

# Plateau–Stein manifolds

Research Article

Misha Gromov<sup>1,2\*</sup>

1 Institut des Hautes Études Scientifiques, Route de Chartres 35, 91440 Bures-sur-Yvette, France

2 Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, NY 10012, USA

Received 23 February 2013; accepted 17 December 2013

**Abstract:** We study/construct (proper and non-proper) Morse functions  $f$  on complete Riemannian manifolds  $X$  such that the hypersurfaces  $f(x) = t$  for all  $-\infty < t < +\infty$  have positive mean curvatures at all non-critical points  $x \in X$  of  $f$ . We show, for instance, that if  $X$  admits no such (not necessarily proper) function, then it contains a (possibly, singular) complete (possibly, compact) minimal hypersurface of finite volume.

**MSC:** 31C12, 53B21, 53C20, 53C23, 53C40, 58B20

**Keywords:** Riemannian manifolds • Stein Manifolds • Geometric measure theory

© Versita Sp. z o.o.

## 1. Introduction

### 1.1. Mean curvature convexity

Let  $X$  be a smooth Riemannian manifold. Given a smooth function  $f$  on  $X$ , define the *mean curvatures*  $\text{mn.curv}_x(f)$  at *non-critical* points  $x \in X$ , i.e. where  $df(x) \neq 0$ , as the mean curvatures of the level hypersurfaces  $Y = Y_r = f^{-1}(r)$ , for  $r = f(x)$ ,

$$\text{mn.curv}_x(f) \stackrel{\text{def}}{=} \text{mn.curv}_x(Y_r),$$

where the mean curvatures of the hypersurfaces  $Y$  are defined by evaluation of their second fundamental forms on the normalized *downstream* gradient field  $-\text{grad} f / \|\text{grad} f\|$ . Call a function  $f$  *strictly mean curvature convex* (sometimes we say “ $(n-1)$ -mean curvature convex” instead of just “mean curvature convex” for  $n-1 = \dim Y = \dim X - 1$ ) if

$$\text{mn.curv}_x(f) \geq \varepsilon(x) > 0 \quad \text{for all } f\text{-non-critical points } x \in X, \text{ i.e. where } df(x) \neq 0,$$

for a positive continuous function  $\varepsilon$  on  $X$ .

\* E-mail: gromov@ihes.fr

**Remarks.**

- (a) We are especially concerned with strictly mean curvature convex *Morse* functions, i.e. where the critical points of  $f$  are *non-degenerate*. Even though our “convexity” definition *formally* makes sense for all smooth functions  $f$ , one has, in reality, to impose some, possibly weaker than Morse, constraints on the critical points of  $f$  – we do not want to accept, for example, constant functions.
- (b) In what follows, most our manifolds  $X$  are non-compact, where  $\varepsilon(x)$  may tend to 0 for  $x \rightarrow \infty$ . This happens, for instance, to the squared distance function in the Euclidean space  $\mathbb{R}^n$  from the origin, that is an archetypical example of a strictly mean curvature convex Morse function.
- (c) If we compose  $f: X \rightarrow \mathbb{R}$  with a smooth nowhere locally constant function  $\psi: \mathbb{R} \rightarrow \mathbb{R}$ , then  $\psi \circ f: X \rightarrow \mathbb{R}$  has, at least locally, the same levels as  $f$ . The sign of the mean curvatures of the levels is preserved if the derivative of  $\psi$  is positive,  $\psi' > 0$ ; however, it changes where  $\psi' < 0$ .

Conclude by observing that Morse properties of a function are influenced by the sign of the mean curvature of the levels via the following obvious inequality.

[ind  $\leq n - 2$ ] *The critical points of strictly mean curvature convex Morse functions have their Morse indices  $\leq n - 2$  for  $n = \dim X$ .*

**1.2. Non-proper and proper Plateau–Stein manifolds**

[n-n-proPS]<sub>-</sub> Call a possibly non-complete, Riemannian manifold  $X$  of dimension  $n \geq 2$  without boundary *Plateau–Stein* if it admits a strictly mean curvature convex Morse function  $f: X \rightarrow \mathbb{R}$ .

Sometimes, to emphasize that  $f$  is *not* assumed proper, we call these *non-proper* Plateau–Stein or [n-n-proPS]<sub>-</sub>, where “<sub>-</sub>” stands for “convex” with “ $\wedge$ ” in the next section for “concave” and where *non-proper* must be always understood as *not necessarily proper*.

Three other similar conditions on  $X$  are as follows.<sup>1</sup>

[PS](1) Given a compact subset  $B \subset X$  and  $\varepsilon > 0$ , there exist a *strictly  $(n - 1)$ -volume contracting* continuous map  $\Psi_\varepsilon: B \rightarrow X$ , that is  $\text{vol}_{n-1}(\Psi_\varepsilon(H)) < \text{vol}_{n-1}(H)$  for all smooth hypersurfaces  $H \subset X$ , such that  $\text{dist}_X(\Psi_\varepsilon(x), x) \leq \varepsilon$  for all  $x \in B$ .

(This condition does not truly need any smooth structure in  $X$ .)

[PS](2)  $X$  admits a  $C^1$ -smooth *strictly  $(n - 1)$ -volume contracting* vector field  $V$ ,  $n = \dim X$ , i.e., for every compact subset  $B \subset X$ , there exists  $\varepsilon > 0$  such that  $V$  integrates on  $B$  to a flow up to the time  $t = \varepsilon$  where the flow maps, say  $V_t: B \rightarrow X$ , are strictly  $(n - 1)$ -volume contracting on  $B$  for  $0 < t < \varepsilon$ .

[PS](3)  $X$  admits a *strictly  $(n - 1)$ -mean convex function*  $f$ , i.e., such that the *gradient field* of  $-f$  is strictly  $(n - 1)$ -volume contracting. (See subsection 3.3 for an alternative definition.)

Clearly, [PS](3)  $\Rightarrow$  [PS](2)  $\Rightarrow$  [PS](1) and also [PS](3) implies Plateau–Stein, since every strictly  $(n - 1)$ -mean convex function  $f$  admits an arbitrarily small perturbation that makes it strictly  $(n - 1)$ -mean curvature convex.

What seems non-obvious – I do not see a direct proof of this – is the following corollary to the *Inverse Maximum Principle* stated in the next section.

**Contraction Corollary for Covering.**

*Let  $X$  be an infinite covering of a compact manifold. If  $X$  is [PS](1), i.e., if it admits strictly  $(n - 1)$ -volume contracting continuous maps  $\Psi_\varepsilon: B \rightarrow X$ , for all compact subsets  $B \subset X$  and all  $\varepsilon > 0$ , with  $\text{dist}_X(\Psi_\varepsilon(x), x) \leq \varepsilon$ , then  $X$  is Plateau–Stein.*

<sup>1</sup> Bringing forth these properties was motivated by our conversations with Bruce Kleiner.

### Remark/Question.

Probably, if  $f$  is strictly mean convex, then one can arrange a smooth function  $a$  on  $X$  with a large derivative along the gradient field  $\text{grad } f$ , such that the field  $-e^{a(x)} \text{grad } f$  would be strictly mean contracting. But it is less clear what should be *exactly* the class of (non-covering) manifolds  $X$  where the existence of a strictly  $(n-1)$ -volume contracting field implies Plateau–Stein.

**[proPS]** Say that  $X$  is *proper Plateau–Stein*, if it admits a *proper positive* strictly mean convex Morse function  $f: X \rightarrow \mathbb{R}_+$ , where proper for (a not necessarily positive) function  $f$  on  $X$  means that  $x \rightarrow \infty \Rightarrow |f(x)| \rightarrow \infty$ , where “ $\rightarrow \infty$ ” means “eventually leaves every compact subset”. Sometimes we say that *the Riemannian metric on  $X$  is proper/non-proper Plateau–Stein*.

Proper Plateau–Stein manifolds are reminiscent of *complex Stein manifolds*  $X$  that, by definition, support proper positive *strictly  $\mathbb{C}$ -convex*, traditionally called *plurisubharmonic*, functions. An obvious *necessary* condition for the existence of such a (not necessarily proper) function is the absence of *compact complex submanifolds* of positive dimensions in  $X$ . A theorem by Grauert says that this condition is also *sufficient* if  $X$  can be exhausted by compact domain with strictly  $\mathbb{C}$ -convex (pseudoconvex) boundaries.

We shall prove in this paper a Riemannian counterpart to Grauert’s theorem with “suitable compact minimal hypersurfaces” (“hypersurface” always means a *codimension 1 subvariety*, possibly with singularities) instead of “compact complex submanifolds”.

The possible topologies of Plateau–Stein manifolds are rather transparent. Plateau–Stein manifolds  $X$  are *non-compact* and, if *proper*, they have *zero homology*  $H_{n-1}(X; \mathbb{Z}_2)$  for  $n = \dim X$ . In particular, they are connected at infinity. Moreover, *proper* Plateau–Stein manifolds  $X$  are diffeomorphic to regular neighborhoods of *codimension two* subpolyhedra in  $X$ . This follows from  $[\text{ind} \leq n-2]$ . For example, proper Plateau–Stein surfaces are homeomorphic to the 2-plane  $\mathbb{R}^2$  and proper Plateau–Stein 3-folds are topological handle bodies, while non-proper Plateau–Stein allows a complete (warped product) metric on the topological cylinder  $X^n = X_0^{n-1} \times \mathbb{R}$  for all  $(n-1)$ -manifolds  $X_0^{n-1}$  as a simple argument shows.

