PITT'S INEQUALITIES AND UNCERTAINTY PRINCIPLE FOR GENERALIZED FOURIER TRANSFORM

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ABSTRACT. We study the two-parameter family of unitary operators

$$\mathcal{F}_{k,a} = \exp\Big(\frac{i\pi}{2a}\left(2\langle k\rangle + d + a - 2\right)\Big)\exp\Big(\frac{i\pi}{2a}\,\Delta_{k,a}\Big),$$

which are called (k,a)-generalized Fourier transforms and defined by the a-deformed Dunkl harmonic oscillator $\Delta_{k,a} = |x|^{2-a}\Delta_k - |x|^a$, $a > 0$, where Δ_k is the Dunkl Laplacian. Particular cases of such operators are the Fourier and Dunkl transforms. The restriction of $\mathcal{F}_{k,a}$ to radial functions is given by the a-deformed Hankel transform $H_{\lambda,a}$.

We obtain necessary and sufficient conditions for the weighted (L^p, L^q) Pitt inequalities to hold for the a-deformed Hankel transform. Moreover, we prove two-sided Boas–Sagher type estimates for the general monotone functions. We also prove sharp Pitt's inequality for $\mathcal{F}_{k,a}$ transform in $L^2(\mathbb{R}^d)$ with the corresponding weights. Finally, we establish the logarithmic uncertainty principle for $\mathcal{F}_{k,a}$.

1. INTRODUCTION

Let $\mathbb{R}^d$ be the real space of d dimensions, equipped with a scalar product $\langle x, y\rangle$ and a norm $|x| = \sqrt{\langle x, x\rangle}$. The Fourier transform is defined by

$$\mathcal{F}(f)(y) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} f(x)e^{-i\langle x,y\rangle}\,dx.$$

R. Howe [16] found the spectral description of $\mathcal{F}$ using the harmonic oscillator $-(\Delta - |x|^2)/2$ and its eigenfunctions forming the basis in $L^2(\mathbb{R}^d)$:

$$\mathcal{F} = \exp\Big(\frac{i\pi d}{4}\Big)\exp\Big(\frac{i\pi}{4}\,(\Delta - |x|^2)\Big),$$

where Δ is the Laplace operator. This representation has been widely used to define the fractional Fourier transform and Clifford algebra-valued analogues, see [6].

One of the generalizations of the Fourier transform is the Dunkl transform $\mathcal{F}_k$ [10], which is defined with the help of a root system $R \subset \mathbb{R}^d$, a reflection group $G \subset O(d)$, and multiplicity function $k\colon R \to \mathbb{R}_+$ such that k is G-invariant. If $k \equiv 0$, we have $\mathcal{F}_k = \mathcal{F}$.

The differential-difference operator Δ_k, the Dunkl Laplacian, plays an important role in the Dunkl analysis, see, e.g., [24]. For $k \equiv 0$ we get $\Delta_k = \Delta$.

S. Ben Saïd, T. Kobayashi, and B. Ørsted [4] defined a-deformed Dunkl-type harmonic oscillator as follows

$$\Delta_{k,a} = |x|^{2-a}\Delta_k - |x|^a, \quad a > 0.$$

Following [16], they constructed a two-parameter unitary operator, the (k, a)-generalized Fourier transforms,

$$(1) \qquad \mathcal{F}_{k,a} = \exp\Big(\frac{i\pi}{2a}(2\lambda_k + a)\Big)\exp\Big(\frac{i\pi}{2a}\Delta_{k,a}\Big)$$

in $L^2(\mathbb{R}^d, d\mu_{k,a})$ with a norm

$$\|f\|_{2,d\mu_{k,a}} = \Big(\int_{\mathbb{R}^d}|f(x)|^2\,d\mu_{k,a}(x)\Big)^{1/2},$$

where

$$\lambda_k = \frac{d}{2} - 1 + \langle k\rangle, \qquad \langle k\rangle = \frac{1}{2}\sum_{\alpha\in R}k(\alpha),$$

$$d\mu_{k,a}(x) = c_{k,a}v_{k,a}(x)\,dx, \qquad v_{k,a}(x) = |x|^{a-2}v_k(x),$$

$$v_k(x) = \prod_{\alpha\in R}|\langle\alpha, x\rangle|^{k(\alpha)}, \qquad c_{k,a}^{-1} = \int_{\mathbb{R}^d}e^{-|x|^a/a}v_{k,a}(x)\,dx.$$

If $a = 2$, (1) recovers the Dunkl transform, and if $a = 2$ and $k \equiv 0$ the Fourier transform. For $a \neq 2$, (1) is a deformed Fourier and Dunkl operators. In particular, if $a = 1$ and $k \equiv 0$, the operator $\mathcal{F}_{k,a}$ is the unitary inversion operator of the Schrödinger model of the minimal representation of the group $O(N+1, 2)$, see [20].

The operator $\mathcal{F}_{k,a}$ is a unitary operator, that is, for $a > 0$, $2\langle k\rangle + d + a > 2$, it is a bijective linear operator such that for any function $f \in L^2(\mathbb{R}^d, d\mu_{k,a})$ the Plancherel formula holds [4, Th. 5.1]

$$(2) \qquad \big\|\mathcal{F}_{k,a}(f)(y)\big\|_{2,d\mu_{k,a}} = \big\|f(x)\big\|_{2,d\mu_{k,a}}.$$

The main goal of this paper is to prove Pitt's inequality

$$(3) \qquad \big\||y|^{-\beta}\mathcal{F}_{k,a}(f)(y)\big\|_{2,d\mu_{k,a}} \leq C(\beta, k, a)\big\||x|^{\beta}f(x)\big\|_{2,d\mu_{k,a}}, \qquad f \in \mathcal{S}(\mathbb{R}^d),$$

with the sharp constant

$$C(\beta, k, a) = a^{-2\beta/a}\frac{\Gamma\big(a^{-1}(\lambda_k + a/2 - \beta)\big)}{\Gamma\big(a^{-1}(\lambda_k + a/2 + \beta)\big)},$$

and the logarithmic uncertainty principle

$$(4) \qquad \int_{\mathbb{R}^d}\ln(|x|)|f(x)|^2\,d\mu_{k,a}(x) + \int_{\mathbb{R}^d}\ln(|y|)|\mathcal{F}_{k,a}(f)(y)|^2\,d\mu_{k,a}(y)$$

$$\geq \frac{2}{a}\Big\{\psi\Big(\frac{\lambda_k}{a} + \frac{1}{2}\Big) + \ln a\Big\}\|f\|_{2,d\mu_{k,a}}^2,$$

provided that

$$0 \leq \beta < \lambda_k + \frac{a}{2}, \qquad 4\lambda_k + a \geq 0.$$

Here and in what follows, $\Gamma(t)$ is the gamma function, $\psi(t) = \Gamma'(t)/\Gamma(t)$ the psi function, and $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space.

Inequalities (3) and (4) were proved by W. Beckner [2] for the Fourier transform, by S. Omri [23] for the Dunkl transform on radial functions, by F. Soltani [26] for the one-dimensional Dunkl transform, and by the authors [13] for the general

Dunkl transform. Regarding inequality (3) for the Fourier transform see also [3, 12, 15, 27].

