Pitt inequalities and restriction theorems for the Fourier transform

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ABSTRACT. We prove new Pitt inequalities for the Fourier transforms with radial and non-radial weights using weighted restriction inequalities for the Fourier transform on the sphere. We also prove new Riemann–Lebesgue estimates and versions of the uncertainty principle for the Fourier transform.

1. Introduction

Weighted inequalities for the Fourier transform provide a natural balance between functional growth and smoothness. On $\mathbb{R}^n$ it is important to determine quantitative comparisons between the relative size of a function and its Fourier transform at infinity. We will let $\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{ix\xi} f(x)\, dx$, $\xi \in \mathbb{R}^n$, be the Fourier transform in $L^1(\mathbb{R}^n)$, and $\|\cdot\|_p$ be the standard norm in $L^p(\mathbb{R}^n)$. We consider Pitt type inequalities

$$(1.1) \qquad \|u^{\frac{1}{q}}\widehat{f}\|_q \le C\|v^{\frac{1}{p}}f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n).$$

Here and throughout the paper, u and v are non-negative measurable functions on $\mathbb{R}^n$, and $1 \le p, q \le \infty$ unless otherwise specified. We will use C to denote numeric constants that may change from line to line. We will let $p' = \frac{p}{p-1}$ be the conjugate exponent of $1 \le p \le \infty$, and we will often let $x = \rho\omega$, with $\omega \in \mathbb{S}^{n-1}$ and $\rho = |x|$. We denote by $|E|$ the Lebesgue measure of a set E and by $\chi_E(x)$ be the characteristic function of E.

In 1983, Heinig [16], Jurkat–Sampson [17] and Muckenhoupt [19], [20] proved

THEOREM 1.1. *Let $n \geq 1$. If the weights u and v satisfy*

$$(1.2) \qquad \sup_{s>0} \left(\int_0^s u^*(t)\, dt \right)^{\frac{1}{q}} \left(\int_0^{\frac{1}{s}} [(1/v)^*(t)]^{\frac{1}{p-1}}\, dt \right)^{\frac{1}{p'}} = C < \infty,$$

for $1 < p \leq q < \infty$, where g^* is the non-increasing rearrangement of g, then (1.1) holds.

To formulate necessary conditions for inequality (1.1) to hold we recall the definition of polar set. If $A \subset \mathbb{R}^n$,

$$A^* = \left\{ \xi \in \mathbb{R}^n \colon |x\xi| \leq 1,\ x \in A \right\}$$

is the polar set of A (see [25, § 4]). We prove the following

THEOREM 1.2. *Let $n \geq 1$. Suppose that the Pitt inequality (1.1) holds for any $f \in C_0^\infty(\mathbb{R}^n)$ and for $1 < p, q < \infty$.*

(1) *Let a convex body $A \subset \mathbb{R}^n$ be centrally symmetric with respect to the origin. Then*

$$(1.3) \qquad \sup_A \left(\int_{cA^*} u(\xi)\, d\xi \right)^{\frac{1}{q}} \left(\int_A v^{1-p'}(x)\, dx \right)^{\frac{1}{p'}} = C < \infty,$$

where $c < \pi/2$ and A^* is a polar set of the set A.

(2) *Let the weights $u(x) = u_0(|x|)$ and $v(x) = v_0(|x|)$ be radial, then*

$$(1.4) \qquad \sup_{s>0} \left(\int_{|x|<s} u(x)\, dx \right)^{\frac{1}{q}} \left(\int_{|x|<\frac{c_n}{s}} v^{1-p'}(x)\, dx \right)^{\frac{1}{p'}} = C < \infty,$$

where c_n is any positive number less than $q_{n/2-1}$, the first zero of the Bessel function $J_{n/2-1}(t)$. In particular, $q_{n/2-1} \geq \pi/2$.

(3) *Results of the part (1) also hold if one replaces the sets A and cA^* by a union of their disjoint translations, that is, by the sets $A_1 = \bigcup_{j=1}^{N_1}(A+x_j)$ and $A_2 = \bigcup_{j=1}^{N_2}(cA^* + \xi_j)$ for any x_j and ξ_j.*

Note that in this theorem we do not assume $q \geq p$.

Part (2) of the theorem is known with a smaller constant c; see the proof of Theorem 3.1 in [16]. Moreover, part (3) generalizes the following necessary condition (see [4, Th. 3]):

$$(1.5) \qquad \left(\int_{Q_1} u(\xi)\, d\xi \right)^{\frac{1}{q}} \left(\int_{Q_2} v^{1-p'}(x)\, dx \right)^{\frac{1}{p'}} = C < \infty,$$

for all cubes Q_1 and Q_2 such that $|Q_1|\,|Q_2| = 1$.

We should also mention [18, Theorem 2.1] where a necessary condition similar to (1.5), with u replaced by a measure $d\mu$, was proved.

When $u(x) = u_0(|x|)$ and $v(x) = v_0(|x|)$ are radial, with $u_0(\cdot)$ non-increasing and $v_0(\cdot)$ non-decreasing, then (1.4) is necessary and sufficient for the validity of (1.1) (see [16]). In particular, if $u(x)$ and $v(x)$ are locally

integrable power weights, i.e., in the form of $u = |x|^b$ and $v = |x|^a$, with $a, b > -n$, we get that the classical Pitt inequality

$$(1.6) \qquad \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^q |\xi|^b \, d\xi \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p |x|^a \, dx \right)^{\frac{1}{p}},$$

where $f \in C_0^\infty(\mathbb{R}^n)$, holds if and only if

$$(1.7) \qquad \frac{a}{p} + \frac{b}{q} = n \left(1 - \frac{1}{p} - \frac{1}{q} \right),$$

$$(1.8) \qquad -n < b \leq 0, \quad \text{and} \quad 0 \leq a < n(p-1);$$

see [**21, 24, 3**].

Pitt type inequalities with power weights that satisfy less restrictive conditions than those in (1.8) are only valid on special subspaces of $L^p(\mathbb{R}^n)$. We have proved in [**11**] that if f is a product of a radial function and a spherical harmonics of degree $k \geq 0$, then (1.1) is satisfied with $u = |x|^a$ and $v = |x|^b$ if and only if a and b satisfy (1.7) and

$$(1.8') \qquad (n-1) \left(\frac{1}{2} - \frac{1}{p} \right) + \max \left\{ \frac{1}{p'} - \frac{1}{q}, 0 \right\} \leq \frac{b}{p} < \frac{n}{p'} + k,$$

which is less restrictive than the conditions in (1.8) even for $k = 0$

In this paper we prove L^p–L^q Pitt inequalities for radial and non-radial weights u and v. Our main tools are weighted restriction inequalities for the Fourier transform in $\mathbb{R}^n$, $n \geq 2$. That is,

$$(1.9) \qquad \left(\int_{\mathbb{S}^{n-1}} |\widehat{f}(\omega)|^q U(\omega) \, d\sigma(\omega) \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) \, dx \right)^{\frac{1}{p}},$$

where U and v are non-negative and measurable on $\mathbb{S}^{n-1}$ and $\mathbb{R}^n$, respectively, and $f \in C_0^\infty(\mathbb{R}^n)$.

