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Planar Feedback Vertex Set and Face Cover: Combinatorial Bounds and Subexponential Algorithms

Athanassios Koutsonas · Dimitrios M. Thilikos

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Abstract The PLANAR FEEDBACK VERTEX SET problem asks whether an n-vertex planar graph contains at most k vertices meeting all its cycles. The FACE COVER problem asks whether all vertices of a plane graph G lie on the boundary of at most k faces of G. Standard techniques from parameterized algorithm design indicate that both problems can be solved by sub-exponential parameterized algorithms (where k is the parameter). In this paper we improve the algorithmic analysis of both problems by proving a series of combinatorial results relating the branchwidth of planar graphs with their face cover. Combining this fact with duality properties of branchwidth, allows us to derive analogous results on feedback vertex set. As a consequence, it follows that PLANAR FEEDBACK VERTEX SET and FACE COVER can be solved in $O(2^{15.11\cdot\sqrt{k}}+n^2)$ and $O(2^{10.1\cdot\sqrt{k}}+n^2)$ steps, respectively.

Keywords Branchwidth · Parameterized algorithms · Feedback vertex set · Face cover

1 Introduction

In this paper, we offer an improved algorithmic analysis for two widely studied combinatorial problems on planar graphs. The first is the planar version of FEEDBACK VERTEX SET that asks, given a graph G and a non-negative integer k, whether all cycles of G can be blocked by a set of at most k vertices. The second is FACE COVER

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A. Koutsonas · D.M. Thilikos (✉)
National and Kapodistrian University of Athens, Panepistimioupolis, 15784 Athens, Greece
e-mail: sedthilk@math.uoa.gr

A. Koutsonas
e-mail: akoutson@math.uoa.gr

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that asks, given a plane graph G and a non-negative integer k whether the boundaries of at most k faces contains all the vertices of G. Our aim is to show that both problems are closely related and to use this fact to improve the analysis of algorithms for both problems.

The FEEDBACK VERTEX SET problem, as well as its directed version, are among the most studied *NP*-complete problems, mainly due to their numerous applications (see [16]). A wide range of algorithmic results on FEEDBACK VERTEX SET have been proposed including approximation algorithms [7, 21, 22], exact algorithms [20] and heuristics [27].

In our study, we focus our attention on the parameterized complexity of both PLANAR FEEDBACK VERTEX SET and FACE COVER.

In a parameterized problem the input is seen as a pair (I, k), where I is the main part of the problem and k is a parameter. A *fixed parameter* algorithm (or simply FPT-*algorithm*) is one that solves the problem in $O(f(k) \cdot |I|^{O(1)})$ steps where f is a function depending exclusively on the parameter k (for more on parameterized complexity and algorithms, see [17, 28]).

For both FEEDBACK VERTEX SET and FACE COVER, we consider their parameterized versions, where k is the parameter. Many FPT-algorithms were proposed for FEEDBACK VERTEX SET. The best current results in this direction are the $O(4^k kn)$ step probabilistic algorithm in [1] and the $O(5^k kn^2)$ step algorithm in [6] (throughout the paper and for both problems, we denote by n the number of vertices of the input graph).

When restricted to planar graphs, both PLANAR FEEDBACK VERTEX SET and FACE COVER are solvable by subexponential FPT-algorithms, i.e. algorithms running in $O(2^{o(k)} \cdot n^{O(1)})$ steps. The first results of this kind were given by Kloks et al. in [26] for both problems. Furthermore, Fernau and Juedes proved that FACE COVER can be solved in $O(2^{24.551\sqrt{k}} \cdot n)$ steps [15].

All previous results can be improved using the win/win meta-algorithmic framework emerging from the bidimensionality theory in [8]. Combining this framework with the dynamic programming algorithms in [10, 11] is easy to derive $O(2^{28.48\sqrt{k}} \cdot n + n^3)$ and $O(2^{19.04\sqrt{k}} \cdot n + n^3)$ step algorithms for PLANAR FEEDBACK VERTEX SET and FACE COVER respectfully.

In this paper, we proceed to an improved "taylor made" analysis of the complexity of both PLANAR FEEDBACK VERTEX SET and FACE COVER. In fact, we unify the analysis of both problems, by exploiting a duality relation between them. As a consequence, we prove that PLANAR FEEDBACK VERTEX SET and FACE COVER can be solved in $O(2^{15.11\cdot\sqrt{k}} + n^2)$ and $O(2^{10.1\cdot\sqrt{k}} + n^2)$ steps, respectively. Our analysis resides in a thorough analysis of the structure of face covers in planar graphs, which leads to combinatorial bounds of independent interest. Moreover, the obtained running times use kernelization techniques as developed in [2–5, 31].

The presentation of the paper is organized as follows. In Sect. 2, we present the main definitions and some preliminary results. In Sect. 3, we present the main algorithmic techniques, as well as our approach for proving the claimed complexity bounds. The algorithmic analysis of Sect. 3 is supported by a series of combinatorial results presented in Sect. 4. Finally, in Sect. 5, we conclude with the discussion of our results.

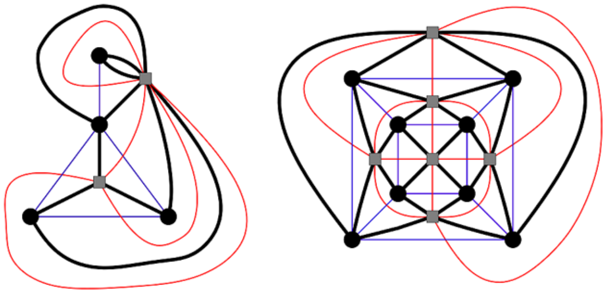

Fig. 1 Two graphs embedded along with their duals (the vertices of the duals are depicted with *squares*). Their radial graphs are depicted with *bold lines*

2 Preliminaries

2.1 Definitions

We consider graphs that may have loops or multiple edges. If a graph has no multiple edges or loops we call it *simple*. Given a graph G, we denote as $V(G)$ its vertex set and as $E(G)$ its edge multiset. For any set $S \subseteq V(G)$, we denote as $G[S]$ the subgraph of G induced by the vertices in S. We also denote as $G \setminus S$ the graph $G[V(G) - S]$ and if $v \in V(G)$ we also write $G \setminus v$ instead of $G \setminus \{v\}$. Finally, if $e \in E(G)$, we write $G \setminus e$ instead of $(V(G), E(G) - \{e\})$. Given a sphere $\mathbb{S}_0$ and a subset $\Delta \subseteq \mathbb{S}_0$, the *closure* of Δ is denoted by $\overline{\Delta}$, and the boundary of Δ is $\widehat{\Delta} = \overline{\Delta} \cap \overline{\mathbb{S}_0 - \Delta}$.

