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PLANE ELASTICITY IN SECTORIAL DOMAIN AND THE HAMILTONIAN SYSTEM*

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Abstract

The governing equations of plane elasticity in sectorial domain are derived to be in Hamiltoinan form via variable substitutes and variational principles. The method of separation of variables and eigenfunction expansion method are derived to solve the finite element analytically for the sectorial domain elasticity problem, so that such kind of analytical element can be installed into FEM program systems. It demonstrates the potential of the Hamiltonian system theory and symplectic mathematics.

key words elasticity, Hamiltonian system, symplectic

I. Introduction

Plane elasticity is a classical field[1,2], but there are still some problems openning to further researches. From the analogy theory of computational structural mechanics and optimal control[3-5], the theory of Hamiltonian system can be applied to the problems of elasticity in prismatic domain, and the method of transverse eigenfunction expansion of the Hamiltonian operator matrix[6,7] can be applied to the analysis of Saint Venant problems. The present paper extends such method to the problems in sectorial domain, see Fig. 1a. The radial coordinate is selected as the longitudinal direction via an appropriate variable transformation to simulate the "time coordinate", so that the problem is derived to be in Hamiltonian system form and then the symplectic algebraic method can be applied to the problems in sectorial domain. It can solve the exact singularity at the tip of sectorial domain, which is very important in applications[8]. Deriving the analytical singular finite element of sectorial domain and then installing it into FEM program system can expand the structural analysis with singular elements. The present paper describes only such analytical element especially its singular solutions.

For simplicity and convenience, the present paper gives only the homogeneous isotropic plane elasticity problem. However, the method can also be applied to homogeneous anisotropic problems, and to different materials adhesive at a radial line (Fig. 1b) or 3D problems.

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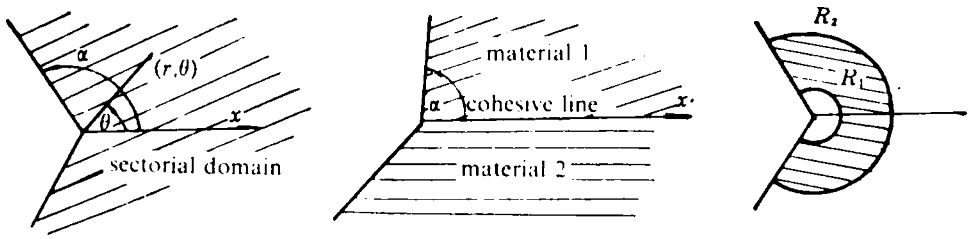

(a) 2α angle (b) Two materials cohesion (c) Ring area

Fig. 1 Sectorial domains

II. The Fundamental Equations and Variational Principle

Let the domain be a ring sector as shown in Fig. 1c, and the material being isotropic homogeneous, described by E, ν as usual[1,2]. Now the polar coordinate is selected, and the fundamental equations are given as:

Equalibrium:

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}(\sigma_r-\sigma_\theta)+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}=0 \tag{2.1a}$$

$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{2}{r}\sigma_{r\theta}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}=0 \tag{2.1b}$$

Stress-displacement relation:

$$\left.\begin{aligned}&\frac{\partial u}{\partial r}=\frac{1}{E}(\sigma_r-\nu\sigma_\theta),\quad \frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}=\frac{1}{E}(\sigma_\theta-\nu\sigma_r)\\&\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}=\frac{2(1+\nu)}{E}\sigma_{r\theta}\end{aligned}\right\} \tag{2.2}$$

where the notations are as usual. The boundary conditions must be given appropriately.

The above equations can be derived from the variational equation

$$\delta\int_{-\alpha}^{\alpha}\int_{R_1}^{R_2}\Big[\frac{\partial u}{\partial r}\sigma_r+\Big(\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\Big)\sigma_\theta+\Big(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\Big)\sigma_{r\theta}$$
$$-\frac{1}{2E}(\sigma_r^2+\sigma_\theta^2-2\nu\sigma_r\sigma_\theta+2(1+\nu)\sigma_{r\theta}^2)\Big]rdrd\theta=0 \tag{2.3}$$

where the variables u, v, σ_r, σ_θ, $\sigma_{r\theta}$ are considered as mutually independent in the variational operation. The free boundary conditions are treated as natural, and the displacement boundary conditions must be satisfied beforehand. These are well-known results[9,10].