We recall several known restriction theorems in Section 2. In Section 3 we present new Pitt inequalities using restriction inequalities. In particular, we prove the following

THEOREM 1.3. *Let* $1 \leq p < \frac{2(n+2)}{n+4}$ *and* $1 \leq q \leq \frac{n-1}{n+1} p'$, *with* $n \geq 2$. *Suppose that* $u(x) = u_0(|x|)$ *satisfies*

$$(1.10) \qquad \int_0^\infty \rho^{n-1-\frac{qn}{p'}} u_0(\rho) \, d\rho < \infty.$$

Then for every $f \in C_0^\infty(\mathbb{R}^n)$

$$(1.11) \qquad \| u^{\frac{1}{q}} \widehat{f} \|_q \leq C \| f \|_p.$$

REMARK 1.1. (i) The proof of Theorem 3.1, of which Theorem 1.3 is a special case, shows that the constant C in (1.11) equals $C' \left(\int_0^\infty \rho^{n-1-\frac{qn}{p'}} u_0(\rho) \, d\rho \right)^{1/q}$, where C' depends on n, p, q.

(ii) When $u \in L^p(\mathbb{R}^n)$ with $1 \le p \le 2$ and $q = 1$, (1.11) is valid also when u is not radial; indeed, by Hausforf–Young inequality,

$$\|u\widehat{f}\|_1 \le \|u\|_p \|\widehat{f}\|_{p'} \le C\|f\|_p.$$

(iii) Theorem 1.1 and most of the Pitt inequalities in the literature are proved for $1 < p \le q < \infty$. Theorem 1.3 provides a rather simple sufficient condition for (1.11) that applies either when $p \le q$ or $p > q$. Note that the known sufficient conditions for (1.11) are usually quite difficult to verify especially in the case $p > q$ (see for example [3]).

Theorem 1.3 applies in cases where Theorem 1.1 does not: In Section 4 we construct a radial weight u for which the inequality (1.2) does not hold, but (1.10) holds for u_0 and therefore (1.11) is valid.

The rest of the paper is organized as follows. In Section 5 we prove necessary conditions for the Pitt inequality (1.1) to hold (Theorem1.1), necessary conditions for the weighted restriction inequality (1.9) to hold (Proposition 2.2), and sufficient conditions from Section 3. These are the main results of the paper.

In Section 6 we prove new versions of the uncertainty principle for the Fourier transform.

In Section 7 we apply our new Pitt's inequality to get a quantitative version of the Riemann–Lebesgue lemma, which provides an interrelation between the smoothness of a function and the growth properties of the Fourier transforms.

Finally, we would like to mention make the following interesting observation which perhaps is not new: the Pitt inequality (1.1) holds if and only if, for some $s \ge p$, we have $\|u^{\frac{1}{q}}\widehat{f}\|_q \le C\|w^{-1}\|_{\frac{sp}{s-p}}^{\frac{1}{s}} \|w^{\frac{1}{s}} v^{\frac{1}{p}} f\|_s$ whenever $w^{-1} \in L^{\frac{p}{s-p}}(\mathbb{R}^n)$. In particular, the inequality $\|\widehat{f}\|_{p'} \le C\|w^{\frac{1}{s}} f\|_s$ holds for every $1 \le p \le 2$ whenever $w^{-1} \in L^{\frac{p}{s-p}}(\mathbb{R}^n)$, $s \ge p$. We will prove this fact in Section 5.

2. Restriction theorems for the Fourier transform

The Tomas–Stein restriction inequality for the Fourier transform on the unit sphere states that, for every $f \in C_0^\infty(\mathbb{R}^n)$, $n \ge 2$,

$$(2.1) \qquad \left(\int_{\mathbb{S}^{n-1}} |\widehat{f}(\omega)|^q \, d\sigma(\omega)\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{\frac{1}{p}},$$

where $d\sigma(\omega)$ is the induced Lebesgue measure on $\mathbb{S}^{n-1}$, $1 \le q \le \frac{n-1}{n+1} p'$, and $1 \le p \le \frac{2(n+1)}{n+3}$ [27, 25].

The same inequality holds also if $d\sigma(\omega)$ is replaced by $\chi(\omega) \, d\sigma(\omega)$, with $\chi \in C_0^\infty(\mathbb{S}^{n-1})$ [25]. So, if $T(f) = \widehat{f}|_{\mathbb{S}^{n-1}}$ is the restriction operator, T maps $L^p(dx)$ into $L^q(d\sigma)$ boundedly when p, q are as in the Tomas–Stein theorem.

Note that (2.1) is trivial when $p = 1$ because

$$\left(\int_{\mathbb{S}^{n-1}} |\widehat{f}(\omega)|^q \, d\sigma(\omega) \right)^{\frac{1}{q}} \leq \omega_{n-1}^{\frac{1}{q}} \|\widehat{f}\|_\infty \leq \omega_{n-1}^{\frac{1}{q}} \|f\|_1,$$

where $\omega_{n-1} = |\mathbb{S}^{n-1}|$.

The restriction conjecture states that inequality (2.1) is valid for all $1 \leq q \leq \frac{n-1}{n+1} p'$ and $1 \leq p < \frac{2n}{n+1}$. When $n = 2$ the restriction conjecture has been proved by C. Fefferman [13]. When $n \geq 3$, T. Tao [28] has proved that (2.1) is valid for $1 \leq p < \frac{2(n+2)}{n+4}$. Note that $\frac{2(n+2)}{n+4} = \frac{2n}{n+1}$ when $n = 2$.

Weighted versions of the restriction inequality (2.1) in the form of

$$(2.2) \qquad \left(\int_{\mathbb{S}^{n-1}} |\widehat{f}(\omega)|^q \, U(\omega) \, d\sigma(\omega) \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p v(x) \, dx \right)^{\frac{1}{p}}$$

have been proved by several authors. In most of the theorems in the literature, $1 \leq p \leq q \leq \infty$ and $U(\omega)$ is the restriction of a function $\widetilde{U}(x) \in C^\infty(\mathbb{R}^n)$, often with compact support.

The following duality argument will be used in the proof of the theorems in the next section. The technique is well known, but we state and prove Lemma 2.1 in this paper for the sake of completeness.

LEMMA 2.1. *Assume* $U(x/|x|) = U(\omega) \in L^1(\mathbb{S}^{n-1})$. *Inequality (2.2) is equivalent to*

$$(2.3) \qquad \left\| \int_{\mathbb{S}^{n-1}} g(\omega) e^{i\omega y} U^{\frac{1}{q}}(\omega) \, d\sigma(\omega) \right\|_{L^{p'}(v^{1-p'} dy)} \leq C \|g\|_{L^{q'}(\mathbb{S}^{n-1})}.$$

In Section 5 we prove necessary conditions for the weighted restriction inequality (2.2) to hold. To the best of our knowledge these results are new.

PROPOSITION 2.2. *Assume that the inequality (2.2) holds with* $U^{1-q'}(\omega) \in L^1(\mathbb{S}^{n-1})$. *Then*

$$(2.4) \qquad \int_{\mathbb{R}^n} v^{1-p'}(x) |j_{n/2-1}(|x|)|^{p'} \, dx < C,$$

where $j_\alpha(t) = \Gamma(\alpha + 1)(t/2)^{-\alpha} J_\alpha(t)$ *is the normalized Bessel function.*

A special case of (2.4) is in [5, (3.1)]. In particular, we obtain the following result.

