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BY

A. H. BOOZER

PLASMA PHYSICS LABORATORY

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

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Plasma Equilibrium with Rational Magnetic Surfaces

Allen H. Boozer

Princeton Plasma Physics Laboratory

Princeton University

Princeton, New Jersey 08544

Abstract

The self consistent classical plasma equilibrium with diffusion is studied in a toroidal geometry having a sheared magnetic field. Near each rational surface it is found that the pressure gradient is zero unless the Fourier component of $1/B^2$, which resonates with that surface, vanishes. Despite the resonances, the overall plasma confinement is, in practice, only slightly modified by the rational surfaces.

I. Introduction

Magnetic surfaces in which the rotation transform, $\iota\!\!\!-$, is a rational number have long been known to be associated with singularities in the plasma equilibrium. An especially clear discussion of this problem has been given by H. Grad.[1] On surfaces with irrational $\iota\!\!\!-$, the magnetic field lines ergodically cover the entire surface and the pressure is constant on the surface. On rational surfaces, the magnetic field lines close on themselves. The condition for plasma equilibrium with closed field lines is that the integral $\oint d\ell/B$ be constant on a pressure surface. In systems in which $\iota\!\!\!-$ depends on radius, the pressure surfaces defined by the irrational surfaces do not have $\oint d\ell/B$ constant on rational surfaces except for special cases such as toroidal symmetry.

In this paper we will examine the problem of self-consistent plasma equilibrium in a toroidal system with shear. We find that the plasma equilibrium is controlled by the Fourier transform of $1/B^2$ in the appropriate toroidal and poloidal angles ϕ and θ. Let δ_{nm} be proportional to the n^{th} toroidal harmonic and the m^{th} poloidal harmonic of $1/B^2$, then near a rational surface $\iota\!\!\!-(\psi_R) = n/m$

$$(j_\parallel)_P \propto \frac{\delta_{nm}}{(\psi - \psi_R)} \frac{dP}{d\psi} \cos\,(n\phi - m\theta)$$

$$\Gamma \propto -\,\eta_\parallel \frac{|\delta_{nm}|^2}{(\psi - \psi_R)^2} \frac{dP}{d\psi}$$

$$\Phi - \Phi_o(\psi) \propto \eta_\parallel \frac{\delta_{nm}}{(\psi - \psi_R)^2} \frac{dP}{d\psi} \sin\,(n\phi - m\theta)$$

with $(j_\parallel)_P$ the pressure driven part of the parallel current, P the pressure, Γ the total flux of particles crossing a surface, $\eta_\parallel$ the parallel resistivity, and $\Phi - \Phi_o(\psi)$ the variation of the electrostatic potential in the surface. The smoothness of the particle flux implies $dP/d\psi \propto |\delta_{nm}|^2 (\psi - \psi_R)^2$ near a rational surface. Unless $|\delta_{nm}|$ vanishes, which is equivalent to $\oint d\ell/B$ being constant on the rational surface, $dP/d\psi \propto (\psi - \psi_R)^2$. Assuming $|\delta_{nm}| \neq 0$ we then find $(j_\parallel)_P \propto (\psi - \psi_R)$, that is, it vanishes at $\psi = \psi_R$ while the potential variation remains finite. Since $(j_\parallel)_P$ vanishes everywhere as the plasma pressure goes to zero, any vacuum field configuration with magnetic surfaces and shear gives a plasma equilibrium at low enough plasma pressure.

Other papers which are related to the work reported here are by Kruskal and Kulsrud,[2] Hamada,[3] Greene and Johnson,[4] and Grad.[5]

In Section II of the paper the appropriate coordinate system will be established, in Section III the equation for the parallel plasma current is derived and solved, in Section IV the consequences of Ohm's law are explored, and the conclusions are given in Section V.