We use the term *plane graph* for a planar graph along with an embedding of it in the sphere $\mathbb{S}_0$ without crossings. Given a plane graph $G = (V, E)$, we call *noose*, a Jordan curve in $\mathbb{S}_0$ that meets the drawing only in vertices of G. For a noose N passing through vertices $v_1, v_2, \ldots, v_n$ we will use the same notation we use for a cycle of a graph, i.e. $N = v_1 v_2 \cdots v_n v_1$. The *length* $|N|$ of a noose N is the number of vertices it meets.

To simplify notations, we do not distinguish between a vertex of the graph and the point of $\mathbb{S}_0$ used in the drawing to represent the vertex or between an edge and the open line segment representing it. We denote as $F(G)$, the set of faces of this embedding, i.e. the connected components of $\mathbb{S}_0 \setminus G$ (that are open sets of $\mathbb{S}_0$). We also use the notation G^* to denote an embedding of the dual graph of G, i.e. the graph whose vertices correspond to the faces of G and whose edges correspond to pairs of faces that are incident to a common edge in G.

Given a plane graph G with at least one edge, we define its *radial graph* R_G as the plane graph whose vertex set is $V(G) \cup V(G^*)$ and whose edges are defined as follows: Let $\mathcal{C} = \{C_1, \ldots, C_r\}$ be the connected components of $\mathbb{S}_0 \setminus (G \cup G^*)$ and observe that for $i = 1, \ldots, r$ C_i is an open set whose boundary contains one vertex, say v_i, from $V(G)$ and one vertex, say u_i, from $V(G^*)$. The edge set of R_G is the set $E(R_G) = \{\{v_i, u_i\}, i = 1, \ldots, r\}$ where edge $\{v_i, u_i\}$ has multiplicity 1 if both v_i and u_i have degree at least 2 in G and G^* respectively, otherwise its multiplicity is 2 (clearly, $\{v_i, u_i\}$ can be seen as a subset of the open set C_i). Notice that $R_G = (V(G) \cup F(G), E(R_G))$ is a bipartite graph, whose parts are the vertex and face sets of G, respectively.

A vertex set $S \subseteq V(G)$ is a *feedback vertex set* of a graph G, if the graph $G \setminus S$ is acyclic. The *feedback vertex set number* of a graph G, denoted as $\mathbf{fvs}(G)$, is the minimum size of a feedback vertex set of G.

A *face cover* of a plane graph G is a set $R \subseteq F(G)$ of faces, such that all vertices of G lie on the boundary of some face in R. We define the *face cover number* of a plane graph G, as the minimum size of a face cover of G and we denote it as $\mathbf{fc}(G)$.

We consider the following two parameterized problems.

PLANAR FEEDBACK VERTEX SET
Instance: A planar graph G and a non-negative integer k
Parameter: k
Question: $\mathbf{fvs}(G) \leq k$? FACE COVER

Instance: A plane graph G and a non-negative integer k
Parameter: k
Question: $\mathbf{fc}(G) \leq k$?

2.2 Branch and Sphere-Cut Decompositions

Let G be a graph on n vertices. A *branch decomposition* (T, μ) of a graph G consists of an unrooted ternary tree T (i.e. all internal vertices are of degree three) and a bijection $\mu : L \to E(G)$ from the set L of leaves of T to the edge set of G. We define for every edge e of T the *middle set* $\omega(e) \subseteq V(G)$, as follows: Let T_1 and T_2 be the two connected components of $T \setminus e$. Then, let G_i be the graph induced by the edge set $\{\mu(f) : f \in L \cap V(T_i)\}$ for $i \in \{1, 2\}$. The *middle set* is the intersection of the vertex sets of G_1 and G_2, i.e. $\omega(e) = V(G_1) \cap V(G_2)$. The *width* of (T, μ) is the maximum order of the middle sets over all edges of T (in case T has no edges, then the width of (T, μ) is equal to 0). An optimal branch decomposition of G is defined by the tree T and the bijection μ which give the minimum width, the *branchwidth*, denoted by $\mathbf{bw}(G)$.

Given two graphs H and G, we write $H \leq G$, when H can occur from a subgraph of G after a series of edge contractions. It is known from [29], that if $H \leq G$, $\mathbf{bw}(H) \leq \mathbf{bw}(G)$.

Let G be a plane graph. A branch decomposition (T, μ) of G is called *sphere-cut decomposition* if for every edge e of T there exists a noose N_e, such that

(a) $G_i \subseteq \Delta_i \cup N_e$ for $i = 1, 2$, where G_i the subgraph induced by the edges corresponding to the leaves of the component $T_i(e)$ of $T \setminus e$ and Δ_1, Δ_2 are the two open discs bounded by N_e,
(b) for every face f of G, $N_e \cap f$ is either empty or connected (i.e. if the noose traverses a face then it traverses it once).

Sphere-cut decompositions appeared as a concept in [30] and were studied in [11–13]. As proved in [30, Theorem (5.1)] every planar graph G where $\mathbf{bw}(G) \leq k$, has a sphere-cut decomposition with width at most k. The following theorem follows from Theorem (7.2) of [30] and will be useful for our analysis.

Theorem 1 *Let G be a planar graph that contains as a subgraph a cycle of length at least 2 and let G^* be a dual of G. Then, the branch-width of G is equal to the branch-width of G^*.*

3 The Algorithms

3.1 The Win/Win Approach for Planar Graphs

The standard technique for the design of subexponential parameterized algorithms for graph parameters on planar graphs, relies on two conditions: the existence of a sublinear combinatorial bound for the branchwidth in terms of the parameter and dynamic programming on branch decompositions (for a survey, see [14]). In particular, we refer to any graph parameter $\mathbf{p}$, for which there exist two positive real numbers $\alpha_{\mathbf{p}}$ and $\beta_{\mathbf{p}}$, such that:

(a) For any planar graph G, $\mathbf{bw}(G) \leq \alpha_{\mathbf{p}} \cdot \sqrt{\mathbf{p}(G)}$.
(b) For every planar graph G and given an optimal sphere-cut decomposition (T, μ) of G, $\mathbf{p}(G)$ can be computed in $O(2^{\beta_{\mathbf{p}} \cdot \mathbf{bw}(G)})$ steps.

