1}(x - f(P_{k'}^{(\mathbf{j}')})), e_{\mathbf{j}'}^*)$. Suppose they are equal; we must then have $\mathbf{j} = \mathbf{j}'$. If $k \neq k'$, we can suppose that $k < k'$, so that $P_k^{(\mathbf{j})}$ is a strict prefix of $P_{k'}^{(\mathbf{j})}$. Then, their images by f are distinct, and since $E_0(\sigma)^{-1}$ is one-to-one, this implies that the base vertices of the two unit tips are distinct. ∎

Lemma 3. *Two distinct unit tips have disjoint images*

Proof. Suppose that $(y, e_{\mathbf{i}}^*)$ is in the intersection of $E_1^*(\sigma)(x, e_{\mathbf{j}}^*)$ and $E_1^*(\sigma)(x', e_{\mathbf{j}'}^*)$. Then, by definition, we can find k and k' such that $y = E_0(\sigma)^{-1}(x - f(P_k^{(\mathbf{i})})) = E_0(\sigma)^{-1}(x' - f(P_{k'}^{(\mathbf{i})}))$, hence $x - f(P_k^{(\mathbf{i})}) = x' - f(P_{k'}^{(\mathbf{i})})$ since $E_0(\sigma)$ is not degenerate. If $x = x'$, we must have $k = k'$, and the unit tips are not distinct. If $x \neq x'$, we must have $k \neq k'$; if for example $k < k'$, we have $x + f(W_k^{(\mathbf{i})} \ldots W_{k'-1}^{(\mathbf{i})}) = x'$; since $f(W_k^{(\mathbf{i})}) = e_{\mathbf{j}}$, this implies that x' is not in $P^{<}$, and this contradicts the definition of the unit tip $(x', e_{\mathbf{j}'}^*)$. ∎

Remark that the contradiction comes from the fact that unit tips must be close to the plane $\mathcal{P}$; there is no reason for an arbitrary pair of elements to have disjoint images.

One checks immediately that the elements $(-e_{\mathbf{i}}, e_{\mathbf{i}}^*)$ are unit tips; the sum of all the elements represent the lower faces of the unit cube at the origin, which touches the plane $\mathcal{P}$ at the origin.

The upper faces of this unit cube are given by the sum of the elements $(0, e_{\mathbf{i}}^*)$. Remark that if, in the definition of the unit tips, we exchange strict and large inequalities, replacing $P^{<}$ by $P^{\leq}$ and $P^{\geq}$ by $P^{>}$, then $(0, e_{\mathbf{i}}^*)$ becomes a unit tip; the lemmas 1 and 2 and the proposition stay true, *mutatis mutandis*, in that case, by the same proof, exchanging all large and strict inequalities. It will prove very useful in the sequel to use these upper and lower faces of the unit cube at the origin. We will use the following notation:
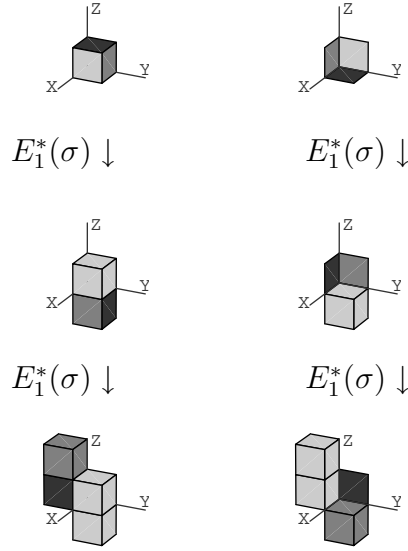
Figure 5: The images $E_1^*(\sigma)(\mathcal{U})$ and $E_1^*(\sigma)(\mathcal{U}')$ for Rauzy substitution

Definition 5. *We denote by $\mathcal{U}$ the union $\sum_{\mathbf{i}\in\mathcal{A}}(-e_{\mathbf{i}}, e_{\mathbf{i}}^*)$, and by $\mathcal{U}'$ the union $\sum_{\mathbf{i}\in\mathcal{A}}(0, e_{\mathbf{i}}^*)$*

Proposition 2. *$E_1^*(\sigma)(\mathcal{U})$ is an element of $\mathcal{G}$ which contains $\mathcal{U}$; $E_1^*(\sigma)(\mathcal{U}')$ contains $\mathcal{U}'$; $E_1^*(\sigma)(\mathcal{U}')$ is not in $\mathcal{G}$, but we have $E_1^*(\sigma)(\mathcal{U}') - E_1^*(\sigma)(\mathcal{U}) = \mathcal{U}' - \mathcal{U}$.*

Proof. Because $P_1^{(\mathbf{j})}$ is, by definition, the empty word, its image by f is zero; suppose that $W_1^{(\mathbf{j})} = \mathbf{i}$, this implies that $(0, e_{\mathbf{j}}^*)$ is contained in $E_1^*(\sigma)(0, e_{\mathbf{i}}^*)$ (we only need to check that, because $W_1^{(\mathbf{j})} = \mathbf{i}$, $(0, e_{\mathbf{j}}^*)$ appears in the sum that defines the image). By taking the sum on all letters, we obtain that $E_1^*(\sigma)(\mathcal{U}')$ contains $\mathcal{U}'$.

In the same way, suppose that the last letter of $W^{\mathbf{j}}$ is $\mathbf{i}$; from the very definition of $E_1^*(\sigma)$, we check that $E_1^*(\sigma)(-e_{\mathbf{i}}, e_{\mathbf{i}}^*)$ contains $(E_0(\sigma)^{-1}(-e_{\mathbf{i}} - f(P_{l_{\mathbf{j}}}^{(\mathbf{j})})), e_{\mathbf{j}}^*)$. But we have $E_0(\sigma)^{-1}(-e_{\mathbf{i}} - f(P_{l_{\mathbf{j}}}^{(\mathbf{j})})) = E_0(\sigma)^{-1}(-f(\sigma(\mathbf{j}))) = -e_{\mathbf{j}}$, so that $E_1^*(\sigma)(-e_{\mathbf{i}}, e_{\mathbf{i}}^*)$ contains $(-e_{\mathbf{j}}, e_{\mathbf{j}}^*)$. This shows that $E_1^*(\sigma)(\mathcal{U})$ contains $\mathcal{U}$.

Remark that, since $\mathcal{U}$ is obviously in $\mathcal{G}$, its image is, by the proposition, in $\mathcal{G}$, hence it is geometric; a similar proof, exchanging large and strict inequalities, shows that $E_1^*(\sigma)(\mathcal{U}')$, while not in $\mathcal{G}$, is geometric.

It remains to compute the difference between the images. Remark that, if $W_k^{(\mathbf{j})} = \mathbf{i}$ and k is less than the length $l_{\mathbf{j}}$ of $\sigma(\mathbf{j})$, we have $e_{\mathbf{i}} + f(P_k^{(\mathbf{j})}) = f(P_{k+1}^{(\mathbf{j})})$; if $k = l_{\mathbf{j}}$, then, as above, $E_0(\sigma)^{-1}(-e_{\mathbf{i}} - f(P_k^{(\mathbf{j})})) = -e_{\mathbf{j}}$. Hence, we easily compute:

$$E_1^*(\sigma)(\mathcal{U}) = \sum_{\mathbf{i} \in \mathcal{A}} E_1^*(\sigma)(-e_{\mathbf{i}}, e_{\mathbf{i}}^*)$$

$$= \sum_{\mathbf{i}} \sum_{\mathbf{j}} \sum_{k; W_k^{(\mathbf{j})} = \mathbf{i}} (E_0(\sigma)^{-1}(-e_{\mathbf{i}} - f(P_k^{(\mathbf{j})})), e_{\mathbf{j}}^*)$$

$$= \sum_{\mathbf{j}} \sum_{k=1}^{l_{\mathbf{j}}} (E_0(\sigma)^{-1}(-e_{W_k^{(\mathbf{j})}} - f(P_k^{(\mathbf{j})})), e_{\mathbf{j}}^*)$$

$$= \sum_{\mathbf{j}} \sum_{k=1}^{l_{\mathbf{j}}-1} (E_0(\sigma)^{-1}(-f(P_{k+1}^{(\mathbf{j})})), e_{\mathbf{j}}^*) + \sum_{\mathbf{j}} (-e_{\mathbf{j}}, e_{\mathbf{j}}^*)$$

$$= \mathcal{U} + \sum_{\mathbf{j}} \sum_{k=2}^{l_{\mathbf{j}}} (E_0(\sigma)^{-1}(-f(P_k^{(\mathbf{j})})), e_{\mathbf{j}}^*)$$