A study of analytical properties of $\mathcal{F}_{k,a}$-transform was first conducted in [4]. Very recently, weighted norm inequalities were obtained in [19]. In particular, the author raises the question on the sharp logarithmic uncertainty principle for $\mathcal{F}_{k,a}$.

The rest of the paper is organized as follows. In Section 2 we study the a-deformed Hankel transforms which are the restriction of $\mathcal{F}_{k,a}$ to radial functions. In particular, we find necessary and sufficient conditions for the Pitt inequalities with power weights to hold and we obtain sharp Pitt's inequality in L^2.

Section 3 deals with boundedness properties of the a-deformed Hankel transform of general monotone functions. In this case we improve the range of parameters in the Pitt inequalities and prove the reverse inequalities. In particular, we obtain two-sided inequalities of the Boas–Sagher type.

Section 4 is devoted to the proof of inequality (3). To show (3), we use the following decomposition

$$(5) \qquad L^2(\mathbb{R}^d, d\mu_{k,a}) = \sum_{n=0}^{\infty} \oplus \mathcal{R}_n^d(v_{k,a}), \qquad \mathcal{R}_n^d(v_{k,a}) = \mathcal{R}_0^d(v_{k,a}) \otimes \mathcal{H}_n^d(v_k),$$

where $\mathcal{R}_0^d(v_{k,a})$ is the space of radial function, and $\mathcal{H}_n^d(v_k)$ is the space of k-spherical harmonics of degree n. Since $\mathcal{R}_n^d(v_{k,a})$ is invariant under the operator $\mathcal{F}_{k,a}$, it is enough to study inequality (3) on $\mathcal{R}_n^d(v_{k,a})$.

In Section 5, we obtain the logarithmic uncertainty principle (4) for $\mathcal{F}_{k,a}$-transform, which follows from (3). It is worth mentioning that the Heisenberg uncertainty principle for $\mathcal{F}_{k,a}$ was proved in [4]. It reads as follows: *for $d \in \mathbb{N}$, $k \geq 0$, $a > 0$, and $2\lambda_k + a > 0$, one has*

$$\left\| |x|^{a/2} f(x) \right\|_{2,d\mu_{k,a}} \left\| |y|^{a/2} \mathcal{F}_{k,a}(f)(y) \right\|_{2,d\mu_{k,a}} \geq (2\lambda_k + a) \|f\|_{2,d\mu_{k,a}}^2.$$

The equality holds if and only if the function f is of the form $f(x) = Ce^{-c|x|^a}$ for some $a, c > 0$. Various uncertainty relations for $\mathcal{F}_{k,a}$ were also studied in [19].

We conclude by Section 6 where we study the uniform boundedness properties of the kernel $B_{k,a}(y,x)$ in the integral transform expression $\mathcal{F}_{k,a}(f)(y) = \int_{\mathbb{R}^d} B_{k,a}(y,x) f(x) \, d\mu_{k,a}(x)$.

2. $\mathcal{F}_{k,a}$-TRANSFORM ON RADIAL FUNCTIONS

Let $\lambda \geq -1/2$, $J_\lambda(t)$ be the classical Bessel function of degree λ, and

$$j_\lambda(t) = 2^\lambda \Gamma(\lambda + 1) t^{-\lambda} J_\lambda(t)$$

be the normalized Bessel function. Let also

$$b_\lambda^{-1} = \int_0^\infty e^{-t^2/2} t^{2\lambda+1} \, dt = 2^\lambda \Gamma(\lambda + 1), \qquad d\nu_\lambda(r) = b_\lambda r^{2\lambda+1} \, dr, \quad r \in \mathbb{R}_+.$$

The norm in $L^p(\mathbb{R}_+, d\nu_\lambda)$, $1 \leq p < \infty$, is given by

$$\|f\|_{p,d\nu_\lambda} = \left(\int_{\mathbb{R}_+} |f(r)|^p \, d\nu_\lambda(r) \right)^{1/p}.$$

Moreover, let $\|f\|_\infty = \operatorname{supvrai}_{r \in \mathbb{R}_+} |f(r)|$.

The Hankel transform is defined as follows

$$H_\lambda(f)(\rho) = \int_{\mathbb{R}_+} f(r) j_\lambda(\rho r) \, d\nu_\lambda(r).$$

It is a unitary operator in $L^2(\mathbb{R}_+, d\nu_\lambda)$ and $H_\lambda^{-1} = H_\lambda$ [1, Chap. 7].

Note that the Hankel transform is a restriction of the Fourier transform on radial functions if $\lambda = d/2 - 1$, and of the Dunkl transforms on radial functions if $\lambda = \lambda_k = d/2 - 1 + \langle k \rangle$.

Let $\mathcal{S}(\mathbb{R}_+)$ be the Schwartz space on $\mathbb{R}_+$. For $f \in \mathcal{S}(\mathbb{R}_+)$, we are interested in the Pitt inequality

$$(6) \qquad \left\| \rho^{-\gamma} H_\lambda(f)(\rho) \right\|_{q,d\nu_\lambda} \leq c_{pq}(\beta, \gamma, \lambda) \left\| r^\beta f(r) \right\|_{p,d\nu_\lambda}$$

with the sharp constant $c_{pq}(\beta, \gamma, \lambda)$. Here and in what follows, we assume that $1 < p \leq q < \infty$.

L. De Carli [8] showed that $c_{pq}(\beta, \gamma, \lambda)$ is finite if and only if

$$\beta - \gamma = 2(\lambda + 1)\Big(\frac{1}{p'} - \frac{1}{q}\Big)$$

and

$$(7) \qquad \Big(\frac{1}{2} - \frac{1}{p}\Big)(2\lambda + 1) + \max\Big\{\frac{1}{p'} - \frac{1}{q}, 0\Big\} \leq \beta < \frac{2(\lambda + 1)}{p'},$$

where p' is the Hölder conjugate of p.

The sharp constant $c_{pq}(\beta, \gamma, \lambda)$ is known only for $p = q = 2$ and $\gamma = \beta$ [27, 23]:

$$c_{22}(\beta, \beta, \lambda) = c(\beta, \lambda) = 2^{-\beta} \frac{\Gamma\big(2^{-1}(\lambda + 1/2 - \beta)\big)}{\Gamma\big(2^{-1}(\lambda + 1/2 + \beta)\big)}, \qquad 0 \leq \beta < \lambda + 1.$$

For $a > 0$ we denote by $L^p(\mathbb{R}_+, d\nu_{\lambda,a})$ the space of complex-valued functions endowed with a norm

$$\|f\|_{p,d\nu_{\lambda,a}} = \left(\int_{\mathbb{R}_+} |f(r)|^p \, d\nu_{\lambda,a}(r) \right)^{1/p}, \qquad 1 \leq p < \infty,$$

$$d\nu_{\lambda,a}(r) = b_{\lambda,a} r^{2\lambda + a - 1} \, dr,$$

where the normalization constant is given by

$$b_{\lambda,a}^{-1} = \int_0^\infty e^{-t^a/a} t^{2\lambda + a - 1} \, dt = a^{2\lambda/a} \Gamma\Big(\frac{2\lambda}{a} + 1\Big).$$

For $4\lambda + a \geq 0$, we define the a-deformed Hankel transform

$$H_{\lambda,a}(f)(\rho) = \int_{\mathbb{R}_+} f(r) j_{2\lambda/a}\Big(\frac{2}{a}(\rho r)^{a/2}\Big) \, d\nu_{\lambda,a}(r).$$

Note that in the paper [4, Sec. 5.5.3] a slightly different definition of the a-deformed Hankel transform has been used (with a different normalization). We find our definition more convenient to use.

