Plane Spanners of Maximum Degree Six

Nicolas Bonichon[1,⋆], Cyril Gavoille[1,⋆,⋆⋆],
Nicolas Hanusse[1,⋆], and Ljubomir Perković[2,⋆⋆⋆]

[1] Laboratoire Bordelais de Recherche en Informatique (LaBRI),
Université de Bordeaux, France
{bonichon,gavoille,hanusse}@labri.fr
[2] School of Computing, DePaul University, USA
lperkovic@cs.depaul.edu

Abstract. We consider the question: "What is the smallest degree that can be achieved for a plane spanner of a Euclidean graph $\mathcal{E}$?" The best known bound on the degree is 14. We show that $\mathcal{E}$ always contains a plane spanner of maximum degree 6 and stretch factor 6. This spanner can be constructed efficiently in linear time given the Triangular Distance Delaunay triangulation introduced by Chew.

1 Introduction

In this paper we focus on the following question:

> "What is the smallest maximum degree that can be achieved for plane spanners of the complete, two-dimensional Euclidean graph $\mathcal{E}$?"

This question happens to be Open Problem 14 in a very recent survey of plane geometric spanners [BS]. It is an interesting, fundamental question that has curiously not been studied much. (Unbounded degree) plane spanners have been studied extensively: obtaining a tight bound on the stretch factor of the Delaunay graph is one of the big open problems in the field. Dobkin *et al.* [ADDJ90] were the first to prove that Delaunay graphs are (plane) spanners. The stretch factor they obtained was subsequently improved by Keil & Gutwin [KG92] as shown in Table 1. The plane spanner with the best known upper bound on the stretch factor is not the Delaunay graph however, but the TD-Delaunay graph introduced by Chew [Che89] whose stretch factor is 2 (see Table 1). We note that the Delaunay and the TD-Delaunay graphs may have unbounded degree.

Just as (unbounded degree) plane spanners, bounded degree (but not necessarily planar) spanners of $\mathcal{E}$ have been well studied and are, to some extent, well understood: it is known that spanners of maximum degree 2 do not exist in general and that spanners of maximum degree 3 can always be constructed (Das & Heffernan [DH96]). In recent years, bounded degree plane spanners have

⋆ Supported by the ANR-project "ALADDIN", the ANR-project "GRAAL", and the équipe-projet INRIA "CÉPAGE".
⋆⋆ Member of the "Institut Universitaire de France".
⋆⋆⋆ Supported by a DePaul University research grant.

S. Abramsky et al. (Eds.): ICALP 2010, Part I, LNCS 6198, pp. 19–30, 2010.

Table 1. Results on plane spanners with maximum degree bounded by Δ

paper	Δ	stretch factor
Dobkin *et al.* [ADDJ90]	∞	$\frac{\pi(1+\sqrt{5})}{2} \approx 5.08$
Keil & Gutwin [KG92]	∞	$C_0 = \frac{4\pi\sqrt{3}}{9} \approx 2.42$
Chew [Che89]	∞	2
Bose *et al.* [BGS05]	27	$(\pi+1)C_0 \approx 10.016$
Li & Wang [LW04]	23	$(1+\pi\sin\frac{\pi}{4})C_0 \approx 7.79$
Bose *et al.* [BSX09]	17	$(2+2\sqrt{3}+\frac{3\pi}{2}+2\pi\sin(\frac{\pi}{12}))C_0 \approx 28.54$
Kanj & Perković [KP08]	14	$(1+\frac{2\pi}{14\cos(\frac{\pi}{14})})C_0 \approx 3.53$
This paper: Section 3	9	6
This paper: Section 4	6	6

been used as the building block of wireless network communication topologies. Emerging wireless distributed system technologies such as wireless ad-hoc and sensor networks are often modeled as proximity graphs in the Euclidean plane. Spanners of proximity graphs represent topologies that can be used for efficient unicasting, multicasting, *and/or* broadcasting. For these applications, spanners are typically required to be planar and have bounded degree: the planarity requirement is for efficient routing, while the bounded degree requirement is due to the physical limitations of wireless devices.

Bose *et al.* [BGS05] were the first to show how to extract a spanning subgraph of the Delaunay graph that is a bounded-degree, plane spanner of $\mathcal{E}$. The maximum degree and stretch factor they obtained was subsequently improved by Li & Wang [LW04], Bose *et al.* [BSX09], and by Kanj & Perković [KP08] (see all bounds in Table 1). The approach used in all of these results was to extract a bounded degree spanning subgraph of the *classical Delaunay triangulation*. The main goal in this line of research was to obtain a bounded-degree plane spanner of $\mathcal{E}$ with the *smallest possible stretch factor*.