To derive the system into Hamiltonian, the longitudinal direction, which simulates the time coordinate, should be identified first. Now r is selected as longitudinal, thus θ is transverse. The transverse force component should be eliminated. Maximizing the functional in Eq. (2.3) with respect to σ_θ gives

$$\sigma_\theta=E\Big(\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\Big)+\nu\sigma_r \tag{2.4}$$

and the variational Eq. (2.3) reduces to

$$\delta\int_{-\alpha}^{\alpha}\int_{R_1}^{R_2}\Big[\sigma_r\Big(\frac{\partial u}{\partial r}+\nu\Big(\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\Big)\Big)+\sigma_{r\theta}\Big(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\Big)$$

$$-\frac{1}{2E}(\sigma_r^2(1-\nu^2)+2(1+\nu)\sigma_{r\theta}^2)+\frac{E}{2}\Big(\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\Big)^2\Big]rdrd\theta=0 \tag{2.5}$$

Now the variable ξ is introduced to substitute r

$$\xi=\ln r \tag{2.6}$$

and the variational Eq. (2.5) becomes

$$\delta\int_{-\alpha}^{\alpha}\int_{\ln R_1}^{\ln R_2}\Big[s_r\Big(\frac{\partial u}{\partial\xi}+\nu u+\nu\frac{\partial v}{\partial\theta}\Big)+s_\theta\Big(\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial\xi}-v\Big)+\frac{E}{2}\Big(u+\frac{\partial v}{\partial\theta}\Big)^2$$

$$-\frac{1}{2E}((1-\nu^2)s_r^2+2(1+\nu)s_\theta^2)\Big]d\xi d\theta=0 \tag{2.7}$$

where

$$s_r=r\sigma_r,\quad s_\theta=r\sigma_{r\theta} \tag{2.8}$$

and u, v, s_r, s_θ are considered as functions of ξ and θ.

Now it can be formulated in Hamiltonian system form, let u, v compose the displacement vector and s_r, s_θ compose the dual vector. Let

$$\boldsymbol{q}=\begin{Bmatrix}u\\ v\end{Bmatrix},\qquad \boldsymbol{p}=\begin{Bmatrix}s_r\\ s_\theta\end{Bmatrix} \tag{2.9}$$

and a dot denotes the differential with respect to ξ, it turns to be

$$\delta\int_{\ln R_1}^{\ln R_2}\int_{-\alpha}^{\alpha}[\boldsymbol{p}^T\dot{\boldsymbol{q}}-\mathscr{H}(\boldsymbol{q},\boldsymbol{p})]d\theta d\xi=0 \tag{2.10}$$

$$-\mathscr{H}(\boldsymbol{q},\boldsymbol{p})=s_r\Big(\nu u+\nu\frac{\partial v}{\partial\theta}\Big)+s_\theta\Big(\frac{\partial u}{\partial\theta}-v\Big)+\frac{E}{2}\Big(u+\frac{\partial v}{\partial\theta}\Big)^2$$

$$-\frac{1}{2E}((1-\nu^2)s_r^2+2(1+\nu)s_\theta^2) \tag{2.11}$$

which is the variational principle of Hamiltonian system for field problems. Expanding the variational equation gives the dual equations

$$\dot{\boldsymbol{q}}=\mathbf{F}\boldsymbol{q}-\mathbf{G}\boldsymbol{p} \tag{2.12a}$$

$$\dot{\boldsymbol{p}}=-\mathbf{Q}\boldsymbol{q}-\mathbf{F}^T\boldsymbol{p} \tag{2.12b}$$

where

$$\left.\begin{aligned}
&\mathbf{G}=\begin{bmatrix}-(1-\nu^2)/E & 0\\ 0 & -2(1+\nu)/E\end{bmatrix},\quad \mathbf{Q}=\begin{bmatrix}-E & -E\dfrac{d\cdot}{d\theta}\\ \dfrac{d}{d\theta}(E\cdot) & \dfrac{d}{d\theta}\Big(E\dfrac{d\cdot}{d\theta}\Big)\end{bmatrix}\\
&\mathbf{F}=\begin{bmatrix}-\nu & -\nu\dfrac{d\cdot}{d\theta}\\ -\dfrac{d\cdot}{d\theta} & 1\end{bmatrix},\qquad \mathbf{F}^T=\begin{bmatrix}-\nu & \dfrac{d\cdot}{d\theta}\\ \dfrac{d}{d\theta}(\nu\cdot) & 1\end{bmatrix}
\end{aligned}\right\} \tag{2.13}$$

and the symmetric or antisymmetric boundary conditions at $\theta=0$, and the free boundary conditions at $\theta=a$

$$s_\theta=0,\quad u+\frac{dv}{d\theta}+\frac{\nu}{E}s_r=0\,,\quad \text{when } \theta=a \tag{2.14}$$

