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Research paper

Planar and Non Planar Construction of γ- Uniquely Colorable Graph

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Abstract

A uniquely colorable graph G whose chromatic partition contains atleast one γ - set is termed as a γ - uniquely colorable graph. In this paper, we provide necessary and sufficient condition for $\overline{G}$and G* to be γ - uniquely colorable whenever G γ- uniquely colorable and also provide constructive characterization to show that whenever G is γ- uniquely colorable such that |P | ≥ 2, G can be both planarand non planar.

Keywords: *Complement; Dual;Non Planar;Planar;Uniquely colorablegraphs.*

1. Introduction

In [1] Bing Zhou investigated the dominating -χ-color number, $d_\chi(G)$, of a graph G. In [2],[3], M. Yamuna et al introduced γ - uniquely colorable graphs and also provided the constructive characterization of γ -uniquely colorable trees and characterized planarity of complement of γ - uniquely colorable graphs. In [4],[5],M. Yamuna et al introduced Non domination subdivision stable graphs (NDSS) andcharacterized planarity of complement of NDSS graphs

2. Terminology

We consider simple graphs G with n vertices and m edges. K_n is a complete graph with n vertices. K_5 and $K_{3,3}$are called Kuratowski's graph. Results related to graph theory we refer to [6].

Chromatic partition of a graph G is partition the vertices into smallest possible umber of disjoint ,independent sets. A graph G = (V, E) is said to be uniquely colorable ifhas a unique chromatic partition.

D is adominating set if every vertex of V – D is adjacent to some vertex of D. Minimum cardinality of D, is said to be a minimum dominating set (MDS). The cardinality of any MDS for G is said to be domination number of G, represented by γ(G). Results related to domination we refer to [7].

3. Result and Discussion

A uniquely colorable graph G whose chromatic partition contains atleast one γ– set is termed as a γ– uniquely colorable graph.

In Fig. 1 G_1 and G_2 are γ– uniquely colorable graphs. $\overline{G_1}$ isγ– uniquely colorable while $\overline{G_2}$ is not γ– uniquely colorable graph. So when G is γ– uniquely colorable, $\overline{G}$need not be γ– uniquely colorable. In Fig. 2 G_1 and G_2 are γ– uniquely colorable graphs. $G_1^*$is γ– uniquely colorable while $G_2^*$is not γ– uniquely colorable graph. So when G is γ - uniquely colorable ,G^*need not be γ– uniquely colorable. In this paper, we determine the condition for$\overline{G}$ and G* to be γ– uniquely colorable whenever G is γ– uniquely colorable. We also provide the constructive characterization to show that whenever G is γ uniquely colorable such that | P | ≥ 2, G can be both planar and non planar.

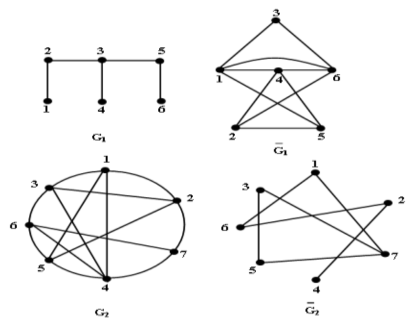

Fig.1

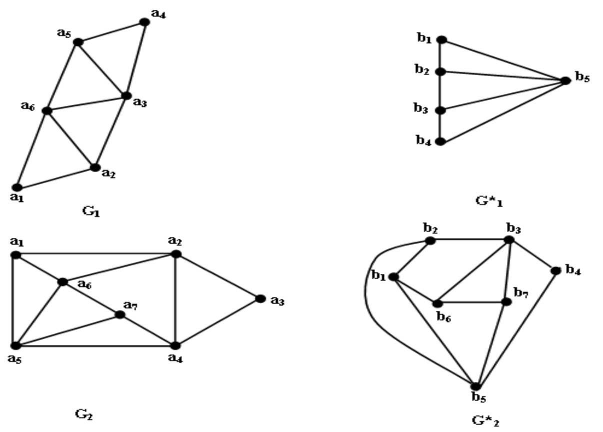

Fig.2

Theorem 1. Let G be a isγuniquelycolorable graph. $\bar{G}$is alsoγ– uniquely colourable if and only if $\exists$ a unique smallestpossible partition P = { V_1, V_2, … V_k } of V(G) ∍
1. every V_i, i = 1 to k is a clique
2. there exist one V_i∍ every vertex in V – { V_i } is notadjacent to atleast one vertex in V_i
3. | V_j|≥| V_i |, for every i ≠ j
4. V_i is the smallest set in G satisfying 2

