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Plane Wave Discontinuous Galerkin Methods: Exponential Convergence of the *hp*-version

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Abstract

We consider the two-dimensional Helmholtz equation with constant coefficients on a domain with piecewise analytic boundary, modelling the scattering of acoustic waves at a sound soft obstacle. Our discretisation relies on the Trefftz-discontinuous Galerkin approach with plane wave basis functions on meshes with very general element shapes, geometrically graded towards domain corners. We prove exponential convergence of the discrete solution in terms of number of unknowns.

Keywords: Helmholtz equation, sound-soft wave scattering, analytic regularity, approximation by plane waves, Trefftz-discontinuous Galerkin method, *hp*-version, a priori convergence analysis, locally refined meshes, exponential convergence.

AMS subject classification: 65N30, 65N15, 35J05.

1 Introduction

This article is concerned with a particular type of Trefftz method for 2D scalar wave scattering problems in the frequency domain, modelled by means of the linear Helmholtz equation with constant coefficients. In general, Trefftz methods try to incorporate information about the exact solution into local approximation spaces by requiring that they are contained in the kernel of the governing differential operator. This policy looks particularly attractive for wave propagation, which usually involves oscillatory solutions.

It is not straightforward to marry the Trefftz idea with classical conforming finite element Galerkin discretisations, *cf.* the partition of unity method [2, 20]. Conversely, discontinuous Galerkin (DG) methods, which do not impose any interelement continuity on the trial functions, offer a very convenient framework for the implementation of Trefftz methods.

For wave propagation problems in homogeneous media, natural Trefftz functions are plane waves, which give rise to plane wave discontinuous Galerkin (PWDG) methods. Their oldest representative is the so-called Ultra Weak Variational Formulation (UWVF), proposed in [5]. It was not recognised as a PWDG method in the beginning, and a comprehensive convergence theory remained elusive for quite some time. Finally, in [4, 8, 9], the UWVF was recast as a DG method, thus paving the way for using the powerful arsenal of DG analysis.

The first fruit was harvested in [9] in the form of a complete convergence analysis of the *h*-version of PWDG. The *h*-version was also tackled independently in [4], based on tools from [29]. It turned out that these tools could also be harnessed to deal with the *p*-version, and this was done in [12]. Algebraic convergence in p could be established, though confined to "quasi-uniform" meshes. Of course, here, instead of designating the polynomial degree, p should be read as the number of plane waves used for local approximation. Later, in [14], the *p*-convergence theory was extended to cover locally refined meshes.

Based on the techniques from [14], in this article we pursue the ultimate goal of establishing *exponential convergence* (with respect to the number of degrees of freedom) of PWDG solutions, when the trial spaces are built following a policy borrowed from standard *hp*-finite

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element methods. Assuming domains and data with sufficient regularity, the idea is to use large mesh cells equipped with many plane waves where the solution is smooth, whereas small cells are employed to resolve singularities of the solution at corners of the boundary. This kind of *hp* approximation with polynomials has seen an amazing development starting from the work of Babuška [1,10]; see [32] for a comprehensive exposition. It has also been adapted to polynomial DG methods by several authors, see, for instance, [16,30,31,35]. Applications to scalar wave propagation are reported in [7,23,24].

Results on the approximation of Helmholtz solutions by plane waves are pivotal. Here, major progress has been achieved in [26,27]. These works made use of Vekua's theory and, thus, could exploit known results about the approximation of harmonic functions by harmonic polynomials. Recently, results in this direction targeting harmonic functions that can be extended analytically were obtained in [15], generalising earlier work by M. Melenk [20]. A proof of exponential convergence of the *hp*-version of (polynomial) Trefftz-DG method for the Laplace problem was included.

The main result of this work (Theorem 6.5, Section 6) is a proof that the L^2-norm of the discretisation error of a special PWDG method on very general, geometrically graded meshes converges exponentially in a root of the number of degrees of freedom. This is the first such result for a numerical method based on plane waves. For the proof, we had to refine the duality arguments of [14], see Section 4, and combine them with novel L^∞-approximation estimates for plane waves given in Section 5. The reason of the restriction to two space dimensions is that the approximation estimates for harmonic functions we rely on (see Proposition 5.1) were derived in [15] using complex analysis arguments, and thus are proved in 2D only. The error is bounded by a negative exponential of the square root of the total number of degrees of freedom employed, while typical polynomial *hp*-schemes in two dimensions only deliver exponential convergence in the cubic root of the same parameter, e.g. see [1, Theorem 5.3]. The results of our analysis hold true also when circular waves are used instead of plane waves.

At this point we emphasise that our focus is on numerical approximation theory. We deliberately ignore the key challenge of ill-conditioning of linear systems arising from PWDG approaches, *cf.* [17,18]. We even acknowledge that an implementation of the method investigated below may severely be affected by numerical instability, see Remark 6.7.

2 Scattering boundary value problem

As in [14, Section 2], let $\Omega_D \subset \mathbb{R}^2$ be a bounded, Lipschitz domain occupied by a sound-soft material, which we assume to be star-shaped with respect to the origin $\mathbf{0}$. We denote by $\Gamma_D := \partial\Omega_D$ its boundary. We introduce another bounded Lipschitz domain Ω_R with boundary Γ_R such that $\overline{\Omega}_D \subset \Omega_R$, and $\mathrm{dist}(\Gamma_D, \Gamma_R) > 0$[1]. We set $\Omega := \Omega_R \setminus \overline{\Omega}_D$ and we assume $\partial\Omega$ to be piecewise analytic. It may have finitely many corners $\mathbf{c}_\nu$, $1 \leq \nu \leq n_c$, which we collect in the set $\mathcal{C} := \{\mathbf{c}_\nu\}_{\nu=1}^{n_c}$.

We focus on the following boundary value problem (BVP) for the Helmholtz equation:

$$\begin{cases} -\Delta u - k^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \nabla u \cdot \mathbf{n} + ik\vartheta u = g_R & \text{on } \Gamma_R, \end{cases} \tag{1}$$

with $g_R \in L^2(\Gamma_R)$, wavenumber $k > 0$, and $\vartheta \in \mathbb{R}$ a non-dimensional, non-zero parameter. We have written $\mathbf{n}$ for the outward-pointing unit normal vector field on $\partial\Omega$.

2.1 Stability and Sobolev regularity

We denote by $\|\cdot\|_{0,D}$ the $L^2(D)$-norm and by $|\cdot|_{\ell,D}$ the $H^\ell(D)$-Sobolev seminorm, $\ell \in \mathbb{N}_0$ ($\mathbb{N}_0 = \{0, 1, 2, \ldots\}$), where D is a Lipschitz domain. For positive non-integer values of s, we consider the $H^s(D)$-seminorm as defined by the Sobolev–Slobodeckij integral (see e.g. [28, Page 43]). On a Lipschitz manifold D we use only the $L^2(D)$-norm and the $H^s(D)$-seminorm for $0 < s < 1$. It is convenient to make use of the following k-weighted Sobolev

[1]For $\mathbf{x} \in \mathbb{R}^2$ and $A, B \subset \mathbb{R}^2$, we denote by $\mathrm{dist}(\mathbf{x}, A)$ the set–point distance $\inf_{\mathbf{y}\in A} |\mathbf{x} - \mathbf{y}|$ and by $\mathrm{dist}(A, B)$ the set–set distance $\inf_{\mathbf{x}\in A, \mathbf{y}\in B} |\mathbf{x} - \mathbf{y}|$.

norms (note that k has the dimension of the inverse of a length):

$$\|v\|^2_{\ell,k,D} := \sum_{j=0}^{\ell} k^{2(\ell-j)} |u|^2_{j,D} \qquad \forall v \in H^\ell(D),\ \ell \in \mathbb{N}.$$

We assume Ω_R to be star-shaped with respect to the ball[2] $B_{\gamma_R d_\Omega}$, for some $\gamma_R > 0$, where $d_\Omega := \mathrm{diam}(\Omega)$.

Theorems 2.1, 2.2, and 2.3 of [14] (see also [11, Propositions 3.3 and 3.4]) give the following stability and elliptic regularity result.

Proposition 2.1. *Let u be the solution of the inhomogeneous boundary value problem*

$$\begin{aligned} -\Delta u - k^2 u &= f && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_D,\\ \nabla u \cdot \mathbf{n} \pm ik\vartheta u &= g_R && \text{on } \Gamma_R. \end{aligned} \tag{2}$$

If $f \in L^2(\Omega)$ and $g_R \in L^2(\Gamma_R)$, the weak formulation of (2) is well-posed in $H^1(\Omega)$. Moreover, if $g_R \in H^r(\Gamma_R)$ for a given $0 < r < 1/2$, then there exists $s_\Omega > 0$ depending only on (the corners of) Ω, such that $u \in H^{\frac{3}{2}+s}(\Omega)$ for every s satisfying

$$0 < s < s_\Omega, \qquad s \le r, \tag{3}$$

and the following bounds hold:

$$\begin{aligned} \|u\|_{1,k,\Omega} &\le C\Big(d_\Omega \|f\|_{0,\Omega} + d_\Omega^{\frac{1}{2}} \|g_R\|_{0,\Gamma_R}\Big),\\ |\nabla u|_{\frac{1}{2}+s,\Omega} &\le C(1+d_\Omega k)\Big(d_\Omega^{\frac{1}{2}-s} \|f\|_{0,\Omega} + d_\Omega^{-s} \|g_R\|_{0,\Gamma_R}\Big) + C\,|g_R|_{s,\Gamma_R}, \end{aligned} \tag{4}$$

where the constant $C > 0$ depends only on s, γ_R and ϑ, but is independent of k, f, g_R and u.

Remark 2.2. In the case of an interior impedance problem (i.e. where $\Omega = \Omega_R$ and $\Omega_D = \emptyset$), k-explicit stability bounds have been proved in [6, Theorem 2.4] and improved in their k-dependence in [33, Theorem 1.6], without assuming Ω to be star-shaped.

At this point we fix a value for s in the admissible range given by the inequalities (3). It will be used throughout the remainder of this article.

2.2 Analytic regularity

In this section, we state an analytic regularity result for the solution u to problem (1). This result is derived within the setting of [21, Chapters 4 and 5], which extends the theory of Babuška and Guo [1] to the case of general elliptic equations with a perturbation parameter. We essentially combine the L^2-estimates of the derivatives of u given in [21, Chapter 5] with the L^∞-estimates of [1, Theorem 2.2].

To translate our problem into the notation of [21], as in [24, proof of Lemma 4.13], we set

$$A(\mathbf{x}) = 1, \qquad f(\mathbf{x}) = 0\ \ (C_f = 0), \qquad b(\mathbf{x}) = 0\ \ (C_b = 0), \qquad c(\mathbf{x}) = 1\ \ (C_c = 1);$$

the perturbation parameter is

$$\varepsilon = \frac{1}{ik},$$

and therefore the length scale is $\mathcal{E} = \dfrac{1}{k+1}$, and $\dfrac{\mathcal{E}}{|\varepsilon|} \le 1$. Comparing the expression of the Robin boundary condition, we also set

$$G_1 = \frac{1}{ik} g_R \ \ \Big(C_{G_1} = \frac{1}{k}\|g_R\|_{H^{1/2}(\Gamma_R)}\Big), \qquad G_2 = -\vartheta\ \ (C_{G_2} = |\vartheta|).$$

Recalling that n_c is the number of corner points of $\partial\Omega$, given $\underline{\beta} \in [0,1)^{n_c}$, let $\mathcal{B}^\ell_{\underline{\beta},\mathcal{E}}(\Omega)$ be the countably normed spaces defined in [21, Chapter 4] (see also [24, Section 1,1]), with weights given by

$$\widehat{\Phi}_{p,\underline{\beta},\mathcal{E}}(\mathbf{x}) = \prod_{\nu=1}^{n_c} \Phi_{p,\beta_\nu,\mathcal{E}}(\mathbf{x} - \mathbf{c}_\nu) \qquad \forall p \in \mathbb{N}_0,$$

[2] We set $B_r(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{R}^2 : \ |\mathbf{x} - \mathbf{x}_0| < r\}$, and $B_r := B_r(\mathbf{0})$.

where

$$\Phi_{p,\beta,\mathcal{E}}(\mathbf{x}) = \min\left\{1, \frac{|\mathbf{x}|}{\min\{1, \mathcal{E}(|p|+1)\}}\right\}^{p+\beta}.$$

We set $\widehat{\Phi}(\mathbf{x}) := \widehat{\Phi}_{1,\underline{0},1}(\mathbf{x}) = \prod_{\nu=1}^{n_c} \min\{1, |\mathbf{x}-\mathbf{x}_c|\}$, which is independent of k.