The same type of computation shows that

$$E_1^*(\sigma)(\mathcal{U}') = \mathcal{U}' + \sum_{\mathbf{j}} \sum_{k=2}^{l_{\mathbf{j}}} (E_0(\sigma)^{-1}(-f(P_k^{(\mathbf{j})})), e_{\mathbf{j}}^*)$$

The result follows. ■

The group $\mathbb{Z}^d$ acts in an obvious way on $\mathcal{F}$ by translation; the following lemma will be useful:

Lemma 4. *The translation of $E_1^*(\sigma)(-e_{\mathbf{i}}, e_{\mathbf{i}}^*)$ by $E_0(\sigma)^{-1}(e_{\mathbf{i}})$ is $E_1^*(\sigma)(0, e_{\mathbf{i}}^*)$*

Proof. From the definition of $E_1^*(\sigma)$, the difference between $E_1^*(\sigma)(0, e_{\mathbf{i}}^*)$ and $E_1^*(\sigma)(x, e_{\mathbf{i}}^*)$ is just a translation of $E_0(\sigma)^{-1}(x)$. ■

Proposition 2 tells us that $E_1^*(\sigma)(\mathcal{U})$ and $E_1^*(\sigma)(\mathcal{U}')$ differ only by the unit cube at the origin; in particular, they have the same projection on the plane $\mathcal{P}$. Lemma 4 gives us another way to recover $E_1^*(\sigma)(\mathcal{U}')$ from $E_1^*(\sigma)(\mathcal{U})$, by translating the different constituent pieces.

4 Tilings and dynamical systems associated to unit tips

From now on, we will identify unit tips and the corresponding subsets of $\mathbb{R}^d$; a geometric element of $\mathcal{F}$ will be considered as a subset of $\mathbb{R}^d$. The results of the preceding section can be applied to σ^n for all n, so we can iterate and obtain a sequence of larger and larger pieces of the stepped surface.

We denote by π the projection on the plane $\mathcal{P}$ along the positive eigendirection of the map $E_0(\sigma)$. We want to project by π the sets $E_1^*(\sigma)^n(\mathcal{U}')$, and establish their properties.

It is clear that the projection of the stepped surface generates a (non periodic) tiling of the plane $\mathcal{P}$ by d tiles, projections of $(-e_{\mathbf{i}}, e_{\mathbf{i}}^*)$; the union $\pi(\mathcal{U})$ of these tiles also generates a periodic tiling, obtained by projecting the stepped surface

associated with the diagonal plane $\mathcal{P}_0$ of equation $\sum x_{\mathbf{i}} = 0$. The main result of this section can be expressed in this way: we can group the tiles so that the non periodic tiling of $\mathcal{P}$ is also generated by the projections of $E_1^*(\sigma)^n(-e_{\mathbf{i}}, e_{\mathbf{i}}^*)$, and the union of these new tiles $\pi(E_1^*(\sigma)^n(\mathcal{U}))$ generates a periodic tiling, obtained from the projection of the stepped surface of the rational plane $E_0(\sigma)^{-n}(\mathcal{P}_0)$. These bigger tiles have an almost self-similar structure, which allow a simple generation of large parts of the tiling by translation of the basic tiles, giving rise to a family of larger and larger dynamical systems induced from each other.

Definition 6. *We denote by D_n the domain $\pi(E_1^*(\sigma)^n(\mathcal{U}))$. We denote by $D_n^{(\mathbf{i})}$ (resp. $D_n^{(\mathbf{i})'}$) the domain $\pi(E_1^*(\sigma)^n(-e_{\mathbf{i}}, e_{\mathbf{i}}^*))$ (resp. $\pi(E_1^*(\sigma)^n(0, e_{\mathbf{i}}^*))$).*

Proposition 3. *The set D_n is the closure of a fundamental domain for the action on $\mathcal{P}$ of the lattice Γ_n generated by the vectors $\pi(E_0(\sigma)^{-n}(e_{\mathbf{i}} - e_{\mathbf{j}}))$, for $1 \leq i < j \leq d$.*

Proof. We start by D_0. It is clear that, if we translate $\mathcal{U}$ by the lattice generated by the vectors $(e_{\mathbf{i}} - e_{\mathbf{j}})$, we obtain a periodic stepped surface over the plane $\mathcal{P}_0$ defined by the equation $\sum_{\mathbf{i}=1}^d x_{\mathbf{i}} = 0$ (this stepped surface consists of lower faces of unit cubes having their lowest vertex on $\mathcal{P}_0$). The projection of this periodic stepped surface gives the required result.

It suffices now to apply $E_1^*(\sigma)^n$ to this periodic stepped surface to obtain a new periodic stepped surface over the plane $E_0(\sigma)^{-n}\mathcal{P}_0$, with lattice generated by $E_0(\sigma)^{-n}(e_{\mathbf{i}} - e_{\mathbf{j}})$, whose projection by π gives the required result. ∎

Lemma 5. *The image of $D_n^{(\mathbf{i})'}$ by the translation of $-\pi(E_0(\sigma)^{-n}(e_{\mathbf{i}}))$ is $D_n^{(\mathbf{i})}$*

Proof. This follows immediately from lemma 4 of the preceding section, replacing σ by σ^n and projecting to the plane $\mathcal{P}$. ∎

It is easy to check that D_n is the disjoint union, up to a set of measure zero, of the sets $D_n^{(\mathbf{i})}$ (and also of the sets $D_n^{(\mathbf{i})'}$). This allows us to define a dynamical system:

Definition 7. *We denote by T_n the map from D_n to itself defined (up to a set of measure 0) by $T_n(x) = x - \pi(E_0(\sigma)^{-n}(e_{\mathbf{i}}))$ if $x \in D_n^{(\mathbf{i})'}$.*

Remark 3. *It could seem more natural to consider the inverse map of T_n, that is, the map defined on $D_n^{(\mathbf{i})}$ by $x \mapsto x + \pi(E_0(\sigma)^{-n}(e_{\mathbf{i}}))$; the reason for our choice is that it turns out that the substitution associated to this map is not σ, but its retrograde (the retrograde of a word $W = W_1 W_2 \ldots W_l$ is the word $\widetilde{W} = W_l \ldots W_2 W_1$, and the retrograde of a substitution σ is the substitution $\tilde{\sigma}$ defined by $\tilde{\sigma}(\mathbf{i}) = \widetilde{\sigma(\mathbf{i})}$.*

The map T_n is not well defined if the point x belongs to two different domains $D_n^{(\mathbf{i})}$, $D_n^{(\mathbf{j})}$. However, there is a canonical projection from D_n onto the torus $\mathcal{P}/\Gamma_n$, which conjugates T_n with the translation by the class modulo Γ_n of any of the vectors $\pi(E_0(\sigma)^{-n}(e_{\mathbf{i}}))$ (all these classes are equal, by definition of Γ_n); after the projection, the map is everywhere well-defined. This proves that the domain exchange we have defined is in fact conjugate to a rotation on a torus.