COROLLARY 2.3. *Assume that the inequality (2.2) holds with* $U^{1-q'}(\omega) \in L^1(\mathbb{S}^{n-1})$; *assume* v *radial and non-negative, and that* $v(x) = v_0(|x|)$ *satisfies either*

$$(2.5) \qquad \int_A v_0^{1-p'}(t - |A|) \, dt \leq C \int_A v_0^{1-p'}(t) \, dt,$$

or

$$(2.6) \qquad \int_A v_0^{1-p'}(t + |A|) \, dt \leq C \int_A v_0^{1-p'}(t) \, dt,$$

for all finite intervals A, with a constant C independent of A. Then

$$\int_{\mathbb{R}^n} v^{1-p'}(x)(1+|x|)^{-\frac{p'(n-1)}{2}} \, dx < C.$$

REMARK 2.1. *If $v_0^{1-p'}$ satisfies a doubling condition, that is,*

$$\int_{2A} v_0^{1-p'}(t) \, dt \le C \int_A v_0^{1-p'}(t) \, dt,$$

for all intervals A, where $2A$ is the interval twice the length of A and with the midpoint coinciding with that of A, then both (2.5) and (2.6) hold. If v_0 is monotonic, then at least one of the conditions (2.5) and (2.6) hold.

Weighted restriction theorems were intensively studied for piecewise power weights, i.e. in the form of

$$(2.7) \qquad\qquad v(x) = \begin{cases} |x|^\alpha, & \text{if } |x| \le 1, \\ |x|^\beta, & \text{if } |x| > 1, \end{cases}$$

see e.g. [**5**]. The method of the proof of [**8**, Cor. 2.8] can be used to prove the following

LEMMA 2.4. *Let $d\mu$ and $d\nu$ be measures on $\mathbb{R}^n$, $n \ge 1$, and let $1 \le p \le q$ and $s \ge p$. An operator T maps $L^p(d\mu) \to L^q(d\nu)$ boundedly if and only T maps $L^s(w \, d\mu) \to L^q(d\nu)$ boundedly whenever $w^{-1} \in L^{\frac{p}{s-p}}(d\mu)$ and*

$$\||T|\|_{L^s(w \, d\mu) \to L^q(d\nu)} \le C \|w^{-1}\|_{L^{\frac{p}{s-p}}(d\mu)}^{\frac{1}{s}}.$$

The proof is in Section 5. If we apply Lemma 2.4 to the restriction operator, with the the Tomas–Stein exponents $s = q = 2$ and $p = \frac{2(n+1)}{n+3}$ we require $w^{-1} \in L^{\frac{n+1}{2}}(\mathbb{R}^n)$. This condition applied to piecewise power weight, allows $\alpha < \frac{2n}{n+1}$ and $\beta > \frac{2n}{n+1}$.

These exponents are not sharp: S. Bloom and G. Sampson have proved in [**5**] a number of restriction theorems with piecewise power weights, and have obtained, in most cases, sharp conditions on α and β. One of the results in [**5**, Thm. 5.6] is the following

THEOREM 2.5. *Let $1 < p \le 2$, $n \ge 2$, $2 \le q \le \frac{n-1}{n+1} p'$. Let $v(x)$ given by (2.7). Then (2.2) with $U = 1$ holds if and only if $\alpha < n(p-1)$ and $\beta \ge 0$. Moreover, (2.2) holds with $p = q = 2$ also when $U = 1$ and $v(x)$ is as in (2.7) with $\alpha < n$ and $\beta > 1$.*

We also notice that weighted restriction theorems have been proved for weights in the Campanato–Morrey spaces: for $0 \le \alpha \le \frac{n}{r}$ and $r \ge 1$, the Campanato–Morrey space $\mathcal{L}^{\alpha,r}$ is defined as

$$\mathcal{L}^{\alpha,r} = \left\{ f \in L^r_{\text{loc}}(\mathbb{R}^n) \colon \|f\|_{r,\alpha} = \sup_{\substack{x \in \mathbb{R}^n \\ \rho > 0}} \rho^\alpha \left(\rho^{-n} \int_{|y-x|<\rho} |f(y)|^r \, dy \right)^{\frac{1}{r}} < \infty \right\}.$$

Note that $\mathcal{L}^{\alpha,\frac{n}{\alpha}} = L^{\frac{n}{\alpha}}(\mathbb{R}^n)$ and $\mathcal{L}^{0,r}(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n)$.

A. Ruiz and L. Vega have proved in [**22**] the following

THEOREM 2.6. *Suppose that* $V \in \mathcal{L}^{\alpha,r}$, *with* $\frac{\alpha}{n} \leq \frac{1}{r} < \frac{2(\alpha-1)}{n-1}$ *and* $\frac{2n}{n+1} < \alpha \leq n$, $n \geq 2$. *Then, the inequality*

$$(2.8) \qquad \left(\int_{\mathbb{S}^{n-1}} |\widehat{f}(\omega)|^2 \, d\sigma(\omega) \right)^{\frac{1}{2}} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^2 V(x) \, dx \right)^{\frac{1}{2}},$$

holds with $C = C'\|V\|_{\alpha,r}^{\frac{1}{2}}$.

In fact, in [**22**] it is proved that

$$\|\widehat{d\sigma} * f\|_{L^2(V)} \leq C'\|V\|_{\alpha,r}\|f\|_{L^2(V^{-1}\mathbb{R}^n)}$$

but we can use Lemma 2.1 to shows that this inequality is equivalent to (2.8). See also [**1**].

Special cases of the restriction inequality in [**22**] are in [**8**] and [**9**]. F. Chiarenza and A. Ruiz have proved in [**9**] a version of (2.8) with special doubling weights; S. Chanillo and E. Sawyer have proved in [**8**, Cor. 2.8], that (2.8) holds when V is in the Fefferman–Phong class F_r, with $r \geq \frac{n-1}{2}$. In particular, (2.8) holds when $V^{-1} \in L^{\frac{n-1}{2}}(\mathbb{R}^n)$.

3. New Pitt inequalities

In this section we obtain new Pitt-type inequalities for the Fourier transforms using restriction inequalities from Section 2.

THEOREM 3.1. *Assume that the restriction inequality* (1.9) *holds for some* $1 \leq p \leq q \leq \infty$. *Let* $w(\rho)$ *be a measurable function for which* $v(\rho x) \leq w(\rho)v(x)$ *for a.e.* $\rho > 0$ *and* $x \in \mathbb{R}^n$. *Suppose that* u *is radial, and* $u(x) = u_0(|x|)$ *satisfies*

$$(3.1) \qquad \int_0^\infty \rho^{n-1-\frac{qn}{p'}} u_0(\rho) w^{\frac{q}{p}}(\rho) \, d\rho < \infty.$$

Then,

$$(3.2) \qquad \left(\int_{\mathbb{R}^n} |\widehat{f}(x)|^q U\left(\frac{x}{|x|}\right) u(x) \, dx \right)^{\frac{1}{q}} \leq C\|v^{\frac{1}{p}}f\|_p.$$

Theorem 1.3 is an easy consequence of Theorem 3.1 (with $U \equiv v \equiv 1$) and the Fefferman–Tao restriction theorem.

In the next section we will show that our theorem can be applied in cases where prior results are not applicable.

