Planetary Waves in Horizontal and Vertical Shear: The Generalized Eliassen-Palm Relation and the Mean Zonal Acceleration[1]

D. G. ANDREWS AND M. E. MCINTYRE

U. K. Universities' Atmospheric Modelling Group, Department of Geophysics, University of Reading, and Department of Applied Mathematics and Theoretical Physics, University of Cambridge

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ABSTRACT

Using a new generalization of the Eliassen-Palm relations, we discuss the zonal-mean-flow tendency $\partial\bar{u}/\partial t$ due to waves in a stratified, rotating atmosphere, with particular attention to equatorially trapped modes. Wave transience, forcing and dissipation are taken into account in a very general way. The theory makes it possible to discuss the latitudinal (y) and vertical (z) dependence of $\partial\bar{u}/\partial t$ qualitatively and calculate it directly from an approximate knowledge of the wave structure. For equatorial modes it reveals that the y profile of $\partial\bar{u}/\partial t$ is strongly dependent on the nature of the forcing or dissipation mechanism. A by-product of the theory is a far-reaching generalization of the theorems of Charney-Drazin, Dickinson and Holton on the forcing of $\partial\bar{u}/\partial t$ by conservative linear waves.

Implications for the quasi-biennial oscillation in the equatorial stratosphere are discussed. Graphs of y profiles of $\partial\bar{u}/\partial t$ are given for the equatorial waves considered in the recent analysis of observational data by Lindzen and Tsay (1975). The y profiles of $\partial\bar{u}/\partial t$ for Rossby-gravity and inertio-gravity modes, in Lindzen and Tsay's parameter ranges, prove extremely sensitive to whether or not small amounts of mechanical dissipation are present alongside the radiative-photochemical dissipation of the waves.

The probable importance of low-frequency Rossby waves for the momentum budget of the descending easterlies is suggested.

1. Introduction

The Lindzen-Holton theory of the quasi-biennial oscillation (QBO) of the equatorial stratospheric mean zonal wind $\bar{u}$ (Lindzen and Holton, 1968; Holton and Lindzen, 1972) convincingly explains many of the gross features of the observed zonal mean accelerations $\partial\bar{u}/\partial t$, by attributing them to the presence of upward-propagating and dissipating equatorial planetary waves. The theory assumes that the effect of the waves on the mean flow is equivalent to a stress $S_{(xz)}$ acting to transfer mean zonal momentum between the levels of wave excitation and dissipation. A heuristically motivated formula for $S_{(xz)}$ is assumed, and only the latitudinal integral of $\partial\bar{u}/\partial t$

$$\langle\partial\bar{u}/\partial t\rangle \equiv \int_{-\infty}^{\infty} (\partial\bar{u}/\partial t)dy$$

is considered. In view of the apparent success of the theory it would seem desirable to place the calculation of $\partial\bar{u}/\partial t$ on a firmer and more detailed theoretical footing. In what follows we show how this can be done without expensive numerical computation.

The discussion depends on new results in the theory of wave mean-flow interaction, which are of interest in a wider context. They substantially generalize and unify the results of Eliassen and Palm (1961; hereafter referred to as EP), Charney and Drazin (1961), Dickinson (1969), Stern (1971), Fels and Lindzen (1974), Holton (1974, 1975), Uryu (1974), Plumb (1975) and others. In some cases they lead to very simple relations between $\partial\bar{u}/\partial t$ and wave forcing or transience. For instance, in the case of an equatorially trapped planetary wave with horizontal phase speed c on an equatorial beta-plane at large Richardson number,

$$\frac{\partial\bar{u}}{\partial t}(y,z,t) \approx -\frac{\partial}{\partial y}\overline{(\eta' X')} + \frac{1}{(c-\bar{u})}\left\{\overline{(u'+\eta'\bar{u}_y)X'} + \overline{v'Y'} + \frac{\overline{\theta' Q'}}{s(z)}\right\} + \frac{1}{2}\frac{\partial}{\partial t}\left[-\frac{\partial}{\partial y}\overline{(\eta' u')} + \frac{1}{(c-\bar{u})}\left\{\overline{(u'+\eta'\bar{u}_y)u'} + \overline{v'^2} + \frac{\overline{\theta'^2}}{s(z)}\right\}\right],$$

where

$$u_y \equiv \partial\bar{u}/\partial y,$$

η' is the latitudinal particle displacement, u', v', θ' the disturbance velocities and potential temperature, $s(z)$

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a measure of the static stability at height z (such that $\overline{\theta'^2}/s$ is eddy available potential energy per unit mass), and $-X'$, $-Y'$, $-Q'$ are unspecified forcing terms on the right of the linearized zonal momentum, meridional momentum and thermal energy equations. These arbitrary forcing terms may represent either wave dissipation or excitation, depending on whether their correlations with the corresponding disturbance motions are positive or negative, respectively.

The above formula for $\partial\bar{u}/\partial t$ also makes clear a fact which appears not to have been appreciated before, namely that latitudinal profiles of $\partial\bar{u}/\partial t$ can vary greatly according as the dissipation or forcing is thermal (due to Q') or mechanical (due to X' or Y'). In a companion paper (Andrews and McIntyre, 1976) the implied behavior of the horizontal Reynolds stress $\overline{u'v'}$ is independently confirmed by direct calculation in some simple cases.

The theoretical developments are given in Sections 1–7 and Appendix A, the new results being presented and discussed in Sections 5, 7 and 8. Examples of y profiles of $\partial\bar{u}/\partial t$ for the simplest models of wave dissipation are presented in Section 9. Section 10 reviews probable wave dissipation mechanisms for real fluids, and Section 11 discusses some implications for the descending easterlies in the equatorial stratosphere.

After submission of this paper for publication, we learned of independent work by Boyd (R. S. Lindzen, personal communication) and by Bretherton (1977) on generalizations of the Eliassen-Palm relations. Bretherton's work illuminates certain fundamentals which will be mentioned in context. A manuscript by Boyd (1976) reached us at a late stage, while the final version of our paper was being typed. We have added some brief comments on the relationship between his results and ours.

2. Equations

Following Lindzen (1971), we start from the Boussinesq, hydrostatic, primitive equations of motion for a beta-plane (mid-latitude or equatorial).[2] The equations are

$$\left.\begin{aligned} u_t+uu_x+vu_y+wu_z-fv+p_x&=-X\\ v_t+uv_x+vv_y+wv_z+fu+p_y&=-Y\\ -\theta+p_z&=\ 0\\ \theta_t+u\theta_x+v\theta_y+w\theta_z&=-Q\\ u_x+v_y+w_z&=\ 0\end{aligned}\right\},\qquad(2.1)$$

where axes are chosen with the zonal coordinate x increasing eastward, y northward (with $y=0$ at the equator) and z upward. Here $f(y)$ is the Coriolis parameter, (u,v,w) is the fluid velocity, and θ is now defined as the total buoyancy acceleration given by minus the gravity acceleration times the fractional departure of the density from its constant reference value ρ_0. The departure of the pressure from the hydrostatic value associated with ρ_0 is denoted by $\rho_0 p$ (p and θ are Lindzen's Φ and $-g\delta\rho/\bar{\rho}$). In Section 9, f will be taken equal to βy, where β is the equatorial planetary vorticity gradient.

The terms $-X$, $-Y$ and $-Q$ represent arbitrary body forces and heating, whose possible physical causes may be left unspecified for the moment.

3. Zonal-mean problem

We let an overbar denote the usual zonal average and a prime the departure therefrom; then the averaged equations are

$$\bar{u}_t+\bar{v}(\bar{u}_y-f)+\bar{w}\bar{u}_z=-(\overline{u'v'})_y-(\overline{u'w'})_z-\bar{X},\qquad(3.1\text{a})$$

$$\bar{v}_t+\bar{v}\bar{v}_y+\bar{w}\bar{v}_z+f\bar{u}+\bar{p}_y=-(\overline{v'^2})_y-(\overline{v'w'})_z-\bar{Y},\qquad(3.1\text{b})$$

$$-\bar{\theta}+\bar{p}_z=0,\qquad(3.1\text{c})$$

$$\bar{\theta}_t+\bar{v}\bar{\theta}_y+\bar{w}\bar{\theta}_z=-(\overline{v'\theta'})_y-(\overline{w'\theta'})_z-\bar{Q},\qquad(3.1\text{d})$$

$$\bar{v}_y+\bar{w}_z=0.\qquad(3.1\text{e})$$

In our analysis all quantities on the right-hand sides will be regarded as known and $O(a^2)$ as $a\to 0$, where a is a measure of disturbance amplitude.[3] If the right-hand sides are zero, then we have a solution representing steady zonal motion, in which $\bar{v}=\bar{w}=\partial/\partial t=0$, and (3.1b) reduces to geostrophic balance. All mean departures from this basic, unforced zonal flow will be taken to be $O(a^2)$. Thus $\bar{v}$ and $\bar{w}$ are $O(a^2)$ and the terms $\bar{v}\bar{v}_y+\bar{w}\bar{v}_z$ in (3.1b) are $O(a^4)$.

The following transformation of (3.1a–e) will prove important. We define $\bar{w}^*$, $\bar{v}^*$ by

$$\bar{w}=\bar{w}^*-(\overline{v'\theta'}/\bar{\theta}_z)_y,\quad \bar{v}=\bar{v}^*+(\overline{v'\theta'}/\bar{\theta}_z)_z,\qquad(3.2)$$

and

$$S_{(xy)}=\overline{u'v'}-\mathcal{B}\overline{v'\theta'}/\bar{\theta}_z,\qquad(3.3\text{a})$$

$$S_{(xz)}=\overline{u'w'}+\mathcal{A}\overline{v'\theta'}/\bar{\theta}_z,\qquad(3.3\text{b})$$

where

$$\mathcal{A}=\bar{u}_y-f,\quad \mathcal{B}=\bar{u}_z.\qquad(3.4)$$

$\mathcal{A}$ and $\mathcal{B}$ are functions of y, z and, at $O(a^2)$, of t. Then

[2] The analysis for a hydrostatic, compressible atmosphere in pressure coordinates requires only slight modifications. Our general results have also been obtained for finite-amplitude disturbances to nonhydrostatic, compressible flow on a sphere (see Andrews and McIntyre, 1977).

[3] The reader is reminded that the O symbol signifies an *upper bound* in the limit, to within an arbitrary constant of proportionality (Lighthill, 1958, p. 3). For instance some of the terms could be zero.

the equations become

$$\bar{u}_t+\mathcal{A}\bar{v}^*+\mathcal{B}\bar{w}^*=-\frac{\partial}{\partial y}S_{(xy)}-\frac{\partial}{\partial z}S_{(xz)}-\bar{X}, \tag{3.5a}$$

$$\bar{v}_t^*+f\bar{u}+\bar{p}_y=-(\overline{v'^2})_y-(\overline{v'w'})_z$$

$$-(\overline{v'\theta'}/\bar{\theta}_z)_{zt}-\bar{Y}+O(a^4), \tag{3.5b}$$

$$-\bar{\theta}+\bar{p}_z=0, \tag{3.5c}$$

and after a little manipulation

$$\bar{\theta}_t+\bar{\theta}_y\bar{v}^*+\bar{\theta}_z\bar{w}^*=-\frac{\partial}{\partial z}(\overline{w'\theta'}+\overline{v'\theta'}\bar{\theta}_y/\bar{\theta}_z)-\bar{Q}, \tag{3.5d}$$

$$\bar{v}_y^*+\bar{w}_z^*=0. \tag{3.5e}$$

4. Disturbance problem

Using the definition $D_t\equiv\partial/\partial t+\bar{u}\partial/\partial x$, the equations for the disturbances are

$$D_tu'+\mathcal{A}v'+\mathcal{B}w'+p_x'=-X', \tag{4.1a}$$

$$D_tv'+fu'+p_y'=-Y', \tag{4.1b}$$

$$-\theta'+p_z'=0, \tag{4.1c}$$

$$D_t\theta'+\bar{\theta}_yv'+\bar{\theta}_zw'=-Q', \tag{4.1d}$$

$$u_x'+v_y'+w_z'=0. \tag{4.1e}$$

If $X'=Y'=Q'=0$, these are the usual linearized equations for conservative waves. But they may also be regarded as the equations for finite a if X', Y' and Q' are defined to include all the nonlinear terms omitted from the left-hand sides. Some specific physical mechanisms, all of which amount to known ways in which these nonlinear terms could contribute systematically to X', Y' and Q', are discussed in Section 10.

Upon multiplying (4.1a–d) by u', v', w' and $\theta'/\bar{\theta}_z$, respectively, averaging, and making use of (4.1e), we obtain the usual wave-energy equation

$$\frac{1}{2}\frac{\partial}{\partial t}(\overline{u'^2}+\overline{v'^2}+\overline{\theta'^2}/\bar{\theta}_z)+(\overline{u'X'}+\overline{v'Y'}+\overline{\theta'Q'}/\bar{\theta}_z)$$

$$=-[(\overline{p'v'})_y+(\overline{p'w'})_z+\bar{u}_y\overline{u'v'}+\bar{u}_z\overline{u'w'}+\bar{\theta}_y\overline{v'\theta'}/\bar{\theta}_z]. \tag{4.2}$$

5. The generalized Eliassen-Palm relation and Charney-Drazin theorem

To find $\partial\bar{u}/\partial t$ for equatorial planetary waves directly from (3.1) or (3.5), it turns out that wave solutions are needed at a higher order of accuracy than usually calculated. The reason lies in an ill-conditioned behavior of the term in $\overline{u'v'}$ (which will be examined further in Section 6e and in the companion paper).