The Hankel transform $H_{\lambda,a}$ is a unitary operator in $L^2(\mathbb{R}_+, d\nu_{\lambda,a})$. Moreover, if $\lambda = \lambda_k$, by Bochner-type identity, the $\mathcal{F}_{k,a}$ transform of a radial function is written by the $H_{\lambda,a}$ transform (see [4, Th. 5.21]): for $f(x) = f_0(r)$, $r = |x|$, $\rho = |y|$, we have

$$\mathcal{F}_{k,a}(f)(y) = (\mathcal{F}_{k,a}(f))_0(\rho), \qquad (\mathcal{F}_{k,a}(f))_0(\rho) = H_{\lambda,a}(f_0)(\rho).$$

Changing variables

$$r = \Big(\frac{a}{2}\Big)^{1/a} s^{2/a}, \quad \rho = \Big(\frac{a}{2}\Big)^{1/a} \theta^{2/a}, \quad f\Big(\Big(\frac{a}{2}\Big)^{1/a} s^{2/a}\Big) = g(s),$$

we arrive at

$$\int_0^\infty r^{\beta p}|f(r)|^p \, d\nu_{\lambda,a}(r) = b_{\lambda,a}\Big(\frac{a}{2}\Big)^{(\beta p+2\lambda)/a} \int_0^\infty s^{2\beta p/a}|g(s)|^p s^{4\lambda/a+1} \, ds$$

$$= b_{2\lambda/a}\Big(\frac{a}{2}\Big)^{\beta p/a} \int_0^\infty s^{2\beta p/a}|g(s)|^p s^{4\lambda/a+1} \, ds$$

$$= \Big(\frac{a}{2}\Big)^{\beta p/a} \int_0^\infty s^{2\beta p/a}|g(s)|^p \, d\nu_{2\lambda/2}(s)$$

and

$$\int_0^\infty \rho^{-\gamma q}|H_{\lambda,a}(f)(\rho)|^q \, d\nu_{\lambda,a}(\rho)$$

$$= \Big(\frac{a}{2}\Big)^{-\gamma q/a} \int_0^\infty \theta^{-2\gamma q/a}\Big|H_{\lambda,a}(f)\Big(\Big(\frac{a}{2}\Big)^{1/a}\theta^{2/a}\Big)\Big|^q \, d\nu_{2\lambda/2}(\theta),$$

where

$$H_{\lambda,a}(f)\Big(\Big(\frac{a}{2}\Big)^{1/a}\theta^{2/a}\Big) = b_{\lambda,a}\Big(\frac{a}{2}\Big)^{2\lambda/a} \int_0^\infty g(s)j_{2\lambda/a}(\theta s)s^{4\lambda/a+1} \, ds$$

$$= b_{2\lambda/a} \int_0^\infty g(s)j_{2\lambda/a}(\theta s)s^{4\lambda/a+1} \, ds = H_{2\lambda/a}(g)(\theta).$$

Therefore,

$$(8) \qquad \frac{\big\|\rho^{-\gamma}H_{\lambda,a}(f)(\rho)\big\|_{q,d\nu_{\lambda,a}}}{\big\|r^\beta f(r)\big\|_{p,d\nu_{\lambda,a}}} = \Big(\frac{a}{2}\Big)^{-(\beta+\gamma)/a} \frac{\big\|\theta^{-2\gamma/a}H_{2\lambda/a}(g)(\theta)\big\|_{q,d\nu_{2\lambda/a}}}{\big\|s^{2\beta/a}g(s)\big\|_{p,d\nu_{2\lambda/a}}}.$$

Hence, the sharp constant $c_{pq}(\beta,\gamma,\lambda,a)$ in Pitt's inequality

$$(9) \qquad \big\|\rho^{-\gamma}H_{\lambda,a}(f)(\rho)\big\|_{q,d\nu_{\lambda,a}} \le c_{pq}(\beta,\gamma,\lambda,a)\big\|r^\beta f(r)\big\|_{p,d\nu_{\lambda,a}}$$

is related to the constant $c_{pq}(\beta,\gamma,\lambda)$ given by (6) as follows

$$c_{pq}(\beta,\gamma,\lambda,a) = \Big(\frac{a}{2}\Big)^{-(\beta+\gamma)/a} c_{pq}\Big(\frac{2\beta}{a},\frac{2\gamma}{a},\frac{2\lambda}{a}\Big).$$

Therefore, using the above mentioned results by De Carli, we arrive at the following two theorems.

Theorem 2.1. *Let $4\lambda + a \ge 0$ and $1 < p \le q < \infty$. Pitt's inequality (9) holds if and only if*

$$1) \quad \beta - \gamma = (2\lambda + a)\Big(\frac{1}{p'} - \frac{1}{q}\Big),$$

$$2) \quad \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big(2\lambda + \frac{a}{2}\Big) + \frac{a}{2}\max\Big\{\frac{1}{p'} - \frac{1}{q}, 0\Big\} \le \beta < \frac{2\lambda + a}{p'}.$$

Theorem 2.2. *Let $4\lambda + a \ge 0$ and $0 \le \beta < \lambda + a/2$. Then Pitt's inequality*

$$\big\|\rho^{-\beta}H_{\lambda,a}(f)(\rho)\big\|_{2,d\nu_{\lambda,a}} \le c(\beta,\lambda,a)\big\|r^\beta f(r)\big\|_{2,d\nu_{\lambda,a}}$$

holds and the constant

$$c(\beta,\lambda,a) = a^{-2\beta/a} \frac{\Gamma\big(a^{-1}(\lambda + a/2 - \beta)\big)}{\Gamma\big(a^{-1}(\lambda + a/2 + \beta)\big)}$$

is sharp.

Let us now verify that $c(\beta,\lambda,a)$ is decreasing with λ.