Proof. Assume that $\bar{G}$is a γ– uniquely colourable graph, implies-there exist a partition P_1 = { V_1, V_2, … V_k } for $\bar{G}$such that

- P_1 is unique and smallest possible set.
- every V_i, i = 1 to k, is independent in $\bar{G}$implies every V_i is a clique in G.
- there exist one V_i such that V_i is a γ– set for $\bar{G}$. Also | V_i | ≥ | V_j|, for every i ≠ j implies there exist one V_i in G such that every vertex in V – { V_i } not adjacent to atleast onevertex in every V_i.

implies P_1 = { V_1, V_2, … V_k } is a γ– chromatic partition for V (G). P_1 is not unique implies $\exists$one P_2 = { W_1, W_2, … W_k } in G such that { W_1, W_2, … W_k } is a clique, implies P_2 is also a γ chromatic partition in $\bar{G}$ such that every W_i is independent and | P_1 | = | P_2|, a contradiction to our assumption that P_1 is unique. P_1 is not smallest, implies one P_3 = { V_1, V_2, … V_k },q < k such that P_3 isa γchromatic partition in G ∍ U_i, i = 1to q is clique implies P_3 isa γchromatic partition in$\bar{G}$ ∍ every U_i is independent and | P_3| < | P_1| , a contradiction. P_1 is a γuniquely colorable partition for $\bar{G}$implies there exist one V_i such that V_i is a γ–set for $\bar{G}$, implies every vertex in V – { V_i } is adjacent to atleast one vertex in V_i, implies P_1 is a γ– chromatic partition in G ∍every vertex in V – { V_i } is not adjacent to atleast one vertex in V_i. Also, we know that |V_i | ≤ | V_j | for every i ≠ j in $\bar{G}$, implies it is true in G also. If V_i is not the smallest set ∍ every vertex in V – { V_i } is ⊥ to atleast one vertex in V_i in $\bar{G}$, implies there exist one W contained in V ($\bar{G}$) such that | W |< | V_i | and every vertex in W - V ($\bar{G}$) is⊥ to atleast one vertexin W, a contradiction ⇒ V_i is the smallest set satisfying the property. Hence P_1 is a γ– chromatic partition in G ∍ the conditions of the theorem are satisfied.

Conversely assume that the conditions of the theorem are satisfied. P is a partition such that it is unique and smallest such that every $V_i$is a clique, implies P_1 is a partition in $\bar{G}$such that every V_i is independent. If P is not a smallest possible partition in $\bar{G}$then there exist one partition P_4 = { R_1, R_2, … R_q }, q < k in $\bar{G}$such that each R_i is independent , implies P_4 is a partition in G such that every $R_i$is a clique such that | P_4 |< | P| , a contradiction. P is not unique in $\bar{G}$, implies there exist a partition P_5 = { S_1, S_2, … S_k} such that each S_i is independent in$\bar{G}$, implies P, P_5 are two possible partition with the same cardinality in G, a contradiction.P is a partition ∍ there exist one V_i, every vertex in V –{ V_i } is not ⊥ to atleast one V_i , | V_j |≥ |V_i | for any i ≠ j implies P is a partitionin $\bar{G}$∍every vertex in V – {V_i } isadjacent toatleast one V_i, | V_i | ≤ | V_j | , i ≠ j, implies $V_i$is a dominating set for $\bar{G}$. Since V_i is the smallest set satisfying this property, implies V_i is the γ - set for $\bar{G}$

Let P = { R_1, R_2, … R_q }, betheset ofregions of G. Let T ={ r_1, r_2, … r_q }, betheset of vertices in the regions R_1, R_2, ..., $R_q$respectively, that is r_1 is the vertex in the region R_1, r_2 is the vertexin the region R_2, ..., r_q is the vertex in the region $R_q$respectively.

We observe that

- There is a 1-1 mapping between S and T, i.e $\forall$ $R_i \in$ S $\exists r_i$in T, i = 1,..., q.
- $\forall$ X ⊆ S $\exists$ a corresponding set in T (say X^*) *, i.e if X ⊆ S ={ R_i, R_p, R_j }, then X ⊆ T = { r_i, r_p, r_j }.
- If a is any edge in G there is a corresponding edge in G^* (say a*).
- Let D ⊆ S ∍ every region in S – D is⊥ to atleast one region in D ⇒ $\exists$ D^*⊆ T ∍ any vertex in T - D^* is ⊥to atleast vertex inD^*.
- D is a smallest cardinality satisfying this property ⇒ D^* is a γ - set for G^*.