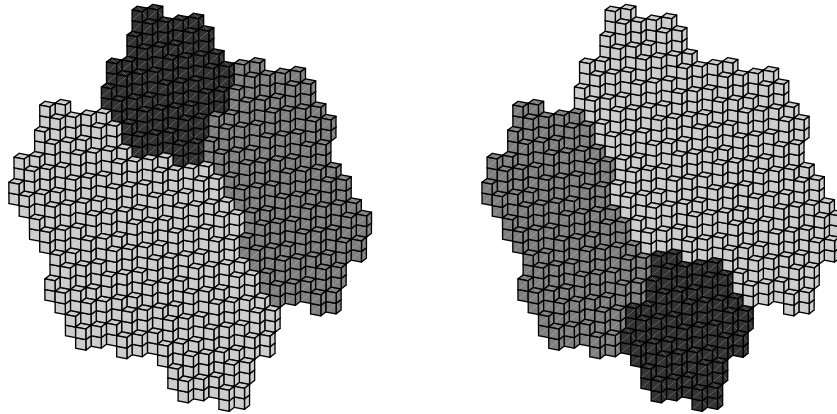These maps T_n, together with the partitions $\{D_n^{(\mathbf{i})}\}$, have remarkable properties, for which we need a definition:

Figure 6: The domain exchanges (D_{10}, T_{10}) for Rauzy substitution

Definition 8. *Let (X, T, μ) be a measured dynamical system, σ a substitution on the alphabet $\mathcal{A}$ such that $\sigma(\mathbf{i}) = W^{(\mathbf{i})}$, and consider a measurable partition $\left\{ X^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$ of X, a subset A of X and a measurable partition $\left\{ A^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$ of A.*

If the partitions $\left\{ X^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$ and $\left\{ A^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$ satisfy the following conditions (up to sets of measure 0):

$$T^k A^{(\mathbf{i})} \subset X^{(W^{(\mathbf{i})}_{k+1})} \quad \text{for all} \quad \mathbf{i} \in \mathcal{A} \quad \text{and} \quad k = 0, 1, \cdots, l_{\mathbf{i}} - 1$$
$$T^k A^{(\mathbf{i})} \cap A = \emptyset \quad \text{for all} \quad \mathbf{i} \in \mathcal{A} \quad \text{and} \quad 0 < k < l_{\mathbf{i}}$$
$$T^{l_{\mathbf{i}}} A^{(\mathbf{i})} \subset A \quad \text{for all} \quad \mathbf{i} \in \mathcal{A}$$
$$X = \bigcup_{\mathbf{i} \in \mathcal{A}} \bigcup_{0 \leq k \leq l_{\mathbf{i}} - 1} T^k A^{(\mathbf{i})}$$

*we say that the measured dynamical system (X, T, μ) has σ-**structure** with respect to the pair of partitions $\left\{ X^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$ and $\left\{ A^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$.*

The simplest example of system with σ-structure is the following: consider a primitive substitution σ which admits a nonperiodic fixed point, let (Ω, S, μ) be the uniquely ergodic system generated by σ (closure of the orbit by the shift map S of the fixed point of the substitution), with μ the unique invariant measure. If we consider the canonical partition of Ω by the elementary cylinders, and take for A the set $\sigma(\Omega)$, also with its canonical partition, one can check that (Ω, S) has σ-structure with respect to these two partitions.

Suppose that the system (X, T, μ) has σ-structure with respect to the partitions $\left\{ X^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$ and $\left\{ A^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$. To almost every point x in A, one can associate two sequences with values in $\mathcal{A}$: the symbolic dynamic with respect to the map T and the partition $\left\{ X^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$, defined as the sequence u such that $T^n(x) \in X^{(u_n)}$, and the symbolic dynamic with respect to the induced map $T_{|A}$ of T on A and the partition $\left\{ A^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$, defined as the sequence v such that $T^n_{|A}(x) \in A^{(v_n)}$. It is then

clear from the definition that, since $x \in A$, we have almost everywhere $u = \sigma(v)$. Call ϕ the map from X to $\mathcal{A}^{\mathbb{N}}$, that takes each point to the corresponding sequence; we see that, if we denote by $[\mathbf{i}]$ the set of sequences in $\mathcal{A}^{\mathbb{N}}$ that begins in $\mathbf{i}$, we have $\phi(X) \subset \cup_{\mathbf{i} \in \mathcal{A}} \cup_{0 \leq k < l_{\mathbf{i}}} S^k \sigma[\mathbf{i}]$.

It is natural to iterate this definition:

Definition 9. *Let (X, T, μ) be a measured dynamical system, σ a substitution on the alphabet $\mathcal{A}$ such that $\sigma(\mathbf{i}) = W^{(\mathbf{i})}$, and consider a decreasing sequence of subsets $A_0 = X, A_1, \ldots, A_n$, of X with associated partitions $\left\{ A_k^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$ for $k = 0, 1, \ldots, n$. We say that the transformation T has σ-**structure of order** n with respect to this sequence of partitions if the first return map $T_{|A_k}$ of T on each A_k, $k = 0, \ldots, n-1$ has σ-structure with respect to the pair of partitions $\left\{ A_k^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$, $\left\{ A_{k+1}^{(\mathbf{i})} | \mathbf{i} \in \mathcal{A} \right\}$.*

*The system (X, T, μ) has σ-**structure of all orders** if we can find such an infinite sequence of subsets and partitions.*

It is easy to check that (X, T, μ) has σ-structure of order n with respect to a sequence of partitions if and only if it has σ^n-structure with respect to the first and last of these partitions. For a primitive substitution σ, there is essentially only one system with σ-structure of all orders:

Lemma 6. *If the measurable dynamical system (X, T, μ) has σ-structure of all orders, and if, either the partition $\left\{ A_0^{(\mathbf{i})} | i \in \mathcal{A} \right\}$ is generating for the dynamical system, or the sequence of partitions $\left\{ T^k A_n^{(\mathbf{i})} | i \in \mathcal{A}, 0 \leq k < |\sigma^n(\mathbf{i})| \right\}$ is generating, the system (X, T, μ) is measurably conjugate to the dynamical system (Ω, S) associated to the substitution σ.*

Proof. First consider $\mathcal{A}^{\mathbb{N}}$ with the product topology. We define, on the set of compact subsets of $\mathcal{A}^{\mathbb{N}}$, a map Σ by:

$$\Sigma(K) = \bigcup_{\mathbf{i} \in \mathcal{A}} \bigcup_{0 \leq k < l_{\mathbf{i}}} S^k \sigma(K \cap [\mathbf{i}])$$

By primitivity of σ, $S^k \sigma$ is strictly contracting on each cylinder $[\mathbf{ij}]$, and so Σ or one of its powers is strictly contracting for the Hausdorff distance; hence it has a unique fixed point, which is the domain Ω of the dynamical system (Ω, S) associated to σ.

The remark above shows that, if (X, T) has σ-structure, the set $\phi(X)$ of symbolic sequences is included in $\Sigma(\mathcal{A}^{\mathbb{N}})$. Hence, if it has σ-structure of all orders, it is included in $\cap_{n \in \mathbb{N}} \Sigma^n(\mathcal{A}^{\mathbb{N}}) = \Omega$.

On the other hand, the partition being measurable, the set $\phi(X)$ is a measurable invariant subset of Ω. But it is well known that (Ω, S), for a primitive substitution, is uniquely ergodic; hence $\phi(X)$ is equal to Ω, up to a set of measure 0.

If the partition $\left\{ A_0^{(\mathbf{i})} | i \in \mathcal{A} \right\}$ is generating with respect to T, ϕ is one-to-one, so it is a measurable isomorphism.

Suppose that the sequence of partitions is generating. To each point of X, we can associate a sequence of sets $T^{k_n} A_n^{\mathbf{i}_n}$ in which it is contained. Except for a set of measure 0, two distinct points cannot be contained in the same sequence. But then, the symbolic dynamic sequence of x is contained in $S^{k_n} \sigma^n([\mathbf{i}_n])$, and these sets are

disjoint, up to measure 0; hence distinct points have distinct associated symbolic sequences, except for a set of measure 0.

In both cases, ϕ is a measurable isomorphism that conjugates (X, T, μ) and (Ω, S, ν), where ν is the unique shift invariant measure on Ω. ∎

Proposition 4. *For each integer k in $\{1, \ldots, n\}$, the dynamical system (D_n, T_n) has σ^k-structure with respect to the partitions $\{D_n^{(\mathbf{i})'} | \mathbf{i} \in \mathcal{A}\}$ and $\{D_{n-k}^{(\mathbf{i})'} | \mathbf{i} \in \mathcal{A}\}$*

Proof. We will check it for D_n and D_0; it suffices in fact to check it for D_1 and D_0, and then to replace σ by σ^n. But the very construction of $D_1^{(\mathbf{i})'}$ implies that the iterates of $D_0^{(\mathbf{j})'}$ are, before their first return to D_0, contained in an element of the partition of D_n, whose index is given by the substitution; this proves the proposition. ∎

To summarize our results, we have now proved that, for any primitive unimodular substitution σ, and any integer k, we can find a domain exchange (and even a translation on the torus $\mathbb{T}^{d-1}$) that has σ^k-structure with respect to a pair of explicit partitions. Our aim in the next section will be to obtain, by a suitable renormalisation, a limit partition, to get, if possible, a system with σ-structure of all orders.