Lemma 2.3. *If $\alpha > 0$, then*

$$(10) \qquad \frac{\Gamma(t+\alpha)}{\Gamma(\tau+\alpha)} < \frac{\Gamma(t)}{\Gamma(\tau)}, \qquad 0 < t < \tau.$$

Proof. If $\alpha = 1/2$, the proof of (10) can be found in [27]. To make the paper self-contained we give the proof for any $\alpha > 0$. Since the function $\psi(t) = \Gamma'(t)/\Gamma(t)$ is increasing, we have

$$\Big(\frac{\Gamma(t+\alpha)}{\Gamma(t)}\Big)' = \frac{\Gamma(t+\alpha)}{\Gamma(t)}\Big[\frac{\Gamma'(t+\alpha)}{\Gamma(t+\alpha)} - \frac{\Gamma'(t)}{\Gamma(t)}\Big] > 0$$

and $\frac{\Gamma(\tau+\alpha)}{\Gamma(\tau)} > \frac{\Gamma(t+\alpha)}{\Gamma(t)}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In this section and in what follows we use the following

Remark 2.1. Let $\mathcal{S}_0(\mathbb{R}_+)$ be a set of functions $f \in \mathcal{S}(\mathbb{R}_+)$ such that $f^{(n)}(0) = 0$ for any $n \in \mathbb{Z}_+$. If $f \in \mathcal{S}_0(\mathbb{R}_+)$, $\alpha \in \mathbb{R}$, $\beta > 0$, then $r^\alpha f(r^\beta) \in \mathcal{S}_0(\mathbb{R}_+)$ and $\mathcal{S}_0(\mathbb{R}_+)$ is dense in $L^p(\mathbb{R}_+, r^\alpha\, dr)$. Therefore, when we assume that $f \in \mathcal{S}(\mathbb{R}_+)$, we may additionally assume that $f \in \mathcal{S}_0(\mathbb{R}_+)$.

3. Boas–Sagher inequalities for general monotone functions

In this section we study boundedness properties of the a-deformed Hankel transform $H_{\lambda,a}$ of the general monotone functions. For the classical Hankel transform

$$H_\lambda(f)(\rho) = \frac{1}{2^\lambda \Gamma(\lambda+1)} \int_{\mathbb{R}_+} f(r) j_\lambda(\rho r) r^{2\lambda+1}\, dr,$$

similar questions were studied in [9].

A function f of locally bounded variation on $[\varepsilon, \infty)$, for any $\varepsilon > 0$, is *general monotone*, written $f \in GM$, if it vanishes at infinity, and there exist $C > 0$ and $c > 1$ such that, for every $r > 0$,

$$(11) \qquad \int_r^\infty |df(r)| \leq C \int_{r/c}^\infty |f(u)| \frac{du}{u},$$

where $\int_a^b |df(u)|$ is the Riemann–Stieltjes integral. The GM class strictly includes the class of monotonic functions. It was introduced in [21] (see also [22]).

By Theorem 1.3 from [9] we have that *if $1 < p \leq q < \infty$ and $f \in GM$, then Pitt's inequality*

$$(12) \qquad \big\|\rho^{-\gamma} H_\lambda(f)(\rho)\big\|_{q,d\nu_\lambda} \leq c\big\|r^\beta f(r)\big\|_{p,d\nu_\lambda}$$

holds if and only if

$$(13) \qquad \beta - \gamma = 2(\lambda+1)\Big(\frac{1}{p'} - \frac{1}{q}\Big)$$

and

$$\Big(\frac{1}{2} - \frac{1}{p}\Big)(2\lambda+1) - \frac{1}{p} < \beta < \frac{2(\lambda+1)}{p'}.$$

It is important to note that the condition on β is less restrictive than the corresponding condition in the general Pitt inequality given by (7).

Moreover, considering general monotone functions allows us to prove the reverse Pitt inequality. First, we note that if

$$(14) \qquad \int_0^1 r^{2\lambda+1}|f(r)|\, dr + \int_1^\infty r^{\lambda+1/2}\, |df(r)| < \infty,$$

then $H_\lambda(f)(\rho)$ is defined as an improper integral (i.e., as $\lim\limits_{a\to 0,\, b\to\infty} \int_a^b$) and continuous for $\rho > 0$ [9]. The reverse Pitt inequality reads as follows: *Let $1 < q \le p < \infty$ and let a non-negative function $f \in GM$ be such that condition* (14) *is satisfied. Then the inequality*

$$(15) \qquad \left\| \rho^{-\gamma} H_\lambda(f)(\rho) \right\|_{q, d\nu_\lambda} \ge c \left\| r^\beta f(r) \right\|_{p, d\nu_\lambda}$$

holds provided that conditions (13) *and*

$$-\frac{2\lambda + 1}{p} - \frac{1}{p} < \beta$$

are satisfied.

Noting that

$$-\frac{2\lambda + 1}{p} - \frac{1}{p} < \left(\frac{1}{2} - \frac{1}{p}\right)(2\lambda + 1) - \frac{1}{p} = \frac{2(\lambda + 1)}{p'} - \frac{2\lambda + 3}{2},$$

inequalities (12) и (15) imply that *if $1 < p < \infty$, $f \in GM$, $f \ge 0$ and* (14) *holds, then*

$$(16) \qquad c_1 \left\| r^\beta f(r) \right\|_{p, d\nu_\lambda} \le \left\| \rho^{-\gamma} H_\lambda(f)(\rho) \right\|_{p, d\nu_\lambda} \le c_2 \left\| r^\beta f(r) \right\|_{p, d\nu_\lambda}$$

if and only if (13) *and*

$$\frac{2(\lambda + 1)}{p'} - \frac{2\lambda + 3}{2} < \beta < \frac{2(\lambda + 1)}{p'}.$$

A study of two-sided inequalities of type (16) for the classical Fourier transform has a long history. In the one-dimensional case for monotone decreasing functions the corresponding conjecture was formulated by Boas [5]. He also obtained some partial results. Boas' conjecture was fully solved by Sagher in [25]. The multidimensional case was studied in [14].

We are going to use the above mentioned results to get direct and reverse Pitt's inequalities for the a-deformed Hankel transform of the general monotone functions. We assume that $4\lambda + a \ge 0$.

Theorem 3.1. *Let $1 < p \le q < \infty$ and $f \in GM$. Then Pitt's inequality*

$$\left\| \rho^{-\gamma} H_{\lambda,a}(f)(\rho) \right\|_{q, d\nu_{\lambda,a}} \le c \left\| r^\beta f(r) \right\|_{p, d\nu_{\lambda,a}}$$

holds if and only if

$$(17) \qquad \beta - \gamma = (2\lambda + a)\left(\frac{1}{p'} - \frac{1}{q}\right)$$

and

$$\left(\frac{1}{2} - \frac{1}{p}\right)\left(2\lambda + \frac{a}{2}\right) - \frac{a}{2p} < \beta < \frac{2\lambda + a}{p'}.$$

Proof. First we show that if $f \in GM$, then the function of the type $f(\alpha r^\beta) = g(r)$ is also a GM function for any $\alpha, \beta > 0$. Indeed, changing variables $\alpha u^\beta = v$ and using inequality (11) for f, we get

$$\int_r^\infty |dg(u)| = \int_{(r/\alpha)^{1/\beta}}^\infty |df(v)| \le C \int_{(r/\alpha)^{1/\beta}/c}^\infty |f(v)| \frac{dv}{v} = C\beta \int_{r/c}^\infty |g(u)| \frac{du}{u}.$$

Now the proof follows from (8) and the change of variables

$$(18) \qquad r = \left(\frac{a}{2}\right)^{1/a} s^{2/a}, \quad \rho = \left(\frac{a}{2}\right)^{1/a} \theta^{2/a}, \quad f\left(\left(\frac{a}{2}\right)^{1/a} s^{2/a}\right) = g(s).$$

$\square$

Theorem 3.2. *Let* $1 < q \le p < \infty$. *Assume that* f *is a non-negative function such that* $f \in GM$ *and*

$$(19) \qquad \int_0^1 r^{2\lambda+a-1}|f(r)|\,dr + \int_1^\infty r^{\lambda+a/4}\,|df(r)| < \infty,$$

is satisfied. Then the reverse Pitt inequality

$$(20) \qquad \left\|\rho^{-\gamma}H_{\lambda,a}(f)(\rho)\right\|_{q,d\nu_{\lambda,a}} \ge c\left\|r^\beta f(r)\right\|_{p,d\nu_{\lambda,a}}$$

holds provided that conditions (17) *and*

$$-\frac{2\lambda+a}{p} - \frac{a}{2p} < \beta < \infty,$$

are satisfied.