Our next result deals with piecewise power weight v defined by (2.7). In order to use Theorem 3.1, we need to find $w(\rho)$ so that $v(\rho x) \leq w(\rho)v(x)$, $\rho > 0$. A straightforward calculation shows that in this case

$$(3.3) \qquad w(\rho) \leq w_0(\rho) := \max\{\rho^{\alpha}, \ \rho^{\beta}\}.$$

Using Theorem 3.1 and weighted restriction inequalities from [**5**] (see Section 3), we have

COROLLARY 3.2. *Let* $1 < p \leq 2$ *and* $2 \leq q \leq \frac{n-1}{n+1} p'$, *with* $n \geq 2$. *Let* v *be a piecewise power weight* $v(x)$ *given by* (2.7) *with* $\alpha < n(p-1)$ *and* $\beta \geq 0$. *Let* u *be a radial weight that satisfies*

$$\int_0^\infty \rho^{n-1-\frac{qn}{p'}} u_0(\rho) w_0^{q/p}(\rho) \, d\rho < \infty,$$

where w_0 *is given by* (3.3). *Then, for every* $f \in C_0^\infty(\mathbb{R}^n)$,

$$(3.4) \qquad \|u^{\frac{1}{q}} \widehat{f}\|_q \leq C \|v^{\frac{1}{p}} f\|_p.$$

REMARK 3.1. This corollary is valid for all piecewise power weights v and exponents p, q for which the restriction theorems in [**5**] hold.

The following result uses weights in a Campanato–Morrey class $\mathcal{L}^{\alpha,r}$ (see Section 2 for a definition).

COROLLARY 3.3. *Let* $V \in \mathcal{L}^{\alpha,r}$, *with* $\frac{2n}{n+1} < \alpha \leq n$ *and* $\frac{\alpha}{n} \leq \frac{1}{r} < \frac{2(\alpha-1)}{n-1}$, $n \geq 2$. *Assume that there exists a measurable function* $w(\rho)$ *for which* $V(\rho x) \leq w(\rho) V(x)$ *for a.e.* $\rho > 0$ *and* $x \in \mathbb{R}^n$, *and that* $u(x) = u_0(|x|)$ *satisfies*

$$(3.5) \qquad \int_0^\infty \rho^{-1} u_0(\rho) w(\rho) \, d\rho < \infty.$$

Then, for every $1 \leq p \leq 2$, *the following weighted Hausforff-Young inequality holds*

$$(3.6) \qquad \|u^{\frac{2}{p'}} \widehat{f}\|_{p'} \leq C \|V^{\frac{2}{p'}} f\|_p \quad f \in C_0^\infty(\mathbb{R}^n).$$

4. Comparison of Theorems 1.1 and 1.3

In this section we give an example of radial weight $u(x) = u_0(|x|)$ that satisfies the conditions of Theorem 1.3 while does not satisfy the conditions (1.2) in Theorem 1.1.

We recall that Theorem 1.3 states the Pitt inequality

$$(4.1) \qquad \|u^{\frac{1}{q}} \widehat{f}\|_q \leq C \|f\|_p$$

holds with $1 \leq q \leq \frac{n-1}{n+1} p'$ and $1 \leq p < \frac{2(n+2)}{n+4}$ whenever $u(x) = u_0(|x|)$ satisfies

$$(4.2) \qquad \int_0^\infty \rho^{-a} u_0(\rho) \rho^{n-1} \, d\rho < \infty, \quad a = \frac{qn}{p'} > 0.$$

On the other hand, when u is radial and $v \equiv 1$, the sufficient condition (1.2) in Theorem 1.1 states that

$$\int_0^s u^*(t) dt \leq C s^{\frac{q}{p'}}.$$

The latter is equivalent to the following condition:

$$(4.3) \qquad \sup_E |E|^{-\frac{q}{p'}} \int_E u \, dx < C,$$

where supremum is taken over all measurable E, $|E| > 0$.

Let now E_0 be a measurable subset of $\mathbb{R}_+$. Consider the radial set $E = \{x \in \mathbb{R}^n : |x| \in E_0\}$. For such set, we can rewrite (4.3) as follows:

$$(4.4) \qquad \int_{E_0} u_0(\rho)\rho^{n-1} \, d\rho \le C|E|^{\frac{q}{p'}} = C \left(\int_{E_0} \rho^{n-1} \, d\rho \right)^{\frac{q}{p'}}.$$

Let $A = \cup_{k=1}^\infty A_k$, where $A_k = (k, k + k^{-n-1})$. Set

$$(4.5) \qquad u_0(\rho)\rho^{n-1} = \sum_{k=1}^\infty k^n \chi_{A_k}(\rho).$$

Then condition (4.2) holds (and so also the Pitt inequality (4.1)) since

$$\int_0^\infty \rho^{-a} u_0(\rho)\rho^{n-1} \, d\rho = \sum_{k=1}^\infty k^n \int_k^{k+k^{-n-1}} \rho^{-a} \, d\rho$$

$$\le \sum_{k=1}^\infty k^n k^{-a} k^{-n-1} = \sum_{k=1}^\infty k^{-1-a} < \infty$$

and $a > 0$.

On the other hand, taking $E_N = \{x \in \mathbb{R}^n : |x| \in \cup_{k=1}^N A_k\}$, we get

$$\int_{E_N} \rho^{n-1} \, d\rho = \sum_{k=1}^N \int_k^{k+k^{-n-1}} \rho^{n-1} \, d\rho \le \sum_{k=1}^\infty (k+1)^{n-1} k^{-n-1} < C.$$

However,

$$(4.6) \qquad \int_{E_N} u_0(\rho)\rho^{n-1} \, d\rho = \sum_{k=1}^N k^n \int_k^{k+k^{-n-1}} d\rho \asymp \ln N.$$

Therefore, (4.4) (and so also (4.3)), do not hold as $N \to \infty$.

It is worthwhile to remark that for the radial weights u, the necessary condition (1.4) for the Pitt inequality (4.2) to hold (see Theorem 1.2) can be written as

$$\sup_{s>0} \left(\int_0^s u_0(\rho)\rho^{n-1} \, d\rho \right)^{\frac{1}{q}} \left(\int_0^{c_n/s} \rho^{n-1} \, d\rho \right)^{\frac{1}{p'}} < C$$

or, equivalently,

$$(4.7) \qquad \sup_{s>0} s^{-a} \int_0^s u_0(\rho)\rho^{n-1} \, d\rho < C,$$

where $a = \frac{qn}{p'} > 0$. For the weight u given by (4.5) it can be easily checked since

$$\int_0^s u_0(\rho)\rho^{n-1}\, d\rho \leq \sum_{k=1}^{[s]+1} k^{-1} \leq 1 + \ln(s+1).$$

This of course implies (4.7) since we only have to consider the case $s \to \infty$.

5. Proofs of the main results

PROOF OF THE THEOREM 1.2. Let us assume that Pitt inequality (1.1) hold.