In fact one can show (we leave this to the reader) the following. *Let  $f: X \rightarrow \mathbb{R}$  be a proper, not necessarily positive, Morse function, where all critical points have indices  $\leq n-2$ . Then there exists a complete Riemannian metric on  $X$  (which is proper Plateau–Stein according to our definition if  $f$  is positive) for which this function is strictly mean curvature convex.*

Unlike the proper Plateau–Stein the non-proper Plateau–Stein condition is not topologically restrictive for open manifolds. *Every open manifold  $X$  admits a (possibly non-complete) non-proper Plateau–Stein Riemannian metric.*

In fact, a simple argument shows that *given a smooth function  $f$  without critical points on a smooth manifold  $X$ , there obviously exists a (possibly non-complete) Riemannian metric on  $X$  such that the level hypersurfaces of  $f$  are convex with respect to this function.*

Probably, every open manifold  $X$  of dimension  $n \geq 3$  admits a *complete* non-proper Plateau–Stein Riemannian metric. (Complete Plateau–Stein surfaces  $X$  are homeomorphic either to  $\mathbb{R}^2$  or to the cylinder  $S^1 \times \mathbb{R}$ , since other *non-compact complete connected* surfaces necessarily contain (non-contractible) closed geodesics that is incompatible with being (complete or not) Plateau–Stein in dimension  $n = 2$ .)

The geometry of Plateau–Stein manifolds is not as apparent as their topology.

### Examples and Questions.

(a) Let  $X = (X, g_0)$  be a complete simply connected  $n$ -manifold,  $n \geq 2$ , of non-positive sectional curvature  $\kappa(g_0) \leq 0$ . Since the spheres  $S_{x_0}(R) \subset X$  around a given point  $x_0 \in X$  are strictly convex, such  $X$  is proper Plateau–Stein and all open subsets  $U \subset X$  are Plateau–Stein.

Even though the inequality  $\kappa(g_0) \leq 0$  is unstable under smooth perturbations of  $g_0$ , the Plateau–Stein may be stable. For example, let the Ricci curvature of  $X$  be bounded from below by  $-\delta g_0$ ,  $\delta > 0$ , e.g.  $X$  is a symmetric space of non-compact type with no Euclidean factor. Then  $\text{mn.curv}(S(R)) \geq \delta > 0$  for all  $R > 0$ . Since this inequality is stable under uniformly  $C^1$ -small perturbations  $g_\epsilon$  of the original metric  $g_0$  on  $X$ , the function  $x \rightarrow \text{dist}_{g_0}^2(x, x_0)$  remains mean curvature convex with respect to  $g_\epsilon$ ; hence, these  $g_\epsilon$  are proper Plateau–Stein.

If a *non-flat* symmetric space  $(X, g_0)$  of non-compact type does have a Euclidean factor, then the perturbed metrics  $g_\varepsilon$  are, obviously, non-proper Plateau–Stein. Probably, they are proper Plateau–Stein.

On the other hand, the Euclidean metric  $g_0$  on  $\mathbb{R}^n$ ,  $n \geq 2$ , admits arbitrarily  $C^\infty$ -small perturbations  $g_\varepsilon$  that are *not* Plateau–Stein. Indeed, one can arrange  $g_\varepsilon$  such that  $g_0 - g_\varepsilon$  is supported in an annulus  $A$  pinched between two large spheres, say  $S^{n-1}(R)$  and  $S^{n-1}(R+1)$  in  $\mathbb{R}^n$ ,  $R \gg 1/\varepsilon$ , and such that  $(A, g_\varepsilon)$  is isometric to the Riemannian product  $S^{n-1}(R) \times [0, 1]$ . It is clear such  $g_\varepsilon$  is not Plateau–Stein, since the mean curvature of a smooth function  $f$  on  $X = (\mathbb{R}^n, g_\varepsilon)$  is, obviously, non-positive at the maximum point of  $f$  on  $S^{n-1}(R)$ .

(b) Let  $G_\varepsilon$  be the space of  $\varepsilon$ -small  $C^\infty$  perturbations  $g_\varepsilon$  of  $g_0$  that are *invariant under the action of  $\mathbb{Z}^n$  on  $\mathbb{R}^n$* , i.e. these  $g_\varepsilon$  correspond to perturbations of the flat metric on the torus  $\mathbb{R}^n/\mathbb{Z}^n$ . Divide the space  $G_\varepsilon$  into three classes:

(b<sub>1</sub>) proper Plateau–Stein;

(b<sub>2</sub>) Plateau–Stein but not proper Plateau–Stein;

(b<sub>3</sub>) not even non-proper Plateau–Stein.

What is the topological structure of this partition? Are all three subsets (b<sub>1</sub>), (b<sub>2</sub>), (b<sub>3</sub>)  $\subset G_\varepsilon$  dense in  $G_\varepsilon$  for small  $\varepsilon$ ? Is any of these (b<sub>1</sub>), (b<sub>2</sub>), (b<sub>3</sub>)  $\subset G_\varepsilon$  a meager set? (I am uncertain of what happens even for  $n = 2$  where the answer might be known, albeit in different terms.)

(c) Let  $p: X_1 \rightarrow X$  be a *Riemannian submersion* between Riemannian manifolds, i.e. the differential  $dp: T(X_1) \rightarrow T(X)$  everywhere has  $\text{rank}(dp) = n = \dim X$  and it is isometric on the *horizontal tangent (sub)bundle*  $T(X_1) \ominus \ker dp \subset T(X_1)$  (normal to the fibers  $p^{-1}(x) \subset X_1$ ). The simplest instance of this is the projection of a Riemannian product  $X_1 = X \times X'$  onto the  $X$  factor. Let the action of the normal holonomy (by the parallel transport along the horizontal bundle) on the fibers be *volume preserving*, e.g.  $p = X \times X' \rightarrow X$ . Then the  $p$ -pullback of hypersurfaces from  $X$  to  $X_1$  preserves their mean curvatures. Therefore, if a function  $f: X \rightarrow \mathbb{R}$  is Morse strictly mean convex, then a *generic* smooth perturbation of  $p \circ f: X_1 \rightarrow \mathbb{R}$  is also Morse as well strictly mean convex. Consequently:

(c<sub>1</sub>) If  $X$  is a Plateau–Stein then so is  $X_1$ .

(c<sub>2</sub>) If  $X$  is a proper Plateau–Stein and the fibers  $p^{-1}(x) \subset X_1$  are closed manifolds for all  $x \in X$ , then  $X_1$  is also proper Plateau–Stein.

(c<sub>3</sub>) If  $X$  and the fibers  $p^{-1}(x)$  are proper Plateau–Stein, if the action of normal holonomy is isometric on the fibers, and if the normal holonomy group is compact, then  $X_1$  is proper Plateau–Stein.

(c<sub>4</sub>) It follows from (a) and (c<sub>2</sub>) that *non-compact semisimple groups with finite centers are proper Plateau–Stein*, while (c<sub>1</sub>) implies that *unimodular solvable, e.g. nilpotent, groups are (not necessarily proper) Plateau–Stein*.

Probably, all non-compact Lie groups  $X$  with left invariant metrics, except for compact  $\times \mathbb{R}$ , are Plateau–Stein but it is less clear which Lie groups, and Riemannian homogeneous spaces in general, are *proper* Plateau–Stein.

(d) The Riemannian cylinders that are product  $\mathbb{R} \times X'$  are Plateau–Stein for many (all?) *open* Riemannian  $X'$ , e.g. for the interiors  $X'$  of compact manifolds with boundaries (this is obvious) and for complete connected manifolds of with infinite volume. (This is not hard.)

(e) What are non-compact, e.g. complete, Riemannian manifolds  $X'$ , such that the Riemannian products  $X \times X'$  are proper Plateau–Stein for all proper Plateau–Stein manifolds  $X$ ?

Conclude with the following questions where topology and geometry are intertwined. Let  $V$  be a closed Riemannian manifold.

(I) When does  $V$  admit a Riemannian metric such that the universal covering  $X$  of  $V$  with this metric is proper, or at least non-proper, Plateau–Stein?

(II) When is the universal covering of  $V$  bi-Lipschitz equivalent to a (proper) Plateau–Stein manifold?

Probably, the answers are invariant under the codimension two surgeries of  $V$ , and even possibly, depend only on the fundamental group  $\Gamma = \pi_1(V)$ . Anyway, the fundamental groups  $\Gamma$  of such manifolds  $V$  (where the answers are positive) may be called (I) or (II) “Plateau–Stein” [proPS]- and [n-n-proPS]-. The best candidates for “Plateau–Stein”  $\Gamma$  are

non-amenable groups with one end. On the other hand, there may exist some “tricky” (forget about virtually cyclic) amenable groups that are not even [n-n-proPS]<sub>-</sub>.

Question (II) makes sense for all complete manifolds  $X$ , not only coverings of compact ones: *When such  $X$  is bi-Lipschitz equivalent to a Plateau–Stein manifold?* (“Bi-Lipschitz” seems too restrictive in this context; one needs something half-way from bi-Lipschitz to *quasiisometric* in the spirit of the *directed Lipschitz metric* [3].)

### 1.3. Inverse Maximum Principle

[n-n-proPS]<sub>min</sub> Say that a Riemannian manifold  $X$  is [n-n-proPS]<sub>min</sub> if it contains no compact minimal hypersurface.

One cannot exclude such a hypersurface being singular. Below is smooth version of this condition with minimal replaced by “almost concave”.

[n-n-proPS]<sub>-</sub> Say that a Riemannian manifold  $X$  is [n-n-proPS]<sub>-</sub> if it admits a continuous positive function  $\varepsilon(x) > 0$  such that every compact smooth domain, i.e. a relatively compact open compact subset  $U \subset X$  with smooth boundary in  $X$ , has a point  $x \in \partial U$  where  $\text{mn.curv}_x(\partial U) \geq \varepsilon(x)$ . Another way to put it is that  $X$  contains *no* bounded domain with  $\varepsilon$ -mean-concave boundary.

[proPS]<sub>min</sub> Say that  $X$  is [proPS]<sub>min</sub> if it is connected at infinity and if, for every compact subset  $B \subset X$ , there is a (larger) compact subset  $C = C(B) \subset X$  such that all compact minimal hypersurfaces  $H_i \subset X$  with boundaries contained in  $B \subset X$  are themselves contained in  $C$ .

A smooth almost concave version of this condition is as follows.