The proof follows from (8) and (18). Note that condition (19) implies condition (14) for the function g given by (18). In particular, condition (19) yields that $H_{\lambda,a}$ is defined as an improper integral and $H_{\lambda,a}(f) \in C(0,\infty)$; see [9, Lemma 3.1].

Since

$$-\frac{2\lambda+a}{p} - \frac{a}{2p} < \left(\frac{1}{2} - \frac{1}{p}\right)\left(2\lambda + \frac{a}{2}\right) - \frac{a}{2p} = \frac{2\lambda+a}{p'} - \frac{4\lambda+3a}{4},$$

we obtain the following Boas–Sagher type equivalence.

Corollary 3.3. *Suppose that* $1 < p < \infty$, $f \in GM$, $f \ge 0$ *and* (19) *holds. Then*

$$c_1\left\|r^\beta f(r)\right\|_{p,d\nu_{\lambda,a}} \le \left\|\rho^{-\gamma}H_{\lambda,a}(f)(\rho)\right\|_{p,d\nu_{\lambda,a}} \le c_2\left\|r^\beta f(r)\right\|_{p,d\nu_{\lambda,a}}$$

if and only if conditions (17) *and*

$$(21) \qquad \frac{2\lambda+a}{p'} - \frac{4\lambda+3a}{4} < \beta < \frac{2\lambda+a}{p'}$$

hold.

Remark 3.1. We note that condition (19) always holds if $\left\|r^\beta f(r)\right\|_{p,d\nu_{\lambda,a}} < \infty$ and β satisfies (21). Indeed, it is easy to check (see, e.g., [14, p. 111]) that any general monotone function satisfies the following property: there is $c > 1$ such that

$$\int_r^\infty u^\sigma|df(u)| \le C \int_{r/c}^\infty u^{\sigma-1}|f(u)|\,du, \qquad \sigma \ge 0.$$

Then using this and Hölder's inequality, we get for $w(r) = \begin{cases} r^{2\lambda+a-1}, & r < 1, \\ r^{\lambda+a/4-1}, & r \ge 1, \end{cases}$ that

$$\int_0^1 r^{2\lambda+a-1}|f(r)|\,dr + \int_1^\infty r^{\lambda+a/4}\,|df(r)| \le C \int_0^\infty w(r)|f(r)|\,dr$$
$$\le C\left\|r^\beta f(r)\right\|_{p,d\nu_{\lambda,a}}I,$$

where

$$I^{p'} = \int_0^\infty r^{(-\beta-\frac{2\lambda+a-1}{p})p'}w^{p'}(r)\,dr.$$

The latter is bounded under condition (21).

4. PITT'S INEQUALITY FOR $\mathcal{F}_{k,a}$ TRANSFORM IN L^2

Recall that $R \subset \mathbb{R}^d$ is a root system, R_+ is the positive subsystem of R, and $k \colon R \to \mathbb{R}_+$ is a multiplicity function with the property that k is G-invariant. Here G is a finite reflection group generated by reflections $\{\sigma_a \colon a \in R\}$, where σ_a is a reflection with respect to a hyperplane $\langle a, x \rangle = 0$.

C. F. Dunkl introduced a family of first-order differential-difference operators (Dunkl's operators) associated with G and k by

$$D_j f(x) = \frac{\partial f(x)}{\partial x_j} + \sum_{a \in R_+} k(a) \langle a, e_j \rangle \frac{f(x) - f(\sigma_a x)}{\langle a, x \rangle}, \quad j = 1, \dots, d.$$

The Dunkl kernel $e_k(x, y) = E_k(x, iy)$ is the unique solution of the joint eigenvalue problem for the corresponding Dunkl operators:

$$D_j f(x) = iy_j f(x), \quad j = 1, \dots, d, \qquad f(0) = 1.$$

The Dunkl transform is given by

$$\mathcal{F}_k(f)(y) = \int_{\mathbb{R}^d} f(x) \overline{e_k(x, y)} \, d\mu_k(x).$$

By $\mathbb{S}^{d-1}$ denote the unit sphere in $\mathbb{R}^d$. Let $x' \in \mathbb{S}^{d-1}$ and dx' be the Lebesgue measure on the sphere. Let

$$a_k^{-1} = \int_{\mathbb{S}^{d-1}} v_k(x') \, dx', \qquad d\omega_k(x') = a_k v_k(x') \, dx',$$

and

$$\|f\|_{2, d\omega_k} = \left(\int_{\mathbb{S}^{d-1}} |f(x')|^2 \, d\omega_k(x') \right)^{1/2}.$$

For $\lambda_k = d/2 - 1 + \langle k \rangle$, we have

$$(22) \qquad c_{k,a}^{-1} = \int_0^\infty e^{-r^a/a} r^{2\lambda_k + a - 1} \, dr \int_{\mathbb{S}^{d-1}} v_k(x') \, dx' = b_{\lambda_k, a}^{-1} a_k^{-1}.$$

Let us denote by $\mathcal{H}_n^d(v_k)$ the subspace of k-spherical harmonics of degree $n \in \mathbb{Z}_+$ in $L^2(\mathbb{S}^{d-1}, d\omega_k)$ (see [11, Chap. 5]). Let $\mathcal{P}_n^d$ be the space of homogeneous polynomials of degree n in $\mathbb{R}^d$. Then $\mathcal{H}_n^d(v_k)$ is the restriction of $\ker \Delta_k \cap \mathcal{P}_n^d$ to the sphere $\mathbb{S}^{d-1}$.