Theorem 2 *If conditions above are satisfied for some parameter $\mathbf{p}$ and some $\alpha_{\mathbf{p}}$ and $\beta_{\mathbf{p}}$, then one can construct an algorithm that, given a graph G and a non-negative integer k, checks whether $\mathbf{p}(G) = k$ in $O(2^{\alpha_{\mathbf{p}} \cdot \beta_{\mathbf{p}} \cdot \sqrt{k}} n + n^3)$ steps.*

Proof Given the optimal sphere-cut decomposition (T, μ) of G, we first check whether $\mathbf{bw}(G) > \alpha_{\mathbf{p}} \cdot \sqrt{k}$. If this is true we respond "no" or "yes" depending on whether $\mathbf{p}$ is a minimization or a maximization parameter. Else, according to the second condition, using the branch decomposition, we can compute $\mathbf{p}(G)$ in $O(2^{\alpha_{\mathbf{p}} \cdot \beta_{\mathbf{p}} \cdot \sqrt{k}} n)$ steps. The $O(n^3)$ overhead corresponds to the time required to construct an optimal sphere-cut decomposition. For this, one can use the $O(n^3)$ step algorithm by Gu and Tamaki [23] (see also [24, 25, 30]). □

According to the results in [8], conditions (a) and (b) are satisfied for both **fvs** and **fc**. Therefore Theorem 2 can be applied for these parameters. We define $\alpha_{\mathbf{fvs}}$ ($\alpha_{\mathbf{fc}}$) and $\beta_{\mathbf{fvs}}$ ($\beta_{\mathbf{fc}}$) as the minimum values for $\alpha_{\mathbf{p}}$ and $\beta_{\mathbf{p}}$, for which conditions (a) and (b), respectively, holds for **fvs** (**fc**). In what follows, we provide bounds to these constants towards improving the time analysis of the algorithm in Theorem 2.

3.2 Estimating $\beta_{\mathbf{fvs}}$ and $\beta_{\mathbf{fc}}$

Regarding condition (b), and in case of **fvs**, it is known that given an optimal sphere-cut decomposition of a n-vertex planar graph G, there is a dynamic programming algorithm that computes $\mathbf{fvs}(G)$ in $O(2^{3.56\mathbf{bw}(G)} \cdot n)$ steps [10]. We conclude, that condition (b) holds for $\beta_{\mathbf{fvs}} \leq 3.56$.

To estimate $\beta_{\mathbf{fc}}$, we use the well known reduction of the FACE COVER problem to the following problem:

PLANAR BLUE-RED DOMINATING SET
Instance: A planar bipartite graph G with parts B and R and a non-negative integer k.
Parameter: k
Question: Is there a vertex set $D \subseteq R$ where $|D| \leq k$ and such that every vertex in B has a neighbour in D?

Observe that (G, k) is a yes-instance of FACE COVER, if and only if (R_G, k) is a yes-instance of PLANAR BLUE-RED DOMINATING SET (set $B = V(G)$ and $R = F(G)$) (see also [15]). From [11, Theorem 2.3.2], PLANAR BLUE-RED DOMINATING SET can be solved in $O(2^{1.19\cdot \mathbf{bw}(G)} \cdot |V(G)|)$ steps, provided that an optimal branch decomposition is given. As we prove in Sect. 4, it holds that $\mathbf{bw}(R_G) \leq 2 \cdot \mathbf{bw}(G)$ (Theorem 5). We conclude that condition (b) is satisfied for $\beta_{\mathbf{fc}} \leq 2.38$.

3.3 Easy Bounds for $\alpha_{\mathbf{fvs}}$ and $\alpha_{\mathbf{fc}}$

Condition (a) follows directly from the theory of bidimensionality introduced in [8]. Applying the meta-algorithmic result of [8] (Theorem 4.14) for both parameters **fvs** and **fc**, condition (a) holds for $\alpha_{\mathbf{fvs}}, \alpha_{\mathbf{fc}} \leq 8$. This implies the existence of an $O(2^{28.48\cdot\sqrt{k}} \cdot n + n^3)$ step algorithm for the PLANAR FEEDBACK VERTEX SET problem (to our knowledge, no other exact bound for this problem exists) and the existence of an $O(2^{19.04\cdot\sqrt{k}} \cdot n + n^3)$ step algorithm for the FACE COVER problem (improving the constants of [15] for the same problem).

The above estimations for $\alpha_{\mathbf{fvs}}$ and $\alpha_{\mathbf{fc}}$ can be easily further improved using known results. Kloks et al. [26] proved that for any planar graph G, there is a planar graph H containing G as a subgraph such that $\mathbf{ds}(H) \leq \mathbf{fvs}(G)$ (here by $\mathbf{ds}(H)$ we denote the minimum size of a dominating set of H). Moreover it holds that for any planar graph H, $\mathbf{bw}(H) \leq 6.364\sqrt{\mathbf{ds}(H)}$ [18]. As $\mathbf{bw}(G) \leq \mathbf{bw}(H)$, we obtain that $\mathbf{bw}(G) \leq 6.364\sqrt{\mathbf{fvs}(G)}$ and this yields condition (a) for $\alpha_{\mathbf{fvs}} \leq 6.364$. For $\alpha_{\mathbf{fc}}$, we need to make the following observation: Suppose that a plane graph G has a face cover $U \subseteq F(G)$ of size $\leq k$. Let H be the graph obtained from G, if for each $f \in U$ we draw a vertex v_f inside f and connect it with the vertices incident to f. Notice that the new vertices constitute a dominating set of H, of size at most k. Again, from the result of [18], we conclude that $\mathbf{bw}(G) \leq \mathbf{bw}(H) \leq 6.364 \cdot \sqrt{k}$, thus $\alpha_{\mathbf{fc}} \leq 6.364$.

According to the above, there is an $O(2^{22.66\cdot\sqrt{k}} \cdot n + n^3)$ step algorithm for the PLANAR FEEDBACK VERTEX SET problem and an $O(2^{15.15\cdot\sqrt{k}} \cdot n + n^3)$ step algorithm for the FACE COVER problem. To our knowledge, these are the fastest algorithms for these problems so far.

3.4 Improved Bounds for $\alpha_{\mathbf{fvs}}$ and $\alpha_{\mathbf{fc}}$

In order to find better bounds for $\alpha_{\mathbf{fvs}}$ and $\alpha_{\mathbf{fc}}$, we should focus our attention to the structure of the corresponding parameters. In fact, face cover and planar feedback vertex set are closely related in dual graphs. Informally speaking, the "dual" version of the face cover number is upper bounded by the feedback vertex set number. Formally, we observe the following:

Lemma 1 *Let G be a plane graph and let G^* be a dual of G. Then,* $\mathbf{fc}(G^*) \leq \mathbf{fvs}(G)$.