III. Eigensolution and Adjoint Symplectic Ortho-Normalization Relation

The dual Eqs. (2.12) and the boundary conditions can be solved by the method of separation of variables

$$q=q_i\exp[\mu_i\xi]\,,\qquad p=p_i\exp[\mu_i\xi] \tag{3.1}$$

where μ_i is the eigenvalue. The dual equations can be combined as

$$\dot{w}=\mathbf{H}w,\quad w=\begin{Bmatrix} q \\ p \end{Bmatrix},\quad \mathbf{H}=\begin{bmatrix} \mathbf{F} & -\mathbf{G} \\ -\mathbf{Q} & -\mathbf{F}^T \end{bmatrix} \tag{3.2}$$

where w can be termed as the whole state function vector, and the eigen-equation can be given as

$$\mu_i\psi_i=\mathbf{H}\psi_i\,,\qquad \psi_i=\begin{Bmatrix} q_i \\ p_i \end{Bmatrix} \tag{3.3}$$

where ψ_i is the eigen-function-vector depending only on θ. It must satisfy the bóundary condition (2.14) and symmetric condition. A rotational exchange operator matrix $\mathbf{J}$ can be introduced as

$$\mathbf{J}=\begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix},\quad \mathbf{J}^T=\mathbf{J}^{-1}=-\mathbf{J},\quad \mathbf{J}^2=\begin{bmatrix} -\mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \tag{3.4}$$

where $\mathbf{I}$ is the identity operator. To describe the behaviour of Hamiltonian operator matrix $\mathbf{H}$, the boundary condition (2.14) should be considered simultaneously. Introduce the operation $\langle\cdot,\cdot,\cdot\rangle$ as

$$\langle w_1^T,\mathbf{P},w_2\rangle=\int_{-a}^{a} w_1^T\mathbf{P}w_2 d\theta \tag{3.5}$$

where $\mathbf{P}$ is an arbitrary operator matrix. It can be verified by integration by parts and the boundary conditions that

$$\langle(\mathbf{J}w_1)^T,\ \mathbf{H},\ w_2\rangle=\langle w_2^T,\mathbf{H}^T,(\mathbf{J}w_1)\rangle \tag{3.6}$$

where w_1 and w_2 are arbitrary whole state vectors satisfying the boundary conditions, and $\mathbf{H}^T$ is given as

$$\mathbf{H}^T=\begin{bmatrix} \mathbf{F}^T & -\mathbf{Q} \\ -\mathbf{G} & -\mathbf{F} \end{bmatrix},\ \text{and } \mathbf{H}^T=\mathbf{JHJ} \tag{3.7}$$

An operator matrix $\mathbf{H}^T$ satisfying Eq. (3.7) is called Hamiltonian by definition.

The eigen-problem (3.3) of Hamiltonian operator matrix has some distinguished behaviour[6,7]. If μ_i is an eigenvalue, $-\mu_i$ is an eigenvalue also. Hence the eigen-solution can be subdivided into two groups of (α) and (β):

$$(\alpha) \qquad \mu_{\alpha i}, \ (i=1,2,\cdots); \ \mathrm{Re}(\mu_{\alpha i})>0 \text{ or } \mathrm{Re}(\mu_{\alpha i})=0 \text{ and } \mathrm{Im}(\mu_{\alpha i})>0 \tag{3.8a}$$

$$(\beta) \qquad \mu_{\beta i}, \ \mu_{\beta i}=-\mu_{\alpha i} \tag{3.8b}$$

and the corresponding eigen-function-vector can be denoted as

$$\boldsymbol{\psi}_{\alpha i} \text{ and } \boldsymbol{\psi}_{\beta i} \tag{3.9}$$

respectively. Between any two of them, there is adjoint symplectic ortho-normalization relation

$$\langle \boldsymbol{\psi}^T_{\alpha i}, \mathbf{J}, \boldsymbol{\psi}_{\beta j}\rangle=\delta_{ij}, \ \langle \boldsymbol{\psi}^T_{\alpha i}, \mathbf{J}, \boldsymbol{\psi}_{\alpha j}\rangle=0, \ \langle \boldsymbol{\psi}^T_{\beta i}, \mathbf{J}, \boldsymbol{\psi}_{\beta j}\rangle=0 \tag{3.10}$$

The expansion solution method based on the adjoint symplectic ortho-normalization relation is of great value for such analytical element formulation.