II. Magnetic Coordinates

Solenoidal vectors such as the magnetic field can always be written in the so-called Clebsch representation

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta_o \tag{1}$$

with a field line defined by constant ψ and θ_o. Since we are assuming a scaler pressure with

$$\vec{\nabla} P = \frac{1}{c} \vec{j} \times \vec{B} , \tag{2}$$

the Clebsch coordinate ψ can be chosen as a function of P alone. The systems we are considering have topologically toroidal pressure surfaces so the function $\psi(P)$ can be choosen with $2\pi\psi$ equal to the magnetic flux inside a pressure surface (i.e., the toroidal flux). This choice of ψ makes θ_o angle-like.

In addition to the Clebsch or contravariant representation, a magnetic field with a scaler pressure can be written in the covariant form[6]

$$\vec{B} = \vec{\nabla}\chi + \beta\vec{\nabla}\psi \tag{3}$$

with ψ, θ_o, χ as coordinates. An important role is played by the arbitrariness in the ψ, θ_o, χ, β representation of $\vec{B}$. Since ψ is defined, this arbitrariness occurs only in θ_o, χ, and β. It is easy to show that if θ_o, χ, and β represent $\vec{B}$ then $\bar{\theta}_o$, $\bar{\chi}$, and $\bar{\beta}$ give a representation if, and only if,

$$\bar{\theta}_o = \theta_o + \theta_*(\psi), \ \bar{\chi} = \chi + \chi_*(\psi), \ \bar{\beta} = \beta - \frac{d\chi_*}{d\psi} . \tag{4}$$

The functions θ_* and χ_* are arbitrary functions of ψ.

Although many fundamental properties of the plasma equations are easily illustrated using θ_o and χ as coordinates, they do obscure the toroidal and the poloidal periodicities of the torus. Angular coordinates θ and ϕ linearly related to θ_o and χ make this periodicity manifest. Suppose we circuit the torus once toroidally and come back to the same physical point In general χ

and θ_o will not return to their original values $\chi(o)$ and $\theta_o(o)$. Rather after a toroidal circuit

$$\chi = \chi(o) + 2\pi g \ , \quad \theta_o = \theta_o(o) - 2\pi \iota \ . \tag{5}$$

Both $\chi(o)$, $\theta_o(o)$, and χ, θ_o are representations of the field at the same physical location so g and ι must be functions of ψ alone. In one poloidal circuit

$$\chi = \chi(o) + 2\pi I \ , \quad \theta_o = \theta_o(o) + 2\pi\sigma \ . \tag{6}$$

Again I and σ must be functions of ψ alone. The function σ will be shown to equal the number one. The periodicities can be simply given by defining the poloidal angle θ and the toroidal angle ϕ so that

$$\theta_o = \sigma\theta - \iota\,\phi, \ \chi = g\phi + I\theta \ . \tag{7}$$

The coordinates of the paper will be ψ, θ, ϕ .

To show σ is unity, remember that $2\pi\psi$ equals the mangetic flux inside a pressure surface or

$$2\pi\psi = \int \vec{B} \cdot d\vec{S}_t \ . \tag{9}$$

The element of surface area in ψ, θ, ϕ coordinates is

$$d\vec{S}_t = \frac{\nabla\phi}{\vec{\nabla}\phi \cdot (\vec{\nabla}\psi\times\vec{\nabla}\theta)} \, d\theta \, d\psi \ . \tag{10}$$

Using Eqs. (1) and (7), $\vec{B} = \sigma\vec{\nabla}\psi\times\vec{\nabla}\theta + \iota\,\vec{\nabla}\phi\times\vec{\nabla}\psi$, which implies

$$\psi = \int \sigma \, d\psi \ . \tag{11}$$

In ψ, θ, ϕ coordinates the contravariant form of the magnetic field is then

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta + \chi(\psi)\vec{\nabla}\phi \times \vec{\nabla}\psi \tag{12}$$

while the covariant form is

$$\vec{B} = g(\psi)\vec{\nabla}\phi + I(\psi)\vec{\nabla}\theta + \beta_* \ \vec{\nabla}\psi \ , \ \beta_* = \beta + (\frac{dg}{d\psi}\ \phi + \frac{dI}{d\psi}\ \theta) \ . \tag{13}$$