Proof We examine the non-trivial case where G is not a forest. Let $S \subseteq V(G)$ be a feedback vertex set in G, of minimum size. As the boundary of any face $f \in F(G)$ contains a cycle of G, it also contains a vertex $v \in S$. This implies that any vertex $f^* \in V(G^*)$ of G^* is in the boundary of some face v^* of S^*, where $S^* \subseteq F(G^*)$ is the set of the duals of the vertices in S. Therefore S^* is a face cover of G^*. □

The above lemma, combined with the duality of branchwidth (Theorem 1), yields that $\mathbf{bw}(G) = \mathbf{bw}(G^*) \leq \alpha_{\mathbf{fc}}\sqrt{\mathbf{fc}(G^*)} \leq \alpha_{\mathbf{fc}}\sqrt{\mathbf{fvs}(G)} \Rightarrow \alpha_{\mathbf{fvs}} \leq \alpha_{\mathbf{fc}}$ (we examine the non-trivial case where G contains a cycle of length at least 2). Therefore, any improvement in the estimation of $\alpha_{\mathbf{fc}}$ reflects to $\alpha_{\mathbf{fvs}}$ as well. In fact we give a better bound for $\alpha_{\mathbf{fc}}$ which is based to the following combinatorial result (the proof occupies the entire Sect. 4).

Theorem 3 *For any planar graph G, $\mathbf{bw}(G) \leq 2 \cdot \sqrt{4.5} \cdot \sqrt{\mathbf{fc}(G)}$.*

We conclude that $\alpha_{\mathbf{fvs}} \leq \alpha_{\mathbf{fc}} \leq 4.243$. This leads to the main result of this paper.

Theorem 4 Planar Feedback Vertex Set *and* Face Cover *can be solved in $O(2^{15.11\cdot\sqrt{k}} + n^2)$ and $O(2^{10.1\cdot\sqrt{k}} + n^2)$ steps, respectively.*

Proof By Theorem 2 and as $\alpha_{\mathbf{fvs}} \leq 4.243$ and $\beta_{\mathbf{fvs}} \leq 3.56$ we derive the an $O(2^{15.11\cdot\sqrt{k}} \cdot n + n^3)$ step algorithm for Planar Feedback Vertex Set. Similarly, taking into account that $\alpha_{\mathbf{fc}} \leq 4.243$ and $\beta_{\mathbf{fc}} \leq 2.38$ we obtain an $O(2^{10.1\cdot\sqrt{k}} \cdot n + n^3)$ step algorithm for Face Cover. Notice that both problems have linear kernels, i.e. they can be reduced in polynomial time to equivalent instances of linear size. It follows from the general meta-algorithmic results of [5] that such kernels can be constructed in $O(n^2)$ steps (for specific kernels for the above problems, see [3, 26], and also [2, 31]). This yields the claimed time bounds. □

3.5 Planar Cycle Packing

Our combinatorial bounds can be useful for computing other parameters that can be bounded by the face cover or the feedback vertex set numbers. An example of such a parameter is the cycle packing number, denoted as $\mathbf{cp}(G)$, that is the maximum number of disjoint cycles in a graph G. The corresponding parameterized problem is the following:

Planar Cycle Packing
Instance: A planar graph G and a non-negative integer k.
Parameter: k
Question: $\mathbf{cp}(G) \geq k$?

According to the results of [10, 11], computing $\mathbf{cp}(G)$ on planar graphs can be done in $O(2^{2.78\cdot\mathbf{bw}(G)} \cdot n + n^3)$ steps. Therefore, condition (b) holds for $\mathbf{cp}$ when $\beta_{\mathbf{cp}} \leq 2.78$. Kloks et al. proved that for any planar graph G, $\mathbf{fvs}(G) \leq 5 \cdot \mathbf{cp}(G)$ [26]. Recall that, by Theorem 3, for any planar G, $\mathbf{bw}(G) \leq 2 \cdot \sqrt{4.5 \cdot \mathbf{fvs}(G)}$. This implies that, for any planar G, $\mathbf{bw}(G) \leq 2 \cdot \sqrt{4.5 \cdot 5 \cdot \mathbf{cp}(G)}$ and thus condition (a) holds for

cp for $\alpha_{\mathbf{cp}} \leq 9.487$. By Theorem 2, the PLANAR CYCLE PACKING can be solved in $O(2^{26.374\cdot\sqrt{k}}n + n^3)$ steps. Combining this with the results in [5] (or, alternatively the specific kernel from [4]) we obtain a $O(2^{26.374\cdot\sqrt{k}} + n^2)$ step algorithm for the problem.

4 Bounding Branchwidth

4.1 Plane Graphs and Hypergraphs

Hypergraphs will be an important ingredient of the proofs of this section. Our first step is to extend some basic concepts for hypergraphs. We use the term *arity* for the number of the vertices of a hyperedge. We insist in calling *edges*, hyperedges of arity equal to two (i.e. those that have only two endpoints), while we confine the term "hyperedge" to hyperedges with arity three or more.

Notice that the definition of branch decomposition and branchwidth applies directly for hypergraphs. Therefore, we use the notation $\mathbf{bw}(H)$, also when H is a hypergraph. The following lemma is useful for gluing together branch decompositions of hypergraphs.

Lemma 2 [18, Lemma 3.1] *Let H_1 and H_2 be hypergraphs with one hyperedge in common, i.e. $V(H_1) \cap V(H_2) = e$ and $\{e\} = E(H_1) \cap E(H_2)$. Then, it holds that*: $\mathbf{bw}(H_1 \cup H_2) \leq \max\{\mathbf{bw}(H_1), \mathbf{bw}(H_2), |e|\}$.

We call *plane hypergraph*, any hypergraph H whose vertices are those of a plane graph G, and whose hyperedges are some of the edges of G, plus some new pairwise distinct hyperedges, each containing the boundary vertices of some of the faces of G. By construction, H has an embedding in $\mathbb{S}_0$ that copies the one of G and where hyperedges are drawn inside the corresponding faces of G (see Fig. 2).

Given a plane simple graph G, consider the hypergraph

$$G^+ = (V(G), E(G) \cup \{\widehat{f} \cap V(G) \mid f \in F(G)\})$$

and notice that G^+ can be embedded in $\mathbb{S}_0$ in a way that its edges are embedded as in G and the rest of is hyperedges are embedded as open discs inside the corresponding faces. Thus a hypergraph H is plane if it is the sub-hypergraph of G^+ for some plane graph G where $V(H) = V(G)$.

We say that the plane graph G *generates* the plane hypergraph H if H can be obtained by G^+ after first removing some hyperedges of arity ≥ 3 and then removing edges that are subsets of remaining hyperedges. The next lemma follows easily from Lemma 2.