Theorem 2.3. *There exists a weight vector* $\underline{\beta} \in (0,1)^{n_c}$ *such that, if* $g_R \in \mathcal{B}^1_{\underline{\beta},\mathcal{E}}(\Gamma_R)$*, the solution* u *to problem* (1) *belongs to* $\mathcal{B}^2_{\underline{\beta},\mathcal{E}}(\Omega)$*. Moreover, there exist two constants* $C, \gamma > 0$ *independent on* k *such that, for all* $\mathbf{x}_0 \in \Omega$*,* $\boldsymbol{\alpha} \in \mathbb{N}_0^2$*,* $|\boldsymbol{\alpha}| = j \geq 1$*,*

$$|(D^{\boldsymbol{\alpha}}u)(\mathbf{x}_0)| \leq C \max\{k^{-1}, k^2\} \cdot \begin{cases} k^2 \left(\dfrac{k\gamma(k+1)^{n_c}}{\widehat{\Phi}(\mathbf{x}_0)}\right)^j & \text{if } j < k-2, \\ \left(\dfrac{\gamma(k+1)^{n_c}}{\widehat{\Phi}(\mathbf{x}_0)}\right)^j (j+2)^{j+2} & \text{if } j \geq k-2. \end{cases} \tag{5}$$

Thus, u *admits a real analytic continuation to the set*

$$\mathcal{N}(u) := \bigcup_{\mathbf{x}_0 \in \overline{\Omega} \setminus \bigcup_{\nu=1}^{n_c} \mathbf{c}_\nu} \left\{\mathbf{x} \in \mathbb{R}^2 : \ |\mathbf{x}-\mathbf{x}_0| < \frac{\widehat{\Phi}(\mathbf{x}_0)}{4e\,\gamma(k+1)^{n_c}}\right\} \subset \mathbb{R}^2. \tag{6}$$

Proof. (Sketch) Within the general setting of [21, Chapter 5], since $\Gamma_D \cap \Gamma_R = \emptyset$ (thus Dirichlet and Robin boundaries do not affect one another), Theorem 5.3.10 and Proposition 5.4.5 (see also Remark 5.3.11 and Remark 5.4.6) of [21] can be applied and, taking into account the first bound in (4), one can conclude that $ku \in \mathcal{B}^2_{\underline{\beta},\mathcal{E}}(\Omega)$ for some $\underline{\beta} \in (0,1)^{n_c}$. In particular, denoting by ∇^ℓ the derivatives of order ℓ (more precisely, $\left|\nabla^\ell u(\mathbf{x})\right|^2 = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^2, |\boldsymbol{\alpha}|=\ell} \frac{\ell!}{\boldsymbol{\alpha}!} |D^{\boldsymbol{\alpha}}u(\mathbf{x})|^2$),

$$\left\|\widehat{\Phi}_{p,\underline{\beta},\mathcal{E}} \nabla^{p+2}u\right\|_{0,\Omega} \leq C\left(\gamma \max\{p+2, k\}\right)^{p+2} k^{-1} \qquad \forall p \in \mathbb{N}_0,$$

in addition to

$$\|u\|_{0,\Omega} \leq Ck^{-1}, \qquad \|\nabla u\|_{0,\Omega} \leq C$$

(see also [24, Lemma 4.13]); here and in the remainder of this proof, C and γ are positive constants independent of k (C depends on the norm of the boundary datum g_R), whose value might change at each occurrence.

Along the lines of the proof of [1, Theorem 2.2], making use of the property of the weight functions stated in Equation (4.2.4) of [21], and of the Sobolev embedding of [21, Lemma 4.2.5], one obtains that, for any $\mathbf{x}_0 \in \Omega$,

$$|D^{\boldsymbol{\alpha}}u(\mathbf{x}_0)| \leq C \max\left\{k^{-1}, k^2\right\}\left(\gamma \max\{j+2, k\}\right)^{j+2}\left(\widehat{\Phi}_{j-1,\underline{\beta},\mathcal{E}}(\mathbf{x}_0)\right)^{-1} \qquad \forall \boldsymbol{\alpha} \in \mathbb{N}_0^2,\ |\boldsymbol{\alpha}| = j \geq 1.$$

Since $\widehat{\Phi}_{j-1,\underline{\beta},\mathcal{E}}(\mathbf{x}_0) \geq \widehat{\Phi}_{j,\underline{0},\mathcal{E}}(\mathbf{x}_0)$ and, from [21, Lemma 4.2.3, point 4], $\widehat{\Phi}_{j,\underline{0},\mathcal{E}}(\mathbf{x}_0) \geq C(\gamma(k+1)^{n_c})^{-j}\widehat{\Phi}_{j,\underline{0},1}(\mathbf{x}_0) = C(\gamma(k+1)^{n_c})^{-j}\left(\widehat{\Phi}(\mathbf{x}_0)\right)^j$, we have

$$|D^{\boldsymbol{\alpha}}u(\mathbf{x}_0)| \leq C \max\left\{k^{-1}, k^2\right\}\left(\frac{\gamma(k+1)^{n_c}}{\widehat{\Phi}(\mathbf{x}_0)}\right)^j \max\left\{j+2, k\right\}^{j+2} \qquad \forall \boldsymbol{\alpha} \in \mathbb{N}_0^2,\ |\boldsymbol{\alpha}| = j \geq 1,$$

from which (5) follows. By Stirling's formula, $(j+2)^{j+2} \leq 2e^j(j+2)^2 j!$ which, for large j, gives $(j+2)^{j+2} \leq 2(2e)^j j!$. For the analytic continuation to the set in (6), see [1, Page 841]. □

3 Trefftz discontinuous Galerkin method

We start from a general mesh $\mathcal{T}_h$ on Ω, whose elements are curvilinear Lipschitz polygons. For any element $K \in \mathcal{T}_h$, we denote by h_K its diameter, and set $h_{\max} := \max_{K\in\mathcal{T}_h} h_K$. Moreover, we define various sets of interfaces $\mathcal{F}_h := \cup_{K\in\mathcal{T}_h}\partial K$, and $\mathcal{F}_h^I := \mathcal{F}_h \setminus \partial\Omega$.

On the mesh $\mathcal{T}_h$, we introduce the *Trefftz space*

$$T(\mathcal{T}_h) := \Big\{v \in L^2(\Omega) : \ \exists s>0 \text{ s.t. } v \in H^{\frac{3}{2}+s}(\mathcal{T}_h) \text{ and } \Delta v + k^2 v = 0 \text{ in each } K \in \mathcal{T}_h\Big\},$$

with $H^r(\mathcal{T}_h)$ a shorthand notation for elementwise H^r-functions on $\mathcal{T}_h$. The solution u of the BVP (1) belongs to $T(\mathcal{T}_h)$ and will be approximated in a finite-dimensional Trefftz-DG trial and test space $V_p(\mathcal{T}_h) \subset T(\mathcal{T}_h)$. At this stage we need not worry about the details of constructing $V_p(\mathcal{T}_h)$; these are postponed to Section 6.2.

We fix bounded functions $\alpha, \beta > 0$, $0 < \delta \le 1/2$, bounded away from zero and defined on appropriate subsets of $\mathcal{F}_h$. Alluding to the construction of the Trefftz-DG method in [9, Section 2], we call them *flux parameters*. We introduce the following sesquilinear form and antilinear functional defined on $T(\mathcal{T}_h)$, *cf.* [14, Section 3.2], [12, Section 2], [9, Section 2],

$$\begin{aligned}
\mathcal{A}_h(u,v) :=& \int_{\mathcal{F}_h^I}\Big(\{\!\{u\}\!\} - \frac{\beta}{ik}[\![\nabla_h u]\!]_N\Big)[\![\nabla_h \overline{v}]\!]_N \,\mathrm{d}S - \int_{\mathcal{F}_h^I}\Big(\{\!\{\nabla_h u\}\!\} - \alpha i k [\![u]\!]_N\Big)\cdot[\![\overline{v}]\!]_N \,\mathrm{d}S \\
&+ \int_{\Gamma_R}\Big(u - \frac{\delta}{ik\vartheta}\left(\nabla_h u\cdot\mathbf{n} + ik\vartheta u\right)\Big)\left(\overline{\nabla_h v\cdot\mathbf{n} - ik\vartheta v}\right)\mathrm{d}S - \int_{\Gamma_D}\Big(\nabla_h u\cdot\mathbf{n} - \alpha i k u\Big)\overline{v}\,\mathrm{d}S,\\
\ell_h(v) :=& -\int_{\Gamma_R}\delta\,(ik\vartheta)^{-1} g_R\, \nabla_h\overline{v}\cdot\mathbf{n}\,\mathrm{d}S + \int_{\Gamma_R}(1-\delta)\, g_R\,\overline{v}\,\mathrm{d}S.
\end{aligned}$$

These are the building blocks of the Trefftz-DG variational problem:

$$\text{find } u_{hp}\in V_p(\mathcal{T}_h) \quad \text{such that} \quad \mathcal{A}_h(u_{hp}, v_{hp}) = \ell_h(v_{hp}) \quad \forall v_{hp}\in V_p(\mathcal{T}_h). \tag{7}$$

For its analysis it is convenient to make use of the mesh-dependent DG-norms:

$$\begin{aligned}
\|v\|_{DG}^2 :=& k^{-1}\left\|\beta^{\frac12}[\![\nabla_h v]\!]_N\right\|^2_{0,\mathcal{F}_h^I} + k\left\|\alpha^{\frac12}[\![v]\!]_N\right\|^2_{0,\mathcal{F}_h^I}\\
&+ k^{-1}\left\|\delta^{\frac12}\vartheta^{-\frac12}\nabla_h v\cdot\mathbf{n}\right\|^2_{0,\Gamma_R} + k\left\|(1-\delta)^{\frac12}\vartheta^{\frac12}v\right\|^2_{0,\Gamma_R} + k\left\|\alpha^{\frac12}v\right\|^2_{0,\Gamma_D},\\
\|v\|_{DG^+}^2 :=& \|v\|_{DG}^2 + k\left\|\beta^{-\frac12}\{\!\{v\}\!\}\right\|^2_{0,\mathcal{F}_h^I} + k^{-1}\left\|\alpha^{-\frac12}\{\!\{\nabla_h v\}\!\}\right\|^2_{0,\mathcal{F}_h^I}\\
&+ k\left\|\delta^{-\frac12}\vartheta^{\frac12}v\right\|^2_{0,\Gamma_R} + k^{-1}\left\|\alpha^{-\frac12}\nabla_h v\cdot\mathbf{n}\right\|^2_{0,\Gamma_D}.
\end{aligned}$$

Here, as in [9, 12, 14], we have used the standard DG notation for averages $\{\!\{\cdot\}\!\}$ and normal jumps $[\![\cdot]\!]_N$ across interelement boundaries, and ∇_h designates the element-wise gradient. Since α, β, δ and $(1-\delta)$ are positive, $\|\cdot\|_{DG}$ (and thus also $\|\cdot\|_{DG^+}$) is actually norm in $T(\mathcal{T}_h)$, see [12, Proposition 3.2].

In [14, Propositions 4.1 and 4.3] (see also [12, Section 3.1]), we proved the following consistency, continuity and coercivity properties for the variational problem (7): for u solution of the BVP (1) and for all $v, w \in T(\mathcal{T}_h)$

$$\mathcal{A}_h(u,v) = \ell_h(v), \qquad |\mathcal{A}_h(v,w)| \le 2\,\|v\|_{DG^+}\|w\|_{DG}, \qquad \mathrm{Im}[\mathcal{A}_h(v,v)] = \|v\|_{DG}^2.$$

This ensures that (7) is well-posed, stable and that the Trefftz-DG method enjoys quasi-optimality in the DG-norm, i.e.,

$$\|u - u_{hp}\|_{DG} \le 3 \inf_{v_{hp}\in V_p(\mathcal{T}_h)} \|u - v_{hp}\|_{DG^+}, \tag{8}$$

where u_{hp} is the solution of the discrete variational problem (7). The Trefftz-DG is therefore *unconditionally stable*, i.e. the quasi-optimality bound (8) holds with the same constant for any wavenumber $k > 0$, any mesh $\mathcal{T}_h$, any discrete Trefftz space $V_p(\mathcal{T}_h)$ and any admissible choice of the flux parameters; on the other hand, the DG-norms used to measure the error in (8) depend on $k, \mathcal{T}_h, \alpha, \beta$ and δ (but not on the specific discrete space $V_p(\mathcal{T}_h)$).

4 L^2-Estimates

Our principal goal is to study the convergence of the discretisation error of the Trefftz-DG method not only in the mesh-dependent DG-norm $\|\cdot\|_{DG}$, but also in the $L^2(\Omega)$-norm. This is made possible by a key duality technique originally introduced in [29, Theorem 3.1] and improved in [12, Section 3.2] and [14, Section 4.2]. In Lemma 4.4 we further modify this duality argument to allow for different flux parameters.

4.1 Assumptions on the meshes

We study the convergence of Trefftz-DG methods for an infinite family of meshes $\mathfrak{T} := \{\mathcal{T}_h\}$ whose members enjoy certain properties *uniformly*:

(M1) *star-shapedness*: there exist $0 < \rho_0 < \rho \leq 1/2$ such that, for all the meshes $\mathcal{T}_h \in \mathfrak{T}$ and for all $K \in \mathcal{T}_h$, there exists $\mathbf{x}_K \in K$ such that $B_{\rho h_K}(\mathbf{x}_K) \subset K$, and K is star-shaped with respect to $B_{\rho_0 h_K}(\mathbf{x}_K)$;

(M2) *local quasi-uniformity*: there exists a constant $\tau \geq 1$ such that, for all the meshes $\mathcal{T}_h \in \mathfrak{T}$,

$$\tau^{-1} \leq \frac{h_{K_1}}{h_{K_2}} \leq \tau \qquad \forall K_1, K_2 \in \mathcal{T}_h \text{ s.t. } |\partial K_1 \cap \partial K_2| \neq 0;$$

(M3) *boundedness of the skeleton measure*: there exists a constant $C_{\mathcal{F}} > 0$ such that, for all the meshes $\mathcal{T}_h \in \mathfrak{T}$,

$$|\mathcal{F}_h^I| \leq C_{\mathcal{F}}.$$

Here and in the following, we adopt the notation $|\cdot|$ for the volume (area or length) of one- or two-dimensional sets. Assumptions (M1)–(M3) are instrumental for achieving abstract error estimates in the $L^2(\Omega)$-norm in Section 4.4. In Section 6.1 they will be supplemented with more specific requirements for *hp*-approximation.

An important tool is the similarity transformation $\mathbf{x} \mapsto \hat{\mathbf{x}} := h_K^{-1}(\mathbf{x} - \mathbf{x}_K)$, which takes an element $K \in \mathcal{T}_h$ to a domain $\hat{K}$ with $\operatorname{diam}(\hat{K}) = 1$, which contains B_ρ and is star-shaped with respect to the ball B_{ρ_0}.

4.2 Flux parameters

We still have the freedom to fix the so-called flux parameters α, β, δ entering $\mathcal{A}_h$ and ℓ_h. Linking them to the local mesh width in a judicious fashion was essential for coping with locally refined meshes in [14]. Hardly surprising, the right choice of the flux parameters is also key to a successful analysis of the *hp*-version of the Trefftz-DG method. It differs slightly from what was used in [14, Formula (21)].