IV. Expansion Theorem with the Eigen-Function-Vectors

An arbitrary whole state function-vector $\boldsymbol{w}$ can always be expressed by the linear combination of the eigen-function-vectors as

$$\boldsymbol{w}=\sum_{i=1}^{\infty}(a_i\boldsymbol{\psi}_{\alpha i}+b_i\boldsymbol{\psi}_{\beta i}) \tag{4.1}$$

where $\boldsymbol{\psi}_i$ are functions of θ only, and the coefficients a_i, b_i are functions of ξ. Using the adjoint symplectic ortho-normalization relation gives

$$a_i=-\langle \boldsymbol{\psi}^T_{\beta i}, \mathbf{J}, \boldsymbol{w}\rangle, \ b_i=\langle \boldsymbol{\psi}^T_{\alpha i}, \mathbf{J}, \boldsymbol{w}\rangle \tag{4.2}$$

Substituting Eq. (4.1) into Eq. (3.2) and using Eq. (3.3) gives

$$\dot{a}_i=\mu_{\alpha i}a_i, \ \dot{b}_i=-\mu_{\alpha i}b_i \tag{4.3}$$

Hence (written μ_i instead of $\mu_{\alpha i}$).

$$\dot{a}_i=a_{i0}\exp[\mu_i\xi], \qquad \dot{b}=b_{i0}\exp[-\mu_i\xi] \tag{4.4}$$

where the integration constants a_{i0} and b_{i0} should be solved by the boundary conditions at $\xi_1=\ln(R_1)$ and $\xi_2=\ln(R_2)$. Now let $R_1\to0$, i.e. $\xi_1\to-\infty$, the problem reduces to the analysis of singularity at the tip of sectorial domain.

V. The Analytical Eigen-Solutions

Expanding the Eq. (3.3) gives (drop the subscripts i)

$$\left.\begin{aligned}
&-(\mu+\nu)u-\nu\frac{dv}{d\theta}+\frac{(1-\nu)^2}{E}s_r+0=0\\
&-\frac{du}{d\theta}+(1-\mu)v+0+\frac{2(1+\nu)}{E}s_\theta=0\\
&Eu+E\frac{dv}{d\theta}+(\nu-\mu)s_r-\frac{ds_\theta}{d\theta}=0\\
&-\frac{d}{d\theta}(Eu)-\frac{d}{d\theta}\left(E\frac{dv}{d\theta}\right)-\frac{d}{d\theta}(\nu s_r)-s_\theta(1+\mu)=0
\end{aligned}\right\} \tag{5.1}$$

Assuming E and ν are independent on θ, to solve Eq.(5.1) the determinant equation

$$\begin{vmatrix} -(\mu+\nu) & -\nu\lambda & (1-\nu^2)/E & 0 \\ -\lambda & (1-\mu) & 0 & 2(1+\nu)/E \\ E & E\lambda & (\nu-\mu) & -\lambda \\ -E\lambda & -E\lambda^2 & -\nu\lambda & -(1+\mu) \end{vmatrix} = 0$$

should be solved first. Expanding it gives the equation for λ

$$\left.\begin{aligned} &\lambda^4+\lambda^2(2+2\mu^2)+(1-\mu^2)^2=0 \\ &\lambda_{1,2}=\pm i(1+\mu), \quad \lambda_{3,4}=\pm i(1-\mu) \end{aligned}\right\} \tag{5.2}$$

Finding the solution symmetric with respect to the line $\theta=0°$ gives

$$\left.\begin{aligned} u&=A_u\cos(1+\mu)\theta+C_u\cos(1-\mu)\theta \\ v&=A_v\sin(1+\mu)\theta+C_v\sin(1-\mu)\theta \\ s_r&=A_r\cos(1+\mu)\theta+C_r\cos(1-\mu)\theta \\ s_\theta&=A_\theta\sin(1+\mu)\theta+C_\theta\sin(1-\mu)\theta \end{aligned}\right\} \tag{5.3}$$