Theorem 2. Let G be a γ - uniquely colourable graph. G* isalso γ - uniquely colourable graph if and only if there exist a unique smallest partition P = { R_1, R_2, …, R_k } of R (G) such that
1. everyR_i, i = 1, 2, ...,k is independent.
2. there exist one R_i such that every region in R – { R_i} is adjacent to atleast one region in R_i.
3. | R_j | ≥ |R_i|.

Proof. Assume that G* is γuniquelycolourable graph. If G* is γ - uniquely colourable graph, then there exist a partition P = { V_1, V_2, …, V_k } such that P is a γ - chromatic partition, ⇒
1. every V_i is independent.
2. V_1 is a γ - set for G^*.

1 implies, there exist a set of regions R_1, R_2, …,R_k in G such that every R_i is independent.
2 implies, there exist R_1∍ every region in R – { R_1} is adjacent to atleast one region in R_1 and R_1 is the smallest set satisfying this property implies the conditions of the theorem are satisfied.
Conversely, assume that the conditions of the theorem are satisfied. P = { V_1, V_2, …, V_k } is a partition of R (G), implies there exist a partition P_1 = { V_1, V_2, …, V_k } of V(G^*).
1 implies, every V_i, i = 1, 2, ...,k is independent.
2 implies, there exist one V_i∍ every vertex in V – { V_i} is ⊥to atleast one region in V_i.
3 implies | V_j | ≥|V_i | for all i ≠ j
Since P is a unique partition there exist no other partition of V (G^*) that satisfies all these conditions implies, P_1 is a γ - chromaticpartition for G*.

Planar and Non planar Construction

In this section, we provide constructive characterization to showthat whenever G is γuniquelycolorable such that | P | ≥ 2, G canbe both planarand nonplanar.
Planar Construction when | P | = 2.
Let γ(G) = k_1. Let P = { V_1, V_2}, where V_1 = { a_1,a_2, …,a_{k1}} V_2 ={ b_1, b_2, …,b_{k2}},k_2≥ k_1, k_1≥ 3, k_2≥4.
Constructagraph G_1 as follows
1. V (G_1) = V (G)
2.Considerk$_1$ vertices in V_1 and V_2 say { a_1, a_2, …,a_{k1}} and { b_1, b_2, …,b_{k2}}.
Construct a comb graph with 2k_1 vertices. Label the vertices of this comb as seen in Fig. 3

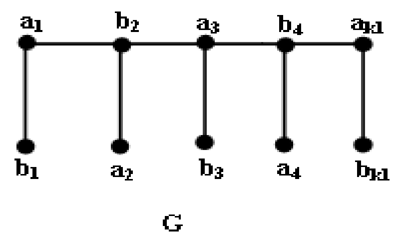

Fig.3

Include the remaining k_2 - k_1 vertices of V_2 as pendant vertices with a_{k1} as the support vertex. The general structure of graph G_1 is as seen in the Fig.4.

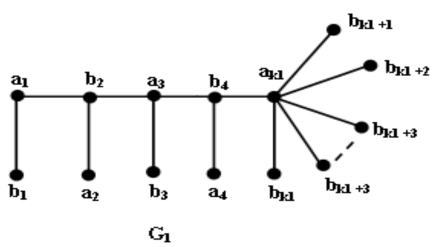

Fig.4

Since we have atleast k_1 pendant vertices, $\gamma(G_1) \geq k_1$, $\{a_1, a_2, \ldots, a_{k1}\}$ is a dominating set for G_1, implies $\gamma(G_1) = k_1$. Since $\langle a_1, a_2, \ldots, a_{k1}, b_1, b_2, \ldots, b_{k2}\rangle$ isacomb, the only possible maximal independent sets are $\{a_1, a_2, \ldots, a_{k1}\}and\{b_1, b_2, \ldots, b_{k2}\}$. P = $\{V_1, V_2\}$ is a partition for G_1 suchthat

1. V_1 is aγ- set for G_1
2. P is the only possible partition for G_1, $\Rightarrow G_1$ is a γ– uniquely colorable graph.

Non Planar Construction when $|P| = 2$.

Let $\gamma(G) \geq k_1$. $k_1 \geq 6$, P = $\{V_1, V_2\}$ $V_1 = \{a_1, a_2, \ldots, a_{k1}\}$;$V_2 = \{b_1, b_2, \ldots, b_{k2}\}$, $k_2 \geq 6$

Construct a graph G_1 as follows

1.V (G_1) = V (G)

2.Considerk$_1$ vertices in V_1 and k_1 vertices in V_2 say $\{a_1, a_2, a_3, a_4, a_5, a_6\}$, $\{b_1, b_2, b_3, \ldots, b_{k2}\}$. Let $\langle a_1, a_2, a_3, b_1\ b_2, b_3\rangle$ is $K_{3,3}$. Include the remaining a_i, b_i, i = 1, 2, 3. Include the remaining b_i, i = 6,7,..,k_2 as arbitrary pendant vertices adjacent to any a_i, i = 1, 2, 3. Graph G_1 is as seen in Fig.5.