We fix the function α on any face $f \subset \mathcal{F}_h^I \cup \Gamma_D$ as follows:

$$\alpha|_f := \mathtt{a}\,\frac{h_{\max}}{h_f}, \tag{9}$$

where $\mathtt{a}$ is a positive *universal constant*, in particular independent of the local mesh sizes, the local Trefftz spaces, and the wavenumber k. The symbol h_f stands for the local mesh width at the interface f defined as

$$h_f := \begin{cases} \min\{h_{K_1}, h_{K_2}\} & \text{if } f = \partial K_1 \cap \partial K_2, \\ h_K & \text{if } f = \partial K \cap \partial\Omega. \end{cases}$$

Notice that this definition works also in the case of hanging nodes (compare with assumption (M2)). Moreover, we choose

$$\beta,\ \delta \quad \text{as fixed positive } \textit{universal constants}, \tag{10}$$

of course, with the additional constraint $\delta \leq 1/2$.

Remark 4.1. The choice of β and δ independent of the local mesh sizes, as opposed to $\beta|_f, \delta|_f \simeq \frac{h_{\max}}{h_f}$ as in [14], ensures that the coefficients in front of the gradient terms in the *DG*-norm do not blow up in regions where the mesh is refined. This permits us to accomplish convergence estimates on strongly locally refined meshes in Section 6. To that end, in Section 4.4 we modify the duality argument of [14].

4.3 Trace inequalities

As technical tools we use the following trace inequalities:

$$\|v\|^2_{0,\partial K} \le C_1\Big(h_K^{-1}\,\|v\|^2_{0,K} + h_K\,|v|^2_{1,K}\Big) \qquad \forall v\in H^1(K), \tag{11}$$

$$\|\nabla v\|^2_{0,\partial K} \le C_2\Big(h_K^{-1}\,\|\nabla v\|^2_{0,K} + h_K^{2s}\,|\nabla v|^2_{\frac12+s,K}\Big) \qquad \forall v\in H^{\frac32+s}(K), \tag{12}$$

where C_1 depends only on ρ_0, and C_2 on ρ_0, ρ and s. Taking $v = 1$ in (11), we can also see that

$$|\partial K| \le C_1\, h_K, \tag{13}$$

with the same C_1 as above, depending only on ρ_0.

Remark 4.2. The dependences of the constants show that the parameters ρ, ρ_0 and h_K capture all the geometrical information that is relevant for the trace inequalities, since both the "roughness" of ∂K (i.e., its Lipschitz constant in some parametrisation) and the "fatness" of K (i.e., the maximal distance of the interior points from the boundary and the relation between its measure and that of its boundary) are controlled by their values.

The bound (11) is standard (see, e.g., [3, Theorem (1.6.6)]), while (12), for simplicial elements, can be proved using [22, Theorem A.2]. Under our Assumption (M1) on the star-shapedness of the mesh element K, the trace inequalities (11) and (12), with explicit dependence of the constants on ρ, ρ_0 and s, readily follow from the following lemma by scaling arguments.

Lemma 4.3. *Let $\hat K\subset\mathbb{R}^2$ be such that* $\mathrm{diam}(\hat K) = 1$ *and let there exist* $0 < \rho_0 < \rho \le 1/2$ *such that $B_\rho \subset \hat K$, and $\hat K$ is star-shaped with respect to B_{ρ_0}. Then,*

$$\|v\|^2_{0,\partial\hat K} \le \frac{1+\sqrt2}{\rho_0}\big(\,\|v\|^2_{0,\hat K} + |v|^2_{1,\hat K}\big) \qquad \forall v\in H^1(\hat K), \tag{14}$$

$$\|w\|^2_{0,\partial\hat K} \le C_{B_1}\frac{1}{\rho^2}\left(\frac{3}{\rho_0\rho^2}\right)^{4+2s}\big(\,\|w\|^2_{0,K} + |w|^2_{\frac12+s,\hat K}\big) \qquad \forall w\in H^{\frac12+s}(\hat K), \tag{15}$$

where C_{B_1} depends on s but not *on $\hat K$.*

Proof. We start with (14). Denoting by $\mathbf{n}_K$ the outward normal unit vector to $\partial\hat K$, since $\hat K$ is star-shaped with respect to B_{ρ_0}, we have

$$\mathbf{n}_K(\mathbf{x})\cdot\mathbf{x} \ge \rho_0 \qquad \text{a.e. on } \partial\hat K, \tag{16}$$

where the inequality is meant to hold for every point $\mathbf{x}$ at which $\mathbf{n}_K(\mathbf{x})$ is defined (see [13, Lemma 3.1]). Thus,

$$\begin{aligned}
\|v\|^2_{0,\partial\hat K} &= \int_{\partial\hat K}|v|^2\,\mathrm{d}S\\
&\overset{(16)}{\le} \frac{1}{\rho_0}\int_{\partial\hat K}\mathbf{n}_K\cdot\mathbf{x}\,|v|^2\,\mathrm{d}S\\
&= \frac{1}{\rho_0}\int_{\hat K}\mathrm{div}(\mathbf{x}\,|v|^2)\,\mathrm{d}\mathbf{x} = \frac{1}{\rho_0}\int_{\hat K}(2\,|v|^2 + \mathbf{x}\cdot\nabla\,|v|^2)\,\mathrm{d}\mathbf{x}\\
&= \frac{1}{\rho_0}\int_{\hat K}(2\,|v|^2 + 2\mathbf{x}\cdot\mathrm{Re}\{v\,\nabla\overline{v}\})\,\mathrm{d}\mathbf{x}\\
&\overset{\substack{\mathrm{diam}(\hat K)=1\\ \Rightarrow|\mathbf{x}|\le1}}{\le} \frac{2}{\rho_0}\Big(\|v\|^2_{0,\hat K} + \|v\|_{0,\hat K}\,\|\nabla v\|_{0,\hat K}\Big)\\
&\le \frac{2}{\rho_0}\left(\|v\|^2_{0,\hat K} + \frac{1}{2(1+\sqrt2)}\,\|v\|^2_{0,\hat K} + \frac{(1+\sqrt2)}{2}\,\|\nabla v\|^2_{0,\hat K}\right)\\
&= \frac{1+\sqrt2}{\rho_0}\Big(\|v\|^2_{0,\hat K} + \|\nabla v\|^2_{0,\hat K}\Big),
\end{aligned}$$

which gives (14).

For the bound (15) we recall Assumption (M1) and without loss of generality, centre K at the origin, that is, $\mathbf{x}_K = \mathbf{0}$. We identify $\mathbb{R}^2$ and $\mathbb{C}$ and make use of the polar parametrisation $\Psi : \mathbb{C} \to \mathbb{C}$ such that

$$\Psi(B_1) = \hat{K}, \qquad \Psi(re^{i\theta}) = \psi(\theta)re^{i\theta}, \qquad \psi : [-\pi, \pi) \to [\rho, 1-\rho].$$

The function ψ is Lipschitz continuous with constant L_ψ satisfying

$$L_\psi = \sup_{\theta\in[-\pi,\pi]} \psi'(\theta) \le \frac{(1-\rho)^2}{\rho_0} \tag{17}$$

(see [15, Lemma 4.1]), and the function $\Psi^{-1} : \mathbb{C} \to \mathbb{C}$ is Lipschitz continuous as well, with constant $L_{\Psi^{-1}}$ satisfying

$$L_{\Psi^{-1}} = \sup_{w,v\in\mathbb{C},w\neq v} \frac{|w-v|}{|\Psi(w)-\Psi(v)|} \le \frac{2(2\rho + L_\psi)}{\rho^2} \tag{18}$$

(see [15, Lemma 4.2]).

We have

$$\begin{aligned}
\|w\|^2_{0,\partial\hat{K}} &= \int_{\partial\Psi(B_1)} |w|^2 \,\mathrm{d}S = \int_{-\pi}^{\pi} \left|(w\circ\Psi)(e^{i\theta})\right|^2 |\psi'(\theta)| \,\mathrm{d}\theta \\
&\le L_\psi \int_{-\pi}^{\pi} \left|(w\circ\Psi)(e^{i\theta})\right|^2 \mathrm{d}\theta \overset{(17)}{\le} \frac{(1-\rho)^2}{\rho_0} \int_{-\pi}^{\pi} \left|(w\circ\Psi)(e^{i\theta})\right|^2 \mathrm{d}\theta \\
&= \frac{(1-\rho)^2}{\rho_0} \|w\circ\Psi\|^2_{0,\partial B_1} \le \frac{(1-\rho)^2}{\rho_0} C_{B_1} \left(\|w\circ\Psi\|^2_{0,B_1} + |w\circ\Psi|^2_{\frac{1}{2}+s,B_1}\right),
\end{aligned}$$

where the last inequality can be proved using [22, Theorem A.2]; clearly, the constant C_{B_1}, which corresponds to that appearing in the analogous of the trace inequality (15) for the unit ball B_1, depends on s and not on $\hat{K}$.

By definition of the $(\frac{1}{2}+s)$-seminorm by the Sobolev–Slobodeckij integral, the Lipschitz property of Ψ^{-1}, and by changing variables within integrals, we obtain

$$\begin{aligned}
|w\circ\Psi|^2_{\frac{1}{2}+s,B_1} &= \int_{B_1}\int_{B_1} \frac{\left|(w\circ\Psi)(\mathbf{x}_B) - (w\circ\Psi)(\mathbf{y}_B)\right|^2}{|\mathbf{x}_B - \mathbf{y}_B|^{3+2s}} \,\mathrm{d}\mathbf{x}_B \,\mathrm{d}\mathbf{y}_B \\
&\le \int_{B_1}\int_{B_1} L^{3+2s}_{\Psi^{-1}} \frac{\left|(w\circ\Psi)(\mathbf{x}_B) - (w\circ\Psi)(\mathbf{y}_B)\right|^2}{|\Psi^{-1}(\mathbf{x}_B) - \Psi^{-1}(\mathbf{y}_B)|^{3+2s}} \,\mathrm{d}\mathbf{x}_B \,\mathrm{d}\mathbf{y}_B \\
&\le \int_{\hat{K}}\int_{\hat{K}} L^{3+2s}_{\Psi^{-1}} \frac{|w(\mathbf{x}) - w(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{3+2s}} \left|\det D\Psi^{-1}(\mathbf{x})\right| \left|\det D\Psi^{-1}(\mathbf{y})\right| \mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y},
\end{aligned}$$

and

$$\|w\circ\Psi\|^2_{0,B_1} = \int_{B_1} |w\circ\Psi(\mathbf{x}_B)|^2 \,\mathrm{d}\mathbf{x}_B = \int_{\hat{K}} |w|^2 \left|\det D\Psi^{-1}(\mathbf{x})\right| \mathrm{d}\mathbf{x}.$$

From the expression of the Jacobian $D\Psi^{-1}$ in Cartesian coordinates given in the proof of [15, Lemma 4.2], we compute

$$\left|\det D\Psi^{-1}\right| = \frac{1}{\psi(\theta)^2} \le \frac{1}{\rho^2}.$$

Therefore,

$$\begin{aligned}
\|w\|^2_{0,\partial\hat{K}} &\le C_{B_1} \frac{(1-\rho)^2}{\rho_0\rho^2} \left(\|w\|^2_{0,\hat{K}} + \frac{L^{3+2s}_{\Psi^{-1}}}{\rho^2} |w|^2_{\frac{1}{2}+s,\hat{K}}\right) \\
&\overset{(18),(17)}{\le} C_{B_1} \frac{(1-\rho)^2}{\rho_0\rho^2} \left(\|w\|^2_{0,\hat{K}} + \frac{1}{\rho^2}\left(\frac{3}{\rho_0\rho^2}\right)^{3+2s} |w|^2_{\frac{1}{2}+s,\hat{K}}\right),
\end{aligned}$$

from which we get (15). ∎

4.4 Duality argument

By using a similar argument as in [4,12,14,29], we bound the L^2-norm of any Trefftz function by its DG-norm, with explicit dependence of the bounding constant on the wavenumber. The first part of the proof of the following lemma is identical to that of [14, Lemma 4.4]. We report the whole proof for completeness.