If l_n is the dimension of $\mathcal{H}_n^d(v_k)$, we denote by $\{Y_n^j \colon j = 1, \dots, l_n\}$ the real-valued orthonormal basis $\mathcal{H}_n^d(v_k)$ in $L^2(\mathbb{S}^{d-1}, d\omega_k)$. A union of these bases forms an orthonormal basis in $L^2(\mathbb{S}^{d-1}, d\omega_k)$, which consists of k-spherical harmonics, i.e., we have

$$(23) \qquad L^2(\mathbb{S}^{d-1}, d\omega_k) = \sum_{n=0}^\infty \oplus \mathcal{H}_n^d(v_k).$$

For $\lambda > -1$, we denote the Laguerre polynomials by

$$L_s^{(\lambda)}(t) = \sum_{j=0}^s \frac{(-1)^j \Gamma(\lambda + s + 1)}{(s - j)! \, \Gamma(\lambda + j + 1)} \frac{t^j}{j!}.$$

Set $\lambda_{k,a,n} = 2(\lambda_k + n)/a$. In [4], the authors constructed an orthonormal basis in $L^2(\mathbb{R}^d, d\mu_{k,a})$

$$\Phi_{n,j,s}^a(x) = \gamma_{n,j,s}^a Y_n^j(x) L_s^{(\lambda_{k,a,n})}\left(\frac{2}{a}|x|^a\right)e^{-|x|^a/a},$$

$$\gamma_{n,j,s}^a > 0, \quad n, s \in \mathbb{Z}_+, \quad j = 1, \cdots, l_n,$$

which consists of the eigenvalues of the operator $\Delta_{k,a} = |x|^{2-a}\Delta_k - |x|^a$, $a > 0$. This helps to define two-parameter unitary operator $\mathcal{F}_{k,a}$ given by (1).

Note that the system $\{\Phi_{n,j,s}^a(x)\}$ is the eigensystem of $\mathcal{F}_{k,a}$, i.e.,

$$\mathcal{F}_{k,a}(\Phi_{n,j,s}^a)(y) = e^{-i\pi(s+n/a)}\Phi_{n,j,s}^a(y).$$

This and (23) imply the decomposition of $L^2(\mathbb{R}^d, d\mu_{k,a})$ given by (5).

To prove Pitt's inequality, we use the following Bochner-type identity [4] for functions of the type $f(x) = Y_n^i(x')\psi(r) \in \mathcal{S}(\mathbb{R}^d)$, $x = rx'$:

$$(24) \quad \mathcal{F}_{k,a}(f)(y) = e^{-i\pi n/a}\rho^n Y_n^j(y')\int_{\mathbb{R}_+}\psi(r)r^{-n}j_{2(\lambda_k+n)/a}\left(\frac{2}{a}(\rho r)^{a/2}\right)d\nu_{\lambda_k+n,a}(r)$$

$$= e^{-i\pi n/a}\rho^n Y_n^j(y')H_{\lambda_k+n,a}(\psi(r)r^{-n})(\rho), \qquad y = \rho y'.$$

We are now in a position to prove inequality (3).

Theorem 4.1. *Let $\lambda_k = d/2 - 1 + \langle k\rangle$, $a > 0$, $4\lambda_k + a \geq 0$, $0 \leq \beta < \lambda_k + a/2$. For any $f \in \mathcal{S}(\mathbb{R}^d)$, the Pitt inequality*

$$\left\||y|^{-\beta}\mathcal{F}_{k,a}(f)(y)\right\|_{2,d\mu_{k,a}} \leq C(\beta, k, a)\left\||x|^{\beta}f(x)\right\|_{2,d\mu_{k,a}}$$

holds with the sharp constant

$$C(\beta, k, a) = a^{-2\beta/a}\frac{\Gamma\left(a^{-1}(\lambda_k + a/2 - \beta)\right)}{\Gamma\left(a^{-1}(\lambda_k + a/2 + \beta)\right)}.$$

Proof. For $\beta = 0$ we have $C(\beta, k, a) = 1$ and Pitt's inequality (3) becomes Plancherel's identity (2). The rest of the proof follows [13]. Let $0 < \beta < \lambda_k + a/2$. If $f \in \mathcal{S}(\mathbb{R}^d)$, then by (23)

$$f_{nj}(r) = \int_{\mathbb{S}^{d-1}}f(rx')Y_n^j(x')\,d\omega_k(x') \in \mathcal{S}(\mathbb{R}_+),$$

$$f(rx') = \sum_{n=0}^{\infty}\sum_{j=1}^{l_n}f_{nj}(r)Y_n^j(x'),$$

$$\int_{\mathbb{S}^{d-1}}|f(rx')|^2\,d\omega_k(x') = \sum_{n=0}^{\infty}\sum_{j=1}^{l_n}|f_{nj}(r)|^2.$$

Using spherical coordinates, decomposition of $L^2(\mathbb{R}^d, d\mu_{k,a})$ (5), formulas (22) and (24), we get that

$$(25) \quad \int_{\mathbb{R}^d}|x|^{2\beta}|f(x)|^2\,d\mu_{k,a}(x) = b_{\lambda_k,a}\int_0^{\infty}r^{2\beta+2\lambda_k+a-1}\int_{\mathbb{S}^{d-1}}|f(rx')|^2\,d\omega_k(x')\,dr$$

$$= b_{\lambda_k,a}\int_0^{\infty}r^{2\beta+2\lambda_k+a-1}\sum_{n=0}^{\infty}\sum_{j=1}^{l_n}|f_{nj}(r)|^2\,dr$$

$$= \sum_{n=0}^{\infty}\sum_{j=1}^{l_n}\int_0^{\infty}|f_{nj}(r)|^2 r^{2\beta}\,d\nu_{\lambda_k,a}(r),$$

$$\mathcal{F}_{k,a}(f)(y) = \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} e^{-i\pi n/a} \rho^n Y_n^j(y') H_{\lambda_k+n,a}(f_{nj}(r)r^{-n})(\rho),$$

and

(26) $\displaystyle\int_{\mathbb{R}^d} |y|^{-2\beta} |\mathcal{F}_{k,a}(f)(y)|^2 \, d\mu_{k,a}(y)$

$$\leq \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} \int_0^\infty \left| H_{\lambda_k+n,a}(f_{nj}(r)r^{-n})(\rho) \right|^2 \rho^{-2\beta+2n} \, d\nu_{\lambda_k,a}(\rho).$$

By Theorem 2.2 with $n \in \mathbb{Z}_+$ and $0 \le \beta < \lambda_k + n + a/2$, we have

(27) $\displaystyle\int_0^\infty \left| H_{\lambda_k+n,a}(f_{nj}(r)r^{-n})(\rho) \right|^2 \rho^{-2\beta+2n} \, d\nu_{\lambda_k,a}(\rho)$

$$\leq c^2(\beta, \lambda_k + n, a) \int_0^\infty \left| f_{nj}(r)r^{-n} \right|^2 r^{2\beta+2n} \, d\nu_{\lambda_k,a}(r)$$

$$= c^2(\beta, \lambda_k + n, a) \int_0^\infty |f_{nj}(r)|^2 \, r^{2\beta} \, d\nu_{\lambda_k,a}(r).$$

Since $c(\beta, \lambda_k + n, a)$ is decreasing with n (see Lemma 2.3), then using (25), (26), and (27), we arrive at

(28) $\displaystyle\int_{\mathbb{R}^d} |y|^{-2\beta} |\mathcal{F}_{k,a}(f)(y)|^2 \, d\mu_{k,a}(y)$

$$\leq \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} c^2(\beta, \lambda_k + n, a) \int_0^\infty |f_{nj}(r)|^2 r^{2\beta} \, d\nu_{\lambda_k,a}(r)$$

$$\leq c^2(\beta, \lambda_k, a) \sum_{n=0}^{\infty} \sum_{j=1}^{l_n} \int_0^\infty |f_{nj}(r)|^2 r^{2\beta} \, d\nu_{\lambda_k,a}(r)$$

$$= c^2(\beta, \lambda_k, a) \int_{\mathbb{R}^d} |x|^{2\beta} |f(x)|^2 \, d\mu_{k,a}(x).$$

$\square$

In the proof of Theorem 4.1 we obtained the following result.