[proPS]<sub>-</sub> Say that  $X$  is [proPS]<sub>-</sub> if it is connected at infinity and there are continuous positive function  $\varepsilon(x) > 0$  and a proper continuous function  $\phi: X \rightarrow \mathbb{R}_+$  such that: given a compact subset  $V \subset X$  and a smooth domain  $U \subset X$ , where

$$\sup_{x \in U} \phi(x) > \sup_{x \in V} \phi(x),$$

there exists a point  $x \in \partial U \setminus V$ , where  $\text{mn.curv}_x(\partial U) \geq \varepsilon(x)$ .

One immediately sees by looking at the *maxima points* of strictly mean curvature convex functions  $f(x)$  on the boundaries  $\partial U$  that proper/non-proper Plateau–Stein manifolds satisfy the corresponding  $[\cdot]_{\text{min}}$ -conditions. Namely,

$$[n-n-proPS]_{-} \implies [n-n-proPS]_{-}, \quad [proPS]_{-} \implies [proPS]_{-}.$$

Also it is not hard to see by a simple approximation argument (see Step 2 in the next section and subsections 3.4, 5.6, 5.7) that

$$[n-n-proPS]_{-} \implies [n-n-proPS]_{\text{min}}, \quad [proPS]_{-} \implies [proPS]_{\text{min}}.$$

**IMP for Thick Manifolds.** The main purpose of the present paper is proving inverse implications for Riemannian manifolds  $X$  that are *thick at infinity* (see subsection 2.1). Examples of these include:

- <sub>conv</sub> complete manifolds  $X$  where the balls of radii  $\leq \varepsilon$  are *convex* for some  $\varepsilon > 0$ ;
- <sub>Ricc</sub> complete manifolds  $X$  where  $\text{Ricci}(X) \geq -\text{const} \cdot \text{Riem.metric}(X)$  and, at the same time, the volumes of the unit balls  $B_x(1) \subset X$  for all  $x \in X$  are bounded from below by some  $\varepsilon > 0$ ;
- <sub>Lip</sub> complete manifolds  $X$  where the  $\varepsilon$ -balls  $B = B_x(\varepsilon) \subset X$  for some  $\varepsilon > 0$  admit  $\lambda$ -*bi-Lipschitz* embeddings  $B \rightarrow \mathbb{R}^n$  for  $n = \dim X$  and for some constant  $\lambda$  independent of  $B$ .

Notice that coverings of compact manifolds are thick at infinity by either of these conditions.

**Main Theorem: Inverse Maximum Principle.**

Let  $X$  be a complete  $C^2$ -smooth Riemannian  $n$ -manifold,  $n \geq 2$ . If  $X$  is thick at infinity, then

IMP[non-proper] [n-n-proPS]<sub>min</sub>  $\Rightarrow$  [n-n-proPS]<sub>-</sub>;

IMP[proper] [proPS]<sub>min</sub>  $\Rightarrow$  [proPS]<sub>-</sub>.

**Convex/Minimal Existence Alternative.**

Observe that IMP[non-proper] says, in effect, that either  $X$  can be “filled” by strictly mean convex hypersurfaces (that are the levels of a strictly mean curvature convex Morse function  $f(x)$ ) or, alternatively,  $X$  contains a compact minimal hypersurface  $Y$  and IMP[proper] encodes a similar alternative.

**1.4.  $\phi$ -bubbles, plan of the proof of IMP and Trichotomy Theorem**

Let  $\mu$  be a Borel measure  $\mu$  in  $X$  and define the  $\mu$ -area of a domain  $U \subset X$  with boundary  $Y = \partial U$  as

$$\text{area}_{-\mu}(U) \stackrel{\text{def}}{=} \text{vol}_{n-1}(Y) - \mu(U),$$

where “domain” means either a closed subset  $U \subset X$  with the interior  $\text{int}(U) \subset U$  being dense in  $U$  or an open subset that equals the interior of its closure.

Call  $U$  a  $\mu$ -bubble if it locally minimizes the function  $U \mapsto \text{area}_{-\mu}(U)$  where “local” may be understood at this point relative to the Hausdorff metric in space of pairs  $(U, Y = \partial U)$ . (We return to this in Section 2.) For instance, if  $\mu = 0$  then  $\mu$ -bubbles are domains bounded by stable minimal hypersurfaces in  $X$ .

If  $\mu$  is given by a measurable density function  $\phi(x)$ ,  $x \in X$ , i.e.  $\mu = \phi dx$  for the Riemannian  $n$ -volume (measure)  $dx$ , then we speak of  $\phi$ -bubbles and observe that if  $\phi$  is a continuous function, then the mean curvatures of the boundaries  $Y = \partial U$  of  $\phi$ -bubbles satisfy  $\text{mn.curv}_x(Y) = \phi(y)$  for all regular points  $x$  of  $Y$ . In particular, if  $\phi \geq 0$ , then  $\phi$ -bubbles are mean convex at all regular points of their boundaries, i.e.  $\text{mn.curv}_x(Y) \geq 0$  at all regular  $x \in Y$  and strictly mean convex at such points if  $\phi > 0$ . In sequel, if  $\phi$  is not specified, these are called just (strictly) mean convex bubbles.

(By definition,  $Y$  is regular at a point  $x \in Y$  if  $Y$  is a  $C^2$ -smooth hypersurface in a neighbourhood of this point.)

**Bubbles with Obstacles.** If  $\phi_0 = \infty_B$  equals  $+\infty$  on some, say, compact subset  $B \subset X$  and zero outside  $B$ , then the boundaries of  $\phi$ -bubbles  $U \supset B$  are, almost by definition, minimal hypersurfaces  $Y$  in the closure of the complement  $X \setminus B$  that solve the Plateau problem with the obstacle  $B$ .

We shall often use positive continuous functions  $\phi = \phi_\varepsilon > 0$  that approximate such  $\phi_0$ , being large, say  $1/\varepsilon$ , on  $B$  and  $\varepsilon > 0$  away from  $B$ . Then the corresponding  $\phi_\varepsilon$ -bubbles  $Y = Y_\varepsilon$  lie close to  $\phi_0$ -bubbles for small  $\varepsilon \rightarrow 0$  that helps to understand their overall geometry, while the continuity of  $\phi$  makes the local (quasi)regularity of these  $Y_\varepsilon$  similar to that of minimal varieties.

We divide the proof of IMP into five steps.

**Step 0: Excluding “Narrow Ends”.** The representative case of the theorem is where  $X$  is one-ended, i.e. connected at infinity, and where this end has infinite area. This means that the boundaries  $Y_j = \partial V_j$  of an arbitrary exhaustion of  $X$  by bounded domains  $V_j \in X$  satisfy  $\text{vol}_{n-1}(Y_j) \rightarrow \infty$ ,  $j \rightarrow \infty$ , for  $n = \dim X$ . (In fact, one needs a slightly more general version of this condition as we shall explain in subsections 2.2 and 2.3.)

**Step 1: Mean Convex Exhaustion.** Let  $X$  be one-ended complete with infinite area at infinity. Then there exist strictly mean convex compact bubbles  $U_j \subset X$ ,  $j = 1, 2, \dots$ , that exhaust  $X$ ,

$$U_1 \subset U_2 \subset \dots \subset U_j \subset \dots \quad \text{and} \quad \bigcup_j U_j = X.$$

Notice that the thickness at infinity condition is not needed beyond this point. On the other hand, it is essential for the existence of the bubbles  $U_j$ . In fact, such  $\phi_j$ -bubble  $U_j$  is obtained for  $\phi_j > 0$  that is large on the ball  $B = B_{x_0}(j) \subset X$  of radius  $j$  around a fixed point  $x_0 \in X$  and small away from  $B$ .

The existence of  $U_j$  is proven in subsections 2.2 and 2.3 by the standard minimization argument of the *geometric measure theory*, that works on our (non-compact)  $X$  because of the thickness condition that is designed exactly for this purpose (see subsection 2.1) in order to prevent *partial escape* of minimizing sequences of (boundaries of) bounded domains  $U$  in  $X$  to infinity. (Such an escape can be imagined as a “long narrow tentacle” protruding from the “main body” of  $U$ .)

**Step 2: Mean Convex Regularization.** One can not guaranty at this point that the boundary hypersurfaces  $Y_j = \partial U_j$  are smooth for  $n = \dim X \geq 7$ . Yet, they do have positive mean curvatures in a generalized sense. Moreover, (this proves the implication  $[\text{n-n-proPS}]_{\sim} \Rightarrow [\text{n-n-proPS}]_{\min}$ ) these  $Y_j$  can be *approximated by  $C^2$ -smooth hypersurfaces  $Y'_j \subset U_j$  with positive mean curvatures.*

In fact, since (the boundaries of)  $\phi$ -bubbles  $U$  are *quasiregular* (as defined in subsection 3.2) for all continuous functions  $\phi$  by the *Almgren–Allard regularity theorem*, the minus distance function  $d(x) = -\text{dist}(x, Y = \partial U)$  can be *regularized* almost without loss of the lower mean curvature bound near the boundary of  $U$ . Namely, we shall see in subsection 3.4, and, from a different angle, in subsection 5.6, that *every strictly mean convex bubble  $U \subset X$  admits a continuous function  $d': U \rightarrow (-\infty, 0]$  such that*

- $d'(x) = 0$  for  $x \in Y = \partial U$ ,
- $d'(x)$  is smooth strictly negative in the interior  $\text{int}(U) \subset U$  and all critical points of  $d'$  in  $\text{int}(U) \subset U$  are non-degenerate,
- there exists a continuous function  $\delta$  in  $U$  that vanishes on the boundary  $Y = \partial U$ , such that  $\text{mn.curv}_x(d') \geq \phi(x) - \delta(x)$  for all  $d'$ -non-critical points  $x \in \text{int}(U)$ .

### Remarks and Questions.

- (a) The above regularization is non-essential at this stage of the proof; yet, it will become relevant later on.
- (b) This regularization, along with the simple but “non-elementary” minimization argument at Step 1 in the framework of the geometric measure theory provides an exhaustion of  $X$  by compact domains  $U'_j \subset X$  with smooth strictly mean convex boundaries  $Y'_j$ . *Is there an “elementary” proof of this?*
- (c) Our regularization procedure (see subsections 3.3, 5.6), however simple, requires  $C^2$ -smoothness of the Riemannian metric in  $X$  and it does not work for  $C^1$ -manifolds with the sectional curvatures bounded from above and from below. But the inverse maxima principles, if properly formulated, may hold for  $C^1$ -smooth manifolds and for some singular spaces, e.g. for *Alexandrov spaces* with curvatures bounded from below.