Lemma 3 *Let G be a plane graph and let H be a hypergraph generated by G. Then* $\mathbf{bw}(G) \leq \mathbf{bw}(H)$.

Given a plane hypergraph H, we define its dual H^* as the hypergraph whose vertices are the faces of H and where each hyperedge e of H corresponds to a hyperedge

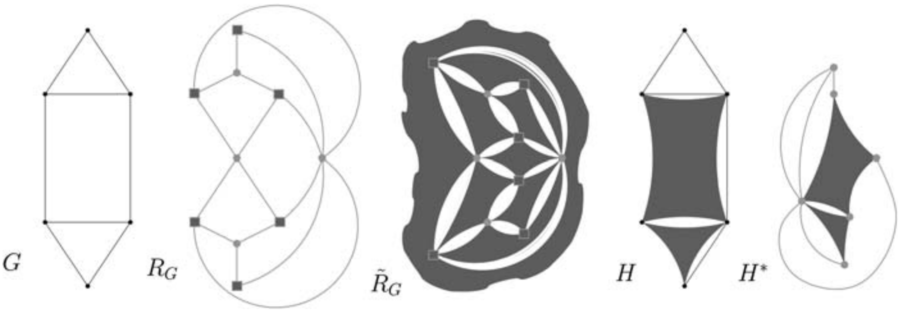

Fig. 2 A plane graph G, its radial graph R_G, a plane hypergraph H generated by G, and the plane hypergraphs H^*, and $\tilde{R}_G$

e^* of H^* whose endpoints are the faces of H that are incident to e. By drawing each vertex of H^* inside the corresponding face of H, one can see that H^* is also a plane hypergraph (see Fig. 2).

In the rest of this section, we will consider only plane hypergraphs generated by simple 3-connected planar graphs.

This permits us to consider the hyperedges of a plane hypergraph and its dual as closed discs whose boundary vertices are their endpoints. As we did for graphs, while working with plane hypergraphs, we will not distinguish between a vertex of the graph and the point of the sphere $\mathbb{S}_0$ used in the drawing to represent the vertex, or between a edge (hyperedge) and the closed line segment (closed disk) representing it in the embedding. Using this convention, we can define the set of faces of a hypergraph H as the set of connected components of $\mathbb{S}_0 \setminus H$. It is now clear that the notion of a face cover naturally extends for plane hypergraphs.

4.2 The Branchwidth of Radial Graphs

Let G be a 2-connected plane graph and let R_G be its radial graph. Notice that, as G is 2-connected, all faces of R_G are squares (i.e. their boundaries are cycles of length 4). We define $\tilde{R}_G$ as the plane hypergraph generated by R_G, if we first add a hyperedge for each face of R_G and then remove all the edges of R_G.

Lemma 4 *For any* 2*-connected plane graph* G*, it holds that* $\mathbf{bw}(\tilde{R}_G) \leq 2 \cdot \mathbf{bw}(G)$.

Proof Suppose that $\mathbf{bw}(G) \leq k$. From [30, Theorem (5.1)] there exists a sphere-cut decomposition (T, μ) of width at most k. By definition, the middle set of e in (T, μ) is equal to $N_e \cap V(G)$ and thus $|N_e| \leq k$. Observe also that the noose N_e can be seen as a cycle C_e of the radial graph G_R of length twice the length of N_e. Recall now that the definitions of R_G and $\tilde{R}_G$ imply the existence of a bijection $\rho : E(G) \to E(\tilde{R}_G)$ between the edges of G and the hyperedges of $\tilde{R}_G$. This permits us consider the branch decomposition (T, σ) of $\tilde{R}_G$ where $\sigma = \rho \circ \mu$ is the composition of the bijections μ and ρ. Observe that for any $e \in E(T)$, the middle set of e in (T, σ) consists of the vertex set of the cycle C_e. Therefore, (T, σ) of $\tilde{R}_G$ has width at most twice the width of (T, μ) and the lemma follows. $\square$

We are now ready to prove the following theorem.

Theorem 5 *For any plane graph G containing a vertex of degree at least* 2, *it holds that* $\mathbf{bw}(R_G) \leq 2 \cdot \mathbf{bw}(G)$.

Proof We first notice that Lemmata 3 and 4 imply that the result holds if G is 2-connected. We also assume that G is not a forest (it is easy to see that forests with at least one vertex of degree ≥ 2 have branchwidth 1 or 2 while their radials have branchwidth at most 2).

We apply induction on the number of biconnected components of G. Let $S \subseteq V(G)$ such that $|S| \leq 1$ and $G \setminus S$ is disconnected and let C be the vertex set of some of its connected components. We set $G_1 = G[C \cup S]$ and $G_2 = G \setminus C$. Let $f \in F(G)$ be the unique face of G whose boundary contains vertices of all connected components of $G \setminus S$. Observe that $S' = S \cup \{f\} \subseteq V(R_G)$ induces a clique in R_G of at most 2 vertices and $R_G \setminus S'$ is disconnected. Notice also that one of the connected components of $R_G \setminus S'$ contains all the vertices of C and we denote as C' its vertex set. We observe that $R_{G_1} = R_G[C' \cup S']$ and $R_{G_2} = R_G \setminus C'$. As $S' = V(R_{G_1} \cap R_{G_2})$ and $R_G = R_{G_1} \cup R_{G_2}$, it follows, using Lemma 2, that $\mathbf{bw}(R_G) \leq \max\{\mathbf{bw}(R_{G_1}), \mathbf{bw}(R_{G_2}), |S'|\}$. Observe that all graphs containing cycles have branchwidth at least 2 and radial graphs are such graphs. Therefore, as $|S'| \leq 2$, we obtain that $\mathbf{bw}(R_G) \leq \max\{\mathbf{bw}(R_{G_1}), \mathbf{bw}(R_{G_2})\}$. As G_1 and G_2 are both subgraphs of G we have $\max\{\mathbf{bw}(G_1), \mathbf{bw}(G_2)\} \leq \mathbf{bw}(G)$.

As G is not acyclic, not both G_1 and G_2 are acyclic. This implies that the induction hypothesis applies for at least one of G_1 and G_2 that should have branchwidth at least 2. We conclude that

$$\begin{aligned}\mathbf{bw}(R_G) &\leq \max\{\mathbf{bw}(R_{G_1}), \mathbf{bw}(R_{G_2})\}\\ &\leq 2 \cdot \max\{\mathbf{bw}(G_1), \mathbf{bw}(G_2)\} \leq 2 \cdot \mathbf{bw}(G)\end{aligned}$$

which concludes the proof. □

4.3 Normalization

Given a plane graph G and a face cover S_G of it, we will refer to the faces in $S_G \subseteq F(G)$ as *$\mathcal{FC}$-faces*. We say that two $\mathcal{FC}$-faces f_1 and f_2 *touch* if , $\widehat{f_1} \cap \widehat{f_2} \neq \emptyset$. Two vertices will be called a *pair*, if they are adjacent and lie on the same $\mathcal{FC}$-face. We call a face of G *triangle* if its boundary is a cycle of length 3. We call an edge of G *bridge* if there are $\mathcal{FC}$-faces f_1 and f_2 such that e it is the unique edge whose endpoints belong in the boundaries of f_1 and f_2.