Lemma 4.4. *There exists a constant $C > 0$ depending only on the shape of Ω, ϑ, ρ_0, ρ, s, a, β and δ (in particular independent of $V_p(\mathcal{T}_h)$, $\mathcal{T}_h$ and k) such that, for any $w \in T(\mathcal{T}_h)$,*

$$\|w\|_{0,\Omega} \leq C \frac{(C_{\mathcal{F}} + |\Gamma_R|)d_\Omega^2}{|\Omega|} \left[\frac{1}{kh_{\max}} + d_\Omega k + (d_\Omega k)^3\right] \|w\|_{DG}.$$

Proof. Let ϕ be in $L^2(\Omega)$. Let v be the solution to the (adjoint) problem (2) with $f = \phi$, $g_R = 0$ and "$-$" in the impedance condition on Γ_R. From Proposition 2.1, we know that $v \in H^{\frac{3}{2}+s}(\Omega)$, and that

$$|v|_{1,\Omega} + k\|v\|_{0,\Omega} \leq C d_\Omega \|\phi\|_{0,\Omega}, \qquad |\nabla v|_{\frac{1}{2}+s,\Omega} \leq C\,(1 + d_\Omega k) d_\Omega^{\frac{1}{2}-s} \|\phi\|_{0,\Omega}, \tag{19}$$

with $C > 0$ depending only on s, γ_R and ϑ, but independent of k, ϕ and v. Multiplying by $w \in T(\mathcal{T}_h)$, integrating by parts twice the first equation of (2) element by element (using $\Delta w + k^2 w = 0$ in each $K \in \mathcal{T}_h$), and taking into account that $\nabla v \cdot \mathbf{n} = ik\vartheta v$ on Γ_R and $v = 0$ on Γ_D, we obtain

$$\begin{aligned}
|(w,\phi)_{0,\Omega}| &= \left|\sum_{K\in\mathcal{T}_h} \int_{\partial K} \left(\nabla w \cdot \mathbf{n}\,\overline{v} - w\,\overline{\nabla v \cdot \mathbf{n}}\right) \mathrm{d}S\right| \\
&= \left|\int_{\mathcal{F}_h^I} \left([\![\nabla_h w]\!]_N \overline{v} - [\![w]\!]_N \cdot \overline{\nabla v}\right) \mathrm{d}S + \int_{\Gamma_R} \left(\nabla_h w \cdot \mathbf{n} + ik\vartheta w\right) \overline{v}\,\mathrm{d}S - \int_{\Gamma_D} w\,\overline{\nabla v} \cdot \mathbf{n}\,\mathrm{d}S\right|,
\end{aligned}$$

from which, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
&|(w,\phi)_{0,\Omega}| \\
&\leq \sum_{f\subset\mathcal{F}_h^I} \left(k^{-\frac{1}{2}} \left\|\beta^{\frac{1}{2}}[\![\nabla_h w]\!]_N\right\|_{0,f} k^{\frac{1}{2}} \left\|\beta^{-\frac{1}{2}} v\right\|_{0,f} + k^{\frac{1}{2}} \left\|\alpha^{\frac{1}{2}}[\![w]\!]_N\right\|_{0,f} k^{-\frac{1}{2}} \left\|\alpha^{-\frac{1}{2}}\nabla v \cdot \mathbf{n}\right\|_{0,f}\right) \\
&\quad + \sum_{f\subset\Gamma_R} \left(k^{-\frac{1}{2}} \left\|\delta^{\frac{1}{2}}\vartheta^{-\frac{1}{2}}\nabla w \cdot \mathbf{n}\right\|_{0,f} k^{\frac{1}{2}} \left\|\delta^{-\frac{1}{2}}\vartheta^{\frac{1}{2}} v\right\|_{0,f} + k^{\frac{1}{2}} \left\|\delta^{\frac{1}{2}}\vartheta^{\frac{1}{2}} w\right\|_{0,f} k^{\frac{1}{2}} \left\|\delta^{-\frac{1}{2}}\vartheta^{\frac{1}{2}} v\right\|_{0,f}\right) \\
&\quad + \sum_{f\subset\Gamma_D} k^{\frac{1}{2}} \left\|\alpha^{\frac{1}{2}} w\right\|_{0,f} k^{-\frac{1}{2}} \left\|\alpha^{-\frac{1}{2}}\nabla v \cdot \mathbf{n}\right\|_{0,f} \\
&\leq \|w\|_{DG}\, \mathcal{G}(v)^{\frac{1}{2}},
\end{aligned}$$

where we have set

$$\begin{aligned}
\mathcal{G}(v) := \sum_{f\subset\mathcal{F}_h^I} &\left(k \left\|\beta^{-\frac{1}{2}} v\right\|_{0,f}^2 + k^{-1} \left\|\alpha^{-\frac{1}{2}}\nabla v \cdot \mathbf{n}\right\|_{0,f}^2\right) \\
&+ \sum_{f\subset\Gamma_R} 2k \left\|\delta^{-\frac{1}{2}}\vartheta^{\frac{1}{2}} v\right\|_{0,f}^2 + \sum_{f\subset\Gamma_D} k^{-1} \left\|\alpha^{-\frac{1}{2}}\nabla v \cdot \mathbf{n}\right\|_{0,f}^2.
\end{aligned}$$

We need to bound $\mathcal{G}(v)$ with $\|\phi\|_{0,\Omega}^2$. We exploit the fact that $v \in L^\infty(\Omega)$, together with the Assumption (M3) on the mesh family, in order to bound the terms containing β and δ. Since ∇v does not necessarily belongs to $L^\infty(\Omega)$, we can not use the same argument for the terms containing α. We report, for completeness, the estimate of the terms containing α from [14].

Using the trace inequality (12), and taking into account the local quasi-uniformity assumption (M2), we obtain

$$\sum_{f\subset\mathcal{F}_h^I} k^{-1} \left\|\alpha^{-\frac{1}{2}}\nabla v \cdot \mathbf{n}\right\|_{0,f}^2 + \sum_{f\subset\Gamma_D} k^{-1} \left\|\alpha^{-\frac{1}{2}}\nabla v \cdot \mathbf{n}\right\|_{0,f}^2$$

$$\leq C \sum_{K\in\mathcal{T}_h} \left\|\alpha^{-\frac{1}{2}}\right\|^2_{L^\infty\left(\partial K\cap(\mathcal{F}_h^I\cup\Gamma_D)\right)} \left[\frac{1}{kh_K}\|\nabla v\|^2_{0,K} + \frac{h_K^{2s}}{k}|\nabla v|^2_{\frac{1}{2}+s,K}\right],$$

with $C>0$ depending only on ρ_0, ρ and s. The assumption (9) on α implies

$$\left\|\alpha^{-\frac{1}{2}}\right\|^2_{L^\infty\left(\partial K\cap(\mathcal{F}_h^I\cup\Gamma_D)\right)} \leq \frac{h_K}{\mathsf{a}\,h_{\max}},$$

which leads to the estimate

$$\sum_{f\subset\mathcal{F}_h^I} k^{-1}\left\|\alpha^{-\frac{1}{2}}\nabla v\cdot\mathbf{n}\right\|^2_{0,f} + \sum_{f\subset\Gamma_D} k^{-1}\left\|\alpha^{-\frac{1}{2}}\nabla v\cdot\mathbf{n}\right\|^2_{0,f}$$
$$\leq C\sum_{K\in\mathcal{T}_h}\left[\frac{1}{kh_{\max}}\|\nabla v\|^2_{0,K} + \frac{h_K^{2s+1}}{kh_{\max}}|\nabla v|^2_{\frac{1}{2}+s,K}\right],$$

where, now, C also depends on a. By definition, $h_K\leq h_{\max}$, and therefore (19) gives

$$\sum_{f\subset\mathcal{F}_h^I} k^{-1}\left\|\alpha^{-\frac{1}{2}}\nabla v\cdot\mathbf{n}\right\|^2_{0,f} + \sum_{f\subset\Gamma_D} k^{-1}\left\|\alpha^{-\frac{1}{2}}\nabla v\cdot\mathbf{n}\right\|^2_{0,f}$$
$$\leq C\,d_\Omega^2\left(k^{-1}h_{\max}^{-1} + d_\Omega^{1-2s}kh_{\max}^{2s}\right)\|\phi\|^2_{0,\Omega}\,. \tag{20}$$

We proceed now with the estimate of the terms in $\mathcal{G}(v)$ containing β and δ. Let us start with the term containing β. From the Sobolev embedding $H^{\frac{3}{2}+s}(\Omega)\subset C^0(\Omega)$ (see e.g. [19, Theorem 3.26]), we have $v\in L^\infty(\Omega)$ and

$$\sum_{f\subset\mathcal{F}_h^I} k\left\|\beta^{-\frac{1}{2}}v\right\|^2_{0,f} \leq k\left|\mathcal{F}_h^I\right|\left\|\beta^{-\frac{1}{2}}v\right\|^2_{L^\infty(\mathcal{F}_h^I)} \leq \beta^{-1}k\left|\mathcal{F}_h^I\right|\|v\|^2_{L^\infty(\Omega)}\,,$$

Since $v\in H^{\frac{3}{2}+s}(\Omega)$, there exists $C>0$ depending only on the shape of Ω and s such that

$$\|v\|^2_{L^\infty(\Omega)} \leq C\,|\Omega|^{-1}\left(\|v\|^2_{0,\Omega} + d_\Omega^2\|\nabla v\|^2_{0,\Omega} + d_\Omega^{3+2s}|\nabla v|^2_{\frac{1}{2}+s}\right).$$

By using (19) we obtain

$$\begin{aligned}\|v\|^2_{L^\infty(\Omega)} &\leq C\,|\Omega|^{-1}\left(\frac{d_\Omega^2}{k^2} + d_\Omega^4 + d_\Omega^4(1+d_\Omega^2k^2)\right)\|\phi\|^2_{0,\Omega}\\ &\leq C\,|\Omega|^{-1}d_\Omega^4\left(\frac{1}{d_\Omega^2k^2} + 1 + d_\Omega^2k^2\right)\|\phi\|^2_{0,\Omega}\\ &\leq C\,|\Omega|^{-1}d_\Omega^2\left(\frac{1}{k^2} + d_\Omega^4k^2\right)\|\phi\|^2_{0,\Omega}\,,\end{aligned}$$

and thus

$$\sum_{f\subset\mathcal{F}_h^I} k\left\|\beta^{-\frac{1}{2}}v\right\|^2_{0,f} \leq C\left|\mathcal{F}_h^I\right||\Omega|^{-1}d_\Omega^2\left(k^{-1} + d_\Omega^4k^3\right)\|\phi\|^2_{0,\Omega}\,, \tag{21}$$

with C only depending on the shape of Ω, ϑ, s and β.

We bound the term containing δ similarly:

$$\sum_{f\subset\Gamma_R} 2k\left\|\delta^{-\frac{1}{2}}\vartheta^{\frac{1}{2}}v\right\|^2_{0,f} \leq 2k\,|\Gamma_R|\left\|\delta^{-\frac{1}{2}}\vartheta^{\frac{1}{2}}v\right\|^2_{L^\infty(\Gamma_R)} \leq 2k\,\delta^{-1}\|\vartheta\|_{L^\infty(\Gamma_R)}|\Gamma_R|\,\|v\|^2_{L^\infty(\Omega)}\,,$$
$$\leq C\,|\Gamma_R|\,|\Omega|^{-1}d_\Omega^2\left(k^{-1} + d_\Omega^4k^3\right)\|\phi\|^2_{0,\Omega}\,, \tag{22}$$

with C only depending on the shape of Ω, ϑ, s and δ.

Thus, collecting the bounds (20), (21) and (22) on the terms containing α, β and δ in the definition of $\mathcal{G}(v)$, for all $\phi\in L^2(\Omega)$, we have

$$\mathcal{G}(v) \leq C\,d_\Omega^2\left((k^{-1}h_{\max}^{-1} + d_\Omega k) + \left|\mathcal{F}_h^I\cup\mathcal{F}_h^R\right|\frac{d_\Omega}{|\Omega|}(d_\Omega^{-1}k^{-1} + d_\Omega^3k^3)\right)\|\phi\|^2_{0,\Omega}\,,$$

and thus, due to assumption (M3),

$$\frac{|(w,\phi)_{0,\Omega}|}{\|\phi\|_{0,\Omega}} \le C\,\frac{(C_{\mathcal{F}}+|\Gamma_R|)d_\Omega^2}{|\Omega|}\left[\frac{1}{kh_{\max}}+d_\Omega k+(d_\Omega k)^3\right]\|w\|_{DG}\,,$$

and the result readily follows. □

Since $u-u_{hp}\in T(\mathcal{T}_h)$, from Lemma 4.4 and the quasi-optimality (8), we immediately deduce the following result.

Theorem 4.5. *Assume the mesh properties (M1)–(M3) and that the solution u of (1) belongs to $T(\mathcal{T}_h)$, and let u_{hp} be the solution of (7). Then there exists a constant $C>0$ depending only on the shape of Ω, ϑ, ρ_0, ρ, s, $\mathtt{a}$, β and δ (in particular independent of $V_p(\mathcal{T}_h)$, $\mathcal{T}_h$ and k) such that*

$$\|u-u_{hp}\|_{0,\Omega}\le C\,\frac{(C_{\mathcal{F}}+|\Gamma_R|)d_\Omega^2}{|\Omega|}\left[\frac{1}{kh_{\max}}+d_\Omega k+(d_\Omega k)^3\right]\inf_{v_{hp}\in V_p(\mathcal{T}_h)}\|u-v_{hp}\|_{DG^+}\,.$$

5 Approximation properties of plane wave spaces

In this section we consider a Helmholtz solution u defined in the neighbourhood

$$K_\eta:=\left\{\mathbf{x}\in\mathbb{R}^2,\ \mathrm{dist}(\mathbf{x},K)<\eta h_K\right\},\qquad 0<\eta\le 1/2,$$

of an (open) element K satisfying the star-shapedness assumption (M1); for simplicity we take K to be centred at the origin, i.e. $B_{\rho h_K}\subset K$ and $\mathbf{n}(\mathbf{x})\cdot\mathbf{x}\ge\rho_0 h_K$ a.e. on ∂K. We note that K_η contains $B_{(\rho+\eta)h_K}$ and is star-shaped with respect to $B_{(\rho_0+\eta)h_K}$. Following the theory developed in [15, 25, 26] we prove approximation bounds for finite dimensional spaces made of circular and plane wave functions.

The main ingredients are three: *(i)* the explicit approximation bounds for harmonic functions and harmonic polynomials proved in [15] (improving on [20]) and reported in Proposition 5.1; *(ii)* the Vekua operators, which permit to transfer these approximation properties to Helmholtz solutions and circular waves (see a detailed discussion [27] and the continuity bounds in Lemma 5.2 below); *(iii)* the approximate inversion of the Jacobi–Anger expansion, which allows to prove bounds for plane waves (see (31) below, which was proved in [26, Lemma 4.3]). The interplay of these ingredients is outlined in Figure 1.

We consider only $W^{j,\infty}$-type norms (as opposed to H^j-type) in our bounds; moreover, since u is analytic in K_η, its possible singularities lie at least at distance η from K: these two facts make the proofs easier than those in [26] (even though here we obtain exponential convergence as opposed to algebraic). On the other hand, we want to control the dependence of the constants on the geometry of K, through ρ and ρ_0, thus we need the sharper bounds of [15].

In the following, for any $j\in\mathbb{N}_0$ and for a Lipschitz open set $D\subset\mathbb{R}^2$, we define the Sobolev seminorms $|\phi|_{W^{j,\infty}(D)}:=\sup_{\boldsymbol{\alpha}\in\mathbb{N}_0^2,|\boldsymbol{\alpha}|=j}\|D^{\boldsymbol{\alpha}}\phi\|_{L^\infty(D)}$.