The total toroidal current inside a flux surface is

$$\int \vec{J} \cdot d\vec{S}_T = \frac{c}{4\pi} \int \vec{B} \ \ d\vec{\ell}_p = \frac{c}{4\pi} \int \vec{B} \cdot \frac{\vec{\nabla}\phi \times \vec{\nabla}\psi}{\vec{\nabla}\phi \ \cdot \ (\vec{\nabla}\psi \times \vec{\nabla}\theta)} \ d\theta = \frac{c}{2} \ I \ . \tag{14}$$

The total poloidal current outside a flux surface can similarly be shown to be $cg(\psi)/2$.

III. The Current Density

The covariant representation of $\vec{B}$, Eq. (13), gives a simple expression for the current density

$$\vec{J} = \frac{c}{4\pi} \ \vec{\nabla}\times\vec{B} = \frac{c}{4\pi} \ [(\vec{\nabla}\phi\times\vec{\nabla}\psi) \ \frac{\partial\beta}{\partial\phi} - (\vec{\nabla}\psi\times\vec{\nabla}\theta) \ \frac{\partial\beta}{\partial\theta} \ ] \ . \tag{15}$$

The cross product of this expression with the contravariant representation of $\vec{B}$, Eq. (12), gives an equation for β

$$\vec{\nabla}P = \frac{1}{c}\,\vec{j} \times \vec{B} = \frac{1}{4\pi}\,\vec{\nabla}\phi \cdot (\vec{\nabla}\psi \times \vec{\nabla}\theta)\,\left(\frac{\partial\beta}{\partial\phi} + \iota\,\frac{\partial\beta}{\partial\theta}\right)\,\vec{\nabla}\psi\ . \tag{16}$$

The inverse of the ψ, θ, ϕ Jacobian can be found by dotting together the covariant, Eq. (13), and contravariant, Eq. (12), representations of $\vec{B}$

$$\vec{\nabla}\phi \cdot (\vec{\nabla}\psi \times \vec{\nabla}\theta) = \frac{B^2}{g + \iota I}\ . \tag{17}$$

The equation for β is then

$$\frac{\partial\beta}{\partial\phi} + \iota\,\frac{\partial\beta}{\partial\theta} = \frac{4\pi}{B^2}\,(g + \iota\, I)\,\frac{dP}{d\psi}\ . \tag{18}$$

The parallel component of $\vec{j}$ can be found by dotting the covariant expression for $\vec{B}$, Eq. (13), into $\vec{j}$, Eq. (15), to obtain

$$\frac{4\pi}{c}\,\frac{j_\parallel}{B} = \frac{1}{g + \iota\, I}\,\left(I\,\frac{\partial\beta}{\partial\phi} - g\,\frac{\partial\beta}{\partial\theta}\right)\ . \tag{19}$$

Let us now solve the equations for β and $j_\parallel/B$. The function β need not be periodic in θ and ϕ, however, $j_\parallel/B$ must be. One finds a homogeneous solution for β.

$$\beta_H = \mu_H(\psi)(\iota\phi - \theta) \tag{20}$$

with μ_H an arbitrary function of ψ. To this solution the inhomogeneous solution to Eq. (18) must be added. To find the inhomogeneous solution let

$$\frac{1}{B^2} = \frac{1}{B_o^2}\,[(1 + \sum_{n,m}{}' \delta_{nm} \cos\,(n\phi - m\theta + \lambda_{nm})]\tag{21}$$

with the prime on the sum implying that the term n = o, m = o is eliminated. That is, we assume the field strength in each magnetic surface is known and that it can be appropriately expanded in a Fourier series. Then one finds

$$\beta = \mu_H(\psi)\ (\iota\phi - \theta) + \frac{4\pi}{B_o^2}\,(g + \iota I)\,\frac{dP}{d\psi}\,\phi + \beta_* \tag{22}$$

with β_*, which we will presently show is the β_* of Eq. (13), equal to

$$\beta_* = \frac{4\pi}{B_o^2}\,\frac{dP}{d\psi}\,\sum_{n,m}{}' \frac{g + \iota I}{n - \iota m}\,\delta_{nm}\,\sin\,(n\phi - m\theta + \lambda_{nm})\ . \tag{23}$$