Now EP showed that for any $f(y)$

$$\frac{\partial}{\partial y}S_{(xy)}+\frac{\partial}{\partial z}S_{(xz)}=0 \tag{5.1}$$

exactly, for steady, conservative, linear waves—hence the celebrated result that $\partial\langle S_{(xz)}\rangle/\partial z=0$ exactly, where $\langle(\ )\rangle\equiv\int(\ )dy$. Our central result is similarly an exact formula for $(\partial S_{(xy)}/\partial y+\partial S_{(xy)}/\partial z)$, but valid for *any* disturbance satisfying Eqs. (4.1). Moreover we can easily simplify it for the case of equatorial planetary waves by introducing appropriate scaling assumptions, since it turns out that there are no important terms having an ill-conditioned behavior like $\overline{u'v'}$.

We define particle displacements $\xi'(x,y,z,t)$, $\eta'(x,y,z,t)$ and $\zeta'(x,y,z,t)$ such that $\xi_x'+\eta_y'+\zeta_z'=0$, $\overline{\xi'}=\overline{\eta'}=\overline{\zeta'}=0$, and

$$D_t\eta'=v',\quad D_t\zeta'=w',\quad D_t\xi'=u^l, \tag{5.2}$$

where (e.g., Hayes 1970)

$$u^l=u'+\bar{u}_y\eta'+\bar{u}_z\zeta'. \tag{5.3a}$$

[Note that (5.2) implies $D_t(\xi_x'+\eta_y'+\zeta_z')=D_t(\overline{\xi'})=D_t(\overline{\eta'})=D_t(\overline{\zeta'})=0$, etc., so that the definitions are self-consistent.] We also define

$$q'=-\theta'-\bar{\theta}_y\eta'-\bar{\theta}_z\zeta', \tag{5.3b}$$

so that $\overline{q'}=0$ and, from (4.1d),

$$D_tq'=Q'. \tag{5.4}$$

Then it follows quite generally from Eqs. (4.1) that

$$\frac{\partial}{\partial y}S_{(xy)}+\frac{\partial}{\partial z}S_{(xz)}=(\overline{\eta'X'})_y+(\overline{\zeta'X'})_z$$

$$+\overline{\xi_x'X'}+\overline{\eta_x'Y'}+\overline{\zeta_x'q'}-\mathcal{B}(\overline{\eta'Q'}/\bar{\theta}_z)_y+\mathcal{A}(\overline{\eta'Q'}/\bar{\theta}_z)_z$$

$$+\frac{\partial'}{\partial t}\Bigg[(\overline{\eta'u'})_y+(\overline{\zeta'u'})_z+\overline{\xi_x'(u^l-f\eta')}+\overline{\eta_x'v'}$$

$$-\left\{\mathcal{B}\left[\frac{\overline{\eta'\theta'}+\frac{1}{2}\bar{\theta}_y(\overline{\eta'^2})}{\bar{\theta}_z}\right]-\tfrac{1}{2}\mathcal{A}(\overline{\eta'^2})\right\}_y$$

$$+\left\{\mathcal{A}\left[\frac{\overline{\eta'\theta'}+\frac{1}{2}\bar{\theta}_y(\overline{\eta'^2})}{\bar{\theta}_z}+\overline{\eta'\zeta'}\right]+\tfrac{1}{2}\mathcal{B}(\overline{\zeta'^2})\right\}_z\Bigg]$$

$$+(\bar{\theta}_y+f\bar{u}_z)(\overline{\zeta_x'\eta'}). \tag{5.5a}$$

The term on the last line is $O(a^4)$, by (3.1b, c), since the departure from thermal wind balance is $O(a^2)$. The symbol $\partial'/\partial t$ denotes time-differentiation of primed quantities only (and not of $\mathcal{A}$, $\mathcal{B}$ and $\bar{\theta}$); but when $O(a^4)$ is neglected, $\partial'/\partial t$ may be replaced by $\partial/\partial t$. The derivation of (5.5a) is given in Appendix A.

Using (5.3b) we may rewrite the last group of terms within braces in (5.5a) as

$$\frac{\partial'}{\partial t}\left\{-\mathcal{A}\left[\frac{\overline{\eta' q'}+\frac{1}{2}\bar{\theta}_y(\overline{\eta'^2})}{\bar{\theta}_z}\right]+\tfrac{1}{2}\mathcal{B}(\overline{\zeta'^2})\right\}. \tag{5.6}$$

The asymmetry between y and z in the $\mathcal{A}$ and $\mathcal{B}$ terms is due to the transformation (3.2), which invokes a nonvanishing stratification $\bar{\theta}_z$. [But (5.5a) holds whether or not $f=0$, and so applies, e.g., to internal gravity waves, after adding terms $\overline{\zeta'_x Z'}$ and $(\overline{\zeta'_x w'})_t$ if the waves are not hydrostatic.]

The right-hand side of (3.5d) may be written in a form similar to (5.5a); multiplying (4.1d) by θ' and taking the average leads immediately to a relation whose z derivative is

$$\frac{\partial}{\partial z}(\overline{w'\theta'}+\overline{v'\theta'}\bar{\theta}_y/\bar{\theta}_z)$$
$$=-(\overline{\theta' Q'}/\bar{\theta}_z)_z+\frac{\partial'}{\partial t}[-\tfrac{1}{2}(\overline{\theta'^2}/\bar{\theta}_z)_z], \tag{5.5b}$$

and which has a familiar interpretation in terms of the rate of change of disturbance available potential energy.

Taken together with Eqs. (3.5), the two relations (5.5) expose the essential role of wave dissipation, excitation and transience in the mean-flow problem. Moreover they show how wave transience by itself cannot give rise to permanent $O(a^2)$ mean-flow changes. When $X=Y=Q=q'=0$, (3.5) and (5.5) imply that the mean-flow problem takes the very special form

$$\bar{u}_t+\mathcal{A}\bar{v}^*+\mathcal{B}\bar{w}^*=\partial_t(\overline{\ \ })+O(a^4), \tag{5.7a}$$
$$\bar{v}^*_{tt}+f\bar{u}_t+\bar{p}_{ty}=\partial_t[(\overline{\ \ })+O(a^4)], \tag{5.7b}$$
$$-\bar{\theta}_t+\bar{p}_{tz}=0, \tag{5.7c}$$
$$\bar{\theta}_t+\bar{\theta}_y\bar{v}^*+\bar{\theta}_z\bar{w}^*=\partial_t(\overline{\ \ })+O(a^4), \tag{5.7d}$$
$$\bar{v}^*_y+\bar{w}^*_z=0, \tag{5.7e}$$

where $\partial_t\equiv\partial/\partial t$, and the overbar stands for various linear combinations of disturbance covariances, with coefficients involving only mean-flow quantities; in (5.7a, d) the expressions are minus those within square brackets in (5.5a, b), respectively. Eqs. (5.7a–e) comprise a complete set of partial differential equations for the set of mean-flow quantities

$$\{\bar{u}_t,\bar{\theta}_t,\bar{p}_t,\bar{v}^*,\bar{w}^*\}. \tag{5.8}$$

The statement that all the $O(a^2)$ forcing terms on the right of (5.7) have the exact form $\partial_t(\overline{\ \ })$ constitutes a new generalization, in two senses, of the earlier results of Charney and Drazin (1961), Dickinson (1969) and Holton (1974, 1975). Those important results were in the first place restricted to steady waves $[\partial_t(\overline{\ \ })=0]$, and in the second place assumed either small Rossby or large Richardson number. No such assumptions are needed here.[4]

For reference, we note the forms taken by (5.5a, b) in spherical coordinates r, λ, φ, in place of z, x, y, where λ is longitude and φ is latitude:

$$\frac{1}{R\cos^2\varphi}\frac{\partial}{\partial\varphi}[\cos^2\varphi(\overline{u'v'}-\mathcal{B}\overline{v'\theta'}/\bar{\theta}_r)]+\frac{\partial}{\partial r}[\overline{u'w'}+\mathcal{A}\overline{v'\theta'}/\bar{\theta}_r]$$
$$=\frac{1}{R\cos^2\varphi}(\overline{\eta'X'}\cos^2\varphi)_\varphi+(\overline{\zeta'X'})_r+\frac{1}{R\cos\varphi}(\overline{\xi'_\lambda X'}+\overline{\eta'_\lambda Y'}+\overline{\zeta'_\lambda q'})-\frac{\mathcal{B}}{R\cos\varphi}\left[\frac{\overline{\eta'Q'}\cos\varphi}{\bar{\theta}_r}\right]_\varphi+\mathcal{A}\left[\frac{\overline{\eta'Q'}}{\bar{\theta}_r}\right]_r$$
$$+\frac{\partial'}{\partial t}\Bigg[\frac{1}{R\cos^2\varphi}(\overline{\eta'u'}\cos^2\varphi)_\varphi+(\overline{\zeta'u'})_r+\frac{1}{R\cos\varphi}\{\overline{\xi'_\lambda(u^l-\mathcal{C}\eta')}+\overline{\eta'_\lambda v'}\}$$
$$-\frac{1}{R\cos^2\varphi}\frac{\partial}{\partial\varphi}\cos^2\varphi\left\{\mathcal{B}\left(\frac{\overline{\eta'\theta'}+\frac{1}{2}\bar{\theta}_\varphi\overline{\eta'^2}/R}{\bar{\theta}_r}\right)-\tfrac{1}{2}\mathcal{A}\overline{\eta'^2}\right\}+\frac{\partial}{\partial r}\left\{\mathcal{A}\left(\frac{\overline{\eta'\theta'}+\frac{1}{2}\bar{\theta}_\varphi\overline{\eta'^2}/R}{\bar{\theta}_r}+\overline{\eta'\zeta'}\right)+\tfrac{1}{2}\mathcal{B}\overline{\zeta'^2}\right\}\Bigg]$$
$$+\frac{1}{R\cos\varphi}\left(\frac{\bar{\theta}_\varphi}{R}+(2\mathcal{C}-f)\bar{u}_r\right)\overline{\eta'\zeta'_\lambda}, \tag{5.9a}$$

and

$$\frac{\partial}{\partial r}\{\overline{w'\theta'}+\overline{v'\theta'}\bar{\theta}_\varphi/(R\bar{\theta}_r)\}=-(\overline{\theta'Q'}/\bar{\theta}_r)_r+\frac{\partial'}{\partial t}[-\tfrac{1}{2}(\overline{\theta'^2}/\bar{\theta}_r)_r]. \tag{5.9b}$$

Here R denotes the radius of the earth, (w,u,v) the velocity components in the (r,λ,φ) directions, and $(-X, -Y)$ the forcing terms in the (λ,φ) momentum equations. The definitions of θ, Q and q' are analogous to the beta-plane case. With $D_t\equiv\partial/\partial t+(\bar{u}/R\cos\varphi)\partial/\partial\lambda$, the displacements are still defined by (5.2) except that $D_t\xi'-R^{-1}\bar{u}\tan\varphi\cdot\eta'=u^l$, where instead of (5.3a) we

[4] The significance of the $\partial_t(\overline{\ \ })$ form of the forcing when the waves are conservative but not steady may be further brought out

have

$$u^l = u' + R^{-1}\bar{u}_\varphi \eta' + \bar{u}_r \zeta'.$$

Also

$$\mathfrak{A} = \frac{(\bar{u}\cos\varphi)_\varphi}{R\cos\varphi} - f, \quad \mathfrak{B} = \bar{u}_r, \quad \mathfrak{C} = \frac{\bar{u}\tan\varphi}{R} + f,$$

where $f = 2\Omega\sin\varphi$, Ω denoting the earth's angular velocity. We omit the equations analogous to (5.7), since an even more general result for a compressible, nonhydrostatic fluid on a sphere is derived in our forthcoming (1977) paper.

Boyd [1976, Eq. (3.9)] has independently found a result equivalent to a special case of (5.9a) [but in the approximate pressure-coordinate system presented by Holton (1975)]. Boyd's result is for a linear wave disturbance with a single, constant zonal wavenumber and a phase speed exactly constant in time. The wave amplitude is allowed to grow or decay exponentially in time at fixed phase (see end of Appendix A). Boyd also obtains for the case of steady waves our extension of the Charney-Drazin theorem to arbitrary Rossby and Richardson numbers [the case $\partial_t(\overline{\ }) = 0$ in Eqs. (5.7) above].