Corollary 4.2. *Let $n \in \mathbb{N}$, $\lambda_k = d/2 - 1 + \langle k \rangle$, and $0 \le \beta < \lambda_k + a/2 + n$. Then Pitt's inequality for the transform $\mathcal{F}_{k,a}$ holds for $f \in \mathcal{S}(\mathbb{R}^d) \cap \mathcal{R}_n^d(v_{k,a})$ with sharp constant $c(\beta, \lambda_k + n, a)$.*

5. Logarithmic uncertainty principle for $\mathcal{F}_{k,a}$ transform

Theorem 5.1. *Suppose that $a > 0$, $\lambda_k = d/2 - 1 + \langle k \rangle$, and $4\lambda_k + a \ge 0$. Then the inequality*

$$\int_{\mathbb{R}^d} \ln(|x|) |f(x)|^2 \, d\mu_{k,a}(x) + \int_{\mathbb{R}^d} \ln(|y|) |\mathcal{F}_{k,a}(f)(y)|^2 \, d\mu_{k,a}(y)$$

$$\geq \frac{2}{a} \left(\psi\left(\frac{\lambda_k}{a} + \frac{1}{2}\right) + \ln a \right) \|f\|_{2, d\mu_{k,a}}^2$$

holds for any $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. Let us write inequality (3) in the following form

$$\int_{\mathbb{R}^d} |y|^{-\beta} |\mathcal{F}_{k,a}(f)(y)|^2 \, d\mu_{k,a}(y) \leq c^2(\beta/2, \lambda_k, a) \int_{\mathbb{R}^d} |x|^\beta |f(x)|^2 \, d\mu_{k,a}(x),$$

where $0 \leq \beta < 2\lambda_k + a$. For $\beta \in (-(2\lambda_k + a), 2\lambda_k + a)$, we define the function

$$\varphi(\beta) = \int_{\mathbb{R}^d} |y|^{-\beta} |\mathcal{F}_{k,a}(f)(y)|^2 \, d\mu_{k,a}(y) - c^2(\beta/2, \lambda_k, a) \int_{\mathbb{R}^d} |x|^\beta |f(x)|^2 \, d\mu_{k,a}(x).$$

Since $|\beta| < 2\lambda_k + a$ and $f, \mathcal{F}_{k,a}(f) \in \mathcal{S}(\mathbb{R}^d)$, then

$$\int_{|x| \leq 1} |\ln(|x|)| |x|^\beta v_{k,a}(x) \, dx = \int_0^1 |\ln(r)| r^{\beta + 2\lambda_k + a - 1} \, dr \int_{\mathbb{S}^{d-1}} v_k(x') \, dx' < \infty,$$

which implies

$$|y|^{-\beta} \ln(|y|) |\mathcal{F}_{k,a}(f)(y)|^2 v_{k,a}(y) \in L^1(\mathbb{R}^d) \text{ and } \ln(|x|) |x|^\beta |f(x)|^2 v_{k,a}(x) \in L^1(\mathbb{R}^d).$$

Hence,

$$(29) \quad \varphi'(\beta) = -\int_{\mathbb{R}^d} |y|^{-\beta} \ln(|y|) |\mathcal{F}_{k,a}(f)(y)|^2 \, d\mu_{k,a}(y)$$

$$- c^2(\beta/2, \lambda_k, a) \int_{\mathbb{R}^d} |x|^\beta \ln(|x|) |f(x)|^2 \, d\mu_{k,a}(x)$$

$$- \frac{dc^2(\beta/2, \lambda_k, a)}{d\beta} \int_{\mathbb{R}^d} |x|^\beta |f(x)|^2 \, d\mu_{k,a}(x).$$

Pitt's inequality and Plancherel's theorem imply that $\varphi(\beta) \leq 0$ for $\beta > 0$ and $\varphi(0) = 0$ correspondingly, hence

$$\varphi'(0) = \lim_{\beta \to 0+} \frac{\varphi(\beta) - \varphi(0)}{\beta} \leq 0.$$

Combining (29) and

$$-\frac{dc^2(\beta/2, \lambda_k, a)}{d\beta}\bigg|_{\beta=0} = \frac{2}{a} \left\{ \psi\left(\frac{\lambda_k}{a} + \frac{1}{2}\right) + \ln a \right\},$$

we conclude the proof. $\qquad \square$

Remark 5.1. In the proof of Theorem 2.1 of the paper [13], sharp Pitt's inequality in L^2 for the Hankel transform H_λ was proved for $\lambda > -1$. Therefore, in Theorems 2.2, 4.1, and 5.1 the conditions $4\lambda + a \geq 0$ and $4\lambda_k + a \geq 0$ can be replaced by the condition $2\lambda + a > 0$ and $2\lambda_k + a > 0$ respectively.

6. FINAL REMARKS

The unitary operator $\mathcal{F}_{k,a}$ on $L^2(\mathbb{R}^d, d\mu_{k,a})$ can be expressed as an integral transform [4, (5.8)]

$$\mathcal{F}_{k,a}(f)(y) = \int_{\mathbb{R}^d} B_{k,a}(y, x) f(x) \, d\mu_{k,a}(x)$$

with a symmetric kernel $B_{k,a}(x, y)$. In particular, $B_{0,2}(x, y) = e^{-i\langle x, y \rangle}$.

A study of properties of the kernel $B_{k,a}(x, y)$ and, in particular, the conditions for its uniform boundedness is an important problem. To illustrate, note that if $|B_{k,a}(x, y)| \leq M$, then the Hausdorff–Young inequality holds:

$$\left\| \mathcal{F}_{k,a}(f) \right\|_{p', d\mu_{k,a}} \leq M^{2/p - 1} \left\| f \right\|_{p, d\mu_{k,a}}, \qquad 1 \leq p \leq 2.$$

Therefore, it is important to know when

$$|B_{k,a}(x,y)| \le B_{k,a}(0,y) = 1, \qquad x, y \in \mathbb{R}^d, \tag{30}$$

which guaranties the accuracy of the Hausdorff–Young inequality with constant 1. Moreover, one can define the generalized translation operator, which allows to define the convolution [24], the notion of modulus of continuity [17, 18], and different constructive and approximation properties.