The equations are simpler if the force-free current part of β is singled out by defining

$$\mu = \mu_H + \frac{4\pi}{B_o^2}\,\frac{dP}{d\psi}\,I \tag{24}$$

then

$$\beta = \mu(\psi)\ (\iota\phi - \theta) + \frac{4\pi}{B_o^2}\,\frac{dP}{d\psi}\,(g\phi + I\theta) + \beta_*\ . \tag{25}$$

The parallel current is given by Eq. (19)

$$\frac{4\pi}{c}\,\frac{j_\parallel}{B} = \mu(\psi) + \frac{4\pi}{B_o^2}\,\frac{dP}{d\psi}\,\sum_{n,m}{}' \frac{nI + mg}{n - \iota m}\,\delta_{nm}\,\cos\,(n\phi - m\theta + \lambda_{nm})\ . \tag{26}$$

The first term on the right side of this equation represents the force-free current and the second term the Pfirsch-Schlüter current. The poloidal and toroidal currents can be evaluated using Eqs. (15) and (25).

$$\frac{dI}{d\psi} = \mu - \left(\frac{4\pi}{B_o^2}\frac{dP}{d\psi}\right) I \tag{27}$$

$$\frac{dq}{d\psi} = - \kappa\mu - \left(\frac{4\pi}{B_o^2}\frac{dP}{d\psi}\right) q \ . \tag{28}$$

Consequently β of Eq. (25) can be rewritten as

$$\beta = - \frac{dq}{d\psi}\,\phi - \frac{dI}{d\psi}\,\vartheta + \beta_* \tag{29}$$

which identifies the β_* of Eq. (23) with that of Eq. (13).

It is of interest to note that in force-free magnetic fields that the plasma is minimum average B stable if B_o increases away from the magnetic by the V^{**} criterion of Johnson and Greene.[7]

IV. Consequences of Ohm's Law

The Ohm's Law of magnetohydrodynamics,

$$\vec{E} + \frac{1}{c}\,\vec{v} \times \vec{B} = \overleftrightarrow{\eta} \cdot \vec{j} \ , \tag{30}$$

allows us to evaluate a plasma diffusion coefficient. The electric field $\vec{E}$ is the sum of a potential, $-\vec{\nabla}\,\Phi$, and a solenoidal, $\vec{\varepsilon}$, part

$$\vec{E} = - \vec{\nabla}\,\Phi + \vec{\varepsilon} \ . \tag{31}$$

The perpendicular part of $\vec{\varepsilon}$ can be written in terms of a velocity $\vec{u}$

$$\vec{\varepsilon}_{\perp} + \frac{1}{c}\,\vec{u} \times \vec{B} = 0 \; . \tag{32}$$

The velocity $\vec{u}$ represents an overall pinching of the field and the plasma. *The Ohm's Law* can be rewritten as

$$- \vec{\nabla}\Phi + \vec{\varepsilon}_{\parallel} + \frac{1}{c}\,(\vec{v} - \vec{u}) \times \vec{B} = \overleftrightarrow{\eta} \cdot \vec{j} \; . \tag{33}$$

The parallel component of this equation gives

$$\frac{\partial \Phi}{\partial \phi} + \varkappa \frac{\partial \Phi}{\partial \theta} = (g + \varkappa I)\left(\frac{\varepsilon_{\parallel}}{B} - \eta_{\parallel} \frac{j_{\parallel}}{B}\right) \; . \tag{34}$$

This equation with the expression for $j_{\parallel}/B$, Eq. (26), implies

$$\vec{\varepsilon}_{\parallel} = \frac{\eta_{\parallel} c}{4\pi}\, \mu(\psi)\, \vec{B} \tag{35}$$

$$\Phi = \Phi_o(\psi) - \eta_{\parallel} c \,\frac{(g + \varkappa I)}{B_o^2} \frac{dP}{d\psi} {\sum_{n,m}}' \frac{nI + mg}{(n - \iota m)^2}\, \delta_{nm} \sin\,(n\phi - m\theta + \lambda_{nm}) \quad , \tag{36}$$

with Φ_o an arbitrary function of ψ.