In this paper we propose a new goal and a new approach. Our goal is to obtain a plane spanner with the *smallest possible maximum degree*. We believe this question is fundamental. The best known bound on the degree of a plane spanner is 14 [KP08]. In some wireless network applications, such a bound is too high. Bluetooth scatternets, for example, can be modeled as spanners of $\mathcal{E}$ where master nodes must have at most 7 slave nodes [LSW04].

Our approach consists of two steps. We first extract a maximum degree 9 spanning subgraph H_2 from Chew's *TD-Delaunay graph* instead of the classical Delaunay graph. Graph H_2 is a spanner of the TD-Delaunay graph of stretch factor 3, and thus a spanner of $\mathcal{E}$ of stretch factor 6. With this fact, combined with a recent result of [BGHI10], we derive *en passant* the following: Every Θ_6-graph contains a spanner of maximum degree 6 that has stretch factor 3. Secondly, by the use of local modifications of H_2, we show how to decrease the maximum degree from 9 to 6 without increasing the maximum stretch while preserving planarity.

Our approach leads to a significant improvement in the maximum degree of the plane spanner, from 14 down to 6 (see Table 1). Just as the Delaunay graph, the TD-Delaunay graph of a set of n points in the plane can be computed in time $O(n \log n)$ [Che89]. Given this graph, our final spanner H_4 can be constructed in $O(n)$ time. We note that our analysis of the stretch factor of the spanner is tight: we can place points in the plane so that the resulting degree 6 spanner has stretch factor arbitrarily close to 6.

2 Preliminaries

Given points in the two-dimensional Euclidean plane, the complete Euclidean graph $\mathcal{E}$ is the complete weighted graph embedded in the plane whose nodes are identified with the points. In the following, given a graph G, $V(G)$ and $E(G)$ stand for the set of nodes and edges of G. For every pair of nodes u and w, we identify with edge uw the segment $[uw]$ and associate an edge length equal to the Euclidean distance $|uw|$. We say that a subgraph H of a graph G is a *t-spanner* of G if for any pair of vertices u, v of G, the distance between u and v in H is at most t times the distance between u and v in G; the constant t is referred to as the *stretch factor* of H (with respect to G). We will say that H is a spanner if it is a t-spanner of $\mathcal{E}$ for some constant t.

A *cone* C is the region in the plane between two rays that emanate from the same point. Let us consider the rays obtained by a rotation of the positive x-axis by angles of $i\pi/3$ with $i = 0, 1, \ldots, 5$. Each pair of successive rays defines a cone whose apex is the origin. Let $\mathcal{C}_6 = (\overline{C}_2, C_1, \overline{C}_3, C_2, \overline{C}_1, C_3)$ be the sequence of cones obtained, in counter-clockwise order, starting from the positive x-axis. The cones C_1, C_2, C_3 are said to be *positive* and the cones $\overline{C}_1, \overline{C}_2, \overline{C}_3$ are said to be *negative*. We assume a cyclic structure on the labels so that $i+1$ and $i-1$ are always defined. For a positive cone C_i, the clockwise next cone is the negative cone $\overline{C}_{i+1}$ and the counter-clockwise next cone is the negative cone $\overline{C}_{i-1}$.

For each cone $C \in \mathcal{C}_6$, let ℓ_C be the bisector ray of C (in Figure 1, for example, the bisector rays of the positive cones are shown). For each cone C and each point u, we define $C^u := \{x + u : x \in C\}$, the translation of cone C from the origin to point u. We set $\mathcal{C}_6^u := \{C + u : C \in \mathcal{C}_6\}$, the set of all six cones at u. Observe that $w \in C_i^u$ if and only if $u \in \overline{C}_i^w$.

Let v be a point in a cone C^u. The *projection distance* from u to v, denoted $d_P(u, v)$, is the Euclidean distance between u and the projection of v onto ℓ_{C^u}.

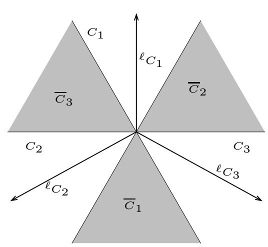

Fig. 1. Illustration of notations used for describing cones. Positive cones are white and negative cones are grey. Bisector rays of the three positive cones are shown.

For any two points v and w in C^u, v is *closer* to u than w if and only if $d_P(u,v) < d_P(u,w)$. We denote by $parent_i(u)$ the closest point from u belonging to cone C_i^u.

We say that a given set of points S are in *general position* if no two points of S form a line parallel to one of the rays that define the cones of $\mathcal{C}_6$. For the sake of simplicity, in the rest of the paper we only consider sets of points that are in general position. This will imply that it is impossible that two points v and w have equal projective distance from another point u. Note that, in any case, ties can be broken arbitrarily when ordering points that have the same distance (for instance, using a counter-clockwise ordering around u).