6. Approximations valid for "tall" mean-flow scaling and large Richardson number

a. Assumptions

Let **U**, **H** and $\mathbf{N}^2$ be scales or typical magnitudes for $\bar{u}$, z and $\bar{\theta}_z$ in the mean flow equations (3.1) such that the inverse square root $\bar{u}_z/\bar{\theta}_z^{\frac{1}{2}}$ of the Richardson number

$$\bar{u}_z/\bar{\theta}_z^{\frac{1}{2}} \lesssim \mathbf{U}/\mathbf{NH}.$$

as follows:

1) If $\bar{v}^*_{tt}$ may be neglected in (5.7b), i.e., if significant zonally-symmetric inertio-gravity oscillations are not excited in the zonal-mean flow, then the left-hand sides of (5.7) involve partial differentiation with respect to y and z only, of the basic set of dependent variables (5.8). Moreover, the "coefficients" $\mathfrak{A}$, $\mathfrak{B}$, $\bar{\theta}_y$ and $\bar{\theta}_z$ on the left of (5.7) may be taken independent of t with error $O(a^4)$. It follows that a solution of (5.7) may be found in the *time-differentiated form*

$$\bar{u}_t = \partial_t F + O(a^4),$$

where F is zero at all times when the wave amplitude, as defined by the displacements ξ', η', ζ', is zero.

2) Even if $\bar{v}^*_{tt}$ is not negligible, we may formally take it onto the right of (5.7b) and conclude that a solution exists such that

$$\bar{u}_t = \partial_t\{F + G(\bar{v}^*{}_t)\} + O(a^4)$$

with F as before, and G a time-independent linear operator. Thus a wave which propagates past a given point in a finite time interval (t_0,t_1), and leaves no net particle displacement, forces no *permanent* $O(a^2)$ mean-flow change:

$$[\bar{u}]_{t_0}^{t_1} = [G(\bar{v}^*{}_t)]_{t_0}^{t_1} + O(a^4),$$

which must oscillate about zero, since G is linear and $\bar{v}^*_t$ must so oscillate after the disturbance covariances have returned to zero. (Otherwise $\bar{v}^*$ and therefore $\bar{v}$ would have to increase without bound against the stratification and Coriolis constraints in the absence of forcing!)

We define

$$\mu \equiv \mathbf{h}/\mathbf{H},$$

where **h** is a scale for z in the disturbance equations (4.1). A self-consistent and useful set of approximations for equatorial waves [and also for the mid-latitude, quasi-geostrophic, vertically-propagating Rossby waves studied by Uryu (1974)] can be found if we assume that

$$\mathbf{U}/\mathbf{NH} \lesssim \mu \ll 1. \tag{6.1}$$

These are equivalent to assumptions used by Lindzen (1971, 1972) and Holton (1975, p. 43), and will make it possible for the standard equatorial wave solutions to be used later as a first approximation. Then **h** will be identified as the vertical wavelength divided by 2π. In cases of practical interest **h** and **H** may vary with altitude, but they can often do so in such a way that (6.1) remains true (Lindzen, 1971, Figs. 5, 10; Appendix B), provided that **U** is also allowed to vary with height like **H** times a typical value of $\bar{u}_z$.

We shall further assume that the waves have y-scale **L** and x-scale $\gtrsim \mathbf{L}$, such that

$$\mathbf{fL}/\mathbf{Nh} = 1, \tag{6.2}$$

where **f** is a typical magnitude for f; in the equatorial problem $\mathbf{f} = \beta\mathbf{L}$. That is, the disturbance height and length scales are in Prandtl's ratio $\mathbf{f}/\mathbf{N}$. The mean flow will be assumed to satisfy

$$\bar{u}_y \lesssim \mathbf{U}/\mathbf{L}. \tag{6.3}$$

Next, typical magnitudes $\mathbf{u}'$, $\boldsymbol{\theta}'$, etc., are postulated for the disturbance fields:

$$u' \sim \mathbf{u}', \quad \theta' \sim \boldsymbol{\theta}', \quad \xi' \sim \boldsymbol{\xi}', \quad \eta' \lesssim \boldsymbol{\eta}', \quad \zeta' \sim \boldsymbol{\zeta}',$$

with the understanding that $\boldsymbol{\eta}'$ denotes the typical magnitude of η' except in the case of Kelvin waves for which $\eta' \ll \boldsymbol{\eta}'$. They are assumed to be such that disturbance kinetic and available potential energies are of comparable magnitudes:

$$\mathbf{u}'^2 = \boldsymbol{\theta}'^2/\mathbf{N}^2. \tag{6.4}$$

We also assume

$$\boldsymbol{\zeta}' = \boldsymbol{\theta}'/\mathbf{N}^2, \quad \boldsymbol{\eta}' = \mathbf{L}\mathbf{u}'/\mathbf{U}, \tag{6.5a, b}$$

$$\xi'_x \lesssim \mathbf{u}'/\mathbf{U}, \quad q' \lesssim \mathbf{Q}'/\mathbf{U}, \quad v' \lesssim \mathbf{u}', \quad \eta' \lesssim \boldsymbol{\xi}', \tag{6.6a–d}$$

where $\mathbf{Q}'$ is a typical magnitude for Q'. Relations (6.2) and (6.4)–(6.6) are true (as may be verified from Section 9) for all equatorially trapped waves of low latitudinal mode number, and of zonal radian wavelength $\lesssim \mathbf{L}$; they are also typically true of vertically propagating, quasi-geostrophic Rossby waves. Intrinsic horizontal phase speeds $\gtrsim \mathbf{Nh} \gtrsim \mathbf{U}$ (phase speeds $\gg \mathbf{U}$ for equatorial Rossby-gravity or inertio-gravity waves of zonal wavelength $\gg 2\pi\mathbf{L}$); thus (6.6a, b) are consistent with Eqs. (5.2c) and (5.4). Together with (6.1), relations (6.4) and (6.5) imply that

$$\boldsymbol{\zeta}'/\boldsymbol{\eta}' \lesssim (\mathbf{U}/\mathbf{NH})\mathbf{H}/\mathbf{L} \lesssim \mathbf{h}/\mathbf{L}. \tag{6.7}$$

[The quasi-geostrophic case actually has $\zeta'/\eta' \ll \mathbf{h}/\mathbf{L}$, since $\mathbf{U}/\mathbf{fL} \ll 1$ there, and $\mathbf{U}/\mathbf{NH} \sim \mu \mathbf{U}/\mathbf{fL}$ by (6.2).]

An additional assumption restricting $\mathbf{Q}'$ will be made in Section 6d, relation (6.25).

b. Simplification of the mean-flow problem

We wish to concentrate attention upon the mean-flow tendency due to the waves, and so from now on set

$$\bar{X}=\bar{Y}=\bar{Q}=0. \tag{6.8}$$

Let $\boldsymbol{\tau}$ be a time scale for wave dissipation, excitation or transience, so that typical magnitudes for X', Y' and Q' satisfy

$$\mathbf{X}'=\mathbf{u}'/\boldsymbol{\tau}, \quad \mathbf{Y}'=\mathbf{v}'/\boldsymbol{\tau}, \quad \mathbf{Q}'=\boldsymbol{\theta}'/\boldsymbol{\tau}. \tag{6.9}$$

Then in (5.5a) the typical magnitude of both $(\overline{\eta' X'})_y$ and $(\overline{\eta' u'})_{yt}$ is

$$\mathbf{A}=\boldsymbol{\eta}'\mathbf{X}'/\mathbf{L}=\boldsymbol{\eta}'\mathbf{u}'/\mathbf{L}\boldsymbol{\tau}. \tag{6.10}$$

It will emerge that $\mathbf{A}$ is in fact the magnitude for the mean acceleration $\bar{u}_t$. From Eqs. (6.4), (6.5b) and (6.9) it is seen that alternative expressions for $\mathbf{A}$ are

$$\mathbf{A}=\boldsymbol{\theta}'\mathbf{Q}'/\mathbf{N}^2\mathbf{U}=\boldsymbol{\theta}'^2/\mathbf{N}^2\mathbf{U}\boldsymbol{\tau}. \tag{6.11}$$

Eqs. (3.1b, c) imply that thermal-wind balance holds with error $O(a^2)$:

$$\bar{\theta}_y=-f\bar{u}_z+O(a^2); \tag{6.12}$$

so

$$\bar{\theta}_y \lesssim \mathbf{fU}/\mathbf{H}. \tag{6.13}$$

Hence, by (6.1) and (6.2), $\bar{\theta}_y \lesssim \mu^2\mathbf{N}^2\mathbf{H}/\mathbf{L}$. That is, the basic stability $\bar{\theta}_z$ can in Eqs. (3.1d) and (4.1d) be taken as a function of z only, with relative error $O(\mu^2)$, over our domain of width $\mathbf{L}$, $|y| \lesssim \mathbf{L}$, say. We shall assume moreover that

$$\bar{\theta}_{yt} \sim -f\bar{u}_{zt}. \tag{6.14}$$

[Approximate *equality* need not hold unless the waves are steady, because the right-hand side of (3.1b) may be comparable with the time-dependent $O(a^2)$ part of $f\bar{u}$.] It is then consistent to take

$$\bar{\theta}_t \lesssim \mathbf{fL}\bar{u}_t/\mathbf{H}. \tag{6.15}$$

We may now simplify (3.5a) to

$$\bar{u}_t=-\frac{\partial}{\partial y}S_{(xy)}-\frac{\partial}{\partial z}S_{(xz)}+O(\mu^2\mathbf{A}) \tag{6.16}$$

for $|y| \lesssim \mathbf{L}$. To show this we first estimate the term $\mathcal{B}\bar{w}^*$ in (3.5a). From (3.5d) and (5.5b),

$$\bar{w}^* \lesssim \mathbf{N}^{-2}\max\{\bar{\theta}_t, \bar{\theta}_y^*\bar{v}, \partial_z[\overline{\theta' Q'}/\bar{\theta}_z, \tfrac{1}{2}(\overline{\theta'^2})_t/\bar{\theta}_z]\}.$$

Respectively invoking (6.15), (6.13) and (6.11) we get

$$\mathbf{N}\bar{w}^* \lesssim \max\left\{\frac{\mathbf{h}}{\mathbf{H}}\bar{u}_t, \frac{\mathbf{U}}{\mathbf{NH}}\mathbf{f}\bar{v}^*, \frac{\mathbf{UA}}{\mathbf{NH}}, \frac{\mathbf{UA}}{\mathbf{NH}}\right\}. \tag{6.17}$$

Therefore, by (3.4) and (6.1),

$$\mathcal{B}\bar{w}^* \lesssim \frac{\mathbf{U}}{\mathbf{NH}}\cdot\mathbf{N}\bar{w}^* \lesssim \mu^2\max\{\mathbf{A}, \bar{u}_t, \mathbf{f}\bar{v}^*\}. \tag{6.18}$$

Since the wave forcing terms on the right of (3.5) vary significantly over the y-scale $\mathbf{L}$, any contribution to $\bar{v}^*$ forced by the waves will do likewise within $y \lesssim \mathbf{L}$ so that by (3.5e)

$$\bar{v}^*/\mathbf{L} \sim \bar{w}^*/\mathbf{H}. \tag{6.19}$$

Since $\bar{u}_y \lesssim \mathbf{f}$ by (6.3), (6.2) and (6.1),

$$\begin{aligned} \mathcal{A}\bar{v}^* &\lesssim \mathbf{f}\bar{v}^*, \\ &\sim \mathbf{Nh}\bar{v}^*/\mathbf{L} \sim \mathbf{Nh}\bar{w}^*/\mathbf{H} \quad \text{[by (6.2, 6.19)]}, \\ &\lesssim \frac{\mathbf{h}}{\mathbf{H}}\max\left\{\frac{\mathbf{h}}{\mathbf{H}}\bar{u}_t, \frac{\mathbf{U}}{\mathbf{NH}}\mathbf{f}\bar{v}^*, \frac{\mathbf{UA}}{\mathbf{NH}}\right\} \quad \text{[by (6.17)]}. \end{aligned}$$

Therefore

$$\mathcal{A}\bar{v}^* \lesssim \mathbf{f}\bar{v}^* \lesssim \mu^2\max\{\mathbf{A}, \bar{u}_t, \mathbf{f}\bar{v}^*\}. \tag{6.20}$$

It now follows from (6.18), (6.20) and the assumption that $\mu \ll 1$ that both $\mathcal{A}\bar{v}^*$ and $\mathcal{B}\bar{w}^* \lesssim \mu^2\max\{\mathbf{A}, \bar{u}_t\}$, whence from (3.5a), after setting $\bar{X}=0$,

$$\bar{u}_t+\partial S_{(xy)}/\partial y+\partial S_{(xz)}/\partial z \lesssim \mu^2\max\{\mathbf{A}, \bar{u}_t\}.$$

But it will be found that $\partial S_{(xy)}/\partial y+\partial S_{(xz)}/\partial z \lesssim \mathbf{A}$ (Section 6d), whence $\bar{u}_t \lesssim \mathbf{A}$, and (6.16) follows. In other words, to two orders in μ the mean meridional circulation is dominated by the $\overline{v'\theta'}$ terms in (3.2), and $\bar{v}^*$ and $\bar{w}^*$ can be ignored in (3.5a).

c. Remarks about the simplified mean-flow problem

The explicitness of (6.16), which gives $\bar{u}_t$ without inversion of a partial differential operator, is a direct consequence of the mean-flow configuration being "tall" by assumption ($\mathbf{fL}/\mathbf{NH} \ll 1$). It is noteworthy in itself that the waves do not, to the first two orders in μ, force a mean meridional circulation $\bar{v}$, $\bar{w}$, extending over a broader latitudinal scale $\mathbf{L}_1 \gg \mathbf{L}$ such that $\mathbf{fL}_1/\mathbf{NH}=1$. The reason lies in the particular mathematical form of the forcing, in that the dominant forcing term $-(\overline{v'\theta'})_y$ on the right of (3.1d) has the form of a y derivative.