Let f_1, f_2 be two $\mathcal{FC}$-faces and let x_1, x_2, y_1, y_2 be four vertices, such that $x_i, y_i \in \widehat{f_i}, i = 1, 2)$; a noose of the form $x_1 y_1 x_2 y_2 x_1$, will be called a *4-noose*. As a Jordan curve, a 4-noose N bounds two closed discs. If one of them contains exactly one hyperedge, whose endpoints are the vertices of N, then we refer to such a 4-noose as *trivial*. We proceed to the first lemma on the structure of the graph.

Lemma 5 *Let G be a simple* 3*-connected plane graph*, *such that* $\mathbf{fc}(G) \leq k$. *Then there exists a plane graph G' and a face cover $S_{G'}$ of G'*, *such that*:

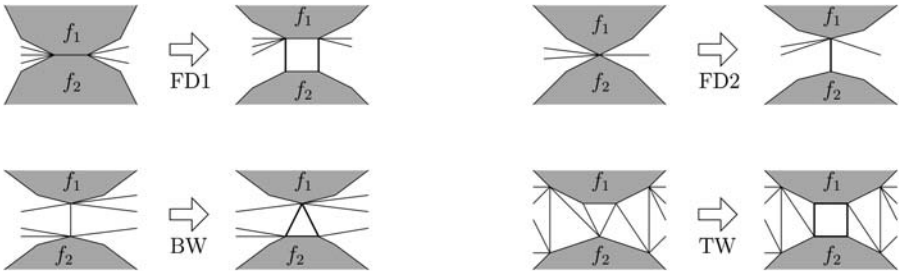

Fig. 3 The transformations of the proof of Lemma 5

(a) $\mathbf{bw}(G) \le \mathbf{bw}(G')$,
(b) $|S_{G'}| \le k$,
(c) *G' is simple and 3-connected*,
(d) *No two different $\mathcal{FC}$-faces touch*,
(e) *G' does not contain any bridge*, *and*
(f) *A face of G' is either a $\mathcal{FC}$-face or a square whose boundary contains two pairs of two different $\mathcal{FC}$-faces or a triangle incident to three different vertices that*, *in turn*, *are incident to three different $\mathcal{FC}$-faces.*

Proof Let S_G be a face cover of G where $|S| \le k$. We will consecutively apply a number of transformations to G that result in the graph G', that has the desired properties. In each step the initial face cover will be altered to a same size face cover of the new graph. The transformations are the following.

Face Detachment Notice that if two $\mathcal{FC}$-faces f_1 and f_2 touch, then the set $\widehat{f_1} \cap \widehat{f_2}$ is either an edge or a vertex of G (otherwise, G cannot be 3-connected). In each case we apply an inverse contraction as depicted in Fig. 3 (transformations FD1 and FD2). Applying this rule as long as there exist touching $\mathcal{FC}$-faces, we obtain a graph G_1, which contains a face cover S_{G_1} with the same size as S_G, satisfies conditions (c), (d) and $G \le G_1$ holds.

Partial Triangulation Add non-parallel edges in G_1, as long as this does not harm the planarity of the resulting graph and does not add edges inside some $\mathcal{FC}$-face. We denote the resulting graph by G_2. Clearly, G_2 contains a face cover S_{G_2} with the same size as S_{G_1}. Moreover, because G_1 was 3-connected and simple, G_2 remains so. Observe then, that G_2 satisfies conditions (c), (d) and $G_1 \le G_2$.

Bridge Widening The third transformation, applies the inverse contraction depicted in Fig. 3 (transformation BW) to any bridge of G_2. The resulting graph, denoted as G_3, contains a face cover S_{G_3} with the same size as S_{G_2}. Observe that G_3 satisfies conditions (c)–(e), and $G_2 \le G_3$ holds.

Triangle Widening From condition (c) and the 3-connectivity of G_3, any of its triangles that is not a $\mathcal{FC}$-face should touch at least two $\mathcal{FC}$-faces. Let f be such a triangle. If f touches 3 $\mathcal{FC}$-faces, then it satisfies the demand of condition (f) for the triangles. Suppose now, f touches two $\mathcal{FC}$-faces, namely f_1 and f_2; we apply the inverse contraction of Fig. 3 (transformation TW), and by repeating this process we obtain a graph G', that contains a face cover $S_{G'}$ with the same size as S_{G_3} and satisfies all demands of condition (f). Moreover notice, that conditions (c)–(e) also hold and $G_3 \leq G'$.

We conclude, that $G \leq G'$ and $\mathbf{fc}(G') \leq \mathbf{fc}(G)$; hence, G' satisfies all required conditions and the lemma follows. □

We call a face of a plane hypergraph H *degenerate* if it is bounded by exactly two hyperedges of H.

Lemma 6 *Let G be a 3-connected simple graph such that $\mathbf{fc}(G) \leq k$. Then, there exists a hypergraph H and a face cover S_H of H with size at most k, such that*:

(a) $\mathbf{bw}(G) \leq \mathbf{bw}(H)$.
(b) *No two different $\mathcal{FC}$-faces touch.*
(c) *H contains no edges and each hyperedge of H has arity 4 and contains two disjoint pairs that are incident to two different $\mathcal{FC}$-faces.*
(d) *A face of H is either a non-degenerate $\mathcal{FC}$-face or a degenerate face or a triangle incident to three different vertices that in turn are incident to three different $\mathcal{FC}$-faces.*

Proof Let G' be a planar hypergraph and $S_{G'}$ a face cover of G', as in Lemma 5. Let also H be the plane hypergraph generated by G' if we add a hyperedge for each square of G' and then remove all edges. Condition (a) follows directly from Lemma 3. Condition (b) follows because it holds for G' and it is invariant under hypergraph generation. Conditions (c) and (d) are consequences of condition (f) in Lemma 5 for G' and the construction of H. □

A plane hypergraph H with a face cover S_H, satisfying properties (b)–(d) of Lemma 6, will be characterized, from now on, as *normalized.*