Our starting point is a geometric graph proposed in [BGHI10]. It represents the first step of our construction.

Step 1. *Every node u of $\mathcal{E}$ chooses $parent_i(u)$ in each non-empty cone C_i^u. We denote by H_1 the resulting subgraph.*

While we consider H_1 to be undirected, we will refer to an edge in H_1 as *outgoing* with respect to u when chosen by u and *incoming* with respect to $v = parent_i(u)$, and we color it i if it belongs to C_i^u. Note that edge uv is in the negative cone $\overline{C}_i^v$ of v.

Theorem 1 ([BGHI10]). *The subgraph H_1 of $\mathcal{E}$:*

- *is a plane graph such that every face (except the outerface) is a triangle,*
- *is a 2-spanner of $\mathcal{E}$, and*
- *has at most one (outgoing) edge in every positive cone of every node.*

Note that the number of incoming edges at a particular node of H_1 is not bounded.

In our construction of the subsequent subgraph H_2 of H_1, for every node u some neighbors of u will play an important role. Given i, let $children_i(u)$ be the set of points v such that $u = parent_i(v)$. Note that $children_i(u) \subseteq \overline{C}_i^u$. In $children_i(u)$, three special points are named:

- $closest_i(u)$ is the closest point of $children_i(u)$;
- $first_i(u)$ is the first point of $children_i(u)$ in counter-clockwise order starting from x axis;
- $last_i(u)$ is the last point of $children_i(u)$ in counter-clockwise order starting from x axis.

Note that some of these nodes can be undefined if the cone $\overline{C}_i^u$ is empty. Let (u,v) be an edge such that $v = parent_i(u)$. A node w is *i-relevant* with respect to (wrt) u if $w \in \overline{C}_i^v = \overline{C}_i^{parent_i(u)}$, and either $w = first_{i-1}(u) \neq closest_{i-1}(u)$, or $w = last_{i+1}(u) \neq closest_{i+1}(u)$. When node w is defined as $first_{i-1}(u)$ or $last_{i+1}(u)$, we will omit specifying "with respect to u". For instance, in Figure 2 **(a)**, the vertices v_l and v_r are i-relevant with respect to w. In Figure 2 **(b)** the vertex $v_r = last_{i+1}(w)$ is not i-relevant since it is not in $\overline{C}_i^u$ and $v_l = first_{i-1}(w)$ is not i-relevant since it is also $closest_{i-1}(w)$.

3 A Simple Planar 6-Spanner of Maximum Degree 9

In this section we describe the construction of H_2, a plane 6-spanner of $\mathcal{E}$ of maximum degree 9. The construction of H_2 is very simple and can be easily distributed:

Step 2. *Let H_2 be the graph obtained by choosing edges of H_1 as follows: for each node u and each negative cone $\overline{C}_i^u$:*

- *add edge $(u, closest_i(u))$ if $closest_i(u)$ exists,*
- *add edge $(u, first_i(u))$ if $first_i(u)$ exists and is $(i+1)$-relevant and*
- *add edge $(u, last_i(u))$ if $last_i(u)$ exists and is $(i-1)$-relevant.*

Note that H_2 is a subgraph of H_1 that is easily seen to have maximum degree no greater than 12 (there are at most 3 incident edges per negative cone and 1 incident edge per positive cone). Surprisingly, we shall prove that:

Theorem 2. *The graph H_2*

- *has maximum degree 9,*
- *is a 3-spanner of H_1, and thus a 6-spanner of $\mathcal{E}$.*

The remainder of this section is devoted to proving this theorem.

The charge of a cone. In order to bound the degree of a node in H_2, we devise a counting scheme. Each edge incident to a node is *charged to* some cone of that node as follows:

- each negative cone $\overline{C}_i^u$ is charged by the edge $(u, closest_i(u))$ if $closest_i(u)$ exists.
- each positive cone C_i^u is charged by $(u, parent_i(u))$ if this edge is in H_2, by edge $(u, first_{i-1}(u))$ if $first_{i-1}(u)$ is i-relevant, and by $(u, last_{i+1}(u))$ if $last_{i+1}(u)$ is i-relevant.

For instance, in Figure 2 **(a)**, the cone C_i^w is charged to twice: once by $v_l w$ and once by $v_r w$; the cone $\overline{C}_{i-1}^w$ is charged to once by its smallest edge. In **(b)**, the cone C_i^w is not charged to at all: $v_l w$ is the shortest edge in $\overline{C}_{i-1}^w$. In **(c)** the cone C_i^w is charged to once by $v_l w$ and once by the edge wu.