To understand the velocity $\vec{u}$ we will consider Faraday's Law

$$\frac{\partial \vec{B}}{\partial t} = - c \vec{\nabla} \times \vec{E}$$

$$= \vec{\nabla} \times (\vec{u} \times \vec{B}) - c\, \vec{\nabla} \times \vec{\varepsilon}_{\parallel} \; . \tag{37}$$

Evaluating $\vec{u} \times \vec{B}$ using the contravariant expression for $\vec{B}$, Eq. (12), one finds

$$\frac{\partial\psi(\vec{x},t)}{\partial t} + \vec{u} \cdot \vec{\nabla}\psi = -\left(\frac{n_{\parallel}c^2}{4\pi}\mu\right) I \qquad (38)$$

$$\frac{\partial\psi_p(x,t)}{\partial t} + \vec{u} \cdot \vec{\nabla}\psi_p = \left(\frac{n_{\parallel}c^2}{4\pi}\mu\right) g \qquad (39)$$

with $d\psi_p/d\psi = ϰ$. One can easily show $2\pi\psi_p$ is the poloidal flux within an additive function of time. One can define the plasma loop voltage $V(\psi,t)$ by

$$V = \frac{\partial}{\partial\psi}\left(\frac{1}{2\pi}\int \vec{\varepsilon} \cdot \vec{B}\, d^3x\right) \qquad (40)$$

in the usual approximations of a tokamak $V \simeq 2\pi RE_\phi$. Evaluating Eq. (40) using Eq. (35) for $\varepsilon_\parallel$, one finds

$$\mu = \frac{2}{c}\frac{V}{(g + ϰ I)n_\parallel} . \qquad (41)$$

This equation plus Eqs. (38) and (39) imply

$$\frac{\partial ϰ(\psi,t)}{\partial t} = \frac{c}{2\pi}\frac{\partial V}{\partial\psi} . \qquad (42)$$

In steadystate one must clearly have the loop voltage a constant, V_o. One then has

$$\vec{u} \cdot \vec{\nabla}\psi = -\frac{c}{2\pi}\frac{I}{g + ϰI}V_o \qquad (43)$$

and the total flux of particles due to $\vec{u}$ with $\rho(\psi)$ the density is

$$\Gamma_p = \int \rho\, \vec{u} \cdot d\vec{S}_\psi$$

$$= -\, 2\pi\, c\, \frac{\rho I}{B_o^2}\, v_o \; . \tag{44}$$

This is the classical pinch effect.

The particle diffusive flux can be evaluated using Ohm's Law, Eq. (33). This equation can be solved for the perpendicular part of $\vec{v} - \vec{u}$ and hence $(\vec{v} - \vec{u})\cdot\vec{\nabla}\psi$. One finds, using Eq. (36),

$$(\vec{v} - \vec{u}) \cdot \vec{\nabla}\psi = \frac{c}{g + \varkappa I}\left(I\, \frac{\partial\Phi}{\partial\phi} - g\, \frac{\partial\Phi}{\partial\theta}\right) - \eta_\perp\, c^2\, \frac{\vec{\nabla}P}{B^2} \cdot \vec{\nabla}\psi$$

$$= -\, \frac{\eta_\parallel c^2}{B_o^2}\, \frac{dP}{d\psi} \sum_{n,m}{}' \left(\frac{nI + mg}{n - \varkappa m}\right)^2 \delta_{nm} \cos\,(n\phi - m\theta + \lambda_{nm}) - \eta_\perp c^2\, \frac{dP}{d\psi}\, \frac{|\nabla\psi|^2}{B^2} \; . \tag{45}$$