Lemma 7 *Let H be a normalized hypergraph with face cover S_H and let N a non-trivial 4-noose bounding the closed discs Δ_1, Δ_2. Let also H_i, $(i = 1, 2)$ be the subgraph of H containing all vertices and edges included in Δ_i, plus the edge $\tilde{e}$ with endpoints the four vertices the 4-noose passes through. Then, H_i (for $i = 1, 2$) is a normalized graph with $\mathbf{fc}(H_i) \leq \mathbf{fc}(H)$ and less vertices than H.*

Proof Let us label the noose N as $x_1y_1y_2x_2x_1$, where $x_j, y_j \in f_j$ (for $j = 1, 2$) and f_1, f_2 two $\mathcal{FC}$-faces. Note that none of the x_j, y_j can be a pair, as otherwise they would both be pairs and all four vertices would lie on a hyperedge, contradicting that N is non-trivial. Let w_j, z_j be two vertices of f_j that keep x_j and y_j from being a pair; they lie, then, in different open discs bounded by N, which implies that H_i (for $i = 1, 2$) has less vertices than H, as wanted. Notice that f_j, $j = 1, 2$ is divided by N into two faces $f_j^i := f_j \cap \Delta_i$ (for $i = 1, 2,\ j = 1, 2$). Faces f_1, f_2 are the only $\mathcal{FC}$-faces touched by N and hence we can choose

$$S_{H_i} = \{f_1^i, f_2^i\} \cup \{f \in S_H : f \subseteq \Delta_i\}, \quad i = 1, 2.$$

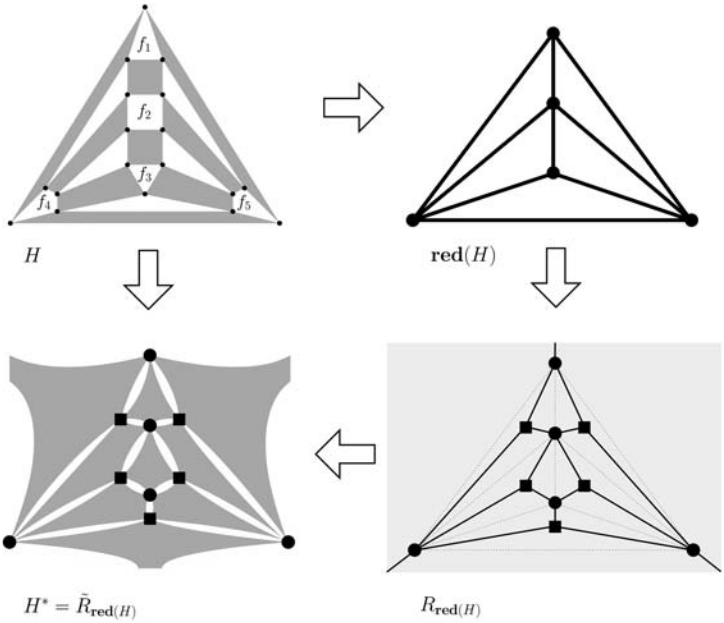

Fig. 4 The prime hypergraph H with face cover $S_H = \{f_1, \dots, f_5\}$, the plane graphs $\mathbf{red}(H)$ and $R_{\mathbf{red}(H)}$ and the isomorphic plane hypergraphs H^* and $\tilde{R}_{\mathbf{red}(G)}$ (in the planar embedding of H^* the vertex corresponding to the infinite face of H is missing)

This guarantees, that $\mathbf{fc}(H_i) \leq \mathbf{fc}(H)$ for $i = 1, 2$. It remains now to verify, that conditions (b)–(d) of Lemma 6 remain invariant in both H_i, $i = 1, 2$ and the lemma holds. Condition (b) follows from the fact that all faces of H_i, $i = 1, 2$ are subsets of faces in H. Observe that no edges are added to H_i, $i = 1, 2$ while the newly added hyperedge $\tilde{e}$ contains the pairs x_1, y_1 and x_2, y_2 that are in turn incident to the new faces f_1, f_2 and this implies that condition (c) holds. Condition (d) follows from the fact that all triangles of H_i, $i = 1, 2$ either are triangles of H or correspond to the triangles of H crossed by N and now are incident to the new hyperedge $\tilde{e}$. $\square$

4.4 Prime Hypergraphs

A normalized hypergraph H will be called *prime*, if every 4-noose is trivial. Let H be a prime hypergraph and S_H a face cover of H with $|S_H| \geq 3$. We define its *reduced* graph $\mathbf{red}(H)$ as the graph whose vertices correspond to the faces of S_H and where two vertices are connected if and only if there is a hyperedge in H with vertices lying on the corresponding faces (see Fig. 4).

Lemma 8 *Let H be a prime hypergraph with* $\mathbf{fc}(H) \geq 3$. *Then, the graphs H^* and $\tilde{R}_{\mathbf{red}(H)}$ are isomorphic.*

Proof Notice that in a prime hypergraph H, all faces are either triangles or $\mathcal{FC}$-faces. Hence, the vertices of H^* can be partitioned to those, that correspond to $\mathcal{FC}$-faces

$$
\begin{array}{ccccccc}
H^* & & H & & \mathbf{red}(H) & & \tilde{R}_{\mathbf{red}(H)} \\
V_{\mathcal{FC}}(H^*) & \longleftrightarrow & S_H & \longleftrightarrow & V(\mathbf{red}(H)) & \longleftrightarrow & V_1(\tilde{R}_{\mathbf{red}(H)}) \\
V_{\mathcal{TR}}(H^*) & \longleftrightarrow & F(H) \setminus S_H & \longleftrightarrow & F(\mathbf{red}(H)) & \longleftrightarrow & V_2(\tilde{R}_{\mathbf{red}(H)}) \\
E(H^*) & \longleftrightarrow & E(H) & \longleftrightarrow & E(\mathbf{red}(H)) & \longleftrightarrow & E(\tilde{R}_{\mathbf{red}(H)})
\end{array}
$$

Fig. 5 The bijections in the proof of Lemma 8.