**Step 3: Inverse Maximum Principle for Manifolds with Boundary.** Let  $V$  be a smooth compact Riemannian manifold with boundary and  $\phi$  be a continuous function on  $V$  such that the boundary of  $V$  is *strictly mean  $\phi$ -convex*, i.e.  $\text{mn.curv}_v(\partial V) > \phi(x)$  for all  $v \in \partial V$ . Then, assuming  $\phi > 0$ ,

- either the interior of  $V$  contains a  $\phi$ -bubble, or
- $V$  admits a Morse function  $f: X \rightarrow \mathbb{R}_- = (-\infty, 0]$  that vanishes on the boundary of  $X$  and that is strictly mean  $\phi$ -convex, i.e.,  $\text{mn.curv}_v(f) > \phi(v)$  for all  $f$ -non-critical points  $v \in V$ .

The proof of this is divided into two half-steps.

**Half-Step 3A: Shrinking Bubbles.** Take a (eventually small) positive  $\rho > 0$ , a (large)  $C > 1$  and a monotone decreasing sequence  $\varepsilon_i > 0$ ,  $i = 0, 1, 2, \dots$ , where  $\varepsilon_1 = \varepsilon_1(V) > 0$  is (very) small and where  $\varepsilon_i \rightarrow 0$  for  $i \rightarrow \infty$ . Construct step by step a sequence

$$U_0 = V, \phi_1, U_1, \phi_2, U_2, \phi_3, \dots, U_i, \phi_{i+1}, U_{i+1}, \dots,$$

where  $\phi_{i+1}$ ,  $i = 0, 1, 2, \dots$ , is a continuous function on  $U_i$  and where  $U_{i+1} \subset U_i$  is a  $\phi_{i+1}$ -bubble and the following three conditions must be satisfied:

$$(*)_{\varepsilon} \quad \phi_i \geq \phi + \varepsilon_i \text{ for all } i = 1, 2, \dots,$$

$$(*)_{\varepsilon, \rho} \quad \phi_i(u) = \phi(u) + \varepsilon_i \text{ for all } u \text{ in the } \rho\text{-neighbourhood of } \partial U_{i-1} \subset U_{i-1},$$

$(*)_{C\rho}$   $\partial U_{i+1}$  is contained in the  $C\rho$ -neighbourhood of  $\partial U_i \subset U_i$  for all  $i = 0, 1, 2, \dots$

This  $(*)_{C\rho}$  is the only non-trivial requirement on our sequence, where the existence of a  $\phi_{i+1}$ -bubble  $U_{i+1} \subset U_i$  with the boundary  $\partial U_{i+1}$  lying close to  $\partial U_i$  needs a suitably chosen  $\phi_{i+1}$  that must be far away from the  $\rho$ -neighbourhood of  $\partial U_i \subset U_i$  (see subsections 2.4, 4.1, 4.2).

**Half-Step 3B: Regularization.** One can show (see Section 2) that if  $V$  contains no  $\phi$ -bubble, the sets  $U_i$  become empty for large  $i$ . Then, if  $\rho > 0$  is sufficiently small, one can construct the required  $f$  by “splicing and regularizing” minus distance functions  $u \mapsto -\text{dist}(u, \partial U_i)$  on the subsets  $U_i$  (see Section 4).

### Remarks.

- “Shrinking bubbles” can be seen as discretization of a some “gradient flow” for a non-Hilbertian norm in the (tangent space to the) space of subvarieties in  $X$ , where the Hilbertian norm leads to the mean curvature flow.
- The condition  $\phi > 0$  can be dropped with a slightly more general notion of “ $\phi$ -bubble”, that would allow, for instance, the central geodesic in the (hyperbolic) Möbius band for the role of a ( $\phi = 0$ )-bubble.

**Step 4. Limits by Exhaustion.** We shall use in subsection 4.2 a simple compactness property (of sets of the boundaries) of our mean convex bubbles  $U_{i,j} \subset U_j$  to construct mean convex functions  $f$  on  $X$  as limits of such functions  $f_j$  in bounded mean convex domains  $U_j \subset X$  that exhaust  $X$ . Looking closer (see Section 4) one obtains with the above argument the following

### Trichotomy Theorem.

Let  $X$  be a complete Riemannian  $C^2$ -smooth manifold (not assumed thick at infinity). Then (at least) one of the three conditions is satisfied.

- $X$  admits a proper (positive, if  $X$  is connected at infinity) strictly mean curvature convex Morse function;
- $X$  contains a complete (possibly compact) minimal hypersurface  $H$  of finite volume;
- $X$  admits a non-proper strictly mean curvature convex Morse function and such that either there is a non-compact minimal hypersurface  $H$  with finite volume that is closed in  $X$  as a subset and that has compact boundary, or there is a sequence of compact minimal hypersurfaces  $H_i \subset X$  with no uniform bound on their diameters, such that the boundaries  $\partial H_i$  are contained in a fixed compact subset in  $X$ .

### Remarks.

- A complete Riemannian manifold  $X$  with two ends that admits a proper strictly mean curvature convex Morse function  $f: X \rightarrow \mathbb{R}$ , may contain, however, arbitrarily large compact minimal hypersurfaces with boundaries in a given compact subset in  $X$ . For instance, the 2D hyperbolic cusp  $X_0$  (the hyperbolic plane divided by a parabolic isometry) has this property and the Cartesian products  $X = X_0 \times V$  for compact  $V$  furnish example of all dimensions.

What are other examples of minimal hypersurfaces protruding toward “concave ends” in complete manifolds? Are there such examples with thick ends, e.g. for manifolds with bounded geometries?

- The above theorem (and, in particular, its special case stated in the abstract to our paper) shows that the inverse maximum principle does not truly need thickness at infinity, but the direct maximum principle, probably, does. Quite likely, there exist complete Plateau–Stein  $n$ -manifolds for  $n \geq 3$  that contain complete minimal hypersurfaces of finite  $(n - 1)$ -volume.

## 1.5. Miscellaneous remarks, questions and corollaries

(A) The most essential ingredient of our proof – the Almgren–Allard regularity theorem for “soap bubble” – is trivial for  $n = \dim X = 2$ : curves with continuous curvature in surfaces are, obviously, smooth. Consequently, our argument is quite elementary for  $n = 2$ . In fact, both IMP hold with (almost) no restrictions on  $X$ , where IMP[non-proper] reduces to the following, most likely known, proposition.

IMP[ $\dim = 2$ ] Let  $X$  be a surface with a complete  $C^2$ -smooth (probably,  $C^1$  will do in this case) Riemannian metric. Then one of the following three properties holds:



- (i)  $X$  contains a simple closed geodesic.
- (ii)  $X$  supports a smooth function  $f: X \rightarrow \mathbb{R}$  which has no critical points and such that the sublevels  $f^{-1}(-\infty, r] \subset X$  are strictly geodesically convex for all  $r \in \mathbb{R}$ . (Such  $X$ , obviously, is homeomorphic to the plane  $\mathbb{R}^2$  or to the cylinder  $S^1 \times \mathbb{R}$ .)
- (iii)  $X$  is homeomorphic to the sphere with three points removed.

(Originally I overlooked (iii); it was pointed to me by Yura Burago that one cannot always ensure a *simple* closed geodesic with this topology where non-simple geodesics are abundant for all metrics. But it should be noticed that the essence of this IMP resides in the surfaces that are homeomorphic to  $\mathbb{R}^2$  and to  $S^1 \times \mathbb{R}$ .)

**(B)** An instance of a corollary to, say, IMP[non-proper], is the validity of the counterparts of the stability of Plateau–Stein under Riemannian submersions (see (c) from the previous section) for the corresponding  $[\cdot]_{\min}$ .

For example, let  $X$  be a complete and thick at infinity. If  $X$  is [n-n-proPS], then so is the Riemannian product  $X \times X'$ , for all closed Riemannian manifolds  $X'$ . This, however, looks almost as obvious as the original Plateau–Stein case and, moreover, “thick” seems unnecessary. Indeed, if a compact smooth domain  $U_1 \subset X \times X'$  is *mean concave*, i.e. its boundary satisfies  $\text{mean.curv}_x(\partial U_1) \leq 0$  for all  $x \in \partial U_1$ , then the boundary of the projection  $U \subset X$  of  $U_1$  to  $X$  is also mean concave at all *regular* points  $x \in \partial U$ , while singular points have generalized mean curvatures  $= -\infty$ . This allows an approximation/regularization of  $\partial U$  with  $\text{mean.curv}(\partial U) \leq \varepsilon$  everywhere for all  $\varepsilon > 0$ .

On the other hand, the  $[\cdot]_{\min}$  counterpart of the above IMP[non-proper] is not fully trivial. If  $X \times X'$  contains a closed minimal (possibly singular) hypersurface then so do  $X$  and  $X'$ , provided  $X$  and  $X'$  are complete and thick at infinity. But the direct proof of this by the geometric measure theory is very simple.

Notice that minimal hypersurfaces in split Riemannian manifolds  $X_1 = X \times X'$  do not always split, e.g. in flat 3-tori. Probably, there are *non-split* compact domains  $U_1$  with *minimal boundaries* in certain Riemannian products  $X_1 = X \times X'$  for open manifolds  $X$  and closed  $X'$ , where “split” means  $U_1 = U \times X'$ .

But it seems unclear, for example, if such non-split  $U_1$  with minimal boundaries exist in the products  $X_1 = X \times S^1$  of a hyperbolic surfaces  $X$  of finite areas by circles and if there are compact domains with minimal boundaries in the products  $X_1 = X \times X'$  of complete hyperbolic surfaces  $X$  and  $X'$  of finite areas. (These  $X_1$  are *not* uniformly locally contractible but some IMP may hold.)