(1) Following [16], consider the function $f = \chi_A v^{1-p'} \in L^p(v)$. For any set $B \subset \mathbb{R}^n$ we get

$$C\|v^{\frac{1}{p}}f\|_p \geq \|u^{\frac{1}{q}}\widehat{f}\|_q \geq \left(\int_B |\widehat{f}(\xi)|^q u(\xi)\, d\xi\right)^{\frac{1}{q}},$$

where

$$\|v^{\frac{1}{p}}f\|_p = \left(\int_A (v^{1-p'}(x))^p v(x)\, dx\right)^{\frac{1}{p}} = \left(\int_A v^{1-p'}(x)\, dx\right)^{\frac{1}{p}} > 0$$

and

$$|\widehat{f}(\xi)| \geq \left|\int_A v^{1-p'}(x) \cos(x\xi)\, dx\right|, \quad \xi \in B.$$

Let $B = c_n A^*$, where $c_n < \pi/2$ and A^* is polar set of the set A. Then for any $x \in A$ and $\xi \in B$ we have $|x\xi| \leq c_n$ and $\cos(x\xi) \geq \cos c_n > 0$. Therefore,

$$|\widehat{f}(\xi)| \geq \cos c_n \int_A v^{1-p'}(x)\, dx, \quad \xi \in B.$$

Hence,

$$C\left(\int_A v^{1-p'}(x)\, dx\right)^{\frac{1}{p}} \geq \left(\int_B |\widehat{f}(\xi)|^q u(\xi)\, d\xi\right)^{\frac{1}{q}}$$

$$\geq \cos c_n \left(\int_A v^{1-p'}(x)\, dx\right)\left(\int_B u(\xi)\, d\xi\right)^{\frac{1}{q}},$$

or, equivalently,

$$\left(\int_{c_n A^*} u(\xi)\, d\xi\right)^{\frac{1}{q}}\left(\int_A v^{1-p'}(x)\, dx\right)^{1/p'} < C.$$

(2) If both weights u and v are radial, then the function $f = \chi_A v^{1-p'}$ and its Fourier transform are also radial. Moreover, taking the ball $A = sB^n$, we get

$$\widehat{f}(\xi) = \omega_{n-1}\int_A v^{1-p'}(x)j_{n/2-1}(|\xi|x)\, dx.$$

Let $q_{n/2-1}$ be the first zero of the normalized Bessel function $j_{n/2-1}(t)$. Note that $q_{n/2-1} \geq q_{-1/2} = \pi/2$ and $q_{n/2-1} \sim n/2$ for $n \geq 1$. Then $j_{n/2-1}(t) \geq$

$j_{n/2-1}(c_n)$, where $c_n \geq t$ can be taken as follows: $\pi/2 < c_n < q_{n/2-1}$ for $n \geq 2$. The rest of the proof is the same as in (1).

(3) To prove this part, we use ideas similar to [**4**]. In order to consider translations of the sets A and $c_n A^*$ by the vectors x_0 and ξ_0 correspondingly, it is enough to consider the function $g(x) = f(x - x_0)e^{-ix\xi_0}$ so that $|g(x)| = |f(x-x_0)|$ and $|\widehat{g}(\xi)| = |\widehat{g}(\xi-\xi_0)|$. The integral condition (1.3) easily applies to unions of disjoint translations of A and cA^*. $\qquad\square$

PROOF OF LEMMA 2.1. Let $A : L^p(v\,dx) \to L^q(\mathbb{S}^{n-1})$ be the operator, initially defined for all $f \in C_0^\infty(\mathbb{R}^n)$, by $Af(\omega) = \widehat{f}(\omega)U^{\frac{1}{q}}(\omega)$. Duality gives

$$\|Af\|_{L^q(\mathbb{S}^{n-1})} = \sup_{\|g\|_{L^{q'}(\mathbb{S}^{n-1})} \leq 1} \left| \int_{\mathbb{S}^{n-1}} Af(\omega)g(\omega)\,d\sigma(\omega) \right|$$

$$= \sup_{\|g\|_{L^{q'}(\mathbb{S}^{n-1})} \leq 1} \left| \int_{\mathbb{R}^n} f(x)A^*g(x)\,dx \right|,$$

where

(5.1) $$A^*g(x) = \int_{\mathbb{S}^{n-1}} g(\omega)e^{i\omega x}U^{\frac{1}{q}}(\omega)\,d\sigma(\omega).$$

By Hölder's inequality

$$\int_{\mathbb{R}^n} f(x)A^*g(x)\,dx \leq \|v^{\frac{1}{p}}f\|_p \|v^{-\frac{1}{p}}A^*g\|_{p'} = \|v^{-\frac{1}{p}}A^*g\|_{p'}\|f\|_{L^p(v\,dx)}.$$

Therefore, the inequality

$$\|v^{-\frac{1}{p}}A^*g\|_{p'} = \left\| \int_{\mathbb{S}^{n-1}} g(\omega)e^{i\omega x}U^{\frac{1}{q}}(\omega)d\sigma(\omega) \right\|_{L^{p'}(v^{1-p'}\,dx)} \leq C\|g\|_{L^{q'}(\mathbb{S}^{n-1})}$$

implies (2.2). A similar argument shows that the inequality (2.2), or $\|Af\|_{L^q(\mathbb{S}^{n-1})} \leq C\|f\|_{L^p(v\,dx)}$, implies

(5.2) $$\|v^{-\frac{1}{p}}A^*g\|_{p'} \leq C\|g\|_{L^{q'}(\mathbb{S}^{n-1})}.$$

$\qquad\square$

PROOF OF PROPOSITION 2.2. Let A and A^* be defined as in Lemma 2.1. Let $g(\omega) = U^{-\frac{1}{q}}(\omega)$. Clearly, $g \in L^{q'}(\mathbb{S}^{n-1})$, and by (5.1)

$$\|v^{-\frac{1}{p}}A^*g\|_{p'} = A^*g(x) = \int_{\mathbb{S}^{n-1}} e^{i\omega x}\,d\sigma(\omega) = \omega_{n-1}j_{n/2-1}(|x|),$$

(see e.g. [**25**]). From (5.2) it follows that

(5.3) $$\int_{\mathbb{R}^n} v^{1-p'}(x)|j_{n/2-1}(|x|)|^{p'}\,dx \leq C \left(\int_{\mathbb{S}^{n-1}} U^{1-q'}(\omega)\,d\sigma(\omega) \right)^{\frac{p'}{q'}}$$

as required. $\qquad\square$

PROOF OF COROLLARY 2.3. Let $q_k = q_{\alpha,k}$, $k \geq 1$, be the positive zeros of the Bessel function $J_\alpha(t)$ in nondecreasing order. It is known (see e.g. [**29**]) that

$$J_\alpha(t) = C_\alpha t^{-1/2} \left(\cos(t - c_\alpha) + O(t^{-1}) \right)$$

as $t \to +\infty$. This gives $|j_\alpha(t)| \leq C(1 + t)^{-\alpha - 1/2}$, $t \geq 0$, and

$$(5.4) \qquad |j_\alpha(t)| \geq C(1 + t)^{-\alpha - 1/2}, \quad t \in I := [0, \infty) - \bigcup_{k=1}^{\infty} I'_k$$

where $I'_k = (q_k - \varepsilon, q_k + \varepsilon)$ and $\varepsilon = \varepsilon_\alpha > 0$ is chosen so that $I'_k \cap I'_l = \emptyset$ when $k \neq l$. We let $I := \cup_{k=0}^{\infty} I_k$ and $I_k = [a_k, b_k]$, with $I_0 = [0, q_1 - \varepsilon]$ and $I_k = [q_k + \varepsilon, q_{k+1} - \varepsilon]$.