5.1 Exponential approximation by circular waves

Corollary 4.11 of [15], after a simple affine scaling, gives the following harmonic approximation estimates.

Proposition 5.1. *Under the above assumptions on ρ,ρ_0,η,K, for any complex-valued, harmonic function $\phi\in W^{1,\infty}(K_\eta)$, there exists a sequence of harmonic polynomials $\{P_N\}_{N\in\mathbb{N}_0}$ of degree at most N such that*

$$|\phi-P_N|_{W^{j,\infty}(K)}\le C\,h_K^{-j}e^{-bN}\Big(\|\phi\|_{L^\infty(K_\eta)}+h_K\|\nabla\phi\|_{L^\infty(K_\eta)}\Big),\tag{23}$$

for all $j\in\mathbb{N}_0$, where $C>0$ and $b>0$ depend only on ρ,ρ_0,η and j. Moreover, P_N interpolates ϕ in at least $(N+1)$ points on ∂K.

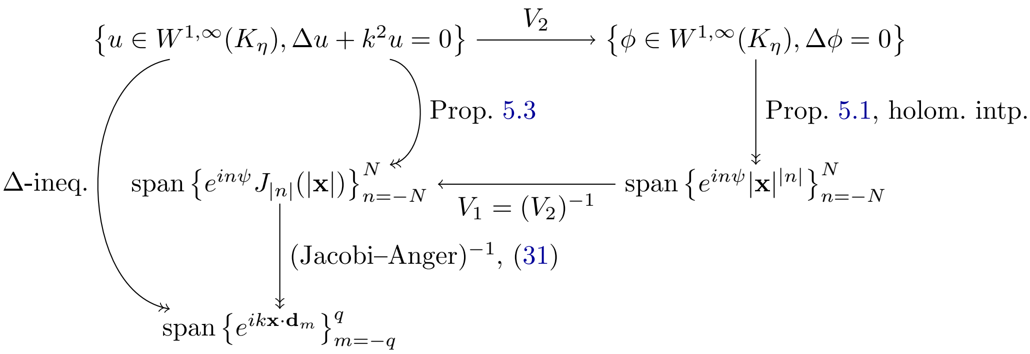

Figure 1: The idea behind the approximation estimates of Section 5: plane waves approximate circular waves (Fourier–Bessel functions), which are Vekua transforms of harmonic polynomials, which approximate harmonic functions, which in turn are inverse Vekua transforms of Helmholtz solutions. The $\rightarrow$ arrow denotes the Vekua operators, which are bijective mappings, and the $\twoheadrightarrow$ arrow can be read as "is approximated by"; the curved arrows are consequences of the straight ones.

The explicit values of the constants C and b can easily be computed following the proofs in [15].

In [27], following [34], the k-dependent Vekua operators $V_1, V_2 : C^0(K) \to C^0(K)$ were introduced. They are inverse of each other, i.e. they satisfy $V_1 = V_2^{-1}$, and are bijective and bicontinuous between the following pairs of spaces (see [27, Theorems 2.5 and 3.1]):

$$\mathcal{H}^j(K) := \{\phi \in H^j(K),\ \Delta\phi = 0\} \quad \underset{V_2}{\overset{V_1}{\rightleftarrows}} \quad \mathcal{H}^j_k(K) := \{u \in H^j(K),\ \Delta u + k^2 u = 0\} \quad \forall j \in \mathbb{N}_0.$$

In [27, Theorem 3.1] the continuity of these operator in $L^\infty(K)$-norm was also stated. Here we generalise this result to higher order $W^{j,\infty}(K)$-norms, maintaining an explicit expression of the continuity constants.

Lemma 5.2. *For any $j \in \mathbb{N}_0$ and $\phi, u \in W^{j,\infty}(K)$ such that $\Delta\phi = \Delta u + k^2 u = 0$ in K, we have the continuity bounds:*

$$\|V_1[\phi]\|_{L^\infty(K)} \le \left(1 + (kh_K)^2\right) \|\phi\|_{L^\infty(K)}\,, \tag{24}$$

$$\|V_2[u]\|_{L^\infty(K)} \le \left(1 + \frac{(kh_K)^2 e^{\frac{1}{2}kh_K}}{4}\right) \|u\|_{L^\infty(K)} \le \left(1 + (1 + kh_K)e^{\frac{3}{4}kh_K}\right) \|u\|_{L^\infty(K)}\,, \tag{25}$$

$$|V_1[\phi]|_{W^{j,\infty}(K)} \le |\phi|_{W^{j,\infty}(K)} + (1+j)\big(j + (kh_K)^2\big)e^j \sum_{\ell=0}^{j} k^{j-\ell}\, |\phi|_{W^{\ell,\infty}(K)}\,, \tag{26}$$

$$|V_2[u]|_{W^{j,\infty}(K)} \le |u|_{W^{j,\infty}(K)} + (1+j)(1+kh_K)e^{j+\frac{3}{4}kh_K} \sum_{\ell=0}^{j} k^{j-\ell}\, |u|_{W^{\ell,\infty}(K)}\,. \tag{27}$$

Proof. The two bounds in $L^\infty(K)$-norms are simpler versions of [27, Equations (18) and (19)]. To prove the remaining ones, we recall that the operators V_ξ, with $\xi = 1, 2$, were defined as $V_\xi[\phi](\mathbf{x}) := \phi(\mathbf{x}) + \int_0^1 M_\xi(\mathbf{x}, t)\phi(t\mathbf{x})\,\mathrm{d}t$ for two suitable kernel functions $M_\xi \in C^\infty(K \times [0,1])$ (see [27, Section 2]). Thus, using the properties of multi-indices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ and the Leibniz rule for multidimensional derivatives $D^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$, we have

$$|V_\xi[\phi]|_{W^{j,\infty}(K)} = \sup_{\boldsymbol{\alpha}\in\mathbb{N}_0^2, |\boldsymbol{\alpha}|=j} \left\| D^{\boldsymbol{\alpha}}\phi + \int_0^1 D^{\boldsymbol{\alpha}}\big(M_\xi(\mathbf{x},t)\phi(t\mathbf{x})\big)\,\mathrm{d}t \right\|_{L^\infty(K)}$$

$$= \sup_{\boldsymbol{\alpha}\in\mathbb{N}_0^2, |\boldsymbol{\alpha}|=j} \left\| D^{\boldsymbol{\alpha}}\phi + \int_0^1 \sum_{\boldsymbol{\beta}\in\mathbb{N}_0^2, \boldsymbol{\beta}\le\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \big(D^{\boldsymbol{\beta}} M_\xi(\mathbf{x},t)\big) t^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|} \big(D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\phi\big)|_{t\mathbf{x}}\,\mathrm{d}t \right\|_{L^\infty(K)}$$

$$\le |\phi|_{W^{j,\infty}(K)} + \sup_{\boldsymbol{\alpha}\in\mathbb{N}_0^2, |\boldsymbol{\alpha}|=j} \sum_{\boldsymbol{\beta}\in\mathbb{N}_0^2, \boldsymbol{\beta}\le\boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \sup_{t\in[0,1]} |M_\xi(\cdot,t)|_{W^{|\boldsymbol{\beta}|,\infty}(K\times[0,1])} |\phi|_{W^{|\boldsymbol{\alpha}-\boldsymbol{\beta}|,\infty}(K)}$$

$$\le |\phi|_{W^{j,\infty}(K)} + \sum_{\ell=0}^{j} (1+\ell) e^j \sup_{t\in[0,1]} |M_\xi(\cdot,t)|_{W^{\ell,\infty}(K\times[0,1])} |\phi|_{W^{j-\ell,\infty}(K)},$$

where in the last step we used $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = \binom{\alpha_1}{\beta_1}\binom{\alpha_2}{\beta_2} \le \frac{\alpha_1^{\beta_1}\alpha_2^{\beta_2}}{\beta_1!\beta_2!} \le e^{|\boldsymbol{\alpha}|}$ and the multi-index count [25, Equation (B.10)]. The final estimates follow from the bounds [25, Equations (2.23) and (2.24)] on the kernels M_ξ. ☐

Some easy improvements of the values of the bounding constants in the assertion of Lemma 5.2 are possible by controlling the various term more carefully, e.g., by using the results of Lemma 2.3.3 in [25] instead of those of Remark 2.3.4 therein.

The results of Lemma 5.2 hold if K is replaced by K_η by substituting h_K with $h_K(1+2\eta)$, since K_η is star-shaped with respect to the origin.

Following [20], we say that $u_N \in C^0(K)$ is a *generalised harmonic polynomial* of degree $N \in \mathbb{N}_0$ if its inverse Vekua transform $V_2[u_N]$ is a harmonic polynomial of degree N. As described in [25, Section 2.4], generalised harmonic polynomials are nothing else than circular waves (often called Fourier–Bessel functions), i.e. smooth solutions of the Helmholtz equation that are separable in polar coordinates: they are linear combinations of

$$\mathbf{x} = (|\mathbf{x}|\cos\psi, |\mathbf{x}|\sin\psi) \quad\longmapsto\quad e^{in\psi} J_{|n|}(k|\mathbf{x}|), \qquad -N \le n \le N,$$

where J_n is a Bessel function of the first kind and order n.

In the next proposition, we exploit the mapping properties of the Vekua operators proved in Lemma 5.2 to transfer the approximation result for harmonic polynomials and harmonic functions of Proposition 5.1 to generalised harmonic polynomials and Helmholtz solutions (compare with [25, Proposition 3.3.3]).

Proposition 5.3. *Under the above assumptions on ρ, ρ_0, η, K, for any $u \in W^{1,\infty}(K_\eta)$ solution of $\Delta u + k^2 u = 0$, there exists a sequence of generalised harmonic polynomials $\{Q_N\}_{N\in\mathbb{N}_0}$ of degree at most N such that*

$$|u - Q_N|_{W^{j,\infty}(K)} \le C\, e^{-bN} h_K^{-j} \big(1 + (kh_K)^{j+4}\big) e^{\frac{3}{2}kh_K} \Big(\|u\|_{L^\infty(K_\eta)} + h_K \|\nabla u\|_{L^\infty(K_\eta)} \Big), \quad (28)$$

for all $j \in \mathbb{N}_0$, where $C > 0$ and $b > 0$ depend only on ρ, ρ_0, η and j.

Proof. For any $N \in \mathbb{N}$, define $Q_N = V_1[P_N]$ where P_N is the harmonic polynomial of degree N associated to $V_2[u]$ by Proposition 5.1. Then, for all $j \ge 0$,

$$\begin{aligned}
|u - Q_N|_{W^{j,\infty}(K)} &\overset{(24),(26)}{\le} (1+j)\big(1+j+(kh_K)^2\big) e^j \sum_{\ell=0}^{j} k^{j-\ell} |V_2[u] - P_N|_{W^{\ell,\infty}(K)} \\
&\overset{(23)}{\le} C\, e^{-bN} k^j \big((kh_K)^2 + (kh_K)^{-j}\big) \Big(\|V_2[u]\|_{L^\infty(K_\eta)} + h_K \|\nabla V_2[u]\|_{L^\infty(K_\eta)} \Big) \\
&\overset{(25),(27),\eta\le 1/2}{\le} C\, e^{-bN} k^j \big((kh_K)^4 + (kh_K)^{-j}\big) e^{\frac{3}{4}kh_K(1+2\eta)} \Big(\|u\|_{L^\infty(K_\eta)} + h_K \|\nabla u\|_{L^\infty(K_\eta)} \Big) \\
&\le C\, e^{-bN} h_K^{-j} \big(1 + (kh_K)^{j+4}\big) e^{\frac{3}{2}kh_K} \Big(\|u\|_{L^\infty(K_\eta)} + h_K \|\nabla u\|_{L^\infty(K_\eta)} \Big).
\end{aligned}$$

☐

5.2 Exponential approximation by plane waves

In Proposition 5.4 we prove approximation bounds for plane wave spaces and Helmholtz solutions. The main result is given by the "inversion" of the Jacobi–Anger expansion obtained in [25, Lemma 3.4.3]; this allows to approximate circular waves with plane waves with more than exponential convergence in the number of plane waves. The final bound is then obtained with a triangular inequality argument, Cauchy's estimates for Helmholtz solutions and Proposition 5.3.

The whole proof is just a modification of those in Sections 3.4.2 and 3.5 of [25] (see in particular Remark 3.5.8 therein). The main differences are: *(i)* here we never use H^j-type Sobolev norms but only $W^{j,\infty}$-type, *(ii)* we aim for exponential convergence and require that the function to be approximated be defined in a neighbourhood of the element, and *(iii)* the bounds coming from [15] allow to reduce the dependence of the bounding constant on the element shape to the parameters ρ and ρ_0 only.

Proposition 5.4. *Fix $q \in \mathbb{N}$ and $p = 2q+1$ different unit vectors (the propagation directions) $\{\mathbf{d}_m = (\cos\theta_m, \sin\theta_m)\}_{m=-q}^{q}$. Assume there exists $0 < \zeta \le 1$ such that*

$$\min_{\substack{m,m'=-q,\dots,q\\ m\neq m'}} \left|\theta_m - \theta_{m'}\right| \ge \frac{2\pi}{p}\,\zeta. \tag{29}$$

Fix $u \in W^{1,\infty}(K_\eta)$ solution of $\Delta u + k^2 u = 0$. Then, under the above assumptions on ρ, ρ_0, η, K, there exists a linear combination of plane waves with propagation directions $\{\mathbf{d}_m\}_{m=-q}^{q}$ which approximates u with the following error bound:

$$\begin{aligned}
&\left|u - \sum_{m=-q}^{q} \alpha_m e^{ik\mathbf{x}\cdot\mathbf{d}_m}\right|_{W^{j,\infty}(K)}\\
&\le C\big(1+(kh_K)^{j+6}\big)e^{3kh_K} h_K^{-j}\left[e^{-bq} + \frac{1+(kh_K)^{q+1}}{\big(c_0\zeta^4(q+1)\big)^{\frac{q}{2}}}\right]\Big(\|u\|_{L^\infty(K_\eta)} + h_K\|\nabla u\|_{L^\infty(K_\eta)}\Big)
\end{aligned}$$

for all $j \in \mathbb{N}_0$, where $C > 0$ and $b > 0$ depend only on ρ, ρ_0, η and j, while $c_0 > 0$ is independent of all the other parameters.