These constants $A_u, C_u, A_r, \cdots, C_\theta$ must satisfy Eq. (5.1), hence

$$\left.\begin{aligned} &-(\mu+\nu)A_u-\nu(1+\mu)A_v+((1-\nu^2)/E)A_r+0=0 \\ &(1+\mu)A_u+(1-\mu)A_v+0+(2(1+\nu)/E)A_\theta=0 \\ &EA_u+E(1+\mu)A_v+(\nu-\mu)A_r-(1+\mu)A_\theta=0 \\ &E(1+\mu)A_u+E(1+\mu)^2A_v+\nu(1+\mu)A_r-(1+\mu)A_\theta=0 \end{aligned}\right\} \tag{5.4}$$

and

$$\left.\begin{aligned} &-(\mu+\nu)C_u-\nu(1-\mu)C_v+\frac{1}{E}(1-\nu^2)C_r+0=0 \\ &(1-\mu)C_u+(1-\mu)C_v+0+\frac{2}{E}(1+\nu)C_\theta=0 \\ &EC_u+E(1-\mu)C_v+(\nu-\mu)C_r-(1-\mu)C_\theta=0 \\ &E(1-\mu)C_u+E(1-\mu)^2C_v+\nu(1-\mu)C_r-(1+\mu)C_\theta=0 \end{aligned}\right\} \tag{5.5}$$

There is one superfluous equation in each of the above equation sets, hence each has one independent coefficient, such as A_θ and C_θ. The eigenvalue μ is to be determined.

Substituting the solution Eq. (5.3) into the boundary condition (2.14), two homogeneous linear equations for A_θ and C_θ are established but trivial solution is useless, hence its determinant must be zero, which gives the eigen-equation for eigenvalues. From (5.4) gives

$$A_r=-A_\theta, \quad A_u=-A_v, \quad A_v=\frac{1+\nu}{E\mu}A_\theta \tag{5.6}$$

and from (5.5) it solves

$$\left.\begin{aligned} &\mu(1-\mu)C_u+\frac{C_\theta}{E}(-3+\nu+\mu+\nu\mu)=0, \quad (1-\mu)C_r-(3-\mu)C_\theta=0 \\ &\mu(1-\mu)C_v+\frac{C_\theta}{E}(3-\nu+\mu+\nu\mu)=0 \end{aligned}\right\} \tag{5.7}$$

The boundary condition (2.14) gives

$$\left.\begin{aligned}&A_\theta\sin(1+\mu)\alpha+C_\theta\sin(1-\mu)\alpha=0\\&\left(A_u+A_v(1+\mu)+\frac{\nu}{E}A_r\right)\cos(1+\mu)\alpha+\left(C_u+C_v(1-\mu)+\frac{\nu}{E}C_r\right)\cos(1-\mu)\alpha=0\end{aligned}\right\} \tag{5.8}$$

Substituting Eqs. (5.6) and (5.7) into the above equation gives

$$\left.\begin{aligned}&A_\theta\sin(1+\mu)\alpha+C_\theta\sin(1-\mu)\alpha=0\\&(1-\mu)A_\theta\cos(1+\mu)\alpha+(1+\mu)C_\theta\cos(1-\mu)\alpha=0\end{aligned}\right\} \tag{5.9}$$

The determinant equals zero gives the eigen-equation

$$\sin 2\mu\alpha+\mu\sin 2\alpha=0 \tag{5.10}$$

from which it is easily seen that both μ and $-\mu$ are eigenvalues.

When $\alpha>\pi/2$, the tip ($\xi\to-\infty$) of sectorial domain has a singularity, i. e. there is eigenvalue in $0<\mu<1$, and it must be $2\mu\alpha\leqslant\pi$. Numerical results are listed in Table 1. Note Eq. (2.8) that the stress singularity is $r^{(\mu-1)}$.

Table 1 **Eigenvalue of symmetric deformation for isotropic material**

$2\alpha/\pi$	2.0	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1	1.0
μ	0.5	0.500310	0.502530	0.508800	0.521711	0.544484	0.581142	0.636728	0.717799	0.833691	1

When $\alpha<\pi/2$, it asserts that there is no singularity at the tip of sector. Because $\sin(2\alpha)>0$, Eq. (5.10) is impossible to have root in $0<\mu<1$, which coincides with the assertion.