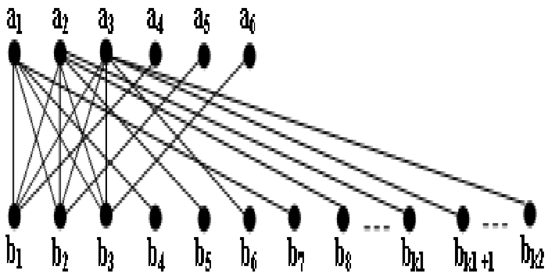

Fig.5.

Since G_1 has atleast k_1 pendant vertices $\{a_4, a_5, \ldots, a_{k1}, b_4, b_5, \ldots, b_{k1}\}$,$\gamma(G_1) \geq k_1$, $\{V_1\}$ dominates G_1. Also $|V_1| = k_1$, implies that V_1isaγ- setfor G_1, since G_1 isabipartite graph P = $\{V_1, V_2\}$ is the onlychromatic partition for G_1 such that V_1 is a γ - set for G_1, implies G_1 is γ - uniquely colorable and non planar.

$\gamma(G) = 3$, P = $\{V_1, V_2\}$, $V_1 = \{a_1, a_2, a_3\}$, $V_2 = \{b_1, b_2, \ldots, b_{k1}\}$,$k_1 \geq 6$,
$\gamma(G) = 4$, P = | P | = 3V_1 = $V_1 = \{a_1, a_2, a_3, a_4\}$, $V_2 = \{b_1, b_2, \ldots, b_{k1}\}$,$k_1 \geq 6$,
$\gamma(G) = 5$, P = $\{V_1, V_2\}$, $V_1 = V_1 = \{a_1, a_2, a_3, a_4, a_5\}$, $V_2 = \{b_1, b_2, \ldots, b_{k1}\}$,$k_1 \geq 6$,are analogus to the above discussion.

| P | = 3 = P = $\{V_1, V_2, V_3\}$, = K_1. $|V_2| = k_2$, $|V_3| = k_3$, $k_2, k_3 \geq k_1$.

Planar Construction when $|P| = 3$.

| P | = 3 = P = $\{V_1, V_2, V_3\}$, = K_1. $|V_2| = k_2$, $|V_3| = k_3$, $k_2, k_3 \geq k_1$.

Consider a wheel graph with k vertices where $k = k_1 + 2k_i$, where$k_i = \min(k_2, k_3)$. Label the vertices of the wheel in the followingfashion as seen in Fig.6.

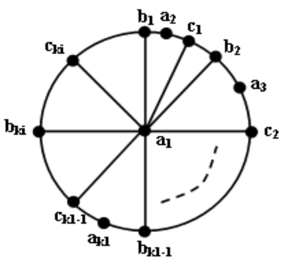

Fig.6

If $k_2 \neq k_3$, then we include the remaining vertices as follows.

Let $k_2 > k_3$. Let $k_2 = k_3 + m$. Label the additional vertices as $\{b_{k3+1}, b_{k3+2}, \ldots, b_{k2}\}$. Include these vertices as seen in Fig.7.

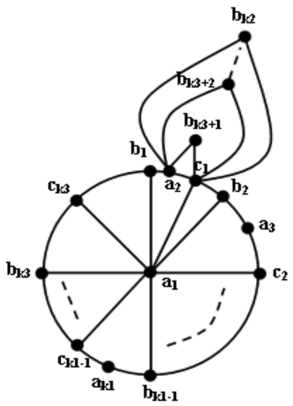

Fig.7

Since $\langle b_1, a_j, c_i\rangle$, i = 1 to k_{1-1}, j = 2 to a_{k1} is P_3eithea$_j$ or b_i or c_i should be included in every possible γ - set for G. $\{a_1, a_2, \ldots, a_{k1}\}$ is a γ- set for G. Also $\{V_1, V_2, V_3\}$ is the only possible chromatic partition for G implies γ - uniquely colorable graph G is planar.

4. Conclusion

In this paper, we provide necessary and sufficient condition for $\overline{G}$and G^* to be γ uniquely colorable and also provide constructive characterization to show that whenever G is γ - uniquely colorable such that $|P| \geq 2$, G can be both planar and non planar.

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