It is well known that $q_k \sim \pi k$, and there exist constants $c_i > 0$, $i = 1, \ldots, 4$, that depend only on $\alpha = n/2 - 1$ so that $c_1 \leq q_{k+1} - q_k \leq c_2$ and, when $k \neq 0$, $c_3 \leq |I_k| = q_{k+1} - q_k - 2\varepsilon \leq c_4$.

Inequalities (5.3) and (5.4) give

$$\int_{|x| \in I} v^{1-p'}(x)(1 + |x|)^{-\frac{p'(n-1)}{2}} \, dx < C.$$

Furthermore,

$$J := \omega_{n-1}^{-1} \int_{\mathbb{R}^n} v^{1-p'}(x)(1 + |x|)^{-\frac{p'(n-1)}{2}} \, dx$$

$$= \int_0^\infty v_0^{1-p'}(t)(1 + t)^{-\frac{p'(n-1)}{2}} t^{n-1} \, dt = \int_{I_0} + \sum_{k=1}^{\infty} \left(\int_{I_k} + \int_{I'_k} \right).$$

Assume that condition (2.5) holds. Then it is clear that

$$\int_{I'_k} v_0^{1-p'}(t) \, dt \leq C \int_{I_k} v_0^{1-p'}(t) \, dt$$

with some constant C. Using this, we get

$$\int_{I'_k} v_0^{1-p'}(t)(1 + t)^{-\frac{p'(n-1)}{2}} t^{n-1} \, dt$$

$$\leq C(1 + b_{k-1})^{-\frac{p'(n-1)}{2}} a_k^{n-1} \int_{I'_k} v_0^{1-p'}(t) \, dt$$

$$\leq C(1 + b_k)^{-\frac{p'(n-1)}{2}} \int_{I_k} v_0^{1-p'}(t) t^{n-1} \, dt$$

$$\leq C \int_{I_k} v_0^{1-p'}(t)(1 + t)^{-\frac{p'(n-1)}{2}} t^{n-1} \, dt,$$

since $b_k = b_{k-1} + |I'_k| + |I_k| \leq b_{k-1} + c \leq C b_{k-1}$.

Thus,

$$J = \int_{I_0} + \sum_{k=1}^{\infty} \left(\int_{I_k} + \int_{I_k'} \right) v_0^{1-p'}(t)(1+t)^{-\frac{p'(n-1)}{2}} t^{n-1} \, dt$$

$$\leq C \sum_{k=0}^{\infty} \int_{I_k} \leq C \int_{|x| \in I} v^{1-p'}(x)(1+|x|)^{-\frac{p'(n-1)}{2}} \, dx < C.$$

If the condition (2.6) is satisfied, the proof is similar.

$\square$

We prove Lemma 2.4 to make the paper self-contained.

PROOF OF LEMMA 2.4. Assume $s > p$, since the proof in the other case is similar. Let $r = \frac{p}{s-p}$. Suppose that $T \colon L^p(d\mu) \to L^q(d\nu)$ is bounded. To show that $T \colon L^s(wd\mu) \to L^q(d\nu)$ is bounded, we observe that $\frac{1}{rs} = \frac{s-p}{sp} = \frac{1}{p} - \frac{1}{s}$. By Hölder's inequality,

$$\|Tf\|_{L^q(d\mu)} \leq C\|f\|_{L^p(d\mu)} = C\|w^{-\frac{1}{s}} w^{\frac{1}{s}} f\|_{L^p(d\mu)}$$

$$\leq C\|w^{-\frac{1}{s}}\|_{L^{rs}(d\mu)} \|w^{\frac{1}{s}} f\|_{L^s(d\mu)} = C\|w^{-1}\|_{L^r(d\mu)}^{\frac{1}{s}} \|w^{\frac{1}{s}} f\|_{L^s(d\mu)},$$

as required.

To prove the other direction we argue as [8] and as in the proof of Proposition 1.10 in [5]. Observe that

$$\|w^{\frac{1}{s}} f\|_{L^s(d\mu)}^s = \int_{\mathbb{R}^n} w|f(x)|^s \, d\mu(x) = \int_{\mathbb{R}^n} |f(x)|^p \, d\mu(x)$$

with $w = |f|^{p-s}$. Since

$$\|w^{-1}\|_{L^r(d\mu)}^{\frac{1}{s}} = \left(\int_{\mathbb{R}^n} |f(x)|^p \, d\mu(x) \right)^{\frac{s-p}{sp}} = \|f\|_{L^p(d\mu)}^{1-\frac{p}{s}},$$

we obtain

$$\|Tf\|_{L^q(d\nu)} \leq C\|w^{-1}\|_{L^r(d\mu)}^{\frac{1}{s}} \|f\|_{L^p(d\mu)}^{\frac{p}{s}} = C\|f\|_{L^p(d\mu)}^{1-\frac{p}{s}} \|f\|_{L^p(d\mu)}^{\frac{p}{s}}$$

$$= C\|f\|_{L^p(d\mu)}.$$

$\square$

PROOF OF THE THEOREM 3.1. Fix $\rho > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$; let $\delta_\rho \psi(x) = \psi(\rho x)$, and let $g(x) = \rho^{-n} \delta_{\frac{1}{\rho}} f(x)$. We apply (1.9) with g in place of f. Recalling that $\rho^{-n} \widehat{\delta_{\frac{1}{\rho}} f} = \delta_\rho \widehat{f}$, we obtain by Lemma 2.4 and (1.9)

with $d\nu = U\,d\omega$ and $d\mu = v\,dx$:

$$\left(\int_{\mathbb{S}^{n-1}} |\delta_\rho \widehat{f}(\omega)|^q U(\omega) d\sigma(\omega)\right)^{\frac{1}{q}} = \left(\int_{\mathbb{S}^{n-1}} |\widehat{g}(\omega)|^q U(\omega)\, d\sigma(\omega)\right)^{\frac{1}{q}}$$

$$\leq C\|g\|_{L^p(v\,dx)} = C\rho^{-n}\|v^{\frac{1}{p}}\delta_{\frac{1}{\rho}}f\|_p$$

$$= C\rho^{-n+\frac{n}{p}}\|(\delta_\rho v)^{\frac{1}{p}}f\|_p.$$

By our assumptions on v we obtain

$$(5.5) \qquad \int_{\mathbb{S}^{n-1}} |\widehat{f}(\rho\omega)|^q U(\omega)\, d\sigma(\omega) \leq C\rho^{-\frac{nq}{p'}} w^{\frac{q}{p}}(\rho)\|v^{\frac{1}{p}}f\|_p^q.$$

We multiply both sides of this inequality by $u_0(\rho)\rho^{n-1}$ and we integrate with respect to ρ. We obtain

$$\int_0^\infty \rho^{n-1}\int_{\mathbb{S}^{n-1}} |\widehat{f}(\rho\omega)|^q u_0(\rho)U(\omega)\,d\sigma(\omega)\,d\rho$$

$$\leq C\int_0^\infty \rho^{n-1-\frac{nq}{p'}} u_0(\rho)w^{\frac{q}{p}}(\rho)\,d\rho\,\|v^{\frac{1}{p}}f\|_p^q$$

which by (3.1) implies $\int_{\mathbb{R}^n} U(\frac{x}{|x|})u(x)|\widehat{f}(x)|^q\,dx \leq C\|v^{\frac{1}{p}}f\|_p^q.$ $\qquad\square$

PROOF OF COROLLARY 3.3. When $p = q = 2$, we use Theorem 2.6. The assumptions of Theorem 3.1 are satisfied, and so the following inequality holds:

$$(5.6) \qquad \|\widehat{f}\|_{L^2(u\,dy)} \leq \|f\|_{L^2(V\,dx)}.$$

To conclude the proof of Corollary 3.3 we use a special case of an interpolation theorem with change of measure proved in [26].