We note in passing that the "residual" mean vertical velocity $\bar{w}^*$ is not the same as the Lagrangian-mean velocity $\bar{w}^L$, which vanishes to leading order in μ, but not generally to the next order [although it does vanish to two orders in cases like that studied by Uryu (1974)]. As shown in Andrews and McIntyre (1977) the Lagrangian-mean meridional circulation forced by the waves is given by

$$\bar{w}^L=(\overline{\eta'\zeta'})_{yt}-(\overline{\eta' Q'}/\bar{\theta}_z)_y+O(\bar{w}^*), \tag{6.21}$$

$$\bar{v}^L=\tfrac{1}{2}(\overline{\eta'^2})_{yt}+(\overline{\eta' Q'}/\bar{\theta}_z)_z+O(\bar{v}^*). \tag{6.22}$$

Note that $\bar{v}^L$ does not vanish even to *leading* order. [This means that $\partial\bar{v}^L/\partial y \gg \partial\bar{w}^L/\partial z$, but no paradox is involved since the waves force a divergence of the Lagrangian-mean velocity at leading order; see also McIntyre (1973, p. 810).]

In (6.16), $S_{(xz)}$ could be called an "effective" stress or momentum flux, purely as a reminder of the fact that it approximately governs the integrated acceleration $\langle\bar{u}_t\rangle$. The quantity

$$S^{\text{heuristic}} \equiv \overline{u'w'} - f\overline{v'\theta'}/\bar{\theta}_z, \qquad (6.23)$$

sometimes quoted in the literature, is relevant only when $\bar{u}_y = 0$.

d. *Simplification of the generalized* EP *relation (5.5a)*

It remains to show that the right-hand side of (5.5a) has typical magnitude $\mathbf{A}$, and to find which terms should be neglected when (5.5a) is used together with (6.16).

Under assumptions (6.1–6.6) it may first be verified that no term in (5.5a) has typical magnitude greater than $\mathbf{A}$, with two exceptions in the quasi-geostrophic case only, namely, the terms $\frac{1}{2}\mathcal{A}(\overline{\eta'^2})_{yt}$ and $-f(\overline{\xi_x'\eta'})_t$. These are separately larger than $\mathbf{A}$ if $\mathbf{f} \gg \bar{u}_y$. But in that case their sum is dominated by

$$-\tfrac{1}{2}f(\overline{\eta'^2})_{yt} - f(\overline{\xi_x'\eta'})_t = -f[\overline{\eta'(\xi_x'+\eta_y')}]_t = f(\overline{\eta'\zeta_z'})_t, \qquad (6.24)$$

which $\lesssim \mathbf{A}$ because of (6.5b), (6.10), and the fact that $\zeta' \lesssim (\mathbf{U}/\mathbf{fL})\boldsymbol{\eta}'\mathbf{H}/\mathbf{L}$ as was indicated below (6.7). The term $\overline{\zeta_x'q'} = -\overline{\zeta' q_x'} \lesssim \mathbf{A}$ in all cases, by (6.11), (6.5a) and (6.6b); the other terms are most easily estimated by comparing them with (6.10) or (6.11), except that $\overline{\eta_x'Y'}$ and $(\overline{\eta_x'v'})_t$ are best compared with $\overline{\xi_x'X'}$ and $(\overline{\xi_x'u'})_t$.

To obtain the most useful simplification of (5.5a) we restrict the time scale $\boldsymbol{\tau}$ for wave excitation, dissipation and transience as follows:

$$\boldsymbol{\tau}^{-1} \lesssim \mu D_t. \qquad (6.25)$$

Then the expression (5.6) is $O(\mu^2\mathbf{A})$ since, invoking (5.4) and (6.9), $D_t q' = Q' \sim \theta'/\boldsymbol{\tau} \lesssim \mu D_t\theta'$, so that

$$q' \lesssim \mu\theta'.$$

Then (5.5a) simplifies to

$$\begin{aligned}\frac{\partial}{\partial y}S_{(xy)} + \frac{\partial}{\partial z}S_{(xz)} = {} & \underset{*}{(\overline{\eta'X'})_y} + (\overline{\zeta'X'})_z + \underset{*}{\overline{\xi_x'X'}} + \underset{*}{\overline{\eta_x'Y'}} + \underset{*}{\overline{\zeta_x'q'}} \\ & - \mathcal{B}(\overline{\eta'Q'}/\bar{\theta}_z)_y + \mathcal{A}(\overline{\eta'Q'}/\bar{\theta}_z)_z \\ & + \frac{\partial}{\partial t}\{\underset{*}{(\overline{\eta'u'})_y} + (\overline{\zeta'u'})_z + \underset{*}{\overline{\xi_x'(u^l - f\eta')}} + \underset{*}{\overline{\eta_x'v'}} \\ & - [\mathcal{B}\overline{\eta'\theta'}/\bar{\theta}_z - \tfrac{1}{2}\mathcal{A}\overline{\eta'^2}]_y\} + O(\mu^2\mathbf{A}) + O(a^4). \end{aligned} \qquad (6.26)$$

The terms marked with asterisks dominate and the rest are $O(\mu\mathbf{A})$.

In the quasi-geostrophic case the terms $(\overline{\zeta'X'})_z$, $(\overline{\zeta'u'})_{zt}$, as well as all the terms in $\mathcal{B}$, are not only $O(\mu\mathbf{A})$ but are in fact $O(\mu^2\mathbf{A})$, and may be neglected.

e. *Further remarks*

Our earlier assertion that $(\overline{u'v'})_y$ is "ill-conditioned" may now be verified. Eqs. (6.25), (6.10) and (5.2a) imply that

$$\mathbf{u}'\mathbf{v}'/\mathbf{L} \gtrsim \mu^{-1}\mathbf{A}. \qquad (6.27)$$

Each term in (6.16) other than the contribution $(\overline{u'v'})_y$ to $S_{(xy)}$ is $O(\mathbf{A})$, and so since $\mathbf{A}$ is the typical magnitude of $\bar{u}_t$ it follows that

$$(\overline{u'v'})_y \lesssim \mu\mathbf{u}'\mathbf{v}'/\mathbf{L}, \qquad (6.28)$$

under the conditions assumed in this section. (For latitudinally-trapped waves this shows that u' and v' are almost in quadrature.)

The ill-conditioned term $(\overline{u'v'})_y$ vanishes from (6.26) if the latitudinal integral, $\langle(6.26)\rangle$, is taken, assuming that $\eta' = v' = 0$ at the sides. Thus $\langle(6.26)\rangle$ can be used, in conjunction with leading-order wave solutions such as those in Section 9, to determine the variation with time and height of the amplitude of latitudinally-trapped waves. Thus to leading order $\langle(6.26)\rangle$ must when $X' = Y' = Q' = 0$ be equivalent to Whitham's wave-action conservation law; see also remark (v) in Section 7. The connection with Bretherton and Garrett's (1968) form of this law will be clarified by Eq. (7.5).

7. Explicit approximate formulas for $\bar{u}_t$

For small μ the simplified mean-flow problem (6.16) and generalized EP relation (6.26) give the mean zonal acceleration explicitly, correct to two orders in μ, when $\bar{X} = \bar{Y} = \bar{Q} = 0\rangle$:

$$\begin{aligned}\bar{u}_t = {} & -\underset{*}{(\overline{\eta'X'})_y} - (\overline{\zeta'X'})_z - \underset{*}{\overline{\xi_x'X'}} - \underset{*}{\overline{\eta_x'Y'}} - \underset{*}{\overline{\zeta_x'q'}} \\ & + \mathcal{B}\left[\frac{\overline{\eta'Q'}}{\bar{\theta}_z}\right]_y - \mathcal{A}\left[\frac{\overline{\eta'Q'}}{\bar{\theta}_z}\right]_z - \frac{\partial}{\partial t}\Bigg\{\underset{*}{(\overline{\eta'u'})_y} + (\overline{\zeta'u'})_z \\ & + \underset{*}{\overline{\xi_x'(u^l - f\eta')}} + \underset{*}{\overline{\eta_x'v'}} - \left[\mathcal{B}\frac{\overline{\eta'\theta'}}{\bar{\theta}_z} - \tfrac{1}{2}\underset{*}{\mathcal{A}\overline{\eta'^2}}\right]_y\Bigg\} \\ & + O(\mu^2\mathbf{A}) + O(a^4). \end{aligned} \qquad (7.1)$$

The terms distinguished by asterisks are of order $\mathbf{A}$, and the rest are $O(\mu\mathbf{A})$ (some being smaller still if the disturbances have the phase relationships characteristic of equatorially-trapped waves).

If we desire only the leading approximation for small μ, and restrict attention to a disturbance with horizontal phase speed c such that

$$D_t=(\bar{u}-c)\partial/\partial x+O(\tau^{-1})=(\bar{u}-c)\partial/\partial x+O(\mu D_t) \quad (7.2)$$

by (6.25), then we have

$$(\bar{u}-c)\{\xi'_x,\eta'_x,\zeta'_x,q'_x\}\approx\{u^l,v',w',Q'\}$$

and hence, since $\overline{\zeta'_x q'}=-\overline{\zeta' q'_x}$, and $-\zeta'\approx\theta'/\bar{\theta}_z$,

$$\bar{u}_t=-(\overline{\eta'X'})_y+\frac{1}{(c-\bar{u})}[\overline{u^lX'}+\overline{v'Y'}+\overline{\theta'Q'}/\bar{\theta}_z]$$

$$+\frac{\partial}{\partial t}\left\{-(\overline{\eta'u'}+\tfrac{1}{2}\mathcal{A}\overline{\eta'^2})_y+\frac{1}{(c-\bar{u})}[\overline{u^l(u^l-f\eta')}+\overline{v'^2}]\right\}$$

$$+O(\mu\mathbf{A})+O(a^4). \quad (7.3)$$

There is an alternative form, for some purposes more useful, but valid only for disturbances (such as equatorially-trapped waves) for which u' and η' may be assumed to be in phase with relative error $O(\mu)$. This differs from (7.3) only in the time-differentiated contribution, and is

$$\bar{u}_t=-(\overline{\eta'X'})_y+\frac{1}{(c-\bar{u})}\left[\overline{u^lX'}+\overline{v'Y'}+\frac{\overline{\theta'Q'}}{\bar{\theta}_z}\right]$$

$$+\frac{1}{2}\frac{\partial}{\partial t}\left\{-(\overline{\eta'u'})_y+\frac{1}{(c-\bar{u})}\left[\overline{u^lu'}+\overline{v'^2}+\frac{\overline{\theta'^2}}{\bar{\theta}_z}\right]\right\}$$

$$+O(\mu\mathbf{A})+O(a^4). \quad (7.4)$$

This is the relation quoted in the Introduction. The equality of the right-hand sides of (7.3) and (7.4) under the stated conditions implies a generalization of the classical "equipartition" law for disturbance energy:

$$\overline{\theta'^2}/\bar{\theta}_z=\overline{u^{l2}}+\overline{v'^2}+\overline{\eta'u^l}(\bar{u}_y-2f)$$

$$+(\bar{u}-c)[\overline{\eta'u'}+\mathcal{A}\overline{\eta'^2}]_y+O(\mu\mathbf{u}'^2)+O(a^4) \quad (7.5)$$

for latitudinally-trapped modes (see also end of Appendix A). This is exact when $\bar{u}_z=0$ and the waves are linear, steady and conservative, and would be useful for checking numerical calculations of latitudinal eigenstructures. For $\bar{u}_y=0$, (7.5) has been verified directly from the solutions given in Section 9, and also for midlatitude Rossby waves.