We will denote by $charge(C)$ the charge to cone C. With the counting scheme in place, we can prove the following lemma, which implies the first part of Theorem 2, since the sum of charges to cones of a vertex is equal to its degree in H_2.

Lemma 1. *Each negative cone of every node has at most 1 edge charged to it and each positive cone of every node has at most 2 edges charged to it.*

Proof. Since a negative cone never has more than one edge charged to it, all we need to do is to argue that no positive cone has 3 edges charged to it. Let C_i^w be a positive cone at some node w.

Let $u = parent_i(w)$. If the edge (w, u) is not in H_2 then clearly $charge(C_i^w) \leq 2$. Otherwise, we consider three cases:

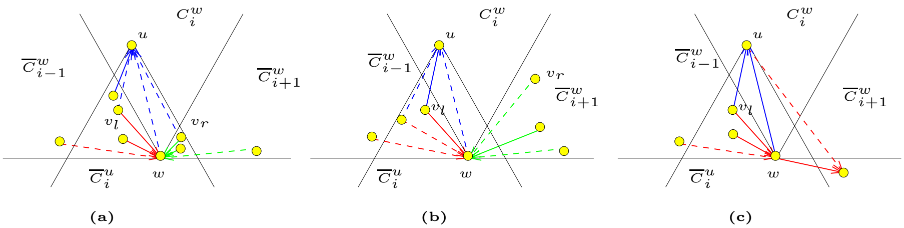

Fig. 2. In all three cases, edge wu is in H_1 but $w \neq \mathit{closest}_i(u)$. Solid edges are edges that are in H_2. **(a)** The edges wv_l and wv_r are, respectively, the clockwise last in C_{i-1}^w and clockwise first in C_{i+1}^w and are i-relevant with respect to w. **(b)** The edge wv_r is not i-relevant because wv_r is not in $\overline{C}_i^u$. The edge wv_l is in H_2 but is not i-relevant because it is the shortest edge in $\overline{C}_{i-1}^w$. **(c)** Edge wv_l is i-relevant. Note that edge wu is in H_2 because it is $(i-1)$-relevant with respect to u.

Case 1: $w = \mathit{closest}_i(u)$. Any point of $R = \overline{C}_i^u \cap \{\overline{C}_{i-1}^w \cup \overline{C}_{i+1}^w\}$ is closer to u than w. Since w is the closest neighbor of u in $\overline{C}_i^u$ the region R is empty. Hence the nodes $\mathit{first}_{i-1}(w)$ and $\mathit{last}_{i+1}(w)$ are not i-relevant. Hence $\mathit{charge}(C_i^w) = 1$.

Case 2: $w = \mathit{last}_i(u)$ and w is $(i-1)$-relevant (with respect to u, see Figure 2 **(c)**). In this case, w, u and $\mathit{parent}_{i-1}(w) = \mathit{parent}_{i-1}(u)$ form an empty triangle in H_1. Therefore, $\overline{C}_i^u \cap \overline{C}_{i+1}^w$ is empty. Hence $\mathit{last}_{i+1}(w)$ is not i-relevant. Hence $\mathit{charge}(C_i^w) \leq 2$.

Case 3: $w = \mathit{first}_i(u)$ and w is $(i+1)$-relevant. Using an argument symmetric to the one in Case 2, $\overline{C}_i^u \cap \overline{C}_{i-1}^w$ is empty. Hence $\mathit{first}_{i-1}(w)$ is not i-relevant. Hence $\mathit{charge}(C_i^w) \leq 2$. □

The above proof gives additional structural information that we will use in the next section:

Corollary 1. *Let $u = \mathit{parent}_i(w)$. If $\mathit{charge}(C_i^w) = 2$ then either:*

1. *(w, u) is not in H_2, and $\mathit{first}_{i-1}(w)$ and $\mathit{last}_{i+1}(w)$ are i-relevant (and are therefore neighbors of w in H_2), or*
2. *$w = \mathit{last}_i(u)$ is $(i-1)$-relevant and $\mathit{first}_{i-1}(w)$ is i-relevant (and thus (w, u) and $(\mathit{first}_{i-1}(w), w)$ are in H_2), or*
3. *$w = \mathit{first}_i(u)$ is $(i+1)$-relevant and $\mathit{last}_{i+1}(w)$ is i-relevant (and thus (w, u) and $(\mathit{last}_{i+1}(w), w)$ are in H_2).*

In case 1 above, note that nodes $\mathit{first}_{i-1}(w)$, w, and $\mathit{last}_{i+1}(w)$ are both in $\overline{C}_i^u$ and that u is closer from both $\mathit{first}_{i-1}(w)$ and $\mathit{last}_{i+1}(w)$ than from w. When the case 1 condition holds, we say that w is *i-distant*.