The total diffusive particle flux crossing a magnetic surface is

$$\Gamma_D = \int \rho(\vec{v} - \vec{u}) \cdot \frac{\vec{\nabla}\psi}{\vec{\nabla}\phi \cdot (\vec{\nabla}\psi \times \vec{\nabla}\theta)}\, d\theta d\phi$$

$$= (g + \varkappa I)\rho \int (\vec{v} - \vec{u}) \cdot \vec{\nabla}\,\psi\, \frac{d\theta d\phi}{B^2} \; . \tag{46}$$

The expression for the total diffusive flux can be rewritten as

$$\Gamma_D = -\,(D_\parallel + D_\perp)\, \frac{dP}{d\psi} \tag{47}$$

with

$$D_{\parallel} = 2\pi^2 \eta_{\parallel} \frac{c^2(q + \lambda I)}{B_o^4} \sum_{n,m}{}' \left(\frac{nI + mg}{n - \lambda m}\right)^2 \delta_{nm}^2 \tag{48}$$

$$D_{\perp} = \eta_{\perp} c^2 \int \frac{\vec{\nabla}\psi}{B^2} \cdot d\,\vec{S}_{\psi}\,, \quad d\,S_{\psi} = \frac{\vec{\nabla}\psi}{\vec{\nabla}\phi \cdot (\vec{\nabla}\psi \times \vec{\nabla}\theta)} d\theta d\phi\,, \tag{49}$$

that is, $d\vec{S}_{\psi}$ is the area element of a flux surface.

Let us assume the plasma is in a steady state. Then particle conservation implies

$$\vec{\nabla} \cdot n\vec{v} = s \tag{50}$$

with s the source of particles per unit volume. The total flux of particles $\Gamma = \Gamma_p + \Gamma_D$ obeys

$$\frac{d\Gamma}{d\psi} = \int s \frac{d\theta d\phi}{\vec{\nabla}\phi \cdot (\vec{\nabla}\psi \times \vec{\nabla}\theta)}$$

$$= S(\psi) \tag{51}$$

with $S(\psi)\, d\psi$ the number of particles added between two differentially separated flux surfaces. Clearly Γ must be a smoothly varying function of ψ.

V. Conclusions

In steadystate the total flux of particles across a magnetic surface is

$$\Gamma = -\, D(\psi) \frac{dP}{d\psi} + \Gamma_p \tag{52}$$

with $D(\psi) = D_\parallel + D_\perp$ given by Eqs. (48) and (49) and Γ_p given by Eq. (44). The parallel current driven differention coefficient $D_\parallel$ is singular at each rational surface. That is, near a rational surface

$$D_\parallel \propto \frac{\delta_{nm}{}^2}{(\psi - \psi_{nm})^2} \tag{53}$$

with ψ_{nm} the value of ψ when $n = \not{x}m$. The total particle flux $\Gamma(\psi)$ must be slowly varying in ψ so we find near a rational surface

$$\frac{dP}{d\psi} \propto (\psi - \psi_{nm})^2/\delta_{nm}{}^2 \; . \tag{54}$$

Consequently, unless $|\delta_{nm}| = 0$ at $\psi = \psi_{nm}$, $dP/d\psi$ vanishes at rational surfaces. The Pfirsch-Schlüter or pressure driven part of parallel current near a rational surface [see Eq. (26)] is

$$(j_\parallel)_P \propto \frac{\delta_{nm}}{\psi - \psi_{mm}} \frac{dP}{d\psi} \cos (n\phi - m\theta)$$

$$\propto (\psi - \psi_{nm}) \; . \tag{55}$$

Consequently the Pfirsch-Schlüter part of $j_\parallel$ vanishes at each rational surface rather than being singular and the pressure driven part of $j_\parallel$ goes to zero everywhere as the plasma pressure goes to zero. The electrostatic potential, interestingly, retains a finite variation on rational magnetic surfaces even though $dP/d\psi$ is zero on these surfaces.