Now turn to look at the solution anti-symmetric with respect to $\theta=0°$. The general solution is

$$\left.\begin{aligned}u&=B_u\sin(1+\mu)\theta+D_u\sin(1-\mu)\theta\\v&=B_v\cos(1+\mu)\theta+D_v\cos(1-\mu)\theta\\s_r&=B_r\sin(1+\mu)\theta+D_r\sin(1-\mu)\theta\\s_\theta&=B_\theta\cos(1+\mu)\theta+D_\theta\cos(1-\mu)\theta\end{aligned}\right\} \tag{5.11}$$

where it must be

$$B_r=B_\theta,\quad B_u=B_v,\quad \mu B_v=(1+\nu)B_\theta/E \tag{5.12}$$

$$\left.\begin{aligned}&(1-\mu)D_r=(\mu-3)D_\theta,\quad E\mu(1-\mu)D_u+(3-\nu-\mu-\nu\mu)D_\theta=0\\&E\mu(1-\mu)D_v+(3-\nu+\mu+\nu\mu)D_\theta=0\end{aligned}\right\} \tag{5.13}$$

and the free boundary condition gives

$$B_\theta\cos(1+\mu)\alpha+D_\theta\cos(1-\mu)\alpha=0$$

$$\left(B_u-B_v(1+\mu)+\frac{\nu}{E}B_r\right)\sin(1+\mu)\alpha+\left(D_u-D_v(1-\mu)+\frac{\nu}{E}D_r\right)\sin(1-\mu)\alpha=0$$

Substituting Eqs. (5.12) and (5.13) in the latter equation gives

$$B_\theta\sin(1+\mu)\alpha+\frac{1+\mu}{1-\mu}D_\theta\sin(1-\mu)\alpha=0$$

Because B_0 and D_0 must not be simultaneously zero, which gives the eigen-equation

$$\sin 2\mu\alpha-\mu\sin 2\alpha=0 \tag{5.14}$$

It is easily seen that μ and $-\mu$ are roots simultaneously.

When $2\alpha>1.430297\pi$, there will be singulareigenvalue in $0<\mu\leqslant 1$, Numerical result is given in Table 2.

Table 2 Eigenvalues of antisymmetric deformation

$2\alpha/\pi$	2.0	1.9	1.8	1.7	1.6	1.5	1.45	1.4303
μ	0.5	0.555202	0.621710	0.701175	0.795785	0.908529	0.972947	0.999999

Next, the case of two different materials adhered together is considered (Fig. 1b). Suppose the material property E being very large and can be regarded as rigid, hence the boundary condition is treated as

$$u=v=0, \quad \text{when } \theta=0 \tag{5.15}$$

and the free condition of Eq. (2.14) still holds.

The general eigen-solution is the sum of Eqs. (5.3) and (5.11), where the coefficients satisfy Eqs. (5.6−7) and (5.12−13). The independent coefficients are A_0, B_0, C_0, D_0, the eigenvalue μ is also to be determined. According to boundary condition (5.15), $A_0=-C_0$, $B_0=-D_0$. Using Eqs. (5.6−7) and (5.12−13)

$$A_0=\frac{(3-\nu-\mu-\nu\mu)}{(1+\nu)(1-\mu)}C_0, \quad B_0=\frac{(3-\nu+\mu+\nu\mu)}{(1+\nu)(1-\mu)}D_0 \tag{5.16}$$

Substituting the general eigen-solution into the free boundary condition (2.14), using Eqs. (5.6−7) and (5.12−13) also gives

$$\left.\begin{aligned}
&A_0\sin(1+\mu)\alpha+B_0\cos(1+\mu)\alpha+C_0\sin(1-\mu)\alpha+D_0\cos(1-\mu)\alpha=0\\
&A_0\cos(1+\mu)\alpha-B_0\sin(1+\mu)\alpha+C_0\frac{1+\mu}{1-\mu}\cos(1+\mu)\alpha\\
&\quad -D_0\frac{1+\mu}{1-\mu}\sin(1-\mu)\alpha=0
\end{aligned}\right\} \tag{5.17*}$$

Substituting Eq. (5.16) into the above equation gives a simultaneous equation set for C_0 and D_0. Its determinant equals zero gives the eigenvalue equation as

$$\begin{vmatrix}
\begin{aligned}&(3-\nu-\mu-\mu\nu)\sin(1+\mu)\alpha\\ &\quad+(1+\nu)(1-\mu)\sin(1-\mu)\alpha\end{aligned} & \begin{aligned}&(3-\nu+\mu+\mu\nu)\cos(1+\mu)\alpha\\ &\quad+(1+\nu)(1-\mu)\cos(1-\mu)\alpha\end{aligned}\\
\begin{aligned}&(3-\nu-\mu-\mu\nu)\cos(1+\mu)\alpha\\ &\quad+(1+\nu)(1+\mu)\cos(1-\mu)\alpha\end{aligned} & \begin{aligned}&-(3-\nu+\mu+\mu\nu)\sin(1+\mu)\alpha\\ &\quad-(1+\nu)(1+\mu)\sin(1-\mu)\alpha\end{aligned}
\end{vmatrix}=0$$