In order to prove that H_2 is a 3-spanner of H_1, we need to show that for every edge wu in H_1 but not in H_2 there is a path from u to w in H_2 whose length is at most $3|uw|$. Let wu be an incoming edge of H_1 with respect to u. Since $wu \notin H_2$, the shortest incoming edge of H_1 in the cone C of u containing wu must be in H_2: we call it vu. Without loss of generality, we assume vu is clockwise from wu with respect to u.

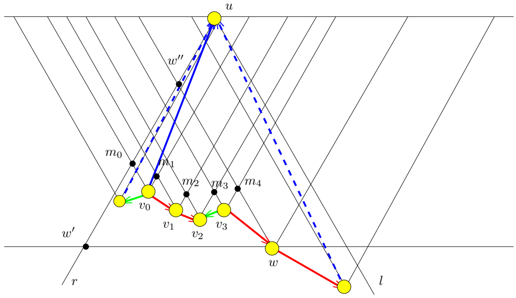

Fig. 3. Canonical path

We consider all the edges of H_1 incident to u that are contained in the cone $\overline{C}_i^u$ and lying in-between vu and wu, and we denote them, in counter-clockwise order, $vu = v_0u$, v_1u, ..., $v_ku = wu$. Because H_1 is a triangulation, the path v_0v_1, v_1v_2, ..., $v_{k-1}v_k$ is in H_1. We call this path the *canonical path with respect to u and w* (see Figure 3). Note that the order – u first, w second – matters.

Lemma 2. *Let $(w, u = parent_i(w))$ be an edge of H_1 and $v = closest_i(u)$. If $w \neq v$ then:*

1. *H_2 contains the edge vu and the canonical path with respect to u and w*
2. *$|uv| + \sum_{i=1}^{k} |v_{i-1}v_i| \leq 3|uw|$.*

The second part of Theorem 2 follows because Lemma 2 shows that for every wu in H_1 but not in H_2, if wu is incoming with respect to u then the path consisting of uv and the canonical path with respect to u and w will exist in H_2 and the length of this path is at most $3|uw|$.

Proof (of Lemma 2). Let $e = (v_j, v_{j+1})$ be an edge of the canonical path with respect to u and w. First assume that e is incoming at v_j. Observe that v_{j+1} is the neighbor of v_j that is just before u in the counter-clockwise ordering of neighbors around v_j in the triangulation H_1. Hence $v_{j+1} = last_{i+1}(v_j)$. Since v_{j+1} is in $\overline{C}_i^u$, v_{j+1} is i-relevant (with respect to v_j) or $v_{j+1} = closest_{i+1}(v_j)$. In both cases, e is in H_2. Now assume that the edge e is incoming at v_{j+1}. We similarly prove that $v_j = first_{i-1}(v_{j+1})$ and that v_j is i-relevant (with respect to v_{j+1}) or $v_j = closest_{i-1}(v_{j+1})$. In both cases, e is in H_2. This proves the first part of the lemma.

In order to prove the second part of Lemma 2, we denote by $C_i^{v_i}$ the cone containing u of canonical path node v_i, for $i = 0, 1, ..., k$. We denote by r_i and l_i the rays defining the clockwise and counter-clockwise boundaries of cone $C_i^{v_i}$. Let r and l be the rays defining the clockwise and counter-clockwise boundaries of cone $\overline{C}_i^u$. We define the point m_o as the intersection of half-lines r and l_0, points m_i as the intersections of half-lines r_{i-1} and l_i for every $1 \leq i \leq k$. Let w' be the intersection of the half-line r and the line orthogonal to ℓ_C $(C = C_i^w)$

passing through w, and let w'' be the intersection of half-lines l_k and r (see Figure 3).

We note that $|uv| = |uv_0| \leq |um_0| + |m_0v_0|$, and

$$|v_{i-1}v_i| \leq |v_{i-1}m_i| + |m_iv_i|$$

for every $1 \leq i \leq k$. Also $|uv_0| \geq |um_0|$. Then

$$\begin{aligned}|uv_0| + \sum_{i=1}^{k} |v_{i-1}v_i| &\leq |um_0| + \sum_{i=0}^{k} |m_iv_i| + \sum_{i=0}^{k-1} |v_im_{i+1}| \\ &\leq |um_0| + |ww''| + |w''m_0| \\ &\leq |uw'| + |ww'| + |w''w'| \\ &\leq |uw'| + 2|ww'|\end{aligned}$$