5 Unimodular substitutions of the Pisot type

We can renormalize the domains $D_n^{(\mathbf{i})}$, using $E_0(\sigma)^n$. One checks that the renormalized domains are, up to a set of measure 0, fundamental domains for the lattice Γ_0, and the associated dynamical systems are all conjugate to the translation by $\pi(e_{\mathbf{i}})$ on $\mathcal{P}/\Gamma_0$. It is natural to ask whether there is a limit dynamical system; we can give an answer in a special case.

Definition 10. *We will say that a substitution is of the Pisot type if the characteristic polynomial of the associated linear map is irreducible over $\mathbb{Q}$ and has all roots, except one, of modulus strictly smaller than 1.*

In that case, the maximal eigenvalue is a Pisot number (hence the name) and the plane $\mathcal{P}$ is the contracting plane associated to $E_0(\sigma)$. From now on, we will suppose that σ is a unimodular substitution of the Pisot type, and that the modulus of the second largest eigenvalue is $|\lambda'| < L < 1$.

Recall ([Bar88]) that the space $\mathcal{H}(\mathcal{P})$ of all compact subsets on $\mathcal{P}$, equipped with the Hausdorff metric d_H, is a complete metric space. We know that, for a good choice of the norm on $\mathcal{P}$,

$$d_H(E_0(\sigma)A, E_0(\sigma)B) \leq L \, d_H(A, B)$$

and we will use the following easy lemma, whose proof we leave to the reader:

Lemma 7. *For any compact sets A, B, C, D, we have:*

$$d_H(A \cup B, C \cup D) \leq \max(d_H(A, C), d_H(B, D))$$

Proposition 5. *The following limit sets exist in the sense of Hausdorff metric:*

$$X := \lim_{n \to \infty} E_0(\sigma)^n D_n$$
$$X^{(\mathbf{i})} := \lim_{n \to \infty} E_0(\sigma)^n D_n^{(\mathbf{i})},$$
$$X^{(\mathbf{i})'} := \lim_{n \to \infty} E_0(\sigma)^n D_n^{(\mathbf{i})'}.$$

Proof. Let us define

$$c_0 := \max_{\mathbf{i} \in \mathcal{A}} d_H \left(\pi(0, e_{\mathbf{i}}^*), E_0(\sigma) \left(\pi(E_1^*(\sigma)(0, e_{\mathbf{i}}^*)) \right) \right)$$

From the preceding lemma, we see

$$d_H(\pi E_1^*(\sigma)^n \mathcal{U}, E_0(\sigma)\pi E_1^*(\sigma)^{n+1}\mathcal{U}) \le c_0$$

Therefore

$$d_H(E_0(\sigma)^n \pi E_1^*(\sigma)^n \mathcal{U}, E_0(\sigma)^{n+1}\pi E_1^*(\sigma)^{n+1}\mathcal{U}) \le c_0 L^n$$

This means the sequence $(E_0(\sigma)^n \pi E_1^*(\sigma)^n \mathcal{U})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}(\mathcal{P})$. Therefore the limit set X in the sense of Hausdorff metric exists. A similar proof shows that $X^{(\mathbf{i})}$ and $X^{(\mathbf{i})'}$ are well defined. ∎

Lemma 8. *The set $X^{(\mathbf{i})'}$ is the image of $X^{(\mathbf{i})}$ by a translation of $\pi(e_{\mathbf{i}})$*

Proof. This is a consequence of lemma 5: the equality $E_0(\sigma)^n(D_n^{(\mathbf{i})'}) + \pi(e_{\mathbf{i}}) = E_0(\sigma)(D_n^{(\mathbf{i})})$ extends to the limit. ∎

We would like to define a domain exchange on the limit set. However, even if the limit set exists, it is not at all clear that it supports a dynamical system. The reason is that the sets $X^{(\mathbf{i})}$ could have an intersection of non zero measure. In the next section we will prove that, if the substitution has strong coincidences for all letters, this cannot happen; in this section, we will only prove, in the general case, that the set X contains a fundamental domain for Γ_0, the vector of measures $(|X^{(\mathbf{i})}|)$ is an eigenvector for $E_0(\sigma)$, and some translates of the $X^{(\mathbf{i})}$ are pairwise disjoint; this last fact will be of critical importance in the next section.

From now on, we will denote by $|K|$ the measure of a compact set K. The measure does not behave very well with respect to Hausdorff metric, as can be seen from the fact that any compact set is the limit of a sequence of finite sets, of measure 0. It is however easy to prove that, by passing to the limit, the measure can only increase:

Lemma 9. *If K is the limit of a sequence $(K_n)_{n \in \mathbb{N}}$ for the Hausdorff metric, we have $|K| \ge \limsup_{n \to \infty} |K_n|$.*

Proof. Define $K + \epsilon = \{x | d(x, K) \le \epsilon\}$. This is a compact set that contains K. By the definition of Hausdorff metric, for all n large enough, K_n is contained in $K + \epsilon$, which implies that $|K + \epsilon| \ge \limsup_{n \to \infty} |K_n|$. Take a sequence (ϵ_n), decreasing, with limit 0. The sets $K + \epsilon_n$ are a decreasing sequence of compact sets, whose intersection is K. This implies that $|K| = \lim_{n \to \infty} |K + \epsilon_n|$, and we get the result by taking the limit in the first inequality. ∎

This result applies in particular to the sets X and $X^{(\mathbf{i})}$. We can even prove a little more:

Lemma 10. *The set X contains a fundamental domain for Γ_0.*

Proof. It is enough to prove that any element of $\mathcal{P}$ is contained in $X + \gamma$, for some $\gamma \in \Gamma_0$. But for any n, it is contained in $E_0(\sigma)^n D_n + \gamma_n$, because the $E_0(\sigma)^n D_n$ contain a fundamental domain for Γ_0. These domains are uniformly bounded, hence the sequence γ_n, even if it is not constant, can take only a finite number of values. Hence we can find a subsequence n_k such that γ_{n_k} is constant, equal to γ, which implies that the point is contained in $X + \gamma$. ∎

We can also be more precise on the ratio of the pieces. Remark that the vector whose coordinates are $|\pi(0, e_{\mathbf{i}}^*)|$ is a positive eigenvector for the maximal eigenvalue λ. Indeed, let v be a positive eigenvector of norm 1. The area of $\pi(0, e_{\mathbf{i}}^*)$ is also the volume of the parallelepiped with basis $\pi(0, e_{\mathbf{i}}^*)$ and height v; but, since π is the projection along v, this is also the volume of the parallelepiped with basis $(0, e_{\mathbf{i}}^*)$ and height v, that is v_i, hence the result. Since one builds the successive sets $D_n^{(\mathbf{i})}$ by applying the substitution, we see that at each step, the vector of measures $^t(|D_n^{(\mathbf{i})}|) \in \mathbb{R}^d$ stays proportional to a positive eigenvector, and after renormalisation, it is constant. Unfortunately, it is not possible to obtain directly the same result for the limit, since we have only an inequality. We will use a property of primitive matrices which is a corollary of Perron-Frobenius theorem.