**(C)** The existence of an exhaustion of a Riemannian manifold  $X$  by compact mean convex domain is an interesting property in its own right, call it *strict mean convexity at infinity*. For instance, a Galois covering  $X$  of a closed Riemannian manifold  $\underline{X}$  is strictly mean convex at infinity unless the Galois group  $\Gamma$  of the covering is virtually cyclic and if, moreover,  $\Gamma$  is non-amenable, then  $X$  can be exhausted by domains with mean curvatures  $\geq \varepsilon > 0$ . (A representative counterexample for cyclic  $\Gamma$  is provided by manifolds  $\underline{X}$  that admit fibrations over the circle  $X \rightarrow S^1$  such that the fibers are *minimal* hypersurfaces.)

Mean convexity at infinity is visibly “cheaper” than Plateau–Stein; yet, there are non-proper Plateau–Stein manifolds that are *not* mean convex at infinity. For instance, let  $X$  be a topological cylinder, i.e. homeomorphic to  $X = X_0 \times \mathbb{R}$ , where  $X_0$  is a compact manifold, and let  $g_0$  be a Riemannian metric on  $X_0$ . Let  $\phi = \phi(t)$ ,  $t \in \mathbb{R}$ , be a positive function and observe that the metric  $\phi(t)g_0 + dt^2$  on  $X$  is *concave* at the  $t \rightarrow -\infty$  end of  $X$  rather than mean convex. But such  $X$ , obviously is Plateau–Stein since  $f(x_0, t) = \phi(t)$  is a strictly mean convex function.

### Question.

Under what conditions does Plateau–Stein imply strict mean convexity at infinity? (An easy instance of such a condition is *thickness at infinity + connectedness at infinity*.)

**(D)** Smoothness conditions we impose on functions and on hypersurfaces in the definitions of Plateau–Stein manifolds and of their  $[\cdot]_{\min}$ -counterparts allows a glib formulation of our results with no need for concept of “minimal hypersurface”. But insistence on this smoothness looks facetious in view of the geometric measure theory techniques that underly the essential part of the argument while “regularization of bubbles” that excludes  $[\cdot]_{\min}$  may strike one as a waste of effort.

In fact, an expected generalization of the IMP-implications must apply to non-smooth objects in *singular* spaces  $X$ . On the other hand, the regularization process we employ delivers — this is implicit in the arguments in Section 4 — a simple but non-trivial information on geometry of singular minimal varieties. (This “information” is by no means new.)

(E) Here another obvious consequence of the inverse maximal principle implications  $[\cdot]_{\min} \Rightarrow [\cdot]_{\sim}$ , where minimal varieties and their singularities do not appear.

### **( $n - 1$ )-Contraction Corollary.**

Let  $X$  be a  $C^2$ -smooth complete Riemannian manifold that is thick at infinity. If  $X$  admits a strictly  $(n - 1)$ -volume contacting vector field  $V$  then  $X$  is Plateau–Stein. If, moreover,  $X$  is connected at infinity and if there are vector fields  $V_i$  such that the supports of  $V_i$  exhaust  $X$  and such that  $V_i$  are strictly  $(n - 1)$ -volume contacting in the complement of their supports, then  $X$  is proper Plateau–Stein.

(Recall, that this theorem was stated for coverings of compact manifolds in subsection 1.2 and that a vector field  $V$  is strictly  $(n - 1)$ -volume contacting if the  $V$ -derivatives of the volumes of all smooth hypersurfaces in  $X$  are negative.)

(F) Let  $X$ , not necessarily thick at infinity, contain no compact minimal hypersurface. Then does it admit a strictly  $(n - 1)$ -volume contracting vector field? (This question is motivated by such a result for 1-volume (i.e. length) contracting fields that was pointed out to me by Bruce Kleiner. Possibly, there is something like that for all  $k$ -volume contracting vector fields.)

## 2. Construction of $\phi$ -bubbles and of minimal hypersurfaces

We shall describe in this section a few standard  $\phi$ -areas (including  $(n - 1)$ -volume) minimization constructions that deliver minimal hypersurfaces, such as  $\phi$ -bubbles, under the thickness condition.

### 2.1. Thickness at infinity

An  $n$ -dimensional Riemannian manifold  $X$  is called *thick at infinity* if it contains no non-compact minimal hypersurface with compact boundary and with finite  $(n - 1)$ -volume. Such a hypersurface  $Y \subset X$  must, by definition, be closed in  $X$  as a subset and be  $\varepsilon$ -locally  $\text{vol}_{n-1}$ -minimizing in  $X$  at infinity. This means that there exists  $\varepsilon > 0$  ( $\varepsilon = 1$  is good for us) and a compact subset  $A = A(Y) \subset X$  (that contains the boundary of  $Y$ ) such that the intersection  $Y \cap B$  with every  $\varepsilon$ -ball  $B = B_x(\varepsilon) \subset X$  for  $x \in X \setminus A$  is  $\text{vol}_{n-1}$ -minimizing in  $X$  in the class of hypersurfaces (integral currents) with the boundary equal  $Y \cap \partial B$ .

#### Examples.

The Paul Lévy (Buyalo–Heintze–Karcher) tube volume bound shows that the condition  $\bullet_{\text{Ric}}$  from subsection 1.3 implies this thickness, while the conditions  $\bullet_{\text{conv}}$  and  $\bullet_{\text{Lip}}$  are taken care by the following corollary to the implication:

$$\text{cone inequality} \quad \implies \quad \text{filling inequality}$$

and the lower bound on volumes of minimal varieties by the filling constant [12].

$\bullet_{\text{fill}}$  If every closed integral  $k$ -chain  $S$  in  $X$ ,  $k = 1, 2, \dots$ , of diameter  $\leq 1$  bounds a  $(k + 1)$ -chain  $T$  such that

$$\text{vol}_{k+1}(T) \leq \text{const} \cdot \text{diam } S \cdot \text{vol}_k(S)$$

for some  $\text{const} = \text{const}(X)$ , then  $X$  is thick at infinity.

**Remark.**

This thickness concept obviously generalizes to all dimensions  $2 \leq m \leq n - 1$  with minimal subvarieties  $Y \subset X$  of dimension  $m$ , where the above criterion remains valid, while  $\bullet_{\text{Ricc}}$  should be replaced by  $\bullet_{\text{sect}}$  with the lower sectional curvature bound instead of such a bound on Ricci.

**2.2. Convex and concave ends**

Let  $X$  be a Riemannian manifold possibly with compact boundary  $\partial X$  and a single end such that  $X$  is *thick at infinity*. Then one of the following three possibilities is realized:

1.  $X$  can be exhausted by compact strictly mean convex bubbles,
2.  $X$  can be exhausted by compact strictly mean concave bubbles,
3. there exists a continuous positive proper function  $h: X \rightarrow \mathbb{R}_+$  such that the levels  $Y_t = h^{-1}(t) \subset X$  are minimal hypersurfaces, that are the boundaries of 0-bubbles, for all  $t \geq t_0 = t_0(X) \geq 0$ .

**Proof.** Start by observing that 1. and 2. are *not* mutually exclusive and if there are these two kinds of exhaustions then there also (obviously) exists an exhaustion by 0-bubbles. But 3. is incompatible with 1. and with 2. by the maximum principle.

Let 3. do not hold and, moreover, assume that  $X$  *cannot* be exhausted by compact 0-bubbles. Then minimization of  $\text{vol}_{n-1}(Y)$  for  $Y = \partial U$ , where  $U \subset X$  is a compact sufficiently large domain, either moves  $Y$  to infinity or brings to a compact region  $X_0 \subset X$ . In the former case, let  $-\phi$  be a negative function on  $X$ , where  $\phi$  is very large at infinity and very small in the vicinity of some  $X_0$  that contains the boundary of  $X$ . Then minimization of  $\phi$ -area brings  $U$  to a strictly mean concave  $-\phi$ -bubble in  $X$ .

Similarly, if volume minimization brings  $Y$  to a fixed compact  $X_1 \subset X$  we use a positive  $\phi$  that is very large on some compact  $X'_1 \supset X_1$  and very small at infinity. Thus we obtain a strictly mean convex bubble. We keep modifying  $\phi$  by shifting the switch from “very large” to “very small” further and further to infinity and, thus, we exhaust  $X$  either by strictly mean convex or by strictly mean concave bubbles.  $\square$

Now, let  $X$  be exhausted by compact 0-bubbles. Then  $X$  contains infinitely many “empty bands”, say  $W$ , between the boundaries of these bubbles say  $U_1$  and  $U_2 \supset U_1$ , i.e.  $W = U_2 \setminus U_1$ , where such  $W$  is bounded by the minimal hypersurfaces  $Y_1 = \partial U_1$  and  $Y_2 = \partial U_2$  with no 0-bubble between the two. Then the obvious adjustment of the above argument delivers both a strictly mean convex and a strictly mean concave bubble pinched between  $U_1$  and  $U_2$ .

Besides, *Almgren’s min-max argument* delivers a non-stable minimal hypersurface in  $W$  that separates  $Y_1$  from  $Y_2$ .

**2.3. Minimal separation of ends**

Here  $X$  has several ends and no boundary, where the set of ends is given its natural topology. Notice that every *isolated* end  $E$  can be represented/isolated by an equidimensional submanifold (domain)  $X_E \subset X$  with compact boundary and a single end; we say “exhaustion of  $E$ ” instead of “exhaustion of  $X_E$  for some  $X_E \subset X$ ”.