Proof. We consider $N \in \mathbb{N}$ such that $N \le \lfloor (q-1)/2 \rfloor$ and, using plane waves, we approximate the circular wave Q_N given by Proposition 5.3.

First we note that Vekua's theory allows to extend Cauchy's estimates for harmonic functions to Helmholtz solutions. In particular, we can control the $W^{j,\infty}(K)$-norm at the left-hand side in the assertion's bound with the $L^\infty(K_\eta)$-norm of the same function: for any $w \in L^\infty(K_\eta)$, $\Delta w + k^2 w = 0$,

$$\begin{aligned}
|w|_{W^{j,\infty}(K)} &\overset{(26)}{\le} (1+j)\big(1+j+(kh_K)^2\big)e^j \sum_{\ell=0}^{j} k^{j-\ell}\,|V_2[w]|_{W^{\ell,\infty}(K)}\\
&\overset{\substack{\text{Cauchy est.}\\ [25,\ (2.29)]}}{\le} (1+j)\big(1+j+(kh_K)^2\big)e^j \sum_{\ell=0}^{j} k^{j-\ell}\left(\frac{2\ell}{\eta h_K}\right)^{\ell} \|V_2[w]\|_{L^\infty(K_\eta)}\\
&\overset{(25),\eta\le 1/2}{\le} (1+j)\big(1+j+(kh_K)^2\big)\big(1+(kh_K)^2 e^{\frac12 kh_K}\big)e^j\,\|w\|_{L^\infty(K_\eta)} \sum_{\ell=0}^{j} k^{j-\ell}\left(\frac{2\ell}{\eta h_K}\right)^{\ell}\\
&\le C\big(1+(kh_K)^{j+4}e^{\frac12 kh_K}\big)\eta^{-j}h_K^{-j}\,\|w\|_{L^\infty(K_\eta)}
\end{aligned} \tag{30}$$

where the constant C depends only on j.

We obtain the order of convergence of the plane wave approximation of Q_N from Lemma 3.4.3 of [25] (together with $K_\eta \subset B_{(1-\rho+\eta)h_K}$, $\|\cdot\|_{L^2(K)} \le h_K\|\cdot\|_{L^\infty(K)}$, and setting $K=0$ in the notation of [25, Lemma 3.4.3]): there exists $\vec{\alpha} \in \mathbb{C}^p$ such that

$$\begin{aligned}
&\left\|Q_N - \sum_{m=-q}^{q}\alpha_m e^{ik\mathbf{x}\cdot\mathbf{d}_m}\right\|_{L^\infty(K_\eta)} \le \left\|Q_N - \sum_{m=-q}^{q}\alpha_m e^{ik\mathbf{x}\cdot\mathbf{d}_m}\right\|_{L^\infty(B_{(1-\rho+\eta)h_K})}\\
&\le \frac{e^3}{\pi^{\frac32}\rho^{N+1}}\left(\frac{e^{\frac52}}{2\sqrt{2}\,\zeta^2}\right)^{q}\left(2^N\sqrt{N+1}\right)\left(1+(kh_K)^{-N}\right)e^{\frac{(1-\rho+\eta)kh_K}{2}}\,\frac{\big((1-\rho+\eta)kh_K\big)^{q+1}}{(q+1)^{\frac{q+1}{2}}}\\
&\quad\cdot\|V_2[Q_N]\|_{L^\infty(K)}\,.
\end{aligned} \tag{31}$$

The norm of the harmonic polynomial $V_2[Q_N]$ is immediately controlled by that of u using the triangle inequality and recalling the definition of Q_N:

$$\begin{aligned} \|V_2[Q_N]\|_{L^\infty(K)} &\leq \|V_2[u]\|_{L^\infty(K)} + \|V_2[u] - V_2[Q_n]\|_{L^\infty(K)} \\ &\overset{(23)}{\leq} C\Big(\|V_2[u]\|_{L^\infty(K_\eta)} + h_K \|\nabla V_2[u]\|_{L^\infty(K_\eta)} \Big) \\ &\overset{(25),(27)}{\leq} C\big(1+(kh_K)^2\big)e^{\frac{3}{4}(1+2\eta)kh_K}\Big(\|u\|_{L^\infty(K_\eta)} + h_K \|\nabla u\|_{L^\infty(K_\eta)} \Big) \end{aligned} \tag{32}$$

where $C > 0$ only depends on ρ, ρ_0 and η.

We now put together the various bounds: the plane wave approximation error is split using the triangle inequality in a Fourier–Bessel approximation error (controlled in Proposition 5.3) and in a remainder term controlled by (31) (using (30) to reduce the order of the norm):

$$\begin{aligned} &\left| u - \sum_{m=-q}^{q} \alpha_m e^{ik\mathbf{x}\cdot\mathbf{d}_m} \right|_{W^{j,\infty}(K)} \leq |u - Q_N|_{W^{j,\infty}(K)} + \left| Q_N - \sum_{m=-q}^{q} \alpha_m e^{ik\mathbf{x}\cdot\mathbf{d}_m} \right|_{W^{j,\infty}(K)} \\ &\overset{(30)}{\leq} |u - Q_N|_{W^{j,\infty}(K)} + C\big(1+(kh_K)^{j+4}e^{\frac{1}{2}kh_K}\big)h_K^{-j} \left\| Q_N - \sum_{m=-q}^{q} \alpha_m e^{ik\mathbf{x}\cdot\mathbf{d}_m} \right\|_{L^\infty(K_\eta)} \\ &\overset{(28),(31)}{\leq} Ce^{-bN}h_K^{-j}\big(1+(kh_K)^{j+4}\big)e^{\frac{3}{2}kh_K}\Big(\|u\|_{L^\infty(K_\eta)} + h_K \|\nabla u\|_{L^\infty(K_\eta)} \Big) \\ &+ C\big((kh_K)^{-N} + (kh_K)^{j+4}\big)h_K^{-j} \left(\frac{3e^{\frac{5}{2}}}{4\sqrt{2}\,\zeta^2} \right)^q \Big(2^N\sqrt{N+1}\Big) e^{\frac{5}{4}kh_K} \frac{(kh_K)^{q+1}}{(q+1)^{\frac{q+1}{2}}} \|V_2[Q_N]\|_{L^\infty(K)} \\ &\overset{(32)}{\leq} C\big(1+(kh_K)^{j+6}\big)e^{3kh_K}h_K^{-j} \\ &\quad \cdot \left[e^{-bN} + \big(1+(kh_K)^{-N}\big) \left(\frac{3e^{\frac{5}{2}}}{4\sqrt{2}\,\zeta^2} \right)^q \Big(2^N\sqrt{N+1}\Big) \frac{(kh_K)^{q+1}}{(q+1)^{\frac{q+1}{2}}} \right] \\ &\quad \cdot \Big(\|u\|_{L^\infty(K_\eta)} + h_K \|\nabla u\|_{L^\infty(K_\eta)} \Big), \end{aligned}$$

where C and b depend on j, ρ, ρ_0, η only. We now fix $N := \lfloor \frac{q-1}{2} \rfloor$ and obtain the assertion (with $c_0 > 0.0119$)

$$\begin{aligned} &\left| u - \sum_{m=-q}^{q} \alpha_m e^{ik\mathbf{x}\cdot\mathbf{d}_m} \right|_{W^{j,\infty}(K)} \\ &\leq C\big(1+(kh_K)^{j+6}\big)e^{3kh_K}h_K^{-j} \left[e^{-\frac{b}{2}q} + \left(\frac{3e^{\frac{5}{2}}}{4\zeta^2} \right)^q \frac{1+(kh_K)^{q+1}}{(q+1)^{\frac{q}{2}}} \right] \\ &\qquad \cdot \Big(\|u\|_{L^\infty(K_\eta)} + h_K \|\nabla u\|_{L^\infty(K_\eta)} \Big). \end{aligned}$$

□

6 Exponential convergence

As in the case of standard polynomial finite elements, we establish exponential convergence of $\|u - u_{hp}\|_{0,\Omega}$ in terms of the number of degrees of freedom for particular families of meshes.

6.1 Geometric meshes

We restrict ourselves to special instances of families of meshes given by sequences $\{\mathcal{T}_L\}_{L\in\mathbb{N}}$ of so-called *geometrically graded meshes* indexed by a refinement level L denoting the number of *element layers* in the mesh, see Assumption 6.1 below. Meshes of this type with simple polygonal or polyhedral elements have universally been used for conventional *hp*-finite element methods [32]. Conversely, we demand only compliance of $\{\mathcal{T}_L\}_{L\in\mathbb{N}}$ with Assumptions (M1) and (M2) from Section 4.1, and, thus, rather general shapes of the elements are admitted. In addition, closely following [15, Section 5.2.1], we impose the following properties on admissible geometrically graded meshes.

Assumption 6.1. Let $0 < \sigma < 1$ be a fixed *grading parameter*. The elements of every mesh $\mathcal{T}_L$ can be grouped into layers $\mathcal{L}_\ell^L$, $1 \le \ell \le L$, that is,

$$\mathcal{T}_L = \bigcup_{\ell=1}^{L} \mathcal{L}_\ell^L, \qquad \mathcal{L}_\ell^L \cap \mathcal{L}_{\ell'}^L = \emptyset \text{ if } \ell \neq \ell',$$

such that:

(GM1) the Lth layer $\mathcal{L}_L^L$ contains the set of elements abutting a corner;

(GM2) the distance of an element from the nearest corner point depends geometrically on its layer index (recalling that $\mathcal{C} = \{\mathbf{c}_\nu\}_{\nu=1}^{n_c}$ is the set of corner points):

$$\exists C > 0: \qquad C^{-1}\sigma^\ell \le \operatorname{dist}(K, \mathcal{C}) \le C\sigma^\ell \qquad \forall K \in \mathcal{L}_\ell^L, \quad 1 \le \ell < L, \quad L \in \mathbb{N}; \quad (33)$$

(GM3) the size of an element depends geometrically on its layer index:

$$\exists C > 0: \qquad C^{-1}\sigma^\ell \le h_K \le C\sigma^\ell \qquad \forall K \in \mathcal{L}_\ell^L, \quad 1 \le \ell \le L, \quad L \in \mathbb{N}; \quad (34)$$

(GM4) for $\ell \ge 2$, $\mathcal{T}_L$ is obtained from $\mathcal{T}_{L-1}$ by refining only elements of $\mathcal{L}_{L-1}^{L-1}$ (i.e., $\mathcal{L}_\ell^L = \mathcal{L}_\ell^{L'}$ for all $\ell < \min\{L, L'\}$).

Here and in the sequel, we adhere to the convention that a "generic constant" $C > 0$ must depend neither on refinement levels ℓ and L, nor on the grading parameter σ, nor on the solution u.

We remind that (GM2) and (GM3) imply that the diameter of an element in the ℓth layer is proportional to its distance from the nearest corner:

$$\exists C > 0: \quad C^{-1} \operatorname{dist}(K, \mathcal{C}) \le h_K \le C \operatorname{dist}(K, \mathcal{C}) \qquad \forall K \in \mathcal{L}_\ell^L, \quad 1 \le \ell < L, \quad L \in \mathbb{N}. \quad (35)$$

Appealing to (M1) and (GM3), we can control the area of the elements in a particular layer:

$$\exists C > 0: \qquad C^{-1}\sigma^{2\ell} \le |K| \le h_K^2 \le C\sigma^{2\ell} \qquad \forall K \in \mathcal{L}_\ell^L, \quad 1 \le \ell \le L, \quad L \in \mathbb{N}.$$

As a consequence of (GM2) and (GM3), we mention that the ℓth layer is contained in a union of balls with radius $\approx \sigma^\ell$ around the corners

$$\exists C > 0: \quad \bigcup_{K \in \mathcal{L}_\ell^L} K \subset \bigcup_{\nu=1}^{n_c} B_{C\sigma^\ell}(\mathbf{c}_\nu), \qquad 1 \le \ell \le L, \quad L \in \mathbb{N},$$

which permits us to bound the area occupied by the ℓth layer by $C\sigma^{2\ell}$. With (GM3), we thus conclude that the number of elements per layer is uniformly bounded

$$\exists C > 0: \qquad \sharp\mathcal{L}_\ell^L \le C, \qquad 1 \le \ell \le L, \quad L \in \mathbb{N}. \quad (36)$$

Immediate from (36) is the fact that geometrically graded meshes satisfy (M3) because, retaining the notation $\mathcal{F}_h^I$ for the set of interior edges of some $\mathcal{T}_L$,

$$\left|\mathcal{F}_h^I\right| \overset{(13)}{\le} C \sum_{K \in \mathcal{T}_L} h_K \le C \sum_{\ell=1}^{L} \sum_{K \in \mathcal{L}_\ell^L} h_K \overset{\text{(GM3)}}{\le} C \sum_{\ell=1}^{L} \sum_{K \in \mathcal{L}_\ell^L} \sigma^\ell \overset{(36)}{\le} C \frac{1}{1-\sigma} =: C_{\mathcal{F}}, \quad (37)$$

with all constants independent of L.