Lemma 11. *Let M be a primitive positive matrix, with maximal eigenvalue λ. Suppose that v is a positive vector such that $Mv \geq \lambda v$. Then the inequality is an equality, and v is an eigenvector with respect to λ.*

Proof. (see [Que87], P. 91) We can suppose, by replacing M and λ by a suitable power, that M is strictly positive. Define, for all positive non zero vector v:

$$r(v) = \inf_i \frac{\sum_j m_{i,j} v_j}{v_i}$$

The function r is homogeneous of degree 0, so we can consider it as defined on the simplex $\sum v_i = 1$, $v_i \geq 0$. By continuity, it is bounded; let θ be its upper bound. There is some v such that $r(v) = \theta$, hence $Mv \geq \theta v$. Let w denote the vector $Mv - \theta v$; if w is not zero, since it is positive, Mw is strictly positive; but then, we have $M(Mv) > \theta Mv$, and Mv contradicts the definition of θ. We conclude that v is a positive eigenvector for θ, hence θ is the Perron eigenvalue λ, and we just proved that the inequality $Mv \geq \lambda v$ implies the equality. ∎

Lemma 12. *The vector of volumes $^t\left(|X^{(\mathbf{i})}|\right)_{\mathbf{i} \in \mathcal{A}}$ satisfies the following inequality:*

$$E_0(\sigma) \begin{pmatrix} |X^{(\mathbf{1})}| \\ \vdots \\ |X^{(\mathbf{d})}| \end{pmatrix} \geq \lambda \begin{pmatrix} |X^{(\mathbf{1})}| \\ \vdots \\ |X^{(\mathbf{d})}| \end{pmatrix}$$

Proof. From the definition of $X^{(\mathbf{i})'}$ and $E_1^*(\sigma)$, we see

$$E_0(\sigma)^{-1}X^{(\mathbf{i})'} = E_0(\sigma)^{-1} \lim_{n\to\infty} E_0(\sigma)^{n+1}\pi\left(E_1^*(\sigma)^{n+1}(0, e_{\mathbf{i}}^*)\right)$$

$$= \lim_{n\to\infty} E_0(\sigma)^n\pi\left(E_1^*(\sigma)^n\left(\bigcup_{\mathbf{j}\in\mathcal{A}}\bigcup_{W_k^{(\mathbf{j})}=\mathbf{i}}\left(E_0(\sigma)^{-1}(-f(P_k^{(\mathbf{j})}), e_{\mathbf{j}}^*)\right)\right)\right)$$

$$= \bigcup_{\mathbf{j}\in\mathcal{A}}\bigcup_{W_k^{(\mathbf{j})}=\mathbf{i}}\left(-E_0(\sigma)^{-1}f(P_k^{(\mathbf{j})}) + \lim_{n\to\infty} E_0(\sigma)^n\pi\left(E_1^*(\sigma)^n(0, e_{\mathbf{j}}^*)\right)\right)$$

$$= \bigcup_{\mathbf{j}\in\mathcal{A}}\bigcup_{W_k^{(\mathbf{j})}=\mathbf{i}}\left(-E_0(\sigma)^{-1}f(P_k^{(\mathbf{j})}) + X^{(\mathbf{j})'}\right).$$

In a similar way, we obtain:

$$E_0(\sigma)^{-1}X^{(\mathbf{i})} = \bigcup_{\mathbf{j}\in\mathcal{A}}\bigcup_{W_k^{(\mathbf{j})}=\mathbf{i}}\left(E_0(\sigma)^{-1}f(S_k^{(\mathbf{j})}) + X^{(\mathbf{j})}\right)$$

Therefore, if we denote by $(m_{\mathbf{i},\mathbf{j}})$ the matrix of $E_0(\sigma)$, we have

$$|E_0(\sigma)^{-1}X^{(\mathbf{i})}| \le \sum_{\mathbf{j}} m_{\mathbf{i},\mathbf{j}}|X^{(\mathbf{j})}|$$

On the other hand, we know that, since the determinant of $E_0(\sigma)^{-1}$ restricted to $\mathcal{P}$ is λ, the volume $|E_0(\sigma)^{-1}X^{(\mathbf{i})}|$ is equal to $\lambda|X^{(\mathbf{i})}|$, hence the conclusion. ∎

Corollary 2. *The vector of volumes* ${}^t(|X^{(\mathbf{i})}|)_{\mathbf{i}\in\mathcal{A}}$ *is an eigenvector with respect to* λ. *Moreover for each* $\mathbf{i}, \mathbf{j}$, *the sets*

$$(E_0(\sigma)^{-1}f(S_k^{(\mathbf{j})}) + X^{(\mathbf{j})}), \mathbf{j} \in \mathcal{A}$$

such that $W_k^{(\mathbf{j})} = i$ *are pairwise disjoint, up to a set of measure 0, as are the sets*

$$(-E_0(\sigma)^{-1}f(P_k^{(\mathbf{j})}) + X^{(\mathbf{j})'}), \mathbf{j} \in \mathcal{A}$$

Proof. This is a direct consequence of lemmas 11 and 12. The disjointness follows from the fact that the measure of the union, in lemma 12, is equal to the sum of the measures. ∎

6   Substitutions with strong coincidences

As we just saw, the difficult question is whether the sets $X^{(\mathbf{i})}$ are disjoint of each other, up to a set of measure zero, and form a partition of X. This is unknown in the general case; however, the last corollary gives us a sufficient combinatorial condition.

Definition 11. *We say that the substitution* σ *has a positive strong coincidence of order 1 for* $\mathbf{i}, \mathbf{j} \in \mathcal{A}$, *if there exists an integer* k *such that* $W_k^{(\mathbf{i})} = W_k^{(\mathbf{j})}$, *and* $f(P_k^{(\mathbf{i})}) = f(P_k^{(\mathbf{j})})$.

The substitution has positive strong coincidence of order n for $\mathbf{i}, \mathbf{j}$ if σ^n has a positive strong coincidence of order 1 for $\mathbf{i}, \mathbf{j}$. It has positive strong coincidence for $\mathbf{i}, \mathbf{j}$ if, for some integer n, it has a positive strong coincidence of order n for $\mathbf{i}, \mathbf{j}$.

The substitution has positive strong coincidences for all letters if, for all pairs $\mathbf{i}, \mathbf{j}$ of letters, it has a positive strong coincidence for $\mathbf{i}, \mathbf{j}$.

Remark 4. *A weaker notion of coincidence was defined by Dekking, see [Dek78]: a substitution σ has a coincidence of order 1 for $\mathbf{i}$ and $\mathbf{j}$ if there exists k such that $W_k^{(\mathbf{j})} = W_k^{(\mathbf{i})} = \mathbf{a}$. The coincidence of order n is defined similarly.*

In the case of a substitution of constant length n, this notion of coincidence is preserved by the substitution, since the images of the prefixes $P_k^{(\mathbf{i})}$ and $P_k^{(\mathbf{j})}$ have same length $n(k-1)$, and they are both followed by $\sigma(\mathbf{a})$; this is the main purpose of the definition. To obtain the same result for a substitution of non constant length, it is not enough to have a coincidence. We need to add a condition on the prefixes, to ensure that $\sigma(P_k^{(\mathbf{i})})$ and $\sigma(P_k^{(\mathbf{j})})$ have the same length.

It is worth remarking that a unimodular primitive substitution can never be of constant length: if this were the case, the length n would be the maximal eigenvalue, and n would divide the determinant.

Remark that, if σ has a positive strong coincidence of order n for $\mathbf{i}, \mathbf{j}$, it has positive strong coincidence of order n' for $\mathbf{i}, \mathbf{j}$ for all $n' > n$; indeed, it is clear that, if $f(V) = f(W)$, then $\sigma(V)$ and $\sigma(W)$ have same length, and same image by f, so there is a positive strong coincidence at the next order. So, if the substitution σ has positive strong coincidences for all letters, there is a power of σ that has positive strong coincidences of order 1 for all letters.

The easiest example of a substitution that has positive strong coincidences for all letters is a substitution such that all the images of the letters begin with the same letter.

Theorem 2. *Suppose that the unimodular primitive substitution σ of the Pisot type has positive strong coincidences for all letters. Then the sets $X^{(\mathbf{i})}$ (resp. $X^{(\mathbf{i})'}$) are disjoint, up to a set of measure 0, and X contains a measurable fundamental domain for the action on $\mathcal{P}$ of the lattice Γ_0. The dynamical system $T : X \to X$, such that*

$$T(x) = x - \pi(e_{\mathbf{i}}) \quad \text{if } x \in X^{(\mathbf{i})'}$$

is well-defined, and has σ-structure of all orders with respect to the partition $X^{(\mathbf{i})'}$ and its images by $E_0(\sigma)^k$, $k \in \mathbb{N}$.