An elementary derivation of (7.4) is given in Appendix A. In (7.3)–(7.5) we may take

$$u^l\approx u'+\eta'\bar{u}_y. \quad (7.6)$$

The following additional points can be made:

(i) It is remarkable that the simple form $\overline{\theta'Q'}$ in (7.3) and (7.4)—precisely equivalent to that found by Fels and Lindzen (1974) for the case of two-dimensional gravity waves [cf. Dickinson 1969, Eq. (13); Plumb 1975, Eq. (18)]—gives not only $\langle\bar{u}_t\rangle$ but also the detailed latitudinal "waveguide structure" of the contribution to $\bar{u}_t$ due to Q', to leading order.[5]

(ii) In Eq. (7.1) the negative of the first two terms,

$$(\overline{\eta'X'})_y+(\overline{\zeta'X'})_z=\overline{\xi'X'_x}+\overline{\eta'X'_y}+\overline{\zeta'X'_z},$$

comprises a "Stokes correction" $\bar{X}^L-\bar{X}$, where $\overline{(\ )}^L$ denotes a Lagrangian mean (approximately the mean following a fluid particle). As a particle is displaced to the north or south of the equator, for example, and if X' is an odd function of y, sgnX' may change in synchronism, contributing systematically to the zonal acceleration of the particle. Similar contributions arise from east-west and vertical displacements. Terms like $\frac{1}{2}\overline{\eta'^2}\bar{X}_{yy}$ are absent since $\bar{X}$ has been set to zero.

Corresponding terms (Andrews and McIntyre, 1977) are present in $\bar{u}^L$ and contribute to the time-differentiated part of (7.1), along with a number of other effects which will not be gone into here.

(iii) Similarly, the terms in $\mathcal{A}$ and $\mathcal{B}$ in the first line of (7.1) can be regarded as due to a mean meridional circulation $\bar{w}^Q=-(\overline{\eta'Q'}/\bar{\theta}_z)_y$, $\bar{v}^Q=(\overline{\eta'Q'}/\bar{\theta}_z)_z$ induced by the leading approximation to the Stokes correction to $\bar{Q}$; see (6.21) and (6.22).

(iv) The quantity $u^l(\mathbf{x},t)$ is approximately the Lagrangian zonal disturbance velocity for the particle associated with the point $\mathbf{x}$.

(v) As the last three remarks suggest, a Lagrangian-mean description is a more natural framework for deriving our results. The required generalization of the work of Bretherton (1969a) and Dewar (1970) is given in our paper. Bretherton (1977) has derived a similar generalized framework independently, and has shown that results like (5.5a) stem fundamentally from Whitham's wave-action law, in the generalized form proposed by Hayes (1970). Hayes' wave-action contains a contribution equal to twice the disturbance kinetic energy, and no available potential energy term [cf. Hayes 1970, p. 205; Bretherton 1977; Eq. (7.5) above. For a full discussion see Andrews and McIntyre (1977)].

(vi) When a critical line $\bar{u}(y,z)=c$ is present it might be thought that (7.3) and (7.4) should contain delta-function contributions independent of X', Y' and Q', as in Dickinson [1969, Eq. (13)]. But the situation

[5] It follows that there is one special case in which $\bar{u}_t$ does have the same y profile as $S^{\text{heuristic}}$ to leading order, namely when Q' and θ' have the same y profiles, $\bar{u}_y=0$ and (except for Kelvin waves) $X'=Y'=\partial'[\quad]/\partial t=0$. The requirement concerning Q' is evidently met for Newtonian cooling. When $\bar{u}_y=0$, it can be shown to be met by the linearized Blake-Lindzen (1973) model for Q' which, in the presence of vertical gradients of ozone, etc., in the basic flow, implies that Q' is given by a complex linear combination of θ' and w' with frequency-dependent but y independent coefficients and associated phase shifts. Further, when $\bar{u}_y=0$ it can be shown with the aid of (4.1c, d), (6.23), (A9) and (A11b) that $\overline{\theta'^2}$, $\overline{w'^2}$ and $S^{\text{heuristic}}$ have the same y profiles.

appears essentially different for vertical propagation at the equator, under the approximations of Section 6; a reason for this is given in Appendix B.

(vii) Note finally that the formula corresponding to (7.4) for the latitudinal integral $\langle\bar{u}_t\rangle$ is

$$\langle\bar{u}_t\rangle=\langle(c-\bar{u})^{-1}\{\overline{(u'+\eta'\bar{u}_y)X'}+\overline{v'Y'}+\overline{\theta'Q'}/\bar{\theta}_z\}\rangle$$
$$+\frac{1}{2}\frac{\partial}{\partial t}\langle(c-\bar{u})^{-1}\{\overline{(u'+\eta'\bar{u}_y)u'}+\overline{v'^2}+\overline{\theta'^2}/\bar{\theta}_z\}\rangle$$
$$+O(\mu\mathrm{AL})+O(a^4). \quad (7.7)$$

8. Profiles of $\bar{u}_t$ as qualitative indicators of forcing or dissipation mechanisms for equatorial waves

Eq. (7.4), when applicable, is particularly useful for qualitative theoretical reasoning. As an example, suppose that $\bar{u}$ has the symmetry

$$\bar{u}(y)=\bar{u}(-y). \quad (8.1)$$

Then if the waves are antisymmetric, like Rossby-gravity waves,[6] entailing vanishing of $\overline{\theta'Q'}$ at the equator $y=0$, Eq. (7.4) immediately suggests that the shape of the y profile of $\bar{u}_t$ depends qualitatively on the relative importance of Q', on the one hand, versus any of X', Y' and wave transience $\partial[\;]/\partial t$ on the other. The latter all make contributions to $\bar{u}_t$ which are not generally zero at the equator.[7]

Thus for free, steady, thermally dissipating Rossby-gravity waves, $\bar{u}_t$ is zero to leading order at the equator, and a double-peaked profile is the simplest possibility, illustrated for instance by the curve $\lambda=0$ in Fig. 1 below. The sign is usually pinned down by Eq. (7.7), which for instance constrains $\langle\bar{u}_t\rangle$ to be westward if $\overline{Q'\theta'}$ is positive, as for dissipating waves, and if c is westward relative to $\bar{u}$ for all y, at the height z of interest.

Similarly, the twin eastward maxima of $\bar{u}_t$ found by Hayashi (1970, Fig. 15) in the wave-CISK problem for Rossby-gravity waves can be attributed via (7.4) to a contribution from $\overline{\theta'Q'}$. Here Q' represents wave excitation by latent heat release. Hayashi also found a substantial westward acceleration near the equator, and this is explained by the fact that $\partial[\;]/\partial t>0$: there is no dissipation (*op. cit.*, p. 154), but the whole disturbance is growing exponentially. Eq. (7.4) shows immediately that Hayashi would, for instance, have obtained hardly any acceleration at the equator had he considered a steady, upward-propagating disturbance dissipated entirely by Newtonian cooling.

We may anticipate, moreover, that *the dependence of the $\bar{u}_t$ profile on X' will be extremely sensitive in the case of "long" Rossby-gravity or inertio-gravity waves.* The reason is that the Stokes correction term $-(\overline{\eta'X'})_y$ then becomes large compared to the term in $(c-\bar{u})^{-1}$, as will become clearer when a dimensionless description is introduced in the next section. It will be seen that this state of affairs is always approached when the wave is sufficiently near a critical level.

It is emphasized that (7.1), (7.3) and (7.4) assume no particular forms for X', Y' and Q'; it is sufficient that their magnitudes satisfy Eqs. (6.9) and (6.25). They may even be *nonlinear*, for instance through contributions of a "half-wave-rectified" character (which could of course also contribute directly to $\bar{u}_t$, especially through $\bar{X}$). Examples of such weak but nonlinear effects might include 1) realistic models of latent-heat contributions to Q' (CISK), 2) contributions to X', Y' and Q' associated with sporadic wave-induced patches of turbulence, and 3) photochemically controlled heating processes involving temperature-sensitive reaction rates (Blake and Lindzen 1973) such that there is significant nonlinearity in the dependence of θ' on Q' and w' even when the remaining terms can be linearized to give Eqs. (4.1). For further discussion of possible mechanisms equivalent to dissipative contributions to X', Y' and Q', see Section 10.

9. Explicit calculations for a simple model

In this section we take

$$X',\, Y',\, Q'=\mu\alpha(\lambda u',\lambda v',\theta'), \quad (9.1)$$

where α and λ are $O(1)$ and independent of y. If α and λ are positive this represents Rayleigh friction and Newtonian cooling, with rate coefficients in the ratio λ. With this model the $\bar{u}_t$ profile for a transient, conservative wave is the same as that for a steady, dissipating wave with $\lambda=1$, from (7.4).

To the scaling assumptions of Section 6 we add an assumption of weak horizontal shear:

$$\bar{u}_y\lesssim\mu\mathrm{U/L}\ll\mathrm{U/L}. \quad (9.2)$$

The standard wave solutions for zero vertical and horizontal shear can then be used as the leading approximation for $\mu\ll1$. When combined with (9.1) and an equation such as ⟨(6.26)⟩ for the height-dependence

[6] If (8.1) holds, the wave solutions of (4.1) can without loss of generality be taken as having the symmetry properties either of Kelvin waves or of Rossby-gravity waves. We follow the usage of tidal theory and call such solutions symmetric and antisymmetric, respectively. Symmetric means that the y dependence of v', η' is odd and of u', w', p', θ' even, while antisymmetric means v', η' even and u', w', p', θ' odd. We shall also assume, unless stated otherwise, that X', Y' and Q' have the symmetry of u', v' and θ'.

[7] The possibility of more than one $\bar{u}_t$ profile is connected with the fact [see (6.27)] that the leading approximation to $\overline{u'v'}$ is not just a function of the leading-order wave solutions but depends also on the $O(\mu)$ corrections, such as those derived in the companion paper. Neglect of this fact appear to be the reason why Fels and Lindzen (1974) mistakenly concluded that antisymmetric modes *always* give $\bar{u}_t=0$ at the equator, and why Lindzen and Tsay (1975) state that "none of the individual modes produce a meridional flux of momentum." [Note that $\overline{u'v'}\neq0$ even for Kelvin waves in models satisfying the special conditions mentioned in footnote 5; see Eqs. (4.2) and (4.3) of the companion paper.]

of wave amplitude, use of these solutions is equivalent in the steady-waves case to the WKB analysis of Lindzen (1971, 1972). Lindzen's illuminating comparisons of such solutions with observational data, and with numerical solutions in which fewer terms were neglected, suggest that the approximations involved may be a good deal better than might have been anticipated from the observational data alone. A reason is that the vertical wavelength $2\pi\mathbf{h}$ diminishes (below observational resolution) as critical-level conditions are approached. This basic behavior tends to keep the scaling of Section 6 locally valid at each height (see Appendix B).

At each height z we thus have a local vertical wavenumber $m(z)=\mathbf{h}^{-1}$ and a length $L(z)$ which we may define such that

$$\beta L^2|m|/N=1, \tag{9.3a}$$

where $N(z)=\bar{\theta}_z^{\frac{1}{2}}$, the local buoyancy frequency. For the purpose of making the equations dimensionless it is convenient to scale the time by

$$T(z)=L|m|/N=(|m|/\beta N)^{\frac{1}{2}}. \tag{9.3b}$$

To leading order for $\mu\ll 1$, the linearized equations—with $x, y, z, t, u', v', w', p', \theta'$ made dimensionless with respect to $L, L, |m|^{-1}, T, L/T, L/T, |m|^{-1}/T, L^2/T^2, NL/T$—are then identical to (4.1) with β and $\bar{\theta}_z$ each replaced by unity, and $\bar{u}_y, \bar{u}_z, \bar{\theta}_y, X', Y', Q'$ ignored. The resulting equations are satisfied by the wave solutions

$$\{u',v',\ldots\}=\mathrm{Re}[\{\hat{u},\hat{v},\ldots\}\exp(ikx-isz-i\omega t)], \tag{9.4}$$

where $s=\pm 1$ and the dimensionless zonal wavenumber k, intrinsic frequency ω and phase speed $(c-\bar{u})$ are defined by the correspondence

$$\partial/\partial x \leftrightarrow +ik, \tag{9.5a}$$

$$D_t \leftrightarrow -i\omega=-ik(c-\bar{u}), \tag{9.5b}$$

where $s=\mathrm{sgn}\omega$ corresponds to an upward group velocity. Observe that k will vary slowly with height if m and therefore L does so, and in fact will tend to zero as a critical level is approached (Appendix B) because L then decreases and the dimensional wavenumber $L^{-1}k$ is a constant.

The y dependences of $\hat{u}$, $\hat{v}$, $\hat{\eta}$ and $\hat{\theta}$ for all the possible wave modes may be summarized as follows:

Kelvin wave

$$\{\hat{u},\hat{v},\hat{\eta},\hat{\theta}\}=a\{i,0,0,s\}e^{-\frac{1}{2}y^2}, \tag{9.6a}$$

where the dispersion relation is

$$\omega=k. \tag{9.6b}$$

All other modes

$$\{\hat{u},\hat{v},\hat{\eta},\hat{\theta}\}=a\{iB_{n+}(y),\ H_n(y),\ i\omega^{-1}H_n(y),\ -sB_{n-}(y)\}e^{-\frac{1}{2}y^2}, \tag{9.7a}$$

where

$$B_{n\pm}(y)\equiv\pm\tfrac{1}{2}(\omega-k)^{-1}H_{n+1}(y)+n(\omega+k)^{-1}H_{n-1}(y)$$

and $H_n(y)$ are the Hermite polynomials $H_{-1}=0$, $H_0=1$, $H_1=2y$, $H_2=4y^2-2$, etc. For the $n=0$ mode,

$$\omega-\frac{1}{\omega}=k. \tag{9.7b}$$

We shall refer to this as the Rossby-gravity mode, at least for westward phase speeds $\omega/k<0$ (Holton 1975, p. 68). For $n\geqslant 1$,

$$\omega^2-k/\omega=k^2+2n+1, \tag{9.7c}$$

where for each k the two largest $|\omega|$ correspond to inertio-gravity modes, and the remaining, third root to the Rossby mode, in the usual terminology.