The singularity of $D(\psi)$ at each rational surface is not as important as it might first appear. Consider a region of the plasma with no sources so $d\Gamma/d\psi = 0$. Define a smoothing function $\Delta(\psi)$ such that $\Delta(\psi)$ goes to zero for $|\psi|$ small but finite and which has a unit integral over ψ.

Then let

$$\bar{P}(\psi) \equiv \int \Delta(\psi - \psi_1)\, P(\psi_1) d\psi_1 \tag{56}$$

$$\frac{d\bar{P}}{d\psi} = - \int \frac{d\Delta}{d\psi_1}\, P(\psi_1)\, d\psi_1$$

$$= + \int \Delta \frac{dP}{d\psi_1}\, d\psi_1$$

$$= - \Gamma \int \frac{\Delta(\psi - \psi_1)}{D(\psi_1)}\, d\psi_1 \; . \tag{57}$$

Defining

$$\frac{1}{\bar{D}(\psi)} \equiv \int \frac{\Delta(\psi - \psi_1)}{D(\psi_1)}\, d\psi_1 \; , \tag{58}$$

one has

$$\Gamma = - \bar{D}(\psi) \frac{d\bar{P}}{d\psi} \; . \tag{59}$$

No matter how narrow the region over which Δ is different from zero, as long as the region is finite, the function $\bar{D}(\psi)$ is finite everywhere. This follows from the fact that the Fourier transform of a smooth function vanishes exponentially for high n or m; hence high order rational surfaces have a exponentially small effect on the integral leading to $\bar{D}$.

An interesting application of the expression for $D_\parallel$ is to derive the well-known *Pfirsch-Schlüter diffusion coefficient for a stellarator.* This is derived by assuming the field strength has the obvious form

$$\frac{1}{B^2} \simeq \frac{1}{B_o^2}\,[1 - 2\varepsilon \cos\theta - 2\delta \cos(N\phi - \ell\theta)]\ . \tag{60}$$

The only terms in δ_{nm} are $\delta_{01} = 2\varepsilon$ and $\delta_{N\ell} = 2\delta$. We assume the plasma has negligible net toroidal current, $I = 0$, and the toroidal field dominates so $g = RB_o$. The diffusion coefficient one is used to seeing $D^*_\parallel$ is $D_\parallel$ divided by the area of the magnetic surface $(2\pi r)(2\pi R)$ and also divided by $d\psi/dr = rB_o$ since the usual $D^*_\parallel$ multiplies dP/dr rather than $dP/d\psi$. Eq. (48) implies with $\varepsilon = r/R$

$$D^*_\parallel = \frac{2}{\iota^2}\,\frac{\eta_\parallel c^2}{B_o^2} \sum_{n,m}' \left(\frac{\delta_{nm}}{2\varepsilon}\right)^2 \left(\frac{\iota}{\iota - \frac{n}{m}}\right)^2\ . \tag{61}$$

The resonance $n = 0$, $m = 1$, with $\delta_{01} = 2\varepsilon$ gives a contribution

$$D^*_{01} = \frac{2}{\iota^2}\,\frac{\eta_\parallel c^2}{B_o^2}\ , \tag{62}$$

the Pfirsch-Schlüter coefficient. The resonance at $n = N$, $m = \ell$, $\delta_{N\ell} = 2\delta$ gives

$$D^*_{N\ell} = \frac{2}{\iota^2}\,\frac{\eta_\parallel c^2}{B_o^2} \left(\frac{\delta}{\varepsilon}\right)^2 \left(\frac{\iota}{\iota - N/\ell}\right)^2\ . \tag{63}$$

Customary stellarator designs have $\delta \sim \varepsilon$ but $N/\ell \gg \iota$ so the Pfirsch-Schlüter coefficient gives an accurate approximation, $D^*_{01} \gg D^*_{N\ell}$.

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