The system (X, T) is conjugate, by the symbolic dynamics with respect to partition $X^{(\mathbf{i})'}$, to the symbolic system (Ω, S), and it is a finite extension of the translation by $-\pi(e_{\mathbf{i}})$ on $\mathcal{P}/\Gamma_0$. The semi-conjugacy map from Ω to $\mathcal{P}/\Gamma_0$ is continuous.

Proof. As we remarked above, we can suppose that some power σ^n of σ has positive strong coincidences of order 1 for all letters; replacing σ by σ^n does not change the eigenvector, and amounts to replacing the sequence (D_n) and the corresponding partitions by a subsequence: this does not change the limit X, so from now on we will suppose that σ has positive strong coincidences of order 1 for all letters.

The corollary 2 shows that, if σ has a positive strong coincidence of order 1 for $\mathbf{i}, \mathbf{j}$, $X^{(\mathbf{i})'} \cap X^{(\mathbf{j})'}$ is of measure 0.

Since X is the disjoint (up to a set of measure 0) union of the $X^{(\mathbf{i})'}$, and is the union of the $X^{(\mathbf{i})}$, it follows that the $X^{(\mathbf{i})}$ are also disjoint up to a set of measure 0.

In that case, we can define the dynamical system T; this system preserves Lebesgue measure. It has, by construction, σ-structure of all orders with respect to the partition $X^{(\mathbf{i})}$ and its images by $E_0(\sigma)^k$, $k \in \mathbb{N}$. This implies, as we saw in section 4, that it is measurably conjugate (using the symbolic dynamics with respect to partition $X^{(\mathbf{i})}$) to the system generated by the substitution σ, because the sequence of partitions $\left\{ T^k A_n^{(\mathbf{i})} \right\}$, being translates of images by $E_0(\sigma)^k$ of the initial one, has a diameter going to 0, hence it is generating.

The set X projects onto the torus $\mathcal{P}/\Gamma_0$, since, by lemma 10, it contains a fundamental domain; after projection, the vectors $\pi(e_{\mathbf{i}})$ become identified, since by definition they differ by an element of Γ_0, hence the transformation T is semi-conjugate to a toral translation.

The continuity of the map from Ω to the torus comes from the fact that a primitive substitution is bilaterally recognizable (cf. [Mos92]); from the initial part of a word u, it is possible to determine n_k such that u is contained in $S^{n_k} \sigma^k(\Omega)$, hence the corresponding point must be contained in $\pi(T^{-n_k} E_0(\sigma)^k(X))$; since the diameter of $E_0(\sigma)^k(X)$ goes to 0, and the translation is an isometry, we get the result (the analogous result for (X, T) could be false, because T is not continuous, hence the diameter of $T^{-n} E_0(\sigma)^k X$ has no reason to tend to 0). $\blacksquare$

Remark 5. *In all the examples we know, the projection from X to the torus $\mathcal{P}/\Gamma_0$ is almost everywhere one-to-one. Since X is compact, we can prove that this projection is always finite-to-one. Furthermore, the number of preimages of a point of the torus is almost everywhere invariant by the translation (it is constant on all orbits whose preimages do not belong to the intersection of two sets $X^{(\mathbf{i})'}$, and this is a set of full measure), hence, by ergodicity of the translation, it is almost everywhere constant.*

One should like to prove that it is almost everywhere one. We do not know a general proof of this fact; however, one can show that, if the projection is not one-to-one, every point of X also belong to some $X + \gamma$, for some non-zero $\gamma \in \Gamma_0$.

If there exists a set K, with non-empty interior, such that $K \subset E_0(\sigma)^n(D_n)$ for all large enough n, this is impossible, since interior points of K cannot belong to translates of $E_0(\sigma)^n(D_n)$ by the lattice; in that case, we can conclude that the substitution system is in fact measurably conjugate to a toral translation.

Remark 6. *This is not the only system we can define on X: there is also $E_0(\sigma)$, but this is not interesting since it is strictly contracting, and its "inverse" A defined by:*

$$A(x) = E_0(\sigma)^{-1}(x + f(P_k^{(\mathbf{j})})) \quad \text{if } x \in X^{(\mathbf{i})'} \text{ and } E_0(\sigma)^{-1}x \in X^{(\mathbf{j})'} + f(P_k^{(\mathbf{j})}).$$

This is a Markov endomorphism, with matrix structure $E_0(\sigma)$.

It is also possible to consider the action of $E_0(\sigma)$ on $\mathbb{R}^d/\mathbb{Z}^d$; we obtain a toral automorphism, and, in the case where the set X is a fundamental domain for the lattice Γ_0, it is not difficult to show that the partition $\left\{ X^{(\mathbf{i})} \right\}$ is the basis for a Markov partition associated to this automorphism. The map A we just considered is the restriction, to the subset X of the stable leaf of 0, of the inverse of this Markov automorphism.
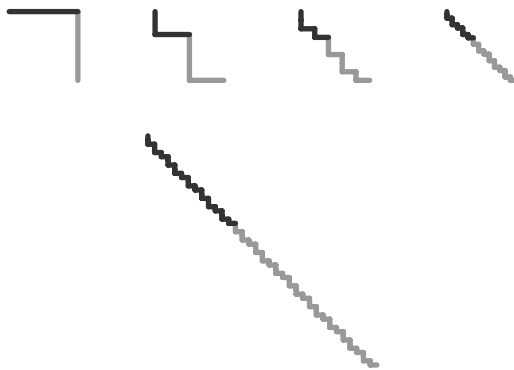
Figure 7: The figure of $E_1^*(\sigma)^n(0, e_{\mathbf{i}}^*)$ for $\sigma = \sigma_{\mathbf{1}}\sigma_{\mathbf{2}}$, $\mathbf{i} = \mathbf{1}, \mathbf{2}$, $n = 0, 1, 2, 3, 4$

This gives us a way to build explicit Markov partitions associated to a Pisot toral automorphism, taking an associated substitution that has strong coincidences for all letters (up to a power, it is always possible to find such a substitution). It remains to understand which substitutions will give good topological properties to the partition (see [FI98]).

Remark 7. *We can of course define negative strong coincidences, using suffixes instead of prefixes; all the proofs work in the same way for negative strong coincidences (we prove first in that case that the $X^{(\mathbf{i})}$, instead of the $X^{(\mathbf{i})'}$, are disjoint, and everything follows)*

7 Examples

In this section, we provide 6 examples; other examples can be found in [FI98], [IO93].

Example 1. *(sturmian substitutions) Let $\sigma_{\mathbf{1}}$ be the substitution $\mathbf{1} \to \mathbf{1}$, $\mathbf{2} \to \mathbf{21}$, and $\sigma_{\mathbf{2}}$ be the substitution $\mathbf{1} \to \mathbf{12}$, $\mathbf{2} \to \mathbf{2}$. Let σ be any product of $\sigma_{\mathbf{1}}$ and $\sigma_{\mathbf{2}}$, where both substitutions occur at least once. It is easy to prove that σ is primitive and unimodular, hence it is Pisot since it cannot have an eigenvalue of modulus 1, and it has negative strong coincidence of order 1 because the images of both letters ends with the same letter. We can apply in that case the results of the paper. (See Fig. 7)*

A combinatorial study shows that all the domain $D_n^{(\mathbf{i})}$ are interval, and in fact one can prove directly that the renormalized domains $E_0(\sigma)D_n^{(\mathbf{i})}$ do not depend on n; at each step, they consist in two intervals whose length vector is the eigenvector of $E_0(\sigma)$.

It is well known that in that case the fixed point of the substitution is a *sturmian* sequence, corresponding to the coding of a rotation by the 2 intervals $X^{(\mathbf{1})}$, $X^{(\mathbf{2})}$. The result extends to all primitive substitutions in the Sturm monoid, corresponding to substitutions that can be extended to automorphisms of the free group on two generators. All matrices in $SL(2, \mathbb{N})$ can be obtained in this way.