Observe that $|uw| = \sqrt{(|uw'|\cos\pi/6)^2 + (|ww'| - |uw'|/2)^2}$. Let $\alpha = |ww'|/|uw'|$; note that $0 \leq \alpha \leq 1$. Then

$$\begin{aligned}\frac{|uw'| + 2|ww'|}{|uw|} &\leq \frac{(1+2\alpha)|uw'|}{\sqrt{(|uw'|\cos\pi/6)^2 + ((\alpha - 1/2)|uw'|)^2}} \\ &\leq \frac{1+2\alpha}{\sqrt{1-\alpha+\alpha^2}} \\ &\leq \max_{\alpha\in[0..1]}\left\{\frac{1+2\alpha}{\sqrt{1-\alpha+\alpha^2}}\right\} \\ &\leq 3\end{aligned}$$

□

4 A Planar 6-Spanner of Maximum Degree 6

We now carefully delete edges from and add other edges to H_2, in order to decrease the maximum degree of the graph to 6 while maintaining the stretch factor. We do that by attempting to decrease the number of edges charged to a positive cone down to 1. We will not be able to do so for some cones. We will show that we can amortize the positive charge of 2 for such cones over a neighboring negative cone with charge 0. By Corollary 1, we only need to take care of two cases (the third case is symmetric to the second).

Before presenting our final construction, we start with a structural property of some positive cones in H_3 with a charge of 2. Recall that a node is *i-distant* if it has two i-relevant neighbors in H_2 (this corresponds to case 1 of Corollary 1). For instance, in Figure 3, the node v_2 is i-distant.

Lemma 3 (Forbidden charge sequence). *If, in H_2, charge$(C_i^w) = 2$ and w is not a i-distant node:*

– *either* $\mathit{first}_{i-1}(w)$ *is* i*-relevant,* $\mathit{charge}(C^w_{i-1}) \leq 1$ *and* $\mathit{charge}(\overline{C}^w_{i+1}) = 0$ *or*
– $\mathit{last}_{i+1}(w)$ *is* i*-relevant,* $\mathit{charge}(C^w_{i+1}) \leq 1$ *and* $\mathit{charge}(\overline{C}^w_{i-1}) = 0$*.*

Proof. By Corollary 1, if w is not an i-distant node, either $\mathit{first}_{i-1}(w)$ or $\mathit{last}_{i+1}(w)$ is i-relevant. We assume the second case, and the first will follow by symmetry.

We first prove the existence of a cone of charge 0. If $u = \mathit{parent}_i(w)$, then by Corollary 1, $w = \mathit{last}_i(u)$ and w is $(i-1)$-relevant (with respect to u). This means that nodes w, u, and $v = \mathit{parent}_{i-1}(u) = \mathit{parent}_{i-1}(w)$ form an empty triangle in H_1 and therefore there is no edge that ends up in $\overline{C}^w_{i+1}$. Hence $\mathit{charge}(\overline{C}^w_{i+1}) = 0$.

Let us prove now by contradiction that $\mathit{charge}(C^w_{i-1}) \leq 1$. Assume that $\mathit{charge}(C^w_{i-1}) = 2$. By Corollary 1 there can be three cases. We have just shown that there are no edges in $\overline{C}^w_{i+1}$, so there cannot be a node $\mathit{first}_{i+1}(w)$ and the first two cases cannot apply. Case 3 of Corollary 1 implies that $w = \mathit{first}_{i-1}(v)$ which is not possible because edge (u, v) is before (w, u) in the counter-clockwise ordering of edges in $\overline{C}_{i-1}(v)$. □

Step 3. *We construct* H_3 *from* H_2 *as follows: for every integer* $1 \leq i \leq 3$ *and for every* i*-distant node* w*:*

– *add the edge* $(\mathit{first}_{i-1}(w), \mathit{last}_{i+1}(w))$ *to* H_3*;*
– *let* w' *be the node among* $\{\mathit{first}_{i-1}(w), \mathit{last}_{i+1}(w)\}$ *which is greater in the canonical path order. Remove the edge* (w, w') *from* H_3*.*

New charge assignments. Since a new edge e is added between nodes $\mathit{first}_{i-1}(w)$ and $\mathit{last}_{i+1}(w)$ in Step 3, we assign the charge of e to $\overline{C}^{\mathit{first}_{i-1}(w)}_{i+1}$ and to $\overline{C}^{\mathit{last}_{i+1}(w)}_{i-1}$. For the sake of convenience, we denote by $\widetilde{\mathit{charge}}(C)$ the total charge, after Step 3, of cone C in H_3 and the next graph we will construct, H_4. The following lemma shows that the application of Step 3 does not create a cone of charge 2 and decreases the charge of cone C^w_i of i-distant node w from 2 to 1.