We now present some y profiles of $\bar{u}_t$ for some of these modes. In selecting examples for graphical presentation we shall focus attention mainly on parameter values corresponding to waves which Lindzen and Tsay (1975) consider important in the descending easterlies for the period April–June 1958.

For the Kelvin mode the $\bar{u}_t$ profile implied by Eq. (7.4) is independent of λ, and is the same for transient, conservative waves as for steady, dissipating waves:

$$\bar{u}_t \propto e^{-y^2}. \tag{9.8}$$

But it should be noted that more realistic dissipation models (Section 10) could imply a different y dependence, such as would arise from taking $\alpha=\alpha(y)$.

For the Rossby-gravity mode, profiles for steady, dissipating waves are given in Fig. 1 for a small negative value of k, and for values of λ ranging from 0 to ∞. The curve for $\lambda=1$ also represents the profile for a transient, conservative wave ($X'=Y'=Q'=0$), as is obvious from (7.4) and (9.1).

In view of the usual modeling assumption which tentatively supposes that $\lambda=0$, i.e., that Q' is the principal cause of wave dissipation, the apparent sensitivity of the $\bar{u}_t$ profile to the value of λ when λ is small is of interest. As mentioned earlier, this sensitivity is to be expected, for small k, from a consideration of the magnitudes of the first and subsequent terms in (7.4). As $k\rightarrow 0$, ω and all the functions in (9.7a) remain finite, but the factor $(c-\bar{u})^{-1}\rightarrow 0$ in the dimensionless form of (7.4). The same is evidently true of all inertio-gravity modes, since they all have ω finite at $k=0$. For the $n=0$ mode, the $\bar{u}_t$ profile turns out to be sensitive to λ at large negative k also, because the v' terms then dominate in (7.4). These results are summarized by the solid curve in Fig. 2 which gives as a function of k, for $\omega>0$, the critical value of λ separating one-peaked from two-peaked profiles for Rossby-gravity waves. The critical value of λ is small both for small and large negative $k\,\mathrm{sgn}\omega$.

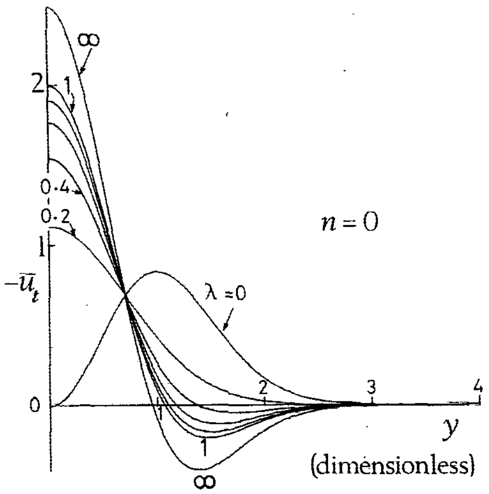

FIG. 1. Profiles of zonal mean westward acceleration $-\bar{u}_t$ for a Rossby-gravity mode ($n=0$), dissipating in accordance with the simple model (9.1) on a basic state of no horizontal shear. The curves are labeled with values of λ, the ratio of mechanical to thermal dissipation rate coefficients, and are normalized to unit area. The profile for a transient, conservative wave is the same as that for $\lambda=1$; the profile of $S^{\text{heuristic}}$ [see (6.23)] is the same as that for $\lambda=0$.

Dimensionless wave parameters are $\omega k=-3/(2\times 4.4)=-0.341$ ($k\approx -0.4$, $\omega\approx 0.8$), which when $\bar{u}=0$ correspond to the wave (of zonal wavenumber 3 and period 4.4 days) considered by Lindzen and Tsay (1975). In their case $\bar{u}\approx 0$ at the equator at a height of about 22 km at which height the scale L for dimensionless distance y from the equator is about 8° latitude, according to their Fig. 6.

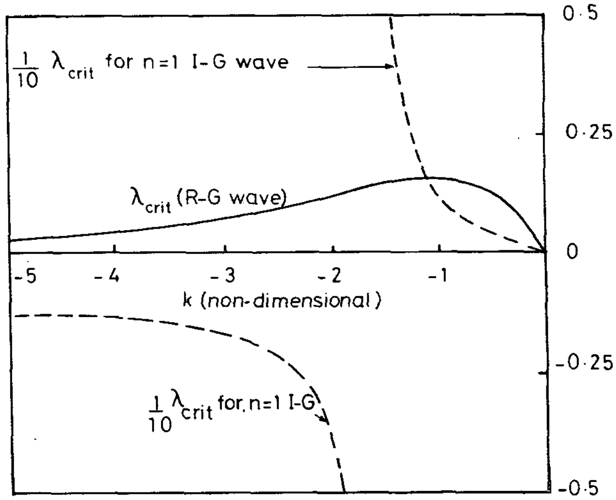

FIG. 2. Critical value of λ, the ratio of mechanical to thermal dissipation rate coefficients, for the Rossby-gravity ($n=0$) and inertio-gravity ($n=1$) modes, as a function of dimensionless zonal wavenumber $k=L/$(radian wavelength). The critical value is defined as the value of λ for which $\partial^2\bar{u}_t/\partial y^2=0$ at the equator $y=0$. At $k=0$, $-d\lambda_{crit}/dk$ equals $\frac{1}{3}$ for Rossby-gravity waves, and equals ω^{-1} ($=0.58$) for inertio-gravity waves.

Fig. 3 gives profiles for a westward-propagating, inertio-gravity mode with $n=1$ and small k. There is again a critical value of λ separating one-peaked from two-peaked profiles, given by the dashed curve in Fig. 2. For inertio-gravity waves the critical value of λ is small and positive only for small negative k sgnω.

The actual parameter values in the examples so far have been chosen to correspond when $\bar{u}=0$ to the waves postulated by Lindzen and Tsay (1975). In the vertical profiles of $\bar{u}$ observed at the equator in their case study, $\bar{u}=0$ at heights around 22 km (see Fig. 5, middle right-hand panel). Above that height, critical-level conditions are approached, so that k diminishes and the sensitivity of the $\bar{u}_t$ profiles to λ is *greater* than indicated in Figs. 1 and 3.

Of course when k is very small some of the higher order terms in (7.1), neglected here, could become relatively important—especially with a dissipation model (e.g., complex α) in which Q' is not approximately in quadrature with η'.

For the Rossby modes there is no reason to expect arbitrarily great sensitivity to λ for small k, but the examples in Fig. 4 (all for small k, again corresponding to conditions near a critical level, as for the 25 mb curves in Lindzen and Tsay's Fig. 10) show considerable variation with λ. This is especially pronounced for $n>1$, when $\overline{\theta'^2}$ has two widely separated peaks near the regions of maximum divergence, which for high-order Rossby modes tend to be concentrated near the critical latitudes.

In the problem studied by Uryu (1974) of transient, conservative, quasi-geostrophic Rossby waves propagating vertically in a mid-latitude channel, on a basic

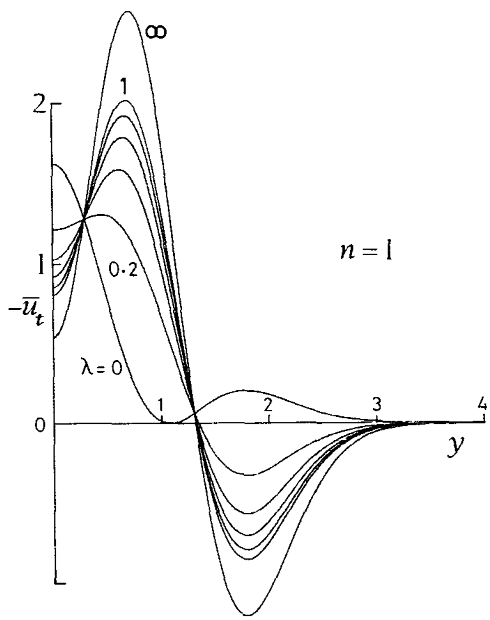

FIG. 3. As in Fig. 1 for an ($n=1$) inertio-gravity wave with $\omega k=-4/(2\times 5)=-0.4$ ($k\approx -0.2$, $\omega\approx 1.7$), which corresponds to the wave (of zonal wavenumber 4 and period 5 days) in Lindzen and Tsay's Fig. 12 when $\bar{u}=0$. (For this wave $L\approx 4°$ latitude at a height of 22 km.)

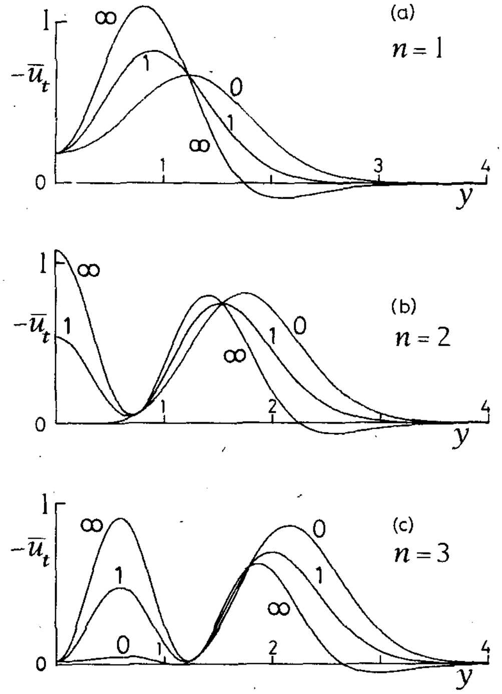

FIG. 4. As in Fig. 1 for Rossby waves in the long-wave limit $k=0-$, $\omega/k=-1/(2n+1)$ for $n=1$ (a), $n=2$ (b), $n=3$ (c). Profiles for $-1<k<0$ (not shown) are found to be qualitatively similar. For $n=1$ the displayed profiles are appropriate to conditions at the 25 mb (25.5 km) level in Lindzen and Tsay's Fig. 10, where k is small because of proximity to a critical level.

flow with no shear, we have v' and $\theta' \propto \sin ly$ for suitable l, and $u' \propto \cos ly$. Since $u'_x+v'_y \approx 0$ in that quasi-geostrophic problem, it can be shown that the sum of the terms involving u' in (7.4) is proportional to $\sin^2 ly$, if $X' \propto u'$. Thus if we add Rayleigh friction and Newtonian cooling to Uryu's problem we find $\bar{u}_t \propto \sin^2 ly$ for all λ.

10. Possible wave dissipation mechanisms in real fluids

Even if the radiative-photochemical contribution to Q' is the most important wave dissipation mechanism in the equatorial stratosphere, the results of Section 9 illustrate how additional, small contributions to wave dissipation from other mechanisms can drastically change $\bar{u}_t$ profiles, especially for Rossby-gravity and inertio-gravity waves near critical levels. The proportion of mechanical to thermal dissipation required to go from a two-peaked to a one-peaked $\bar{u}_t$ profile in Fig. 1, for instance, is far smaller than could be deduced from independent estimates of Q' plus observations of the variation of wave amplitude with height. The latter observable is too sensitive to other factors [especially the dependence of $(c-\bar{u})$ on z] to furnish useful estimates for even a total dissipation rate coefficient in practice (Lindzen, 1972, Section 6; Lindzen and Tsay, 1975, p. 2013).

There are several known ways in which the waves could feel an effective dissipation independent of radiative contributions, particularly by becoming unstable to smaller scales of motion. (Viewed mathematically, all such mechanisms are just ways in which nonlinear terms like $v'u'_y$ can have a systematic effect as contributions to X', Y' and Q'.)

(i) Resonant-interactive instability, due to resonant-triad interaction with pairs of other free modes initially of smaller amplitude (Davis and Acrivos, 1967; Hasselmann, 1967).

(ii) Kelvin-Helmholtz instability or local static instability.

(iii) Three-dimensional, large-scale instabilities such as quasi-barotropic instability associated primarily with the horizontal variation of the disturbance velocity fields.

(iv) Interaction with preexisting disturbance motions not directly caused by the equatorial wave motions. For instance these other motions could be the result of barotropic instability of the $\bar{u}$ profile itself.

Mechanism (i) has been extensively studied both theoretically and experimentally for the case of internal gravity waves (e.g., Bretherton, 1969b; McEwan, 1971; Martin *et al.*, 1972; Olbers, 1976), and can immediately be expected to be a possibility at least for Kelvin waves. This is because of the close dynamical correspondence between Kelvin waves and two-dimensional internal gravity waves, expressed for instance by their having almost the same dispersion relation. In either case the dispersion relation allows plenty of suitable resonant triads, even among the hydrostatic waves ($\mathbf{h} \ll \mathbf{L}$) considered here. Triads involving two antisymmetric and one symmetric wave exist also (R. Griffith-Jones, personal communication).