If $a = 1/r$, $r \in \mathbb{N}$, $2\langle k \rangle + d + a > 2$, then $\mathcal{F}_{k,a}^{-1}(f)(x) = \mathcal{F}_{k,a}(f)(x)$ [4, Th. 5.3], i.e., for $f \in L^2(\mathbb{R}^d, d\mu_{k,a})$,

$$f(x) = \int_{\mathbb{R}^d} B_{k,a}(x,y)\mathcal{F}_{k,a}(f)(y) \, d\mu_{k,a}(y).$$

If condition (30) holds, the generalized translation operator is defined by

$$T^t(f)(x) = \int_{\mathbb{R}^d} B_{k,a}(t,y)B_{k,a}(x,y)\mathcal{F}_{k,a}(f)(y) \, d\mu_{k,a}(y), \qquad t \in \mathbb{R}^d.$$

Similarly, if $a = 2/(2r+1)$, $r \in \mathbb{Z}_+$, $2\langle k \rangle + d + a > 2$, then $\mathcal{F}_{k,a}^{-1}(f)(x) = \mathcal{F}_{k,a}(f)(-x)$ [4, Th. 5.3], that is,

$$f(x) = \int_{\mathbb{R}^d} B_{k,a}(-x,y)\mathcal{F}_{k,a}(f)(y) \, d\mu_{k,a}(y)$$

and

$$T^t(f)(x) = \int_{\mathbb{R}^d} B_{k,a}(-t,y)B_{k,a}(-x,y)\mathcal{F}_{k,a}(f)(y) \, d\mu_{k,a}(y), \qquad t \in \mathbb{R}^d.$$

The operators act in $L^2(\mathbb{R}^d, d\mu_{k,a})$ and $\|T^t\| = 1$.

If $d = 1$, using [4, Sect. 5.4], we can define

$$B_{k,a}^{\mathrm{even}}(x,y) = \frac{1}{2}\left[B_{k,a}(x,y) + B_{k,a}(x,-y)\right].$$

Then

$$B_{k,a}^{\mathrm{even}}(x,y) = j_{(2k-1)/a}\left(\frac{2}{a}|xy|^{a/2}\right).$$

For $2k + 1 + a > 2$ we have $(2k-1)/a > -1$. The inequality $|B_{k,a}^{\mathrm{even}}(x,y)| \le 1$ holds only when $(2k-1)/a \ge -1/2$ or, equivalently, $2k + a/2 \ge 1$. In this case the generalized translation operator can be defined by the formula

$$T^t(f)(x) = \int_{\mathbb{R}^d} B_{k,a}^{\mathrm{even}}(t,y)B_{k,a}(\pm x,y)\mathcal{F}_{k,a}(f)(y) \, d\mu_{k,a}(y), \qquad t \in \mathbb{R}^1.$$

Proposition. *Assume that*

$$2\langle k \rangle + d + a > 2.$$

Inequality (30) may not be true in general.

Cf. [4, Th. 5.11] and [19, L. 2.13]. To prove Proposition, we construct the following

Example. Let $d = 1$, $\langle k \rangle = k \ge 0$, and $2\langle k \rangle > 1 - a$. First, we remark that the kernel $B_{k,a}$ can be given by [4, (5.18)]

$$B_{k,a}(x,y) = j_{(2k-1)/a}\left(\frac{2}{a}|xy|^{a/2}\right) + \frac{\Gamma((2k-1)/a+1)}{\Gamma((2k+1)/a+1)}\frac{xy}{(ai)^{2/a}} j_{(2k+1)/a}\left(\frac{2}{a}|xy|^{a/2}\right).$$

Therefore, if $a = 1$ and $k > 0$, we get

$$B_{k,1}(x, y) = j_{2k-1}(t) - \text{sign}\,(xy) \frac{(t/2)^2}{2k(2k+1)} j_{2k+1}(t), \qquad t = 2|xy|^{1/2}.$$

Let us now investigate when $|B_{k,1}(x, y)| \leq 1$, $x, y \in \mathbb{R}$, for different values of k.

Taking into account the known properties of the Bessel function

$$J_{\nu-1}(t) + J_{\nu+1}(t) = 2\nu t^{-1} J_\nu(t),$$
$$J_{\nu-1}(t) - J_{\nu+1}(t) = 2J_\nu'(t),$$

we get for $\nu = 2k$ that

$$B_{k,1}(x, y) = \begin{cases} j_{2k}(t), & xy \leq 0, \\ 2^{2k}\Gamma(2k)t^{1-2k} J_{2k}'(t), & xy \geq 0. \end{cases}$$

Hence, $|B_{k,1}(x, y)| \leq 1$ for $xy \leq 0$ and for any $k \geq 0$.

Case $0 < k < 1/4$. Using asymptotic formula for the derivative of the Bessel function

$$J_\nu'(t) = -\sqrt{\frac{2}{\pi t}} \left\{ \sin\,(t - \nu\pi/2 - \pi/4) + O(t^{-1}) \right\}, \qquad t \to +\infty,$$

we obtain that the kernel $B_{k,1}(x, y)$ is not bounded for $0 < k < 1/4$, $xy > 0$, and is uniformly bounded for $k \geq 1/4$, $x, y \in \mathbb{R}$.

Case $k = 1/4$. Using $J_{1/2}(t) = (\pi t/2)^{-1/2} \sin t$, we get for $xy > 0$

$$B_{1/4,1}(x, y) = 2(\cos t - t^{-1} \sin t)$$

and then $\max_{x,y\in\mathbb{R}} |B_{1/4,1}(x, y)| \approx 2.13$.

Case $1/4 < k < 1/2$. Easy computer calculations show that $|B_{k,1}(x, y)| \leq M_k$ for $x, y \in \mathbb{R}_+$, where $M_k = \max_{x,y\in\mathbb{R}_+} |B_{k,1}(x, y)| > 1$ for $k \in (1/4, k_0)$ and $M_k = 1$ for $k \in [k_0, 1/2)$. Moreover, $k_0 \approx 0.44$. The number k_0 can be found from the condition that the first minimum of the function $2^{2k}\Gamma(2k)t^{1-2k} J_{2k}'(t)$ for $t > 0$ is equal to -1.

Case $k \geq 1/2$. For the kernel $B_{k,1}$, the following integral representation with a nonnegative weight holds: [4, (5.17), (5.19)]

$$B_{k,1}(x, y) = \frac{\Gamma(k+1/2)}{\Gamma(k)\Gamma(1/2)} \int_{-1}^1 j_{k-1}\big(\sqrt{2|xy|(1 + \text{sign}\,(xy)u)}\big)(1 + u)(1 - u^2)^{k-1}\, du.$$

Since $|j_\lambda(t)| \leq 1$ for $t \in \mathbb{R}$ and $\lambda \geq -1/2$, then $|B_{k,1}(x, y)| \leq B_{k,1}(0, 0) = 1$, $x, y \in \mathbb{R}$ for any $k \geq 1/2$.

We formulate the following

Conjecture. Inequality (30) holds whenever $2\langle k \rangle + d + a \geq 3$.

In particular, we expect that if $d \geq 3$, then inequality (30) always holds. Calculations above for the case $d = 1$ and results of the paper [6] for $d = 2$ show that the condition $2\langle k \rangle + d + a \geq 3$ is only sufficient for (30) to hold.

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