and those that correspond to triangles of H. We denote these two vertex sets of H^* as $V_{\mathcal{FC}}(H^*)$ and $V_{\mathcal{TR}}(H^*)$. On the other hand, the $\mathcal{FC}$-faces of H correspond to vertices of $\mathbf{red}(H)$ and the triangles of H correspond to the faces of $\mathbf{red}(H)$. Moreover, the sets $V(\mathbf{red}(H))$ and $F(\mathbf{red}(H))$ correspond to the two parts of the vertex set of $R_{\mathbf{red}(H)}$, and thus to a bipartition $V_1(\tilde{R}_{\mathbf{red}(H)}), V_2(\tilde{R}_{\mathbf{red}(H)})$ of the vertices of $\tilde{R}_{\mathbf{red}(H)}$. We now have a chain of bijections, that merge into a bijection σ between $V_{\mathcal{FC}}(H^*) \cup V_{\mathcal{TR}}(H^*)$ and $V_1(\tilde{R}_{\mathbf{red}(H)}) \cup V_2(\tilde{R}_{\mathbf{red}(H)})$. We claim that σ is a isomorphism from H^* to $\tilde{R}_{\mathbf{red}(H)}$. To see this, observe that any hyperedge e of H^* has four endpoints containing two anti-diametrical pairs: two corresponding to $\mathcal{FC}$-faces and two corresponding to triangles of H. Notice that these $\mathcal{FC}$-faces and triangles of H correspond to vertices and faces of $\mathbf{red}(G)$ and therefore to the vertices of the hyperedge $\sigma(e)$ of $\tilde{R}_{\mathbf{red}(H)}$ (see Fig. 5). □

Corollary 1 *If H is a prime hypergraph, then* $\mathbf{bw}(H) \leq 2 \cdot \sqrt{4.5 \cdot \mathbf{fc}(H)}$.

Proof If $\mathbf{fc}(H) = 2$, then H is the graph of 6 vertices—three on each disk—with 3 hyperedges of arity four between these vertices. It is $\mathbf{bw}(H) = 4 \leq 2 \cdot \sqrt{4.5 \cdot 2}$. Suppose now, that S_H is a face cover of H where $3 \leq |S_H| = \mathbf{fc}(H)$ and notice that $\mathbf{red}(H)$ contains $|S_H|$ vertices. From the main result in [19], any n-vertex plane graph has branchwidth bounded by $\sqrt{4.5 \cdot n}$. Applying this result on $\mathbf{red}(H)$ we have that $\mathbf{bw}(\mathbf{red}(H)) \leq \sqrt{4.5 \cdot \mathbf{fc}(H)}$. Also, applying [30, Theorem (7.2)] on H and H^* it follows that $\mathbf{bw}(H) = \mathbf{bw}(H^*)$. From Lemmata 4 and 8, we obtain that $\mathbf{bw}(H) = \mathbf{bw}(H^*) = \mathbf{bw}(\tilde{R}_{\mathbf{red}(H)}) \leq 2 \cdot \mathbf{bw}(\mathbf{red}(H)) \leq 2 \cdot \sqrt{4.5 \cdot \mathbf{fc}(H)}$. □

Lemma 9 *Let H be a normalized graph. Then* $\mathbf{bw}(H) \leq 2 \cdot \sqrt{4.5 \cdot \mathbf{fc}(H)}$.

Proof We use induction on the number of vertices of H. In case $|V(H)| = 6$, G has two $\mathcal{FC}$-faces, three vertices on each of them, and three hyperedges. So, indeed $\mathbf{bw}(H) \leq 4 \leq 2 \cdot \sqrt{4.5 \cdot 2}$. We now assume that for any normalized hypergraph H where $6 \leq |V(H)| < n$ it holds that $\mathbf{bw}(H) \leq 2 \cdot \sqrt{4.5 \cdot \mathbf{fc}(H)}$ and we will show that the same upper bound holds for any normalized hypergraph H with n vertices. If H is prime then the result follows directly from Corollary 1. Suppose now that H is not prime, therefore it contains a non-trivial 4-noose N. As N bounds two discs Δ_1, Δ_2, Lemma 7 implies that the graph H_i (for $i = 1, 2$) is a normalized graph with $\mathbf{fc}(H_i) \leq k$ and $|H_i| < n, i = 1, 2$. By the induction hypothesis we have $\mathbf{bw}(H_i) \leq 2 \cdot \sqrt{4.5 \cdot k_i}, i = 1, 2$. Finally, using Lemma 2, we conclude that $\mathbf{bw}(H) = \max\{\mathbf{bw}(H_1), \mathbf{bw}(H_2)\}$, i.e. $\mathbf{bw}(H) \leq 2 \cdot \sqrt{4.5 \cdot k}$. □

We are now ready to prove the main combinatorial result of this paper.

Proof of Theorem 3 We can assume that $\mathbf{fc}(G) \geq 2$, as otherwise G is either a forest or an outerplanar graph, implicating that $\mathbf{bw}(G) \leq 2$ yielding trivially the result. Also, we can assume that G is simple as the removal of loops or multiples edges may reduce the branchwidth of a graph by at most 2 and this only in the case where the resulting graph is a forest. We will use induction on $|V(G)|$. For the smallest graph with $\mathbf{fc}(G)$ at least two, namely the K_4, the upper bound is true. We assume the same for any graph with less than $n > 4$ vertices and we will show that it holds also for any n-vertex graph. If the graph G is 3-connected, then by Lemmata 5 and 6, there is a hypergraph H where $\mathbf{fc}(H) \leq \mathbf{fc}(G)$ and $\mathbf{bw}(G) \leq \mathbf{bw}(H)$ and the result follows from Lemma 9. So, let us assume that G is not 3-connected. Then, it has a separator of at most two vertices. We will describe the case where the minimum separator has two vertices x and y as, otherwise, the result follows by applying Lemma 2 to the (bi-)connected components of G. Let C be some of the connected components of $G[V(G) - \{x, y\}]$. We set $G_1 = G[V(C) \cup \{x, y\}]$ and $G_2 = G[V(G) - V(C)]$ and we add in both G_1 and G_2 the edge $e = \{x, y\}$ (if its does not already exists). Notice that $G_i \leq G$ and therefore $\mathbf{fc}(G_i) \leq \mathbf{fc}(G)$. By the induction hypothesis, we have $\mathbf{bw}(G_i) \leq 2 \cdot \sqrt{4.5 \cdot \mathbf{fc}(G_i)}$ and the result follows by applying Lemma 2 for G_1 and G_2. □

5 Discussion

According to the Win/win approach described in Sect. 3.1, the algorithmic analysis of all problems of this paper is reduced to the problem of bounding the decomposability of a planar graph (i.e. the branchwidth) by a sublinear function of the parameter. While such general (but not optimal) upper bounds are provided by bidimensionality theory [8], better constants (and thus faster algorithms) have been achieved by a "tailor made" analysis of the parameter in the cases of *vertex cover*, *edge dominating set*, and *dominating set* (see [9, 18]). Our results for *feedback vertex set*, *face cover*, and *cycle packing* offer to the same line of research. Furthermore, specific combinatorial similarities between our proofs in Sect. 4 and the proofs in [9, 18], make us believe, that a generic technique for wider families of problems may exist.

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