Mechanism (ii) may well be important for the strongest observed Kelvin waves, as has been argued in some detail by Kousky and Koermer (1974). Lindzen and Tsay (1975, p. 2014) suggest that it could be important for the observed Rossby-gravity waves near critical levels.

Mechanisms (iii) are not contained in Kousky and Koermer's two-dimensional model and have not been studied in detail. The scale analysis of Section 6 implies that plausible heuristic criteria such as $u_{yy}' \sim \beta$ are indistinguishable in terms of order of magnitude from criteria for mechanism (ii) and it is a delicate problem to determine which of (ii) and (iii) will be important first in given circumstances.

Two further points are worth emphasis. First, while it is convenient for heuristic purposes to speak of instability mechanisms with various suggestive names, the distinctions between them may not be sharp in reality. For instance, Gill (1974) traces a continuous transition, as wave amplitude increases, between mechanisms (i) and (iii) for barotropic flow on a beta-plane. Second, while the several mechanisms may be distinct at small wave amplitude, their mutual interaction may be important. The discussion by Bretherton (1969b) and the experimental results of McEwan (1971, 1973) on various instabilities of internal gravity waves show how mechanism (i), by transferring wave-action into smaller scales, can enormously enhance the effectiveness of (ii) and, in our case, (iii). Thus (ii) and (iii) can be effective, sporadically, even when the most prominent waves have small amplitude.

11. Remarks concerning the descending easterlies

a. Barotropic instability of the mean zonal flow

Figs. 1, 3 and 4 suggest that the y profile of $\bar{u}_t$ can be highly inflected. The profile of $\bar{u}$ itself would presumably never become strongly inflected, because of barotropic instability. For instance, simple numerical estimates for Rossby-gravity waves in the descending easterlies show that, for $\bar{u}$ profiles shaped like the $\lambda=0$ curve in Fig. 1, $\beta-\bar{u}_{yy}$ becomes two-signed as soon as differential velocities in the horizontal reach values of several meters per second.

Our guess is that such instabilities would tend to occur intermittently. With $\bar{u}_t$ values of order 5 m s^{-1} per month, for instance, it would take a time presumably of the order of a month or more for an unstable profile to redevelop after being smeared out by an instability.

b. The deficit found by Lindzen and Tsay for April–June 1958

The discussion of mean zonal accelerations in Section 4 of Lindzen and Tsay (1975, denoted hereafter by LT), requires some reassessment in the light of our results. But it is important to note that their main conclusion, that the "observed" waves cannot account for the mean zonal acceleration at all heights, in their case study, stands unaffected. LT postulate a marginally-observable, westward-propagating, $n=1$ inertio-gravity mode which could make up most of the shortfall in $\bar{u}_t$ at the equator. There is an even greater shortfall away from the equator in the layer 20–23 km, which cannot be accounted for by this inertio-gravity wave even with the possible y profile variations suggested by our Fig. 3. For reasons now to be explained we regard this discrepancy as more significant than do LT, and believe that it indicates the probable importance of Rossby modes ($n\geqslant 1$) with periods longer than 6 days.

Inspection of our Figs. 1, 3 and 4 together with LT Figs. 10 and 11 (which as already noted are correct if X', Y' and $\partial\overline{[\ ]}/\partial t$ are all very close to zero, with decreasing tolerance as critical levels are approached) strongly suggests that the discrepancy cannot be resolved simply by invoking additional dissipation mechanisms. These could redistribute the momentum tendency $\rho_0\bar{u}_t$ horizontally and vertically, but would leave a similar shortfall somewhere else in the meridional section. In other words, the total mean momentum change would remain unaccounted for. The shortfall appears even more striking on inspection of the $\bar{u}(y,z)$ sections in Fig. 5 (taken from Newell *et al.*, 1974), for March, May and July 1958, which suggest a latitudinal scale of nearly 40° for the slab, lying between about 20°N and 20°S, and between 20 and 23 km altitude, whose mean momentum change should be accounted for.

It should be noticed that our theoretical results, with their implications concerning the role of $\overline{u'v'}$, make it clear that the equator cannot be considered dynamically isolated from neighboring latitudes, insofar as planetary waves are involved. Thus deficits at and away from the equator call for discussion on an equal footing. LT tentatively suggest that a mean meridional circulation term $f\bar{v}$ could help make up the shortfall away from the equator. This is possible, but plausible only if a suitable zonally symmetric forcing $\bar{X}$, $\bar{Y}$, or $\bar{Q}$ ***independent of the waves*** is available to drive such a circulation. The remaining part of the mean meridional circulation, i.e., the $\bar{v}$ appearing in our analysis and forced mainly by the term $\overline{v'\theta'}$ in Eq. (3.1d), cannot be invoked to explain accelerations additional to that given by the formulas of Section 7. That contribution to $\bar{v}$ is intrinsic to the disturbance dynamics, just as much so as $\overline{u'v'}$; indeed, their effects upon $\bar{u}_t$ are not separately identifiable in our alternative, generalized Lagrangian-mean description (Andrews and McIntyre, 1977). But a sufficiently strong zonally symmetric forcing independent of the waves and fluctuating quasi-biennially does not seem plausible;[8] the reasons are already familiar from the well-known failure of early attempts to explain the QBO purely in terms of such forcing. Thus we are led to expect that large-scale, zonally asymmetric disturbances additional to those considered by LT are important for the momentum budget of the descending easterlies.

The most plausible candidates are Rossby modes with periods longer than those of 4–6 days singled out for special consideration by LT. Rossby modes automatically select westward phase speeds $(c-\bar{u})$ and hence, by (7.3), a tendency for $\bar{u}_t$ to be westward. The broad latitude scale for the descending easterlies in Fig. 5 suggests, perhaps, that modes higher than $n=1$ are significant. It may of course be quite wrong to think of such modes as incident from below: departures from latitudinal normal-mode structure associated with a preponderance of southward or of northward propagation are likely. Indeed the possible importance of vertically and horizontally propagating Rossby waves was the subject of an earlier suggestion by Dickinson (1968, p. 1001) based upon unpublished observational evidence due to J. M. Wallace.

[8] We note, however, the *caveat* implied by the arguments of Newell *et al.* (1974, pp 255–262) for the lowest part of the stratosphere.

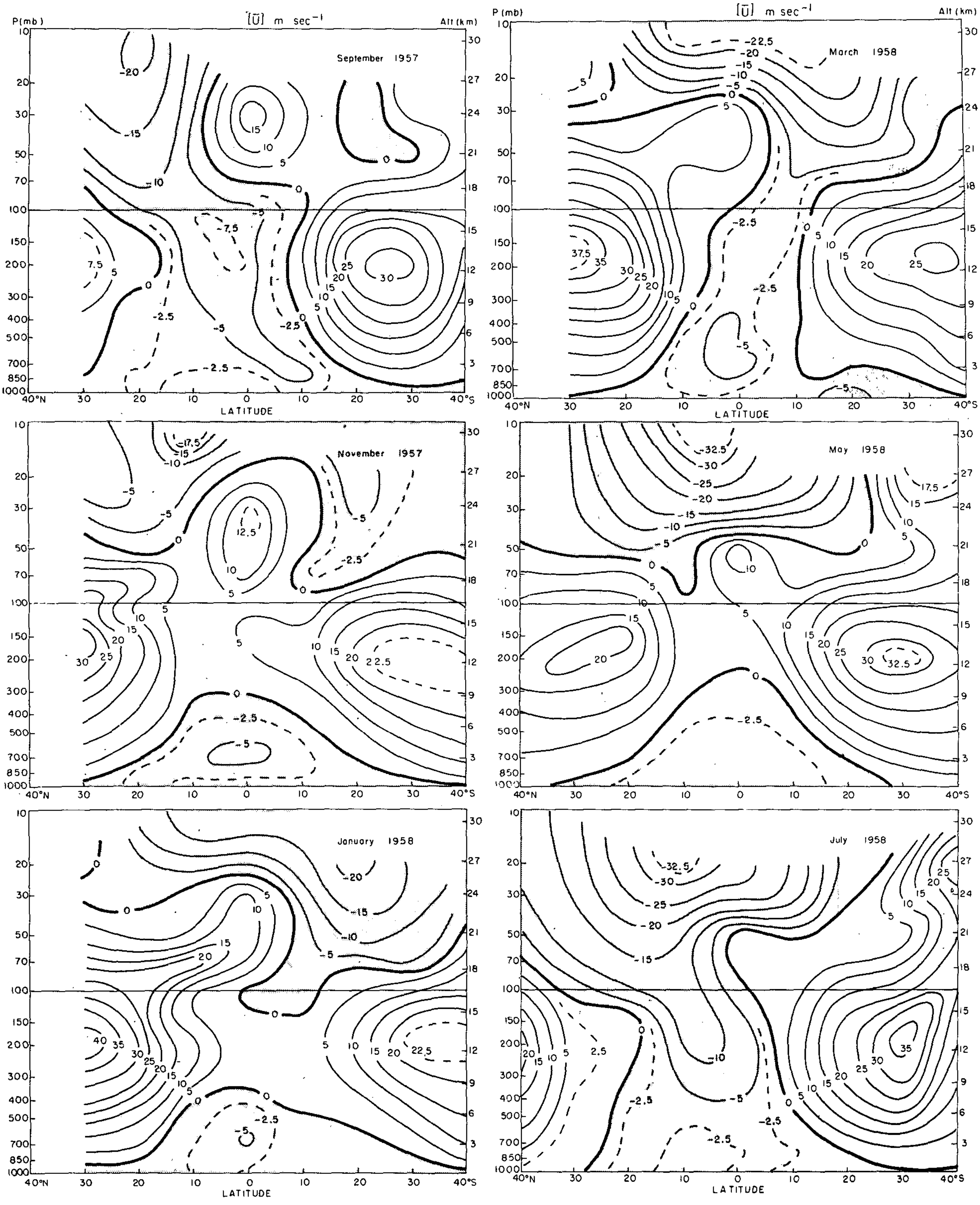

FIG. 5. Meridional sections of the zonally-averaged monthly mean zonal wind $\bar{u}(y,z)$ given at two-month intervals for the period September 1957 to July 1958 (from Newell *et al.*, 1974, Fig. 10.24). Easterlies are shown shaded. The middle right-hand panel corresponds to the time of Lindzen and Tsay's case study.

It might be questioned whether the presence of such Rossby waves is consistent with the observed continuum of low-frequency spectral power. However, the most important waves might well be nearly stationary ($c \simeq 0$) and not detectable in time series at fixed stations. We observe that $c=0$ corresponds to a critical line (the zero isotach in the middle right-hand panel of Fig. 5) which lies just where LT's Fig. 11 indicates the greatest deficit in the momentum budget.

Acknowledgments. We are grateful to R. Griffith-Jones for communicating his preliminary results on resonant triad interactions, and to the following for helpful conversations or correspondence: R. E. Dickinson, R. A. Ebdon, J. R. Holton, R. S. Lindzen, T. Matsuno, J. Pedlosky and A. J. Simmons. F. P. Bretherton kindly showed us a first draft of his independent work on the fundamental conservation relations underlying our results. Fig. 5 is reproduced by permission of The MIT Press, and was kindly supplied by R. E. Newell. C. Chalk, J. Smith and N. F. Spicer gave invaluable assistance with the preparation of the manuscript. D.G.A. thanks the Natural Environment Research Council and Trinity College, Cambridge, for financial support.

APPENDIX A

Derivation of Eqs. (5.5a) and (7.4)

In this Appendix we drop the primes from symbols representing disturbance fields. The following identities hold for any two fields $\phi(x,t)$ and $\psi(x,t)$:

$$\overline{\phi\psi_x}=-\overline{\phi_x\psi}, \tag{A1}$$

$$\overline{\phi D_t\psi}=-\overline{\psi D_t\phi}+(\overline{\phi\psi})_t, \tag{A2}$$

$$\overline{\phi D_t\phi}=\tfrac{1}{2}(\overline{\phi^2})_t. \tag{A3}$$

Thus from

$$v=D_t\eta,\quad w=D_t\zeta, \tag{A4}$$

we deduce that

$$\overline{\eta D_t u}=-\overline{uv}+(\overline{\eta u})_t, \tag{A5}$$

$$\overline{\zeta D_t u}=-\overline{uw}+(\overline{\zeta u})_t. \tag{A6}$$

Further,

$$\overline{\eta v}=\tfrac{1}{2}(\overline{\eta^2})_t, \tag{A7}$$

$$\overline{\zeta w}=\tfrac{1}{2}(\overline{\zeta^2})_t, \tag{A8}$$

$$\overline{v\zeta}=-\overline{\eta w}+(\overline{\eta\zeta})_t. \tag{A9}$$

We now rewrite Eq. (4.1a) as

$$-D_t u-p_x-\mathcal{A}v-\mathcal{B}w=X. \tag{A10}$$

Multiplication of (A10) by η, and then by ζ [Bretherton 1969a, Eq. (18); Uryu, 1973], followed by use of (A1) and (A5)–(A9) yields

$$\overline{uv}+\overline{\eta_x p}-\mathcal{B}\overline{\eta w}=\overline{\eta X}+\tfrac{1}{2}\mathcal{A}(\overline{\eta^2})_t+(\overline{\eta u})_t, \tag{A11a}$$

$$\overline{uw}+\overline{\zeta_x p}+\mathcal{A}\overline{\eta w}=\overline{\zeta X}+\tfrac{1}{2}\mathcal{B}(\overline{\zeta^2})_t+(\overline{\zeta u})_t+\mathcal{A}(\overline{\eta\zeta})_t. \tag{A11b}$$

Viewed physically, these serve to relate the disturbance momentum fluxes as defined by (3.1a), namely $\overline{uv}$ and $\overline{uw}$, to the momentum fluxes

$$-\overline{\eta_x p}\quad\text{and}\quad -\overline{\zeta_x p} \tag{A12}$$

which appear in Lagrangian-mean analogues of (3.1a) and which represent mean tangential forces across *material* surfaces corrugated by the waves (see Bretherton 1969a, Fig. 1).