Example 2. *Let σ be the substitution $\mathbf{1} \to \mathbf{112}$, $\mathbf{2} \to \mathbf{21}$. It is easily checked that σ is unimodular, and it has positive strong coincidence of order 2. (See Fig. 8)*

Figure 8: The figure of $E_1^*(\sigma)^n(0, e_{\mathbf{i}}^*)$ for example 2, $\mathbf{i} = \mathbf{1}, \mathbf{2}$, $n = 0, 1, 2, 3, 4$

Here again, we can apply the theorem, but the situation is more complicated. This substitution is not sturmian; we can however prove that the renormalized domains contain a fixed interval, and it corresponds to the coding of a rotation by a set that is not connected.

Example 3. *(Rauzy substitution) Let σ be the Rauzy substitution defined in the introduction, $\mathbf{1} \mapsto \mathbf{12}$, $\mathbf{2} \mapsto \mathbf{13}$, $\mathbf{3} \mapsto \mathbf{1}$. We check easily that σ is a unimodular substitution of the Pisot type that has positive strong coincidences of order 1 for all letters. (See Fig. 9)*

In that case, we can prove stronger results than those given by the theorem: the domains $X^{(\mathbf{i})}$ are simply connected, and one can compute the Hausdorff dimension of their boundary (cf. [Rau82], [IK91]).

Example 4. *(modified Rauzy substitution) Let σ be the substitution $\mathbf{1} \mapsto \mathbf{21}$, $\mathbf{2} \mapsto \mathbf{13}$, $\mathbf{3} \mapsto \mathbf{1}$. The substitution σ is unimodular, of the Pisot type. It has negative strong coincidences of order 2 for all letters. (See Fig. 10)*

We check easily that σ is a unimodular substitution of the Pisot type, since its matrix is the same as for Rauzy substitution. The substitution does not have all strong coincidences of order 1, but its square $\sigma^2 : \mathbf{1} \mapsto \mathbf{1321}$, $\mathbf{2} \mapsto \mathbf{211}$, $\mathbf{3} \mapsto \mathbf{21}$ does, since the images of all the letters end by $\mathbf{1}$. Hence the system is conjugate to an exchange of pieces; the sets $X^{(\mathbf{i})}$ have a complicated structure, and in particular, they seem not to be simply connected, as appears in Fig. 10.

Example 5. *(generalized Rauzy substitution) Define the substitution $\sigma_{\mathbf{i}}$, $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$, by $\sigma_{\mathbf{i}}(\mathbf{i}) = \mathbf{i}$ and $\sigma_{\mathbf{i}}(\mathbf{j}) = \mathbf{ji}$ if $\mathbf{j} \neq \mathbf{i}$. Let σ be a product of the substitutions $\sigma_{\mathbf{1}}, \sigma_{\mathbf{2}}, \sigma_{\mathbf{3}}$, where all these substitutions occur at least once. The system generated by σ is measurably isomorphic to a translation on the torus $\mathbb{T}^2$. (See Fig. 11)*

It is clear that σ is unimodular, as a product of unimodular substitutions, and it has all negative strong coincidences of order 1, since all the images end with the same letter. A case study shows that the renormalized sets $E_0(\sigma)^n D_n$ contain a
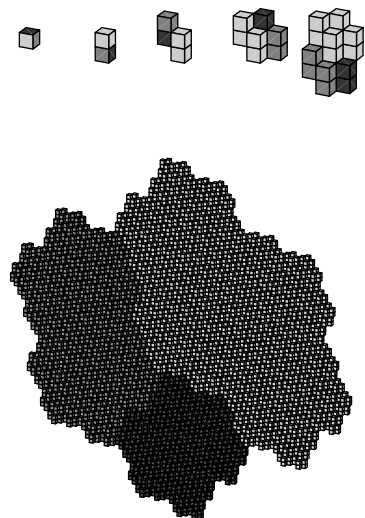
Figure 9: The figure of $E_1^*(\sigma)^n(0, e_{\mathbf{i}}^*)$ for the Rauzy substitution, $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$, $n = 0, 1, 2, 3, 4, 10$
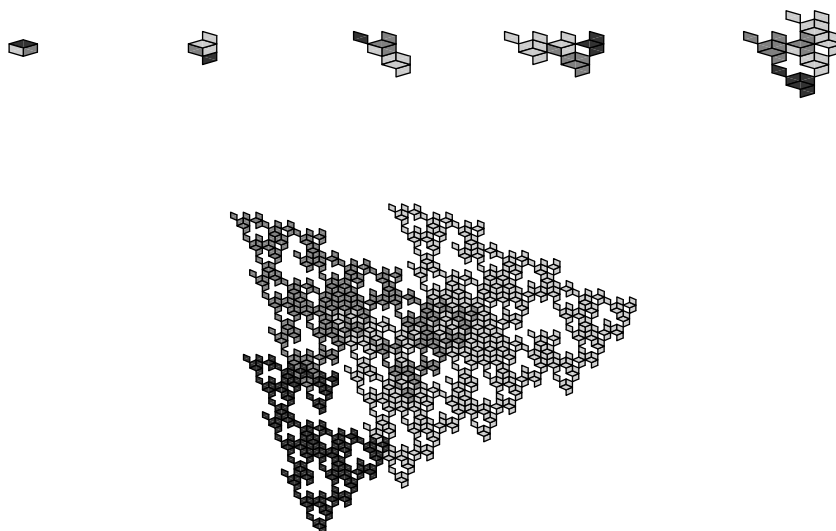
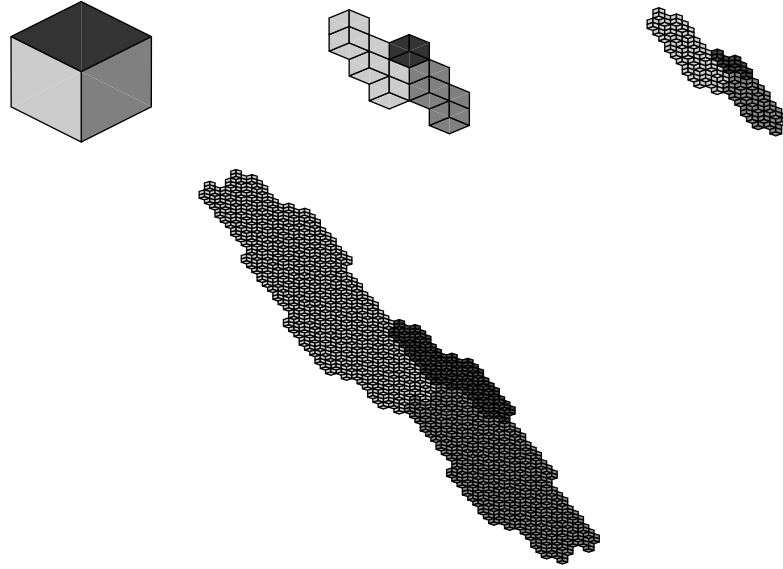Figure 10: The figure of $E_1^*(\sigma)^n(0, e_{\mathbf{i}}^*)$ for the modified Rauzy substitution, $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$, $n = 0, 1, 2, 3, 4, 10$

Figure 11: The figure of $E_1^*(\sigma)^n(0, e_{\mathbf{i}}^*)$ for $\sigma = \sigma_{\mathbf{1}}\sigma_{\mathbf{2}}^2\sigma_{\mathbf{3}}$, $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$, $n = 0, 1, 2, 3$

fixed open set (the basic idea is to prove that, for n large enough, D_n contain a disk of large diameter).

The only thing we have to check is that the substitution is of the Pisot type.

The characteristic polynomial of the matrix M associated to σ is of the type $P(X) = X^3 - \mathrm{Tr}MX^2 + b(M)X - 1$. The trace $\mathrm{Tr}M$ increases when we multiply on the right by one of the matrices associate to the substitutions $\sigma_{\mathbf{i}}$, and its minimum value is 3. Lemma 13 below proves that, for all matrices associated to a non degenerate product of $\sigma_{\mathbf{1}}, \sigma_{\mathbf{2}}, \sigma_{\mathbf{3}}$, we have $|b(M)| < \mathrm{Tr}M$. This implies that $P(-1) = -2 - \mathrm{Tr}M - b(M) < 0$, $P'(-1) = 3 + 2\mathrm{Tr}M + b(M) > 0$, $P(1) = b(M) - \mathrm{Tr}M < 0$, and $P'(1) = 3 - 2\mathrm{Tr}M + b(M) < 0$. Hence P has one real root bigger than 1, and either the other roots are complex conjugate, hence of modulus less than one since the product of all the roots is 1, or the other roots are real and contained in $]-1, 1[$; in each case, the substitution is of the Pisot type. It is associated to a translation on the torus by a vector with coordinates in a cubic field.