If the set of ends of  $X$  contains at least two limit points (e.g. if it has no isolated ends), then  $X$  contains a compact two-sided smooth hypersurface  $H \subset X$  such that there are infinitely many ends of  $X$  on either side of  $H$ . Hence, clearly, *if  $X$  is complete and thick at infinity, then the homology class of  $H$  contains a minimizing hypersurface.*

Now let  $X$  have *at least two isolated* ends, say  $E_1$  and  $E_2$ . *If none of these ends admits a strictly mean convex exhaustion, then  $H$  contains a compact (non-stable) minimal variety  $M \subset X$  where this  $M$  may be a varifold.*

**Proof.** Let  $h: x \mapsto t = h(x) \in (-\infty, +\infty)$  be a smooth proper Morse function  $X \rightarrow \mathbb{R}$  such that  $h(x) \rightarrow -\infty$  for  $x \rightarrow E_1$  and  $h(x) \rightarrow +\infty$  for  $x \rightarrow E_2$ . The  $\text{vol}_{n-1}$ -minimization process starting from the levels  $h^{-1}(t)$  moves some connected component of  $h^{-1}(t)$  for small negative  $t$  (approaching  $-\infty$ ) to  $E_1$ , while some component for positive large  $t$  goes to  $E_2$ . Since the manifold  $X$  is thick at infinity, Almgren’s min-max theorem applies and the proof follows.  $\square$

By combining the above with 1./2./3<sub>±</sub> in the previous section, we conclude to the following. Let  $X$  be a complete Riemannian manifold that is thick at infinity. Then

[✓/–] *Either an isolated end of  $X$  admits a strictly mean convex exhaustion, or  $X$  contains a compact minimal subvariety.*

## 2.4. Shrinking mean convex ends

Let  $X$  be a complete connected Riemannian manifold of dimension  $n$  with non-empty compact boundary  $Y_{\infty} = \partial X$  with  $\text{mn.curv}(Y_{\infty}) > \varepsilon_0 > 0$ , let  $0 < \varepsilon_i < \varepsilon_0$ ,  $i = 1, 2, \dots$ , be a sequence of positive numbers that converges to 0 and let  $\rho(x) > 0$  be a continuous function on  $X$ . Then either

★<sub>A</sub>  $X$  contains a minimal hypersurface  $H \subset X$  of finite  $(n-1)$ -volume that is closed in  $X$  as a subset and that does not meet  $Y_{\infty}$ , or

★<sub>B</sub>  $X$  can be exhausted by an increasing sequence of compact strictly concave bubbles  $U_i$  in  $X$  that contain  $Y_{\infty}$ ,

$$Y_{\infty} \subset U_1 \subset U_2 \subset \dots \subset U_i \subset \dots \subset X, \quad \bigcup_i U_i = X,$$

such that

- $U_i$  is contained in the  $\rho_{i-1}$ -neighborhood of  $U_{i-1}$  for  $\rho_{i-1} = \inf_{x \in U_{i-1}} \rho(x)$  and all  $i = 1, 2, \dots$ , where  $U_0 \stackrel{\text{def}}{=} Y_{\infty}$ ,
- the mean curvatures of the topological boundaries  $Y_i = \partial U_i \subset X$  satisfy  $\text{mn.curv}_x(Y_i) = -\varepsilon_i$  at all those regular  $x \in Y_i$ , where  $\text{dist}(x, U_{i-1}) \leq \rho/2$ .

### Remarks.

If  $X$  is thick at infinity, then, as we know, the above minimal hypersurface  $H$ , if it exists at all, must be compact. If  $X$  is compact and no minimal  $H$  exists, then the sequence  $U_i$  stabilizes and the boundaries  $Y_i$  of the bubbles  $U_i$  become empty for large  $i$ . On the other hand, if  $H$  does exist and  $\rho \leq \rho_0 = \rho_0(H) > 0$ , then no  $U_i$  intersects  $H$ . (If  $\rho$  is large then  $U_i$  may exhaust  $X$  even in presence of  $H$ .)

**Proof of the ★<sub>A</sub>/★<sub>B</sub>-Alternative.** Proceed as at the half-step 3A from subsection 1.4. Namely, granted  $U_{i-1}$  for some  $i$ , let  $\phi_i$  be positive continuous functions on  $X$  such that  $\phi_i = \varepsilon_i$  in the  $\rho_i/2$ -neighbourhood of  $U_{i-1}$ ,  $\phi_i(x)$  is very large for  $\text{dist}(x, U_{i-1}) \geq 2\rho_i/3$ . Then, clearly, there exists a compact  $-\phi_i$ -bubble  $U_i \supset U_{i-1}$  that satisfies all of the above properties. Since  $\varepsilon_i \rightarrow 0$ , the boundary of the union  $\bigcup_i U_i \subset X$  must be a minimal hypersurface  $H$  in  $X$  with  $\text{vol}_{n-1}(H) < \text{vol}_{n-1}(Y_0)$ ; if no such hypersurface exists, then this boundary must be empty and  $\bigcup_i U_i = X$ .  $\square$

If ★<sub>B</sub> holds for all  $\rho(x) > 0$  then, by the maximum principle,  $X$  contains no minimal hypersurface  $H$  of any volume. This leads to the following

### Almgren's min/max Theorem for Non-Compact Manifolds.

*If a complete Riemannian manifold  $X$  with non-empty compact strictly mean convex boundary contains a complete minimal hypersurface, then it also contains a complete minimal hypersurface of finite volume. ("Complete" means being closed in  $X$  as a subset, not intersecting the boundary of  $X$  and having no boundary of its own.)*

## 2.5. Shrinking to concave boundary

The above admits a relative version where  $X$  has a concave component  $Y_{\infty}$  in its boundary (or several such components) that serves as an obstacle for shrinking bubbles and where the minimal hypersurface  $H$  that (if it exists at all) obstructs shrinking of bubbles may have non-empty boundary that is contained in  $Y_{\infty}$ .

Namely, let again  $X$  be a complete connected Riemannian manifold of dimension  $n$  with compact boundary that is now decomposed into two disjoint parts  $\partial X = Y_{\infty} \cup Y_{\infty}$  (these  $Y_{\infty}$  and  $Y_{\infty}$  are unions of connected components of  $\partial X$ ),

where  $Y_{\sim}$  (possibly, empty as in the previous section) is strictly mean concave and where  $Y_{\prec}$  is non-empty and has  $\text{mn.curv}(Y_{\prec}) > \varepsilon_0 > 0$ .

Let  $\rho(x) > 0$  be a continuous function on  $X$ . Then either

$\sim \star_A$   $X$  contains a minimal hypersurface  $H \subset X$  of finite  $(n - 1)$ -volume that is closed in  $X$  as a subset, that does not meet  $Y_{\sim}$ , and that may have boundary contained in  $Y_{\sim}$ , or

$\sim \star_B$  the complement  $X \setminus Y_{\sim}$  can be exhausted by an increasing sequence of compact strictly concave bubbles  $U_i$  in  $X$  that contain  $Y_{\sim}$ ,

$$Y_{\sim} \subset U_1 \subset U_2 \subset \dots \subset U_i \subset \dots \subset X, \quad \bigcup_i U_i = X \setminus Y_{\sim},$$

such that  $U_i$  is contained in the  $\rho_{i-1}$ -neighborhood of  $U_{i-1}$  for  $\rho_{i-1} = \inf_{x \in U_{i-1}} \rho(x)$  and all  $i = 1, 2, \dots$ , where  $U_0 \stackrel{\text{def}}{=} Y_{\sim}$ .

**Remarks.**

We could also impose here additional constrains on the mean curvatures of the boundaries  $Y_i = \partial U_i \subset X$  as we did in the absence of  $Y_{\sim}$  but we postpone doing this until Section 4 where this becomes relevant.

An essential case of the above is where  $X$  is compact with two boundary components,  $Y_{\prec}$  and  $Y_{\sim}$ , and these shrinking bubbles are used (see subsection 1.4) for construction of a strictly mean curvature convex Morse function  $f$  on  $X$  that equals 1 on  $Y_{\prec}$  and 0 on  $Y_{\sim}$ , where *such  $f$  exists if and only if there is no minimal hypersurface  $H$  in  $X$  with  $\partial H \subset Y_{\sim}$* . Here (and everywhere in this kind of context) “only if” follows by the maximum principle, while “if” is what we call the inverse maximum principle.

**Proof of  $\sim \star_A$  and  $\sim \star_B$ .** Proceed as earlier and keep pushing boundaries of bubbles closer and closer to  $Y_{\sim}$ . Then, say in the compact case, we arrive at a *maximal* compact bubble  $U_{\max} \subset X$  the boundary of which cannot be moved closer to  $Y_{\sim}$  anymore. Then either  $U_{\max} = X$  or the topological boundary of  $U_{\max}$  is non-empty. Then this boundary, call it  $H$ , makes our *minimal* hypersurface in the interior of  $X$ : if not “minimal” it could be moved closer to  $Y_{\sim}$ .  $\square$

Notice that this  $H$  is *tangent* (rather than transversal) to  $Y_{\sim}$  where the two hypersurfaces meet.

### 3. Distance functions, equidistant hypersurfaces and $k$ -mean convexity

We fix in this section our terminology/notation and state a few standard facts on distance functions in Riemannian manifolds  $X$ .

#### 3.1. Signed distance function and equidistant hypersurfaces

**Interior domains  $U_{-\rho}^<$ ,  $U_{-\rho} = U_{-\rho}^{\leq}$  and equidistant hypersurfaces  $Y_{-\rho}$ .** Let  $U$  be a (closed or open) domain or an open subset (possibly dense) in a Riemannian manifold  $X$  (or in any metric space for this matter) and denote by  $x \mapsto d(x) = d_U(x) = \text{dist}_{\pm}(x, \partial U)$  the *signed distance function* to the topological boundary  $Y = \partial U$ , i.e.  $d_U(x)$  equals the distance from  $x$  to  $Y$  outside  $U$ ,  $d_U(x)$  equals *minus* the distance from  $x$  to  $Y$  in  $U$ . In writing,

$$d_U(x) = \text{dist}(x, Y) = \inf_{y \in Y} \text{dist}(x, y) \quad \text{for all } x \in X \setminus U,$$

$$d_U(x) = -\text{dist}(x, Y) = -\text{dist}(x, X \setminus U) \quad \text{for all } x \in U,$$

where  $\text{dist}(x, Y = \partial U) = \text{dist}(x, U)$  for  $x \in X \setminus U$ , since the Riemannian distance is a *length metric* being defined via the lengths of curves between pairs of points.