Multiplying (4.1d) by η gives

$$-\overline{v\theta}+(\overline{\eta\theta})_t+\tfrac{1}{2}\bar{\theta}_y(\overline{\eta^2})_t+\bar{\theta}_z\overline{\eta w}=-\overline{\eta Q}. \tag{A13}$$

This allows (A11) to be rewritten as

$$S_{(xy)}+\overline{\eta_x p}=\overline{\eta X}-\mathcal{B}\left[\frac{\overline{\eta Q}+(\overline{\eta\theta})_t+\frac{1}{2}\bar{\theta}_y(\overline{\eta^2})_t}{\bar{\theta}_z}\right]+\tfrac{1}{2}\mathcal{A}(\overline{\eta^2})_t+(\overline{\eta u})_t, \tag{A14a}$$

$$S_{(xz)}+\overline{\zeta_x p}=\overline{\zeta X}+\mathcal{A}\left[\frac{\overline{\eta Q}+(\overline{\eta\theta})_t+\frac{1}{2}\bar{\theta}_y(\overline{\eta^2})_t}{\bar{\theta}_z}+(\overline{\eta\zeta})_t\right]+\tfrac{1}{2}\mathcal{B}(\overline{\zeta^2})_t+(\overline{\zeta u})_t. \tag{A14b}$$

Now

$$(\overline{\eta_x p})_y+(\overline{\zeta_x p})_z=(\overline{\xi_x p})_x+(\overline{\eta_x p})_y+(\overline{\zeta_x p})_z,$$
$$=\overline{\xi_x p_x}+\overline{\eta_x p_y}+\overline{\zeta_x p_z},$$

since $\xi_x+\eta_y+\zeta_z=0$. Taking the scalar product of (ξ_x,η_x,ζ_x) with (4.1a–c) and using (5.3b) and the fact that $D_t u=D_t u^l-(\mathcal{A}+f)v-\mathcal{B}w$, we obtain

$$(\overline{\eta_x p})_y+(\overline{\zeta_x p})_z=-\overline{\xi_x X}-\overline{\eta_x Y}-\overline{\zeta_x q}-\frac{\partial'}{\partial t}[\overline{\xi_x(u^l-f\eta)}+\overline{\eta_x v}]-(\bar{\theta}_y+f\bar{u}_z)\overline{\zeta_x\eta}. \tag{A15}$$

Taking $\partial(\text{A14a})/\partial y+\partial(\text{A14b})/\partial z$ and eliminating the terms in p by means of (A15) now gives (5.5a).

Eq. (7.4) results from a different, approximate, treatment of the pressure terms in (A14), making use of the wave-energy equation as in EP and in Eliassen (1968). Under assumption (7.2) we may write each disturbance field in the form

$$\phi(x-ct,\,y,\,z,\,T),$$

where $\mathcal{T}=\mu t$, a "slow" time. Then

$$D_t=\mathcal{U}\partial/\partial x+\mu\partial/\partial\mathcal{T}, \qquad (A16)$$

where $\mathcal{U}\equiv\bar{u}-c$. The relation [cf. EP Eqs. (10.5) and (10.7)] between the wave-energy flux and the "radiation stress" contributions (A12) is simply

$$\overline{pv}-\mathcal{U}\overline{p\eta_x}=\mu\overline{p\eta_{\mathcal{T}}}, \qquad (A17a)$$

$$\overline{pw}-\mathcal{U}\overline{p\zeta_x}=\mu\overline{p\zeta_{\mathcal{T}}}, \qquad (A17b)$$

an immediate consequence of (A4). We now add $\partial(\text{A17a})/\partial y+\partial(\text{A17b})/\partial z$ to

$$\partial[\mathcal{U}\times(\text{A14a})]/\partial y+\partial[\mathcal{U}\times(\text{A14b})]/\partial z;$$

the left-hand side of the resulting equation is

$$\mathcal{U}\left(\frac{\partial S_{(xy)}}{\partial y}+\frac{\partial S_{(xz)}}{\partial z}\right)+[(\overline{pv})_y+\cdots]+O(a^4), \qquad (A18a)$$

where the expression within square brackets denotes the square brackets in the wave-energy equation (4.2), and where (6.12) has been used. The right-hand side may be simplified, by order-of-magnitude considerations which parallel those used in arriving at (6.26), to

$$[\mu\overline{p\eta_{\mathcal{T}}}+\mathcal{U}\{\overline{\eta X}+\tfrac{1}{2}\mu\mathcal{A}(\overline{\eta^2})_{\mathcal{T}}+\mu(\overline{\eta u})_{\mathcal{T}}\}]_y$$
$$+O(\mu\mathcal{U}\mathbf{A}). \qquad (A18b)$$

In (A18b) p is needed only to leading order. We may therefore use the relation

$$p=-\mathcal{U}(u+\mathcal{A}\eta)+O(\mu\mathbf{U}\mathbf{u})+\text{func}(y,z,t) \qquad (A19)$$

which follows as in Eliassen [1968, Eq. (3.7)] from writing (A10) as

$$p_x=-D_t(u+\mathcal{A}\eta+\mathcal{B}\zeta)-X$$

and invoking (A16), (6.7), (6.9) and (6.25). Substituting (A19) into the first term of (A18b), and (4.2) into (A18a), and equating (A18a, b) we get

$$\mathcal{U}\left(\frac{\partial S_{(xy)}}{\partial y}+\frac{\partial S_{(xz)}}{\partial z}\right)$$
$$=\overline{uX}+\overline{vY}+\overline{\theta Q}/\bar{\theta}_z+\tfrac{1}{2}\mu\frac{\partial}{\partial\mathcal{T}}(\overline{u^2}+\overline{v^2}+\overline{\theta^2}/\bar{\theta}_z)$$
$$+\{\mathcal{U}(\overline{\eta X}+\mu\overline{u_{\mathcal{T}}\eta})\}_y+O(\mu\mathcal{U}\mathbf{A})+O(a^4). \qquad (A20)$$

If

$$\mu\overline{u_{\mathcal{T}}\eta}\approx\mu\overline{\eta_{\mathcal{T}}u} \qquad (A21)$$

to leading order, then both are equal to $\frac{1}{2}\mu\overline{(\eta u)_{\mathcal{T}}}=\frac{1}{2}\overline{(\eta u)_t}$, and if $\mathcal{U}\neq 0$ (7.4) follows, upon noting (7.6) and (6.16).

Eq. (A21) is true if, for instance, each Fourier component $\hat{u}e^{ik(x-ct)}$ is in phase with the corresponding component $\hat{\eta}e^{ik(x-ct)}$, to leading order. Under (7.2), (A16) and (6.9) we may write $\hat{u}\approx a(\mathcal{T})\hat{u}_0(y,z)$, $\hat{\eta}\approx a(\mathcal{T})\hat{\eta}_0(y,z)$, where $\hat{u}_0$ and $\hat{\eta}_0$ are constrained by Eqs. (4.1) to have the structure of a conservative, linear wave and, by the above phase assumption, are such that $\hat{u}_0\hat{\eta}_0^*$ is real. Then (A21) follows since (for each Fourier component)

$$\overline{u_{\mathcal{T}}\eta}=\tfrac{1}{2}\,\text{Re}(a_{\mathcal{T}}\hat{u}_0a^*\hat{\eta}_0^*), \qquad (A22a)$$

$$\overline{\eta_{\mathcal{T}}u}=\tfrac{1}{2}\,\text{Re}(a_{\mathcal{T}}\hat{\eta}_0a^*\hat{u}_0^*). \qquad (A22b)$$

Eq. (A21) would be true without the phase assumption, i.e., for complex $\hat{u}_0\hat{\eta}_0^*$, if it were assumed alternatively that $a(\mathcal{T})$ were real. This amounts to assuming a special form of transience in which $u_{\mathcal{T}}$ is strictly in phase with u, and $\eta_{\mathcal{T}}$ with η. Boyd (1976) makes this assumption—in fact he assumes that $a\propto\exp(bt)$ where b is a real constant. This is the reason why his terms describing transience take the same form as his terms in X, Y and Q, even for μ not small [contrast with our result (5.9a)].

APPENDIX B

Critical Lines

In this Appendix we shall show that for equatorial waves the scaling of Section 6, and hence Lindzen's WKB approximation, remains uniformly valid up to a critical line. In saying this we are presuming a configuration for the critical-line isotach in a meridional section such that it lies well outside the region enclosed by the critical latitudes $k[c-\bar{u}(y,z)]=\pm f(y)$, meeting that region only at its apex P, i.e., its highest point, where $\bar{u}-c=f=0$ on the equator.

Thus the ray time (group-velocity time) to the point P is physically meaningful. Moreover, the ray time is infinite; consequently the wave amplitude goes to zero at P (apart from terms exponentially small in μ) whenever any dissipation is present, no matter how small. The interaction of linear waves with the mean flow must therefore be consistently described by (7.1) without any extra critical-line terms such as seem to be required in the quasi-geostrophic problem studied by Dickinson (1969).

There have been no investigations of the corresponding nonlinear critical-layer problems, but the various finite-amplitude critical-layer effects which have been studied for simpler systems (e.g., Breeding, 1971; Maslowe, 1971; Geisler and Dickinson, 1974; Murakami, 1974) suggest that critical-layer absorption has no more effect, and sometimes much less, on $\bar{u}_t$ than is indicated by linear theory.

The crucial scaling requirement is

$$\mu\equiv\mathbf{h}/\mathbf{H}=(|m|\Delta z)^{-1}\ll 1 \qquad (B1)$$

uniformly as $\mathbf{H}\to 0$; we have identified $\mathbf{H}$ with the vertical distance Δz to the point P, and $\mathbf{h}^{-1}$ with the vertical wavenumber $|m|$. Now all the dispersion relations of interest have a power law behavior for large $|m|$, as can be seen by converting (9.6b) and (9.7b, c) back into dimensional form:

$$(\bar{u}-c)\sim|m|^{-\sigma}, \quad |m|\rightarrow\infty. \qquad \text{(B2)}$$

For Rossby-gravity and equatorial inertio-gravity waves, $\sigma=\frac{1}{2}$. For Kelvin, equatorial Rossby, and two-dimensional internal gravity waves, $\sigma=1$. (The different values correspond to whether or not the relevant dispersion curve meets the origin of the usual dimensionless $k\omega$ diagram.) Near P; $\bar{u}-c\sim\bar{u}_z\Delta z$, and so $|m|\Delta z$ is large as $\Delta z\rightarrow 0$ if either (i) $0<\sigma<1$, or (ii) $\sigma=1$ and $|m|\Delta z\gg 1$ away from P. In case (i), appropriate to Rossby-gravity waves, $|m|\Delta z\rightarrow\infty$. In case (ii), appropriate to Kelvin and equatorial Rossby waves, $|m|\Delta z$ will be large near P if it is already large away from P, as was assumed in Section 6. The different behavior of m for the two cases is well illustrated by Fig. 3 of Lindzen (1971).

The mid-latitude, quasi-geostrophic Rossby waves considered by Dickinson correspond to $\sigma=2$, which violates (i) and (ii). So Dickinson's problem is essentially different from the equatorial problem, in that a WKB approximation cannot be valid near a critical line.

The fact that the ray time is infinite is just a consequence of (B2) when $\sigma>0$. The ray time is defined as $\int dz/\gamma$, where γ is the vertical group velocity, $\propto\partial(\bar{u}-c)/\partial m$, and the integration is taken up to the point P:

$$\int^{\mathrm{P}}\frac{dz}{\gamma}\propto\int^{\mathrm{P}}\left|\frac{\bar{u}_z^{-1}d(\bar{u}-c)}{\partial(\bar{u}-c)/\partial m}\right|\propto\int^{\mathrm{P}}\left|\frac{dm}{\bar{u}_z}\right|=\infty, \qquad \text{(B3)}$$

since $|m|=\infty$ at P when $\sigma>0$.

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