LEMMA 5.1. *Let Tf be a linear operator defined in a space of measurable functions that include $L^{p_1}(V_1 dx)$ and $L^{p_2}(V_2 dx)$; assume that*

$$\|Tf\|_{L^{q_1}(u_1\,dy)} \leq C\|f\|_{L^{p_1}(V_1\,dx)} \quad \text{and} \quad \|Tf\|_{L^{q_2}(u_2\,dy)} \leq C\|f\|_{L^{p_2}(V_2\,dx)}.$$

Then, for every $0 \leq t \leq 1$,

$$(5.7) \qquad \|Tf\|_{L^{q_t}(u_1^t u_2^{1-t}\,dy)} \leq C\|f\|_{L^{p_t}(V_1^t V_2^{1-t}\,dx)}$$

where $\frac{1}{p_t} = \frac{t}{p_1} + \frac{1-t}{p_2}$ and $\frac{1}{q_t} = \frac{t}{q_1} + \frac{1-t}{q_2}$.

We apply Lemma 5.1 with $Tf = \widehat{f}$; we interpolate the inequality (5.6) and the $\|\widehat{f}\|_\infty \leq \|f\|_1$; we let $u = u_1$ and $V = V_1$, and $u_2 = V_2 = 1$; we let $\frac{1}{p_t} = \frac{t}{2} + 1 - t = 1 - \frac{t}{2}$, so that $t = 2\left(1 - \frac{1}{p_t}\right) = \frac{2}{p_t'}$. Note that $q_t = p_t'$. By (5.7), we have

$$\|\widehat{f}\|_{L^{p'}(u^{\frac{2}{p'}}\,dy)} \leq \|f\|_{L^p(V^{\frac{2}{p'}}\,dx)}$$

where we have let $p = p_t$ for simplicity. That concludes the proof of the corollary. $\qquad\square$

6. Applications to the uncertainty principle

The uncertainty principle is a cornerstone in quantum physics and in Fourier Analysis. The simplest formulation of the uncertainty principle in harmonic analysis is *Heisenberg's inequality*, which applies to functions in $L^2(\mathbb{R}^n)$ of norm $= 1$. It states that the product of the variances of f and $\widehat{f}$ is bounded above by a universal constant, i.e.

$$\inf_{a\in\mathbb{R}^n}\int_{\mathbb{R}^n}|x-a|^2|f(x)|^2\,dx\,\inf_{b\in\mathbb{R}^n}\int_{\mathbb{R}^n}|\xi-b|^2|\widehat{f}(\xi)|^2\,d\xi\geq\frac{(2\pi)^n n^2}{4}.$$

One of the many consequences of this inequality is that a nonzero function and its Fourier transform cannot both be compactly supported.

The uncertainty principle for L^p functions is also interesting. Inequalities in the form of $\|f\|_2^2\leq C\|v^{\frac{1}{p}}f\|_p\|w^{\frac{1}{q}}\widehat{f}\|_q$, where v and w are suitable weight functions and $1\leq p,\ q\leq\infty$ are discussed in [**10**]. Power weights are of particular interest: using a standard homogeneity argument, is easy to prove that a necessary condition for the inequality $\|f\|_2^2\leq C\||x|^a f\|_p\||\xi|^b\widehat{f}\|_q$ to hold for all $f\in C_0^\infty(\mathbb{R}^n)$ is that $a+\frac{n}{p}=b+\frac{n}{q}$. See also [**14**] for a survey on uncertainty principle.

We prove the following

THEOREM 6.1. *Let $u,\ v$ be weights for which the Pitt inequality (1.1) holds for some $1\leq p,q\leq\infty$. Then, for every $f\in C_0^\infty(\mathbb{R}^n)$,*

$$\|f\|_2^2\leq C\big\|u^{-\frac{1}{q}}|\xi|\widehat{f}\big\|_{q'}\big\|v^{\frac{1}{p}}|x|f\big\|_p,$$

where C is independent of f.

COROLLARY 6.2. *Let $1\leq p<\frac{2(n+2)}{n+4}$ and $1\leq q\leq\frac{n-1}{n+1}p'$. Let $s(x)=s_0(|x|)$ be a radial weight that satisfies*

$$(6.1)\qquad\int_0^\infty\frac{\rho^{n-1-\frac{qn}{p'}}}{s_0(\rho)}\,d\rho<\infty.$$

Then,

$$(6.2)\qquad\|f\|_2^2\leq C\big\|s_0^{\frac{1}{q}}(|\xi|)|\xi|\,\widehat{f}\big\|_{q'}\big\||x|f\big\|_p,\quad f\in C_0^\infty(\mathbb{R}^n).$$

For example, $s_0(\rho)=\rho^{-m}(1+\rho)^{m+n-\frac{nq}{p'}+\varepsilon}$, with $\varepsilon>0$, and $m+n-\frac{qn}{p'}>0$, satisfies (6.1).

COROLLARY 6.3. *Let*

$$v(x)=\begin{cases}|x|^\alpha,&|x|\leq 1,\\|x|^\beta,&|x|>1,\end{cases}\quad\text{and}\quad w_0(\rho)=\max\{\rho^\alpha,\ \rho^\beta\},$$

Let $1 < p \le 2$, $2 \le q \le \frac{n-1}{n+1} p'$, $\alpha < n(p-1)$, *and* $\beta \ge 0$. *We have*

$$(6.3) \qquad \|f\|_2^2 \le C \big\| s_0^{\frac{1}{q}}(|\xi|)|\xi| \, \widehat{f} \big\|_{q'} \big\| |x| v^{\frac{1}{p}} f \big\|_p, \quad f \in C_0^\infty(\mathbb{R}^n).$$

provided

$$(6.4) \qquad \int_0^\infty \frac{\rho^{n-1-\frac{qn}{p'}} w_0^{\frac{q}{p}}(\rho)}{s_0(\rho)} \, d\rho < \infty.$$

When $\alpha < n$ *and* $\beta > 1$, *we have*

$$(6.5) \qquad \|f\|_2^2 \le C \big\| s_0^{\frac{1}{2}}(|\xi|)|\xi| \widehat{f} \big\|_2 \big\| |x| v^{\frac{1}{2}} f \big\|_2, \quad f \in C_0^\infty(\mathbb{R}^n),$$

provided

$$\int_0^\infty \frac{w_0(\rho)}{\rho \, s_0(\rho)} \, d\rho < \infty.$$

PROOF OF THEOREM 6.1. We use the same idea of the proof of the L^2 Heisenberg principle (see [**14**]). Let $f \in C_0^\infty(\mathbb{R}^n)$. We denote $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ by (x_1, x'), with $x' \in \mathbb{R}^{n-1}$. We integrate by parts the function $|f(x)|^2 = |f(x_1, x')|^2$ with respect to x_1. That is,

$$\int_{-\infty}^\infty |f(x_1, x')|^2 \, dx_1 = x_1 |f(x_1, x')|^2 \big|_{x_1=-\infty}^\infty - \int_{-\infty}^\infty x_1 \frac{\partial |f(x_1, x')|^2}{\partial x_1} \, dx_1.$$