Expanding the determinant derives

$$4-(1+\nu)(3-\nu)\sin^2\mu\alpha-\mu^2(1+\nu)^2\sin^2\alpha=0 \tag{5.18}$$

For the cases of $\alpha=\pi/2$ and $\alpha=\pi$ respectively

*In further research, Associate professor Zhang Hong-wu found an error in Eq. (5.17) of the original text, and proposed the correction text until Table-3. The author sincerely expresses gratitude to him.

$$4-(1+\nu)(3-\nu)\sin^2\frac{\mu\pi}{2}-(1+\nu)^2\mu^2=0, \quad \text{for } \alpha=\frac{\pi}{2} \tag{5.18$'$}$$

$$\sin\mu\pi=\sqrt{\frac{4}{(1+\nu)(3-\nu)}}>1, \quad \text{for} \quad \alpha=\pi \tag{5.18$''$}$$

The Eq. (5.18)$''$ has no real root, but equation (5.18)$'$ does has real root, that means when α locates between $\pi/2$ and π there must be a transition point from real root to complex roots. For the case of $\nu=0.3$, the roots μ versus the α angles are listed in the Table 3. The angle $\tilde{\alpha}\simeq131°$ is the transition point for real and complex roots, that $90°\leqslant\alpha\leqslant\tilde{\alpha}$ gives real root, and there are two real roots when $119°\leqslant\alpha\leqslant\tilde{\alpha}$.

Table 3 Eigenvalues for the sector with one boundary clamped and the another free when $\nu=0.3$

$\alpha°$	180	170	160	150	140	131		126		119		115	108	90
Re(μ)	0.500	0.530	0.567	0.611	0.665	0.730	0.730	0.679	0.849	0.675	0.990	0.680	0.692	0.758
Im(μ)	0.116	0.122	0.123	0.117	0.097	0		0		0		0	0	0

The cohesion of differential materials is quite useful for composite material or in micro-electronics. Although only the case of $\alpha=\pi/2$ is calculated here, however other value of α can also be selected. The eigenvalue can be solved from Eq. (5.18). Selection of best angle α to reduce the stress singularity within the tolerence of technology can be considered as one of the measures to reduce possible cracking.

For homogeneous isotropic plane problem, the Airy stress function method and the method of complex variable can also be applied to solve such problem. However, the Hamiltonian system method can be applied to all auto-modelling problem in linear elasticity, such as anisotropic material or even three dimensional case.

The eigenvalues given above are only for the singular solutions, but there are infinite eigenvalues, which are generally complex conjugates. When substituting back to the simultaneous equations such as Eq. (5.9) and solve the constants, Eqs. (5.3) or (5.11) give the corresponding eigenvectors. These eigenvectors are mutually adjoint symplectic ortho-normalized. The eigenvalue determines only the characteristics of the singularity, but the intensity of the singularity should be determined by other means, such situation is the same to the theory of fracture mechanics. The intensity of singularity depends on the connection of the sector to the surrounding structure and its loading. Currently the structural analysis is mainly by FEM and the sector is treated as a super-element of the structure. The analytical stiffness matrix of the sectorial super-element can be generated via the expansion method of eigen-function-vectors with combination of the variational principle. The method will be given in the next section.

VI. Formulation of Stiffness Matrix of the Sectorial Domain

Generally speaking, assuming there are n_r external pionts on each inner and outer arcs of the ring sectorial domain (Fig. 1c). The inner arc will connect the plastic zone if any, and the outer arc boundary will connect the surrounding structure. For ring domain, both the eigen-solutions of μ and $-\mu$ are both necessary. When only elastic solution is considered, $R_1\to0$, so that only the eigenvalue of Re$(\mu)>0$ is appropriate.