Let  $\rho \geq 0$  and denote by  $U_{-\rho} = U_{-\rho}^{\leq} \subset U$  and  $U_{-\rho}^< \subset U_{-\rho}$ ,  $\rho \geq 0$ , the closed/open  $(-\rho)$ -sublevels of  $d_U$ , that are

$$U_{-\rho}^{\leq} = d_U^{-1}(-\infty, -\rho] \quad \text{and} \quad U_{-\rho}^< = d_U^{-1}(-\infty, -\rho),$$

where, clearly,  $U_0 = U_{-0} = U_0^{\leq}$  equals the topological closure of  $U$  and  $U_{-\rho}^{\leq}$  are closed subsets in  $U$  for  $\rho > 0$  with  $U_{-\rho}^{\leq}$  being equal the interior of  $U_{-\rho}^{\leq}$  for  $\rho > 0$ . Let

$$Y_{-\rho} = d_U^{-1}(-\rho) \subset X, \quad \rho \geq 0,$$

be the (*interior*)  $\rho$ -*equidistant hypersurface* to  $Y$ , that is the subset of points  $u$  in  $U$ , where  $\text{dist}_X(u, Y = \partial U) = \rho$  and that equals the topological boundary  $\partial U_{-\rho}$  since Riemannian manifolds  $X$  are *length metric spaces*. Similarly define  $U_{+\rho} = U_{+\rho}^{\leq}$  and  $U_{+\rho}^{\leq} \subset U_{+\rho}^{\leq}$  or  $\rho \geq 0$  as  $U_{+\rho}^{\leq} = d_U^{-1}(-\infty, \rho]$  and  $U_{+\rho}^{\leq} = d_U^{-1}(-\infty, \rho)$ . Thus,  $U_{+\rho} = U_{+\rho}^{\leq}$  equals the *closed*  $\rho$ -*neighbourhood* of  $U$  in  $X$  and  $U_{+\rho}^{\leq}$  is the *open*  $\rho$ -*neighbourhood*.

**On Hausdorff (dis)continuity.** Clearly, the boundaries of the open sublevels of  $d_U$  satisfy

$$\partial U_{-\rho}^{\leq} \subset Y_{-\rho} = \partial U_{-\rho}^{\leq},$$

where the *local minima* of the  $d_U$  on  $U$  make the difference set  $Y_{-\rho} \setminus \partial U_{-\rho}^{\leq}$ .

The set valued function  $\rho \mapsto U_{-\rho} \subset X$ ,  $\rho \in \mathbb{R}_+$ , is continuous for the Hausdorff metric in the space of subsets in  $X$  at those  $\rho$  where  $\partial U_{-\rho}^{\leq} = Y_{-\rho}$ , or, equivalently, where the closure of the interior of  $U_{-\rho}$  equals  $U_{-\rho}$ . Since  $\rho \mapsto U_{-\rho}$  is a *monotone decreasing* function in  $\rho$  for the inclusion order on subsets, it has *at most countably many* discontinuity points  $\rho$ . Also observe that the function  $\rho \mapsto Y_{-\rho} = d_U^{-1}(-\rho) = \partial U_{-\rho}$  is Hausdorff continuous at the Hausdorff continuity points of the function  $\rho \mapsto U_{-\rho}$  and the word “hypersurface” is justifiably applicable to  $Y_{-\rho}$  at these continuity points  $\rho$ .

### Exercise.

Let  $Z \subset X$  be a compact subset that is contained in a smooth hypersurface in  $X$ . Then, for all sufficiently small  $\rho > 0$ , there exists an open subset  $U \supset Z$  in  $X$  with smooth boundary  $Y$  such that  $Z = Y_{-\rho} = U_{-\rho}$ , i.e.  $Z$  serves in  $U$  as the set of the minima of the (minus distance to  $Y$ ) function  $d_U: U \rightarrow (-\infty, 0)$ .

### Example ( $U_{-\rho}$ as the intersection of translates of $U$ ).

If  $X = \mathbb{R}^n$  then, obviously,  $U_{-\rho}$  equals the intersection of the parallel  $r$ -translates  $U + r \subset \mathbb{R}^n$  for all  $r \in \mathbb{R}^n$  with  $\|r\| \leq \rho$  and  $Y_{-\rho}$  equals the topological boundary of this intersection.

$$U_{-\rho} = \bigcap_{\|r\| \leq \rho} U + r \quad \text{and} \quad Y_{-\rho} = \partial \left( \bigcap_{\|r\| \leq \rho} U + r \right).$$

Thus, *the transformation*  $U \mapsto U_{-\rho}$  *preserves all classes of Euclidean domains (e.g. the class of convex domains) that are closed under intersections.* It is also clear that

$$Y_{-\rho} \subset \partial \left( \bigcap_{\|r\| = \rho} U + r \right)$$

and if the boundary of  $U$  is *connected*, then

$$U_{-\rho} = \bigcap_{\|r\| = \rho} U + r \quad \text{and} \quad Y_{-\rho} = \partial \left( \bigcap_{\|r\| = \rho} U + r \right).$$

More generally, let  $\text{iso}_{\leq \rho}$  denote the set of isometries  $r: X \rightarrow X$  such that  $\text{dist}(x, r(x)) \leq \rho$  for all  $x \in X$ . Then, obviously,

$$U_{-\rho} \subset \bigcap_{r \in \text{iso}_{\leq \rho}} r(U).$$

Furthermore, if  $X$  is a *compact two-point homogeneous space*, i.e. the isometry group of  $X$  is transitive on the unit tangent bundle of  $X$ , then, as in the Euclidean case,

$$U_{-\rho} = \bigcap_{r \in \text{iso}_{\leq \rho}} r(U) \quad \text{and} \quad Y_{-\rho} \subset \partial \left( \bigcap_{r \in \text{iso}_{\leq \rho}} r(U) \right) \quad \text{for} \quad \text{iso}_{=\rho} = \partial(\text{iso}_{\leq \rho}).$$

### 3.2. Accessibility and quasi-regularity

A point  $x$  in the boundary  $Y = \partial U$  is called  $\rho$ -*accessible* (from  $U$ ), if  $x \in (U_\rho)_{+\rho} \subset U \cup Y$ . In other words,  $x$  is contained in some Riemannian  $\rho$ -ball in  $X$  that is contained in the closure of  $U$ . (The referee pointed out to me that this is usually called “with reach  $\rho$ ”, with a possible origin of the concept due to Federer.)

Say that an open subset  $U$  in  $X$  is  $C^2$ -*quasiregular* (at its boundary) if, loosely speaking, the singular locus  $\text{sing}_Y \subset Y = \partial U$  is non-accessible from  $U$ . More precisely, the following two conditions must be satisfied:

- The subsets

$$\text{Acc}_{>\rho_0}(Y) = A_{>\rho_0}(Y) = \bigcup_{\rho > \rho_0} A_\rho(Y) \subset Y \subset X$$

are *open* in  $Y$  for all  $\rho_0 \geq 0$ . Notice that this condition implies that, besides being open, the subsets  $A_{>0}(Y_{-\rho}) \subset Y_{-\rho}$ ,  $\rho > 0$ , are  $C^{1,1}$  in  $C^{1,1}$ -smooth Riemannian submanifolds in  $X$ .

- The subset  $A_{>0}(Y) \subset X$  is a  $C^2$ -*smooth* hypersurface in  $X$ , that is a  $C^2$ -smooth  $(n - 1)$ -submanifold without boundary that, topologically, is a locally closed subset in  $X$ .

If  $U \subset X$  is a *closed* domain then its quasiregularity means that of the *interior*  $\text{int}(U) \subset X$ . On the other hand, “quasiregularity of a hypersurface”  $H \subset X$  is understood as quasiregularity if its complement  $X \setminus H \subset X$ .

#### Almgren–Allard Quasiregularity Theorem.

Let  $X$  be a  $C^2$ -smooth Riemannian manifold. Then  $\phi$ -*bubbles*  $U \subset X$  are  $C^2$ -*quasiregular* for all continuous functions  $\phi(x)$ . Also, all kinds of *minimal hypersurfaces*  $H \subset X$  are *quasiregular*.

(See [13] for a simple prove of this.) Here “minimal hypersurface” is understood as a *minimal varifold* that does not, necessarily, bound any domain in  $X$ .

The following two instances of quasiregularity are, unlike Almgren–Allard theorem, fully obvious.

- (A) *Locally finite intersections of  $C^2$ -smooth domains with transversally intersecting boundaries are  $C^2$ -quasiregular.*
- (B) *If  $U$  a  $C^2$ -quasiregular domain in a  $C^2$ -smooth Riemannian manifold, e.g. the boundary  $Y$  is  $C^2$ -smooth to start with, then the sub-domains  $U_{-\rho}^c$  are also  $C^2$ -quasiregular.*

### 3.3. Smooth and non-smooth $k$ -mean convex functions and hypersurfaces

Let  $X$  be a  $C^2$ -smooth Riemannian manifold and let  $\text{Gr}_k(X)$  be the Grassmann space of the tangent  $k$ -planes  $\tau$  in  $X$ . Define the  $k$ -*Laplacian*  $\Delta_k$  from  $C^2$ -functions  $f$  on  $X$  to functions on  $\text{Gr}_k(X)$  by taking the traces of the Hessian of  $f$  on all  $\tau \in \text{Gr}_k(X)$ ,

$$\Delta_k(f)(\tau) = \text{trace}_\tau \text{Hess}(f).$$

For example  $\Delta_n = \Delta$  is the ordinary Laplacian for  $n = \dim X$  and  $\Delta_1(f)(\tau)$  equals the second derivative of  $f$  on the geodesic in  $X$  tangent to the tangent line  $\tau$ .

Say that  $f$  is  $k$ -*mean  $\varphi$ -convex* for a given continuous function  $\varphi = \varphi(\tau)$  on  $\text{Gr}_k(X)$  if  $\Delta_k(f)(\tau) \geq \varphi$ , where, as usual strictly corresponds to  $> \varphi$  and plain “convex” stands for 0-convex. Observe that the  $k$ -mean convexity says, in effect, that the gradient of  $-f$  is strictly  $k$ -volume contracting. Also notice that

$$\begin{aligned} k\text{-mean convex} &\implies l\text{-mean convex} && \text{for } l \geq k, \\ 1\text{-mean convex} = \text{convex}, &&& n\text{-mean convex} = \text{subharmonic} \end{aligned}$$

and that

$$(n - 1)\text{-mean convex} \implies \text{mean curvature convex},$$

while the converse implication is not, in general true.