A simple calculation shows that

$$\frac{\partial |f(x_1, x')|^2}{\partial x_1} = \frac{\partial}{\partial x_1} \left(f(x_1, x') \overline{f(x_1, x')} \right) = 2 \operatorname{Re} \left(\overline{f(x_1, x')} \frac{\partial f(x_1, x')}{\partial x_1} \right).$$

We obtain

$$\int_{-\infty}^\infty |f(x_1, x')|^2 \, dx_1 = -2 \operatorname{Re} \int_{-\infty}^\infty x_1 \overline{f(x_1, x')} \frac{\partial f(x_1, x')}{\partial x_1} \, dx_1.$$

We integrate the above identity in x', to obtain

$$\|f\|_2^2 = -2 \operatorname{Re} \int_{\mathbb{R}^n} x_1 \overline{f(x)} \frac{\partial f(x)}{\partial x_1} \, dx.$$

We use the identity $\int_{\mathbb{R}^n} f_1 \overline{f_2} \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f_1} \, \overline{\widehat{f_2}} \, d\xi$, and we recall that the Fourier transform of $\frac{\partial f(x)}{\partial x_1}$ is $-i\xi_1 \widehat{f}(\xi)$. Thus,

$$\|f\|_2^2 = 2(2\pi)^{-n} \operatorname{Re} \left(i \int_{\mathbb{R}^n} \xi_1 \widehat{f}(\xi) \, \overline{(\widehat{x_1 f})(\xi)} \, d\xi \right)$$

$$= 2(2\pi)^{-n} \operatorname{Re} \left(i \int_{\mathbb{R}^n} (u^{-\frac{1}{q}} \xi_1 \widehat{f}(\xi)) \overline{(u^{\frac{1}{q}} \, (\widehat{x_1 f})(\xi)} \, d\xi \right)$$

and by Hölder inequality and Theorem 3.1,

$$\|f\|_2^2 \le C \, \|u^{-\frac{1}{q}}\xi_1\widehat{f}\|_{q'}\|u^{\frac{1}{q}}\widehat{x_1 f}\|_q$$

$$\le C\|u^{-\frac{1}{q}}\xi_1\widehat{f}\|_{q'}\|v^{\frac{1}{p}}x_1 f\|_p$$

$$\le C\|u^{-\frac{1}{q}}|\xi|\widehat{f}\|_{q'}\|v^{\frac{1}{p}}|x|f\|_p$$

as required. □

PROOF OF COROLLARY 6.2. Follows from Theorems 1.3 and 6.1, with $v \equiv 1$ and $u_0(\rho) = s_0^{-1}(\rho)$. □

PROOF OF COROLLARY 6.3. follows from Corollary 3.2 and Theorem 6.1, with $u_0(\rho) = s_0^{-1}(\rho)$. □

7. Riemann–Lebesgue estimates via Pitt inequalities

Here we investigate the interrelation between the smoothness of a function and the growth properties of the Fourier transforms. The original result goes back to the Riemann–Lebesgue estimate $|\widehat{f}(\xi)| \to 0$ as $|\xi| \to \infty$, where $f \in L^1(\mathbb{R}^n)$ and its quantitative version given by

$$(7.1) \qquad |\widehat{f}(\xi)| \le C\omega_l\left(f, \frac{1}{|\xi|}\right)_1, \quad f \in L^1(\mathbb{R}^n),$$

where the modulus of smoothness $\omega_l(f,\delta)_p$ of a function $f \in L^p(X)$ is defined by

$$(7.2) \qquad \omega_l\left(f,\delta\right)_p = \sup_{|h|\le\delta}\left\|\Delta_h^l f(x)\right\|_{L^p(\mathbb{R}^n)}, \quad 1 \le p \le \infty,$$

and

$$\Delta_h^l f(x) = \Delta_h^{l-1}\left(\Delta_h f(x)\right), \qquad \Delta_h f(x) = f(x+h) - f(x).$$

Recently this result was extended for L^p-functions. Let us first define the suitable multivariate substitution for the classical modulus of smoothness.

For a locally integrable function f the average on a sphere in $\mathbb{R}^n$ of radius $t > 0$ is given by

$$V_t f(x) := \frac{1}{m_t} \int_{|y-x|=t} f(y)\, dy \quad \text{with} \quad V_t 1 = 1, \quad n \ge 2.$$

For $l \in \mathbb{N}$ we define

$$V_{l,t}f(x) := \frac{-2}{\binom{2l}{l}} \sum_{j=1}^{l} (-1)^j \binom{2l}{l-j} V_{jt}f(x).$$

and set

$$\Omega_l(f,t)_p = \|f - V_{l,t}f\|_p.$$

In [**15**, Th. 2.1 (A), $n \ge 2$] the following Riemann–Lebesgue type estimates was proved.

THEOREM 7.1. *Let $f \in L^p(\mathbb{R}^n)$, $1 < p \leq 2$. Then for $p \leq q \leq p'$ we have*
$|\xi|^{n(1-\frac{1}{p}-\frac{1}{q})}\widehat{f}(\xi) \in L^q(\mathbb{R}^n)$, *and*

$$\left(\int_{\mathbb{R}^n} \left[\min\left(1, t|\xi|\right)^{2l}|\xi|^{n(1-\frac{1}{p}-\frac{1}{q})}|\widehat{f}(\xi)|\right]^q d\xi\right)^{\frac{1}{q}} \leq C\Omega_l(f,t)_p.$$

Note that some partial cases were previously proved in [**7, 12**]; see also [**6**]. The essential step in the proof of Theorem 7.1 is the use of Pitt's inequalities (1.6) under conditions (1.7) and (1.8) in the case when $b = 0$, that is when the right-hand side of (1.6) is the non-weighted L^p-norm.

Here we refine Theorem 7.1 using new Pitt's inequality given by Theorem 1.3.

THEOREM 7.2. *Under the assumption of Theorem 1.3, we have*

$$\left(\int_{\mathbb{R}^n} \left[\min\left(1, \ t|\xi|\right)^{2l}|\widehat{f}(\xi)|\right]^q u(\xi)\, d\xi\right)^{\frac{1}{q}} \leq C\Omega_l(f,t)_p.$$

The proof repeats the proof of Theorem 7.1 with the only modification that one should use the weight $u^{\frac{1}{q}}(\xi)$ in place of $|\xi|^{n(1-\frac{1}{p}-\frac{1}{q})}$ (see [**15**, (2.16)]) and Theorem 1.3.

8. Other applications

Inequality (1.10) in Theorem 1.3 implies $\int_{\mathbb{R}^n} u(\xi)(1 + |\xi|)^{-\frac{qn}{p'}}d\xi < \infty$. In [**2**] it is proved that if (1.1) holds for $1 < p \leq q < \infty$, and if $\int_{\mathbb{R}^n} u(\xi)^{1-q'}(1 + |\xi|)^{-M}d\xi < \infty$ for some $M > 0$, then once can prove a Bernstein-type theorem, which characterizes the Fourier transform on weighted Besov spaces. We leave the generalization of the main Theorem in [**2**] to the interested reader.

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