6.2 Plane wave hp-spaces

The gist of hp-approximation is to raise the number of plane waves used on each element along with refining the mesh. This is reflected in the construction of the plane wave hp-spaces based on a sequence of geometrically graded meshes $\{\mathcal{T}_L\}_{L \in \mathbb{N}}$ as introduced in Section 6.1. To begin with, we fix $\epsilon \ge 0$ and set the dimension of the local plane wave spaces to

$$p(L) := 2\lceil L^{1+\epsilon} \rceil + 1, \quad L \in \mathbb{N}, \quad (38)$$

with $\lceil\cdot\rceil$ selecting the smallest integer greater than or equal to $L^{1+\epsilon}$. The role of ϵ is explained in Section 6.5. For the sake of simplicity, we opt for equi-spaced plane wave directions (i.e., $\zeta = 1$ in Proposition 5.4)

$$\mathbf{d}_m^p = \begin{pmatrix}\cos(\frac{2\pi}{p}m)\\ \sin(\frac{2\pi}{p}m)\end{pmatrix}, \quad 0 \le m < p, \quad p \in \mathbb{N},$$

which give rise to the local plane wave spaces

$$PW_{p,k}(K) := \left\{v \in C^\infty(\mathbb{R}^2):\ v(\mathbf{x}) = \sum_{m=0}^{p-1} \alpha_m \exp\left(ik\mathbf{d}_m^p \cdot (\mathbf{x} - \mathbf{x}_K)\right), \alpha_m \in \mathbb{C}\right\}, \quad p \in \mathbb{N}.$$

where $\mathbf{x}_K$ was defined in Assumption (M1), Section 4.1. Then, the trial and test spaces for the hp-version of the Trefftz-DG method of Section 3 are defined as

$$V_L := \left\{v \in L^2(\Omega):\ v_{|K} \in PW_{p(L),k}(K)\ \forall K \in \mathcal{T}_L\right\},$$

and the corresponding solution will be denoted by $u_L \in V_L$. Obviously, thanks to the bound on the number of elements per layer (36), the total number of degrees of freedom, which is $\dim V_L$, is bounded by

$$\dim V_L \le C\, L\, p(L) \quad \forall L \in \mathbb{N}. \tag{39}$$

According to Theorem 4.5 and the bound on $|\mathcal{F}_h^I|$ (37), an L-uniform bound of the discretisation error $\|u - u_L\|_{0,\Omega}$ is provided by $\|u - v_L\|_{DG^+}$ for a conveniently chosen $v_L \in V_L$. A concrete choice of v_L will rely on particular local approximations of u chosen differently for elements away from corners, see Section 6.3, and elements at corners, see Section 6.4.

Before we give details, we elaborate a simpler bound for $\|u - v_L\|_{DG^+}$. Immediate from the definition of $\|\cdot\|_{DG^+}$ is

$$\begin{aligned}\|u - v_L\|_{DG^+}^2 \le C \sum_{K\in\mathcal{T}_L} \Big(& k^{-1}\left\|\beta^{1/2}\nabla(u - v_L)\cdot\mathbf{n}\right\|_{0,\partial K\setminus\partial\Omega}^2 + k\left\|\alpha^{1/2}(u - v_L)\right\|_{0,\partial K\setminus\Gamma_R}^2 \\ & + k\left\|\beta^{-1/2}(u - v_L)\right\|_{0,\partial K\setminus\partial\Omega}^2 + k^{-1}\left\|\alpha^{-1/2}\nabla(u - v_L)\cdot\mathbf{n}\right\|_{0,\partial K\setminus\Gamma_R}^2 \\ & + k^{-1}\left\|\delta^{1/2}\vartheta^{-1/2}\nabla(u - v_L)\cdot\mathbf{n}\right\|_{0,\partial K\cap\Gamma_R}^2 \\ & + k\left\|(1-\delta)^{1/2}\vartheta^{1/2}(u - v_L)\right\|_{0,\partial K\cap\Gamma_R}^2 + k\left\|\delta^{-1/2}\vartheta^{1/2}(u - v_L)\right\|_{0,\partial K\cap\Gamma_R}^2 \Big).\end{aligned}$$

Thanks to the particular choice of the parameters α, β and δ made in (9) and (10), we thus arrive at the bound

$$\|u - v_L\|_{DG^+}^2 \le C \sum_{K\in\mathcal{T}_L} \left(k^{-1}\|\nabla(u - v_L)\cdot\mathbf{n}\|_{0,\partial K}^2 + \frac{kh_{\max}}{h_K}\|u - v_L\|_{0,\partial K}^2 \right), \tag{40}$$

where we have used the fact that the local quasi-uniformity assumption (M2) implies $h_f \le \tau h_K$ for any face f of the element K; thus, in the estimate (40), C depends on the local quasi-uniformity of the mesh.

6.3 Estimates away from corners

A simple consequence of Theorem 2.3 is the possibility to extend u analytically beyond ∂K, provided that K does not abut a corner. The solution can be extended to a distance from K proportional to the distance from the closest domain corner, thus proportional to the diameter of K itself, thanks to relation (35). The proof is the same as that of [15, Lemma 5.4].

Lemma 6.2. *There exists $\eta_* > 0$ independent of u and $L \in \mathbb{N}$ (but not of k) such that the solution u of (2) is analytic in*

$$K_{\eta_*} := \left\{\mathbf{x} \in \mathbb{R}^2:\ \operatorname{dist}(\mathbf{x}, K) < \eta_* h_K\right\},$$

for all $K \in \mathcal{T}_L \setminus \mathcal{L}_L^L$, that is, for all elements not adjacent to a corner.

From this lemma, it is immediate that $u \in L^\infty(K)$ and $\nabla u \in L^\infty(K)^2$ for every element $K \in \mathcal{T}_L \setminus \mathcal{L}_L^L$. Now we fix such an element K. If $w \in L^\infty(K)$, the consequence (13) of the star-shapedness of K gives

$$\|w\|_{0,\partial K}^2 \leq |\partial K| \, \|w\|_{L^\infty(K)}^2 \leq C \, h_K \, \|w\|_{L^\infty(K)}^2 \, .$$

Hence, the contribution of the elements $K \in \mathcal{T}_L \setminus \mathcal{L}_L^L$ to the right hand side of estimate (40) can be bounded by

$$\sum_{K \in \mathcal{T}_L \setminus \mathcal{L}_L^L} \left(k^{-1} \, \|\nabla(u - v_L) \cdot \mathbf{n}\|_{0,\partial K}^2 + \frac{k h_{\max}}{h_K} \, \|u - v_L\|_{0,\partial K}^2 \right) \\ \leq C \left(\frac{h_K}{k} \, \|\nabla(u - v_L)\|_{L^\infty(K)}^2 + k h_{\max} \, \|u - v_L\|_{L^\infty(K)}^2 \right).$$

Along with Lemma 6.2, this paves the way for using the approximation result of Proposition 5.4 for $\zeta = 1$ (defined in (29)) locally on each element $K \in \mathcal{L}_\ell^L$, $1 \leq \ell < L$: picking $v_L \in PW_{p(L),k}(K)$ as a suitable linear combination of plane waves according to Proposition 5.4, we find

$$\frac{h_K}{k} \, \|\nabla(u - v_L)\|_{L^\infty(K)}^2 + k h_{\max} \, \|u - v_L\|_{L^\infty(K)}^2 \\ \leq C\big(1 + (k h_K)^{-1}\big) \left[e^{-bq} + \frac{1 + (k h_K)^{q+1}}{\big(c_0(q+1)\big)^{\frac{q}{2}}} \right]^2 \Big(\|u\|_{L^\infty(K_{\eta_*})} + h_K \, \|\nabla u\|_{L^\infty(K_{\eta_*})} \Big)^2 . \quad (41)$$

Here, we abbreviated $q := \lceil L^{1+\epsilon} \rceil$. Moreover, an inspection of the bound from Proposition 5.4 reveals that the constant C depends on the product $k h_K$ as an increasing function, and, of course, it also depends on ρ, ρ_0, η_*.

6.4 Estimates at corners

On $K \in \mathcal{L}_L^L$, we can neither take for granted $\nabla u \in L^\infty(K)$, nor analyticity of u beyond ∂K. Fortunately, since the combined area of these elements is very small for large L, simple local estimates suffice. Our aim is to control the terms relative to K in (40) with some bounded function of u, independent of K, multiplied with any positive powers of h_K; then the geometric scaling (34) for $\ell = L$ provides exponential convergence in L.

The first tool we need are the polynomial quasi-interpolation operators Q^m, $m = 1, 2$, introduced in [3, Chapter 4], which project onto the spaces $\mathbb{P}_{m-1}$ of 2-variate polynomials of degree at most $m - 1$. In particular, we make use of $Q^1_{\hat{K}}$ and $Q^2_{\hat{K}}$ for each $\hat{K}$, where $\hat{K}$ is the scaling of the element $K \in \mathcal{T}_L^L$ as introduced in Section 4.1. We remind that the projectors Q^m rely on Taylor expansions averaged over B_{ρ_0}. Then [3, Corollary (4.1.15)] gives us

$$\left| Q^m_{\hat{K}} \hat{w} \right|_{j,B_1} \leq C_{m,j} \, \|\hat{w}\|_{0,B_{\rho_0}} \quad \forall \hat{w} \in H^{m-1}(\hat{K}), \quad j = 0, 1, \quad m = 1, 2, \quad (42)$$

with constants $C_{m,j}$ depending only on ρ_0. Moreover, by the Bramble–Hilbert Lemma from [3, Lemma (4.3.8)] we know

$$\left\| \hat{w} - Q^m_{\hat{K}} \hat{w} \right\|_{0,\hat{K}} \leq C_m \, |\hat{w}|_{m,\hat{K}} \quad \forall \hat{w} \in H^m(\hat{K}), \quad m = 1, 2, \quad (43)$$

where C_m depends on ρ_0 only. By interpolation between $H^2(\hat{K})$ and $L^2(\hat{K})$ of the operator $(\mathrm{Id} - Q^m_{\hat{K}})$ taking values in $L^2(\hat{K})$, we conclude from (42) and (43) for $m = 2$ and $j = 0$

$$\left\| \hat{w} - Q^2_{\hat{K}} \hat{w} \right\|_{0,\hat{K}} \leq C \, |\hat{w}|_{\frac{3}{2}+s,\hat{K}} \quad \forall \hat{w} \in H^{\frac{3}{2}+s}(\hat{K}), \quad (44)$$

with, as before, C depending on ρ_0 only. Next, [3, Lemma (4.1.17)] asserts that $\nabla \circ Q^2_{\hat{K}} = Q^1_{\hat{K}} \circ \nabla$, which yields, by interpolation between $H^1(\hat{K})$ and $L^2(\hat{K})$, applying (42) and (43) to $\nabla \hat{w}$ with $m = 1$ and $j = 0$,

$$\left\| \nabla(\hat{w} - Q^2_{\hat{K}} \hat{w}) \right\|_{0,\hat{K}} \leq C \, |\nabla \hat{w}|_{\frac{1}{2}+s,\hat{K}} \quad \forall \hat{w} \in H^{\frac{3}{2}+s}(\hat{K}). \quad (45)$$

The second tool is a set of special results about the approximation of polynomials by plane waves which can be derived combining Lemma 3.10 and Proposition 3.9 in [9]. In that article, the estimates target a family of triangles and the unit square, here we need the estimates on the unit disk only.

Lemma 6.3. *For odd $p \geq 5$, $\hat{k} > 0$, and any $\hat{p}_1 \in \mathbb{P}_1(B_1)$, we can find $\hat{v}_p \in PW_{p,\hat{k}}(B_1)$ such that*

$$\|\hat{p}_1 - \hat{v}_p\|_{0,B_1} \leq C\hat{k}^2 \|\hat{p}_1\|_{0,B_1}, \tag{46}$$

$$|\hat{p}_1 - \hat{v}_p|_{1,B_1} \leq C(\hat{k}+1)\hat{k}^2 \|\hat{p}_1\|_{0,B_1}, \tag{47}$$

$$|\hat{v}_p|_{2,B_1} \leq C(\hat{k}+1)^2\hat{k}^2 \|\hat{p}_1\|_{0,B_1}. \tag{48}$$

Based on this lemma, we prove other auxiliary estimates.

Lemma 6.4. *Fix odd $p \geq 5$. For every $K \in \mathcal{T}_L$ and $u \in H^{\frac{3}{2}+s}(K)$, we can find $v_p \in PW_{p,k}(K)$ such that*

$$\|u - v_p\|_{0,K}^2 \leq C\left(h_K^{3+2s}|u|_{\frac{3}{2}+s,K}^2 + h_K^4 k^4 \|u\|_{0,K}^2\right), \tag{49}$$

$$|u - v_p|_{1,K}^2 \leq C\left(h_K^{1+2s}|u|_{\frac{3}{2}+s,K}^2 + (kh_K+1)^2 k^4 h_K^2 \|u\|_{0,K}^2\right), \tag{50}$$

$$|\nabla(u - v_p)|_{\frac{1}{2}+s,K}^2 \leq C\left(|u|_{\frac{3}{2}+s,K}^2 + (1+h_K k)^4 h_K^{1-2s} k^4 \|u\|_{0,K}^2\right), \tag{51}$$

with constants $C > 0$ independent of u, K, and L (depending only on ρ_0 and ρ from Assumption (M1)).