Lemma 4 (Distant nodes). *If* w *is an* i*-distant node then:*

– $\widetilde{\mathit{charge}}(C^w_i) = \mathit{charge}(C^w_i) - 1 = 1$*;*
– $\widetilde{\mathit{charge}}(\overline{C}^{\mathit{first}_{i-1}(w)}_{i+1}) = \mathit{charge}(\overline{C}^{\mathit{last}_{i+1}(w)}_{i-1}) = 1$*.*

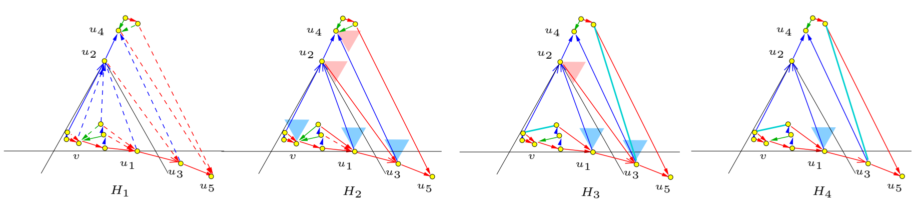

Fig. 4. From H_1 (plain arrows are the closest edges) to H_4. Light blue and pink positive cones have a charge equal to 2. The node v is i-distant and the node u_4 is $i+1$-distant.

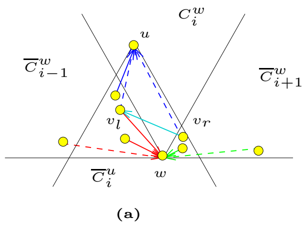

Fig. 5. **(a)** Step 3 applied on the configuration of Figure 2 **(a)**: the edge wv_r is removed because the canonical path of w with respect to u doesn't use it. The edge is then replaced by the edge $v_r v_l$. **(b)** Step 4 applied on the configuration of Figure 2 **(c)**: the edge wv_l is removed. **(c)** the edge $v_l w$ is the shortest in $\overline{C_{i-1}^w}$; the cone C_i^w is thus charged to only once, by wu, and the edge $v_l w$ is not removed during step 4.

Step 4. *We construct H_4 from H_3 as follows: for every integer $1 \leq i \leq 3$ and for every node w such that $\widetilde{charge}(C_i^w) = 2$ and $\widetilde{charge}(\overline{C}_{i-1}^w) = \widetilde{charge}(\overline{C}_{i+1}^w) = 1$, if $w = last_i(parent_i(w))$ then remove the edge $(w, first_{i-1}(w))$ from H_3 and otherwise remove $(w, last_{i+1}(w))$.*

Lemma 5. *There is a 1-1 mapping between each positive cone C_i^w that has charge 2 after step 4 and a negative cone at w that has charge 0.*

Proof. Corollary 1 gives the properties of two types of cones with charge 2 in H_2. If cone C_i^w is one in which w is an i-distant node in H_2, then C_i^w will have a charge of 1 after Step 3, by Lemma 4. If w is not i-distant, there can be two cases according to Lemma 3. We assume the first (the second follows by symmetry); so we assume that $\overline{C}_{i+1}^w$ has charge 0 in H_2. If that charge is increased to 1 in step 3, then step 4 will decrease the charge of C_i^w down to 1. So, if C_i^w still has a charge of 2 after step 4, then $\overline{C}_{i+1}^w$ will still has charge 0 and we map C_i^w to this adjacent negative cone. The only positive cone that could possibly map to $\overline{C}_{i+1}^w$ would be the other positive cone adjacent to $\overline{C}_{i+1}^w$, C_{i-1}^w, but that cone has charge at most 1 by Lemma 3. □

Theorem 3. *H_4 is a plane 6-spanner of $\mathcal{E}$ of maximum degree 6.*

Proof. By Corollary 1, Lemma 4, and Lemma 5, it is clear that H_4 has maximum degree 6.

Let us show H_4 is a 6-spanner of $\mathcal{E}$. By Lemma 2, for every edge wu in H_1 but not in H_2, the canonical path with respect to u and w in H_2 has total length at most $3|wu|$. We argue that the removal, in step 3, of edges on the canonical path from u is compensated by the addition of other edges in step 3. Observe first that while some edges of the canonical path may have been removed from H_2 in step 3, in every case a shortcut has been added. Some edges have also been removed in step 4. The removed edge is always the last edge on the canonical path from u to w, where uw is the first or last edge, in counterclockwise order, in some negative cone at u and $uw \in H_2$. This means that the canonical path edge is only needed to reach w from u, and no other nodes. Therefore it can be removed since $wu \in H_2$. In summary, no "intermediate" canonical path edge is dropped without a shortcut, and "final" canonical path edges will be dropped only when no longer needed. Therefore any canonical path (of length at most $3|wu|$) in H_2 is replaced by a new path (with shortcuts) of length at most $3|wu|$. By Lemma 2, the above argument can also be directly applied for every edge $xy \in H_2$ removed in H_3.