The elastic sector has n_r points at the outer arc $r=R_2$, hence the analytical stiffness matrix of the sector domain will be corresponding to the displacements of these points. For plane problems two displacements u, v for each points, thus the external displacements of the super-element have $2n_r$ degrees of freedom. Hence $2n_r$ eigen-solutions with $\mathrm{Re}(\mu)>0$ will be necessary and the sectorial singular solution will be

$$\boldsymbol{w}(\xi,\theta)=\sum_{i=1}^{2n_r} A_i\boldsymbol{\psi}_i(\theta)\exp(\mu_i\xi) \tag{6.1}$$

where A_i are constants to be determined. For the case of complex eigen-solutions, its complex conjugate will also be eigen-solution and so is A_i. The eigen-function-vector $\boldsymbol{\psi}_i$ are composed of u_i, v_i, s_{ri}, $s_{\theta i}$. The current FEM systems are all based on displacement method, the general solution (6.1) should be transformed to external stiffness matrix. The $2n_r$ displacements u, v of the n_r points can determine the $2n_r$ coefficients A_i (the real and imaginary parts). When the $2n_r$ displacements u, v successively given as $\{1,0;0,0;\cdots;0,0\}^T$, $\{0,1;0,0;\cdots;0,0\}^T$, $\{0,0;1,0;\cdots;0,0\}^T,\cdots$, $\{0,0;0,0;\cdots;0,1\}^T$, totally $2n_r$ independent vectors in turn, the $2n_r$ sets of solutions of contants A_i can be solved. Using these A_i solutions as columns, a $2n_r\times 2n_r$ matrix $\mathbf{T}$ is composed, which transforms the external displacement vector to the vector of A_i.

The element stiffness matrix can be obtained from the strain energy of the element, which is right the functional of the variational Eq. (2.3), or of Eqs. (2.7) or (2.10). Substituting Eq. (6.1) into the functional of Eq. (2.10), by use of integration by parts and noting that the eigen-solution (6.1) satisfies all the differential equations and boundary conditions (except $r=R_2$), the element strain energy can be derived as

$$\begin{aligned} E_e &=\frac{1}{2}\int_{-\alpha}^{\alpha}[s_r(\theta)\cdot u(\theta)+s_\theta(\theta)\cdot v(\theta)]d\theta \\ &=\frac{1}{2}\boldsymbol{a}^T\cdot\mathbf{R}_a\cdot\boldsymbol{a} \end{aligned} \tag{6.2}$$

where $\boldsymbol{a}$ is the vector composed of A_i $(i=1,\ 2,\cdots,\ 2n_r)$, and $\mathbf{R}_a$ is the element matrix corresponding to vector $\boldsymbol{a}$, with size $2n_r\times 2n_r$.

Let $\boldsymbol{d}_e$ denote the external displacement vector of sectorial element so that

$$\boldsymbol{a}=\mathbf{T}\cdot\boldsymbol{d}_e,\ E_e=\frac{1}{2}\boldsymbol{a}^T\mathbf{R}_a\boldsymbol{a}=\frac{1}{2}\boldsymbol{d}_e^T\mathbf{R}_e\boldsymbol{d}_e \tag{6.3}$$

where

$$\mathbf{R}_e=\mathbf{T}^T\cdot\mathbf{R}_a\cdot\mathbf{T} \tag{6.4}$$

is the element matrix desired. The elements of $\mathbf{R}_a$ is

$$(\mathbf{R}_a)_{i,j}=\int_{-\alpha}^{\alpha}[s_{ri}(\theta)u_j(\theta)+s_{\theta i}(\theta)v_j(\theta)]d\theta \tag{6.5}$$

Betti reciprocal principle gives the symmetry of matrix $\mathbf{R}_a$.

VII. On FEM of Singular Element

The computation of the intensity factor, the connection between analytical element and

FEM have been given in last section. Now the FEM application to the singular element itself is discussed. So far analytical method is applied for isotropic plane problem, but for more complicated or 3D problems the pure analytical method will be difficult. The method of separation of veriables for Hamiltonian system can reduce only one dimension to the governing equation, so that applying FEM along the transverse direction can be considered. Usually the formulation for FEM is best via variational method, which is heavily used in this paper, and the free boundary condition is also treated with the variational principle. Note that along the radial coordinate ξ (or r) the formulation is analytical, hence the element is semi-analytical, which is important for identifying the characteristics of singularity. The detail is omitted.

VIII. Concluding Remarks

There are a number of auto-modelling problems in applied mechanics, the sectorial domain problem in elasticity is one of them. For such kind of problems the variable substitution method and variational principle can be used to derive the governing equations to Hamiltonian system, and then the method of separation of variables, the eigen-function-vector expansion method and adjoint symplectic orthonormality and the corresponding mathematical tools can be applied. The present paper demonstrates such mathematical method via the sectorial domain plane elasticity problem, which can be imbedded into some fracture or composite material finite element analysis programs.

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