Proof. Set $\hat{p} := Q_{\hat{K}}^2 \hat{u}$ and write $\hat{v}_p \in PW_{p,\hat{k}}(\hat{K})$, with $\hat{k} := h_K k$, for the plane wave approximation of $\hat{p}$ according to Lemma 6.3. Its transformation back to K provides $v_p \in PW_{p,k}(K)$. Simple transformations of norms yield

$$\begin{aligned}
\|u - v_p\|_{0,K} &= h_K \|\hat{u} - \hat{v}_p\|_{0,\hat{K}} \leq h_K\left(\|\hat{u} - \hat{p}\|_{0,\hat{K}} + \|\hat{p} - \hat{v}_p\|_{0,\hat{K}}\right) \\
&\overset{(44)}{\leq} Ch_K\left(C|\hat{u}|_{\frac{3}{2}+s,\hat{K}} + \|\hat{p} - \hat{v}_p\|_{0,B_1}\right) \\
&\overset{(46),\,(42)}{\leq} Ch_K\left(|\hat{u}|_{\frac{3}{2}+s,\hat{K}} + h_K^2 k^2 \|\hat{u}\|_{0,B_{\rho_0}}\right) \\
&\leq C\left(h_K^{\frac{3}{2}+s}|u|_{\frac{3}{2}+s,K} + h_K^2 k^2 \|u\|_{0,K}\right).
\end{aligned}$$

Rather similar arguments establish the second assertion of the lemma for the *same* v_p:

$$\begin{aligned}
|u - v_p|_{1,K} &= |\hat{u} - \hat{v}_p|_{1,\hat{K}} \leq |\hat{u} - \hat{p}|_{1,\hat{K}} + |\hat{p} - \hat{v}_p|_{1,\hat{K}} \\
&\overset{(45)}{\leq} C\left(|\nabla\hat{u}|_{\frac{1}{2}+s,\hat{K}} + |\hat{p} - \hat{v}_p|_{1,B_1}\right) \\
&\overset{(47)}{\leq} C\left(|\nabla\hat{u}|_{\frac{1}{2}+s,\hat{K}} + (h_K k+1)h_K^2 k^2 \|\hat{p}\|_{0,B_1}\right) \\
&\overset{(42)}{\leq} C\left(|\hat{u}|_{\frac{3}{2}+s,\hat{K}} + (h_K k+1)h_K^2 k^2 \|\hat{u}\|_{0,B_{\rho_0}}\right) \\
&\leq C\left(h_K^{\frac{1}{2}+s}|u|_{\frac{3}{2}+s,K} + (h_K k+1)h_K k^2 \|u\|_{0,K}\right).
\end{aligned}$$

The third estimate follows along the same lines, using $|\nabla\hat{p}|_{\frac{1}{2}+s,\hat{K}} = |\hat{p}|_{2,\hat{K}} = 0$:

$$\begin{aligned}
|\nabla(u - v_p)|_{\frac{1}{2}+s,K} &= h_K^{-\frac{1}{2}-s}|\nabla(\hat{u} - \hat{v}_p)|_{\frac{1}{2}+s,\hat{K}} \\
&\leq h_K^{-\frac{1}{2}-s}\left(|\nabla(\hat{u} - \hat{p})|_{\frac{1}{2}+s,\hat{K}} + |\nabla(\hat{p} - \hat{v}_p)|_{\frac{1}{2}+s,\hat{K}}\right) \\
&\overset{(45)}{\leq} Ch_K^{-\frac{1}{2}-s}\left(|\nabla\hat{u}|_{\frac{1}{2}+s,\hat{K}} + \|\hat{p} - \hat{v}_p\|_{2,\hat{K}}\right) \\
&\overset{(48)}{\leq} Ch_K^{-\frac{1}{2}-s}\left(|\nabla\hat{u}|_{\frac{1}{2}+s,\hat{K}} + (h_K k+1)^2 h_K^2 k^2 \|\hat{p}\|_{0,B_1}\right)
\end{aligned}$$

$$\stackrel{(42)}{\leq} Ch_K^{-\frac{1}{2}-s}\left(|\hat{u}|_{\frac{3}{2}+s,\hat{K}} + (h_K k+1)^2 h_K^2 k^2 \|\hat{u}\|_{0,B_{\rho_0}}\right)$$
$$\leq C\left(|u|_{\frac{3}{2}+s,K} + (h_K k+1)^2 h_K^{\frac{1}{2}-s} k^2 \|u\|_{0,K}\right).$$

□

The natural candidate for a local plane wave approximating u on $K \in \mathcal{L}_L^L$ is $v_{L\,|K} := v_p$ with v_p supplied by the previous lemma. Then we can tackle the terms on the right-hand side of (40) invoking Lemma 6.4 and the trace inequalities (11) and (12), respectively:

$$\begin{aligned}
\frac{kh_{\max}}{h_K}\|u-v_p\|^2_{0,\partial K} &\stackrel{(11)}{\leq} C\frac{kh_{\max}}{h_K}\left(\frac{1}{h_K}\|u-v_p\|^2_{0,K} + h_K|u-v_p|^2_{1,K}\right)\\
&\leq Ckh_{\max}\left(\frac{1}{h_K^2}\|u-v_p\|^2_{0,K} + |u-v_p|^2_{1,K}\right)\\
&\stackrel{(49),\,(50)}{\leq} Ckh_{\max}\left(h_K^{1+2s}|u|^2_{\frac{3}{2}+s,K} + (k^2h_K^2+1)k^4h_K^2\|u\|^2_{0,K}\right),\\
\frac{1}{k}\|\nabla(u-v_p)\|^2_{0,\partial K} &\leq \frac{C}{k}\left(h_K^{-1}\|\nabla(u-v_p)\|^2_{0,K} + h_K^{2s}|\nabla(u-v_p)|^2_{\frac{1}{2}+s,K}\right)\\
&\stackrel{(50),\,(51)}{\leq} \frac{C}{k}\left(h_K^{2s}|u|^2_{\frac{3}{2}+s,K} + (1+h_Kk)^4h_Kk^4\|u\|^2_{0,K}\right).
\end{aligned}$$

Therefore, taking into account the geometric scaling of the elements (GM3), the contribution of K to the right hand side of (40) can be bounded as

$$\begin{aligned}
&\frac{1}{k}\|\nabla(u-v_p)\|^2_{0,\partial K} + \frac{kh_{\max}}{h_K}\|u-v_p\|^2_{0,\partial K}\\
&\qquad \leq C\sigma^{2sL}\left(k^{-1}(1+k^2h_{\max}^2)|u|^2_{\frac{3}{2}+s,K} + h_{\max}^{1-2s}k^3(1+k^4h_{\max}^4)\|u\|^2_{0,K}\right).
\end{aligned} \tag{52}$$

6.5 Main a priori error bound

Now we combine the estimates obtained in Sections 6.3 and 6.4 into a final best approximation estimate for u in V_L in terms of the DG^+-norm, on families of geometric meshes complying with Assumptions (GM1)–(GM4). The focus is on asymptotic behaviour with respect to the depth L of refinement. Hence, we admit additional dependence of the constants on the solution u, the grading parameter σ, and ϵ from (38). Such constants are tagged with a tilde.

Theorem 6.5. *Denote by u the solution of BVP (1), and by u_L its approximation obtained by the method (7) defined on a mesh $\mathcal{T}_L$ with L refinement levels belonging to a family of geometric meshes with grading parameter σ, and with local approximating plane wave spaces of dimension $p(L)$ given by (38).*
If $\epsilon > 0$ in (38), then, there exists a threshold $L_* \in \mathbb{N}$ and two constants $\tilde{C}, \tilde{b} > 0$ such that

$$\|u-u_L\|_{0,\Omega} \leq \tilde{C}e^{-\tilde{b}(\dim V_L)^{\frac{1}{2+\epsilon}}} \qquad \forall L > L^*.$$

If $\epsilon = 0$, the same conclusion holds, provided that $\sigma \in (0,1)$ is sufficiently large.

Proof. Combining the result of Theorem 4.5 with (40), we have

$$\begin{aligned}
\|u-u_L\|^2_{0,\Omega} &\stackrel{\text{Thm. } 4.5}{\leq} \tilde{C}\inf_{v_L\in V_L}\|u-v_L\|_{DG^+}\\
&\stackrel{(40)}{\leq} \tilde{C}\sum_{K\in\mathcal{T}_L}\left(k^{-1}\|\nabla(u-v_L)\cdot\mathbf{n}\|^2_{0,\partial K} + \frac{kh_{\max}}{h_K}\|u-v_L\|^2_{0,\partial K}\right).
\end{aligned}$$

Next, we split the sum into two parts comprising the small cells of layer $\mathcal{L}_L^L$ and cells away from corners, respectively:

$$\sum_{K\in\mathcal{L}_L^L}\left(k^{-1}\|\nabla(u-v_L)\cdot\mathbf{n}\|^2_{0,\partial K} + \frac{kh_{\max}}{h_K}\|u-v_L\|^2_{0,\partial K}\right) \stackrel{(52)}{\leq} \tilde{C}\sigma^{2sL},$$

and

$$\sum_{K\in\mathcal{T}_L\setminus\mathcal{L}_L^L} \left(k^{-1}\left\|\nabla(u-v_L)\cdot\mathbf{n}\right\|_{0,\partial K}^2 + \frac{kh_{\max}}{h_K}\left\|u-v_L\right\|_{0,\partial K}^2\right)$$

$$\overset{(41)}{\le} \tilde{C} \sum_{K\in\mathcal{T}_L\setminus\mathcal{L}_L^L} \left(1+\frac{1}{kh_K}\right)\left[e^{-bq}+\frac{1+(kh_K)^{q+1}}{(c_0(q+1))^{\frac{q}{2}}}\right]^2 \left(\left\|u\right\|_{L^\infty(K_{\eta_*})}+h_K\left\|\nabla u\right\|_{L^\infty(K_{\eta_*})}\right)^2$$

$$\le \tilde{C}\left\{\left(e^{-2bq}\sum_{K\in\mathcal{T}_L\setminus\mathcal{L}_L^L}\left(1+\frac{1}{kh_K}\right)\right)+\frac{1}{(c_0(q+1))^q}\sum_{K\in\mathcal{T}_L\setminus\mathcal{L}_L^L}\left(\frac{1}{kh_K}+(kh_K)^{2q+2}\right)\right\}$$

$$\overset{(36),\,(34)}{\le} \tilde{C}\left\{e^{-2bq}\sum_{\ell=1}^{L-1}\left(1+\frac{1}{k\sigma^\ell}\right)+\frac{1}{(c_0(q+1))^q}\sum_{\ell=1}^{L-1}\left(\frac{1}{k\sigma^\ell}+(k\sigma^\ell)^{2q+2}\right)\right\}$$

$$\le \tilde{C}\left\{e^{-2bq}\left(L+\frac{\sigma}{1-\sigma}\sigma^{-L}\right)+(c_0q)^{-q}\left(\frac{\sigma}{1-\sigma}\sigma^{-q}+\frac{(k\sigma)^{2q+2}}{1-\sigma}\right)\right\}$$

$$\le \tilde{C}\left\{e^{-2bL^{1+\epsilon}}\left(L+\frac{\sigma}{1-\sigma}\sigma^{-L}\right)+(c_0q)^{-\frac{q}{2}}\underbrace{\left(\frac{\sigma}{1-\sigma}(c_0\sigma^2q)^{-\frac{q}{2}}+\frac{k^2\sigma^2}{1-\sigma}(c_0k^{-4}\sigma^{-4}q)^{-\frac{q}{2}}\right)}_{\text{uniformly bounded in } q\ge 0}\right\}.$$

As in (41), we wrote $q := \lceil L^{1+\epsilon}\rceil$, used $L \le q$, and have tacitly incorporated norms of the exact solution u into the constants $\tilde{C}$. We combine these estimates and simplify the expressions by making the constants dependent even on σ and c_0:

$$\left\|u-u_L\right\|_{0,\Omega}^2 \le \tilde{C}\left\{e^{2sL\log\sigma}+e^{-2bL^{1+\epsilon}+\log L}+e^{-L(2bL^\epsilon+\log\sigma)}+(c_0q)^{-\frac{q}{2}}\right\}.$$

Therefore,

$$\left\|u-u_L\right\|_{0,\Omega} \le \tilde{C}\left\{e^{-L(-s\log\sigma)}+e^{-L(bL^\epsilon-\frac{\log L}{2L})}+e^{-L(bL^\epsilon+\frac{\log\sigma}{2})}+e^{-L(\frac{q}{4L}\log(c_0q))}\right\}.$$

If $L \ge L_*$, where L_* is the minimal integer satisfying

$$\frac{\log L_*}{L_*} \le b, \qquad L_* \ge \frac{e^{2b}}{c_0}, \tag{53}$$

then $bL^\epsilon - \frac{\log L}{2L} \ge \frac{b}{2}$ for all $\epsilon \ge 0$, and $\frac{q}{4L}\log(c_0q) \ge \frac{b}{2}$ (where we used again $q \ge L$). We fix

$$b' := \min\left\{-s\log\sigma,\ \frac{b}{2}\right\}.$$

Next, we distinguish two cases:

i) Case $\epsilon > 0$. If $L > L_*$, where L_* is the minimal integer satisfying (53) and

$$L_* \ge \left(\frac{b-\log\sigma}{2b}\right)^{\frac{1}{\epsilon}},$$

so that we also have $bL^\epsilon + \frac{\log\sigma}{2} \ge \frac{b}{2}$, we get

$$\left\|u-u_L\right\|_{0,\Omega} \le \tilde{C}e^{-b'L}. \tag{54}$$

Since, by (39), $L \ge C(\dim V_L)^{\frac{1}{2+\epsilon}}$, with C only depending on the maximum number of element per mesh layer, the assertion of the theorem follows.

ii) Case $\epsilon = 0$. Provided that $\sigma > e^{-b}$, we obtain (54) for all $L > L_*$, where L_* is the minimal integer satisfying (53), and we conclude as before.

□

The proof of Theorem 6.5 shows that the rate $\tilde{b}$ of exponential convergence of the Trefftz-DG method and the layer number threshold L_* only depend on: *(i)* the maximum number of elements per layer, which is bounded (see (36)); *(ii)* the regularity parameter s relative to the solution u; *(iii)* the mesh grading parameter σ; *(iv)* the parameter b from Proposition 5.1 (and [15, Corollary 4.11]), which is the exponential convergence rate for the approximation of certain harmonic functions by harmonic polynomials.

Remark 6.6. The Trefftz-DG method with a basis composed by circular waves (i.e. Fourier–Bessel functions) can be considered in the same setting examined here (graded meshes, flux parameters). Using Proposition 5.3 instead of Proposition 5.4, the same exponential convergence in the square root of the total number of degrees of freedom, as in the plane wave basis case, is achieved.

Remark 6.7. For piecewise polynomial hp-approximation, it is possible to use local degrees on $K \in \mathcal{L}_\ell^L$ linearly increasing with $L - \ell$ without affecting overall exponential convergence [10, 31]. Unfortunately, owing to the presence of negative powers of h_K in the bounds, a reduction of the number of plane wave basis functions on small cells cannot be accommodated by our current analysis. However, such a reduction seems to be inevitable in practice to curb the instability of the plane wave basis, see [17].

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