Lemma 13. *Let $M_{\mathbf{i}}$ denote the matrix associated to $\sigma_{\mathbf{i}}$. Let M be a product of the matrices $M_{\mathbf{i}}$, where all these matrices occur at least once. We have $|b(M)| < \mathrm{Tr}M$.*

Proof. We denote by P_i the orthogonal projection to the plane $x_i = 0$, and by $\|u\|$ the L^1 norm of the vector u in $\mathbb{R}^3$ (sum of the absolute values of coordinates).

We will show that the function $\mathrm{Tr}M - |b(M)|$ is increasing. We use the following lemma:

Lemma 14. *Let $f_n^{\mathbf{i}}$, $n \in \mathbb{N}$, $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$ be three sequences of vectors in $\mathbb{R}^2$ such that $f_0^{\mathbf{1}} = (0,0), f_0^{\mathbf{2}} = (1,0), f_0^{\mathbf{3}} = (0,1)$, and for each n there exists $\mathbf{i}$ such that $f_{n+1}^{\mathbf{i}} = f_n^{\mathbf{i}}$, $f_{n+1}^{\mathbf{j}} = f_n^{\mathbf{i}} + f_n^{\mathbf{j}}$ if $\mathbf{j} \neq \mathbf{i}$. Then, for each n, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ distinct, we have:*

$$|\det(f_n^{\mathbf{i}}, f_n^{\mathbf{j}} - f_n^{\mathbf{k}})| \leq \|f_n^{\mathbf{i}}\|$$

Proof. The property is true by inspection for $n = 0$. If we suppose $f_{n+1}^{\mathbf{i}} = f_n^{\mathbf{i}}$, we have:

$$|\det(f_{n+1}^{\mathbf{i}}, f_{n+1}^{\mathbf{j}} - f_{n+1}^{\mathbf{k}})| = |\det(f_n^{\mathbf{i}}, f_n^{\mathbf{j}} - f_n^{\mathbf{k}})| \leq \|f_n^{\mathbf{i}}\| = \|f_{n+1}^{\mathbf{i}}\|$$

If $\mathbf{j} \neq \mathbf{i}$, we have

$$
\begin{aligned}
|\det(f_{n+1}^{\mathbf{j}}, f_{n+1}^{\mathbf{i}} - f_{n+1}^{\mathbf{k}})| &= |\det(f_n^{\mathbf{j}} + f_n^{\mathbf{i}}, -f_n^{\mathbf{k}})| \\
&= |\det(f_n^{\mathbf{j}} + f_n^{\mathbf{i}}, f_n^{\mathbf{j}} + f_n^{\mathbf{i}} - f_n^{\mathbf{k}})| \\
&\leq |\det(f_n^{\mathbf{j}}, f_n^{\mathbf{i}} - f_n^{\mathbf{k}})| + |\det(f_n^{\mathbf{i}}, f_n^{\mathbf{j}} - f_n^{\mathbf{k}})| \\
&\leq \|f_n^{\mathbf{i}}\| + \|f_n^{\mathbf{j}}\| \\
&= \|f_{n+1}^{\mathbf{j}}\|
\end{aligned}
$$

(Remark that the vectors are all positive, hence we can add the norms). ∎

End of proof of lemma 13. We compute $\mathrm{Tr}MM_{\mathbf{i}} = \mathrm{Tr}M + \|P_i M e_{\mathbf{i}}\|$. On the other end, we can compute explicitly $b(M) = \sum_{\mathbf{i}<\mathbf{j}} M_{\mathbf{i},\mathbf{i}} M_{\mathbf{j},\mathbf{j}} - \sum_{\mathbf{i}<\mathbf{j}} M_{\mathbf{i},\mathbf{j}} M_{\mathbf{j},\mathbf{i}}$. We obtain the relation:

$$|b(MM_{\mathbf{i}}) - b(M)| = \det(P_i M e_{\mathbf{i}}, P_i M e_{\mathbf{j}} - P_i M e_{\mathbf{k}})$$

We can apply lemma 14 to the vectors $P_i M e_{\mathbf{i}}, P_i M e_{\mathbf{j}}, P_i M e_{\mathbf{k}}$, which clearly satisfy the hypothesis, and we obtain: $|b(MM_{\mathbf{i}}) - b(M)| \leq \mathrm{Tr}MM_{\mathbf{i}} - \mathrm{Tr}M$; hence the function $\mathrm{Tr}M - |b(M)|$ does not decrease if we multiply on the right by $M_{\mathbf{i}}$. This function is 0 for the identity; hence it is nonnegative for all products of the $M_{\mathbf{i}}$.

If one of the substitutions $\sigma_{\mathbf{i}}$ does not occur in the product, we can suppose that, up to permutation, it is $\sigma_{\mathbf{1}}$, and the first line of matrix M is $(1, 0, 0)$, hence 1 is an eigenvalue, and $\mathrm{Tr}M = b(M)$. In that case, one checks that the matrix M has the form:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ a+b-1 & a & b \\ c+d-1 & c & d \end{pmatrix}$$

where $ad - bc = 1$ and, if both substitutions $\sigma_{\mathbf{2}}$ and $\sigma_{\mathbf{3}}$ occur, a, b, c, d are strictly positive integers.

We have $\mathrm{Tr}M.M_{\mathbf{1}} = \mathrm{Tr}M + (a + b + c + d - 2)$, and, by a direct computation, $b(M.M_{\mathbf{1}}) = b(M) + a + b - c - d - 2$. This implies that, for the first nondegenerate matrix in a product, $|b(M)| < \mathrm{Tr}M$; the result follows, since $\mathrm{Tr}M - |b(M)|$ does not decrease by right multiplication by $M_{\mathbf{j}}$. ∎

We remark that, in this case, it is possible to find examples where the characteristic polynomial has 3 real roots, as well as examples with one real and two complex conjugate roots.

Example 6. *Let σ be the substitution $\mathbf{1} \mapsto \mathbf{23}$, $\mathbf{2} \mapsto \mathbf{123}$, $\mathbf{3} \mapsto \mathbf{1122233}$. it is unimodular of the Pisot type, and it has all negative strong coincidences of order 1, since all images end by 3 (See Fig. 12).*

This is an exemple of a non-invertible substitution, that is, this substitution extends to an homomorphism of the free group that is not an automorphism.
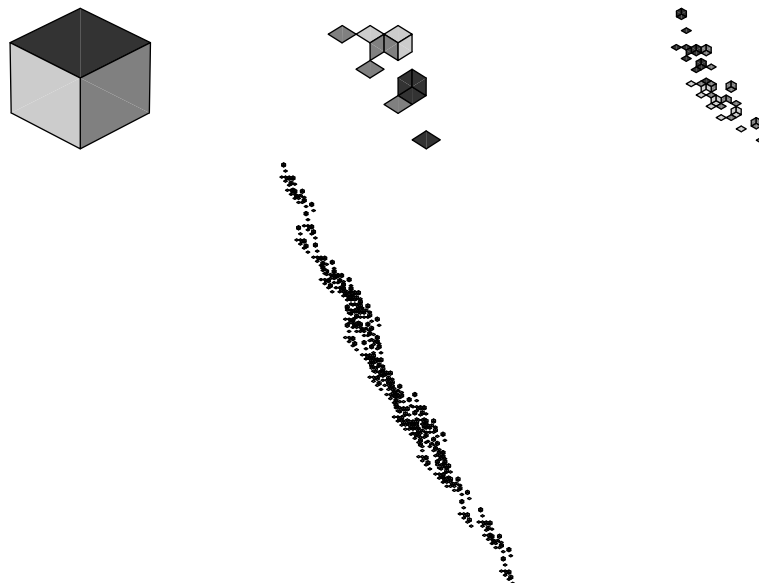
Figure 12: The figure of $E_1^*(\sigma)^n(0, e_{\mathbf{i}}^*)$ for example 6, $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$, $n = 0, 1, 2, 4$

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