It remains to show that H_4 is planar. More precisely we have to show that edges introduced during step 3 do not create crossings in H_3. Let $v_l v_r$ be an edge created during step 3. Observe that in H_1 there are two adjacent triangular faces $f_1 = uv_l w$ and $f_2 = uwv_r$. Since the edge wv_l is in C^w_{i-1} and v_r is in C^w_{i+1} the angle $v_l uv_r$ is less than π. Hence the edge $v_l v_r$ is inside the two faces f_1 and f_2. The only edge of H_1 that crosses the edge $v_l v_r$ is the edge wu. Since the edge wu is not present in H_4 there is no crossing between an edge of $H_1 \cap H_3$ and an edge added during step 3. What now remains to be done is to show that two edges added during step 3 cannot cross each other. Let $v'_l v'_r$ be an edge created during step 3 and let $f'_1 = u'v'_l w'$ and $f'_2 = u'w'v'_r$ the two faces of H_1 containing this edge. If the edges $v_l v_r$ and $v'_l v'_r$ cross each other, then they are supported by at least one common face of H_1, i.e. $\{f_1, f_2\} \cap \{f'_1, f'_2\} \neq \emptyset$. Observe that the edges $v_l u, wu$ and $v_r u$ are colored i, the edge wv_l is colored $i-1$ and the edge wv_r is colored $i+1$. Similarly the edges $v'_l u', w'u'$ and $v'_r u'$ are colored i', the edge $w'v'_l$ is colored $i'-1$ and the edge $w'v'_r$ is colored $i'+1$. Each face f_1, f_2, f'_1 and f'_2 has two edges of the same color, hence $i = i'$. Because of the color of the third edge of each face, this implies that $f_1 = f'_1$ and $f_2 = f'_2$, and so $v_l v_r = v'_l v'_r$. This shows that H_4 has no crossing. □

5 Conclusion

Our construction can be used to obtain a spanner of the unit-hexagonal graph, a generalization of the complete Euclidean graph. More precisely, every unit-hexagonal graph G has a spanner of maximum degree 6 and stretch factor 6. This can be done by observing that, in our construction, the canonical path associated with each edge $e \in G \setminus H_2$ is composed of edges of "length" at most the "length" of e, where the "length" of e is the hexagonal-distance[1] between its end-points.

References

[ADDJ90] Althöfer, I., Das, G., Dobkin, D.P., Joseph, D.: Generating sparse spanners for weighted graphs. In: SWAT, pp. 26–37 (1990)

[BGHI10] Bonichon, N., Gavoille, C., Hanusse, N., Ilcinkas, D.: Connections between Theta-graphs, Delaunay triangulations, and orthogonal surfaces. Technical Report hal-00454565, HAL (February 2010)

[1] The hexagonal-distance between u and v Euclidean distance between u and the projection of v onto the bisector of the cone of $\mathcal{C}^u_6$ where v belongs to.

[BGS05] Bose, P., Gudmundsson, J., Smid, M.: Constructing plane spanners of bounded degree and low weight. Algorithmica 42(3-4), 249–264 (2005)

[BS] Bose, P., Smid, M.: On plane geometric spanners: A survey and open problems (submitted)

[BSX09] Bose, P., Smid, M., Xu, D.: Delaunay and diamond triangulations contain spanners of bounded degree. International Journal of Computational Geometry and Applications 19(2), 119–140 (2009)

[Che89] Paul Chew, L.: There are planar graphs almost as good as the complete graph. Journal of Computer and System Sciences 39(2), 205–219 (1989)

[DH96] Das, G., Heffernan, P.J.: Constructing degree-3 spanners with other sparseness properties. Int. J. Found. Comput. Sci. 7(2), 121–136 (1996)

[KG92] Mark Keil, J., Gutwin, C.A.: Classes of graphs which approximate the complete Euclidean graph. Discrete & Computational Geometry 7(1), 13–28 (1992)

[KP08] Kanj, I.A., Perković, L.: On geometric spanners of Euclidean and unit disk graphs. In: 25^{th} Annual Symposium on Theoretical Aspects of Computer Science (STACS), vol. hal-00231084, pp. 409–420. HAL (February 2008)

[LSW04] Li, X.-Y., Stojmenovic, I., Wang, Y.: Partial Delaunay triangulation and degree limited localized bluetooth scatternet formation. IEEE Trans. Parallel Distrib. Syst. 15(4), 350–361 (2004)

[LW04] Li, X.-Y., Wang, Y.: Efficient construction of low weight bounded degree planar spanner. International Journal of Computational Geometry and Applications 14(1–2), 69–84 (2004)