On the other hand, a  $C^2$ -smooth strictly mean convex co-oriented hypersurface  $Y \subset X$  (e.g. if  $Y = \partial U$ ) can be realized as the zero set of a  $C^2$ -smooth strictly  $(n-1)$ -convex function  $f(x)$  defined in some neighbourhood of  $Y$ .

**Bending  $d(x)$  to a Mean Convex  $f(x)$ .** (Compare [5, 10].) A strictly mean convex function  $f$  can be obtained, for example, by “bending” the signed distance function  $d(x) = \pm \text{dist}(x, Y)$  (that is  $d(x) = d_U(x)$  for  $Y = \partial U$ ), i.e. where bending is achieved by means of a smooth strictly monotone increasing function  $\beta(d)$ ,  $-\infty < d < +\infty$ , that vanishes at  $d = 0$ , that has the first derivative  $d'(0) = 1$  and positive second derivative  $d''(x) > 0$ .

If  $Y$  is compact, then the function

$$f(x) = \beta \circ d(x) = \beta(d(x))$$

is strictly  $(n-1)$ -mean convex in some neighborhood of  $Y$ , provided the second derivative  $d''(0)$  is sufficiently large (compared to the absolute values of the negative principal curvatures of  $Y$ ).

If  $Y$  is non-compact one needs to modify this  $f$  by making its second derivative normal to  $Y$  to be large as a function on  $Y$ .

### Remark

The above remains true (and equally obvious) for  $k$ -mean convex hypersurfaces  $Y \subset X$ ,  $k = 1, 2, \dots, n-1$ , where the traces of the second fundamental forms are positive on the  $k$ -planes tangent to  $Y$ .

The notion of  $k$ -mean convexity extends from  $C^2$ -functions to all continuous ones via *linearity* of the operator  $\Delta_k$  by declaring a *continuous* function  $f(x)$  being  $k$ -mean  $\varphi$ -convex if it is contained in the *localized weak convex hull* of the space of *smooth*  $k$ -mean  $\varphi$ -convex functions. In other words,  $f(x)$  is  $k$ -mean  $\varphi$ -convex if  $\Delta_k(f)(\tau) - \varphi(\tau)$ , *understood as a distribution, is representable by a positive measure on  $\text{Gr}_k(X)$ .*

Then one defines the set of *strictly*  $k$ -mean  $\varphi$ -convex functions as the intersection of the sets of  $(\phi + \varepsilon)$ -convex ones, where the intersection is taken over all positive functions  $\varepsilon = \varepsilon(\tau)$  on  $\text{Gr}_k(X)$ . For example, a continuous function  $f$  is strictly  $k$ -mean convex if there exists a continuous function  $\varepsilon(x) > 0$  such that the restriction of  $f$  to every (local)  $k$ -dimensional submanifold  $Y \subset X$  with principal curvatures  $\kappa_y(Y)$  bounded by  $|\kappa_y(Y)| \leq \varepsilon(y)$  is a *subharmonic* function on  $Y$ .

Also one easily sees that if  $f_i$ ,  $i \in I$ , are strictly  $k$ -mean  $\phi$ -convex functions, then  $f(x) = \max_i f_i(x)$  is also strictly  $k$ -mean  $\phi$ -convex.

**Linearized Definition of  $k$ -Mean Curvature Convexity.** A cooriented hypersurface  $Y$  is called *strictly  $k$ -mean  $\varphi$ -convex* for  $\varphi = \varphi(\tau)$  defined on a neighborhood of the pullback of  $Y$  under the tautological map  $\text{Gr}_k(X) \rightarrow X$ , if  $Y$  is representable as the zero set of a continuous strictly  $k$ -mean  $\varphi$ -convex function  $f(x)$  defined in some neighbourhood of  $Y$ .

Here, “cooriented hypersurface” means that there is a neighbourhood  $X_0$  of  $Y$  where  $Y$  serves as the boundary of a closed domain  $U \subset X_0$ . Then our  $f$  must be positive inside  $U$  and positive outside. We say in this situation that  $U$  itself is *strict  $k$ -mean (curvature)  $\phi$ -convex (at the boundary)*.

**Mean Curvature Convexity of Functions Revisited.** (Compare [5, 10].) A continuous function  $f(x)$  is called *strictly  $k$ -mean  $\varphi$ -curvature convex* if, for every point  $x \in X$ , there exists a convex  $C^2$ -function  $\beta: \mathbb{R} \rightarrow \mathbb{R}$  with strictly positive derivative  $\beta' > 0$  such that the composed function  $x \mapsto \beta \circ f(x) = \beta(f(x))$  is strictly  $k$ -mean  $\varphi$ -convex in some neighbourhood of  $x \in X$ .

Notice that the so defined strict  $k$ -mean curvature  $\varphi$ -convexity is *stable under small  $C^2$ -perturbations of functions*. Also, *maxima of families of strictly  $k$ -mean curvature  $\varphi$ -convex functions are strictly  $k$ -mean curvature  $\varphi$ -convex*, since

$$\beta \circ \max_i f_i = \max_i \beta \circ f_i \quad \text{for monotone increasing } \beta,$$

and since strict  $k$ -mean  $\varphi$ -convexity is stable under taking maxima.



**Remark.**

Probably, little (essentially nothing?) changes if one allows *non-smooth* convex monotone increasing  $\beta$  in this definition.

***k*-Convexity Lemma.**

Let  $X$  be a  $C^2$ -smooth Riemannian manifold and  $U \subset X$  a quasiregular domain with boundary  $Y = \partial U$  which is  $k$ -mean  $\phi$ -convex on the regular locus  $\text{reg}_Y = Y \setminus \text{sing}_Y$  for a positive continuous function  $\phi = \phi(\tau) > 0$ . Then the minus distance function  $d_U(x) = -\text{dist}(x, Y)$  is  $k$ -mean curvature  $(\phi - \varepsilon)$ -convex in the interior of  $U$  for some continuous function  $\varepsilon = \varepsilon(\tau)$  that vanishes on the pullback of  $Y$  in  $\text{Gr}_k(X)$ . Moreover,  $\varepsilon(x)$  is bounded in terms of  $|\inf_x \text{Ricci}_x(X)|$  for  $x$  running over the ball  $B_x(R) \subset X$  for  $r = \text{dist}(x, Y)$ . (For instance,  $\varepsilon(x) = 0$  if the Ricci curvature is non-negative in this ball.)

The proof is quite simple and, I guess, is well known in some quarters. Yet, for the completeness sake, we spell it down in subsection 5.6, where our argument is essentially the same as that in [5, 10].

**Question.**

Is there a meaningful characterization of Plateau–Stein  $n$ -manifolds that admit (proper) strictly  $(n - 1)$ -mean convex functions?

**3.4. Smoothing and approximation**

Continuous strictly  $k$ -mean convex function  $f$  can be approximated by *smooth* strictly  $k$ -mean convex ones, by convolving with the following

**Standard  $\varepsilon$ -Smoothing Kernel.** Such a kernel is a function in two variables  $K_\varepsilon(x_1, x_2)$ ,  $\varepsilon > 0$ , on a Riemannian manifold  $X$  that is defined with some  $\Psi$  by

$$K_\varepsilon(x_1, x_2) = \lambda(x_1) \Psi(\varepsilon^{-1} \text{dist}(x_1, x_2)), \quad \varepsilon > 0,$$

for

$$\lambda(x_1) = \left( \int_X \Psi(\varepsilon^{-1} \text{dist}(x_1, x_2)) dx_2 \right)^{-1}.$$

A *standard  $\varepsilon$ -smoothing* of functions on  $X$  is

$$f(x) \mapsto f_\varepsilon(x) = \int_X f(x_2) \cdot K_\varepsilon(x, x_2) dx_2.$$

It is obvious that if  $f$  is a *continuous* strictly  $k$ -mean  $\phi$ -convex function and  $V \subset X$  is a compact subset, then  $f_\varepsilon$  is strictly  $k$ -mean  $\phi$ -convex on  $V$  for all sufficiently small  $\varepsilon > 0$ .

It follows that  $f$  can be uniformly, and even in the fine  $C^0$ -topology, approximated by  $C^2$ -smooth strictly  $k$ -mean  $\phi$ -convex functions, where, moreover, such approximating functions can be chosen equal  $f$  on a closed subset  $X_0 \subset X$  if  $f$  itself is smooth in a neighborhood of  $X_0$  in  $X$ . Recall that  $C^0$ -fine approximation means that the difference between  $f$  and an approximating function can be made less than a given strictly positive continuous function on  $X$ .

**Curvature Smoothing Corollary.**

Let  $f(x)$  be a continuous strictly  $k$ -mean curvature  $\phi$ -convex function on a  $C^2$ -smooth Riemannian manifold  $X$ . Then  $f$  can be  $C^0$ -finely approximated by smooth strictly mean curvature  $\phi$ -convex functions with non-degenerate critical points.

**Proof.** Locally, in a neighborhood  $U_x \subset U$  of a given point  $x$ , such an approximation is obtained by finely approximating  $f \circ \beta(x) = f(\beta(x))$ , for a suitable  $\beta$ , by a smooth  $k$ -mean  $\phi$ -convex function on  $U_x$ , call such an approximation  $(f \circ \beta)_{\text{appr}}$ , and then applying the inverse  $-\beta$ -function, thus approximating  $f$  by  $f_{\text{appr}} = \beta^{-1} \circ (f \circ \beta)_{\text{appr}}$ . Then the global  $C^2$ -smooth approximation of  $f$  is obtained, by a usual argument with a covering of  $X$  by open subsets  $U_i$ ,  $i = 1, 2, \dots, n + 1 = \dim X + 1$ , where each of  $U_i$  equals the disjoint union of arbitrarily small subsets. Finally, “ $C^2$ -smooth” is upgraded to “generic  $C^\infty$ ” by an arbitrarily  $C^2$ -small perturbation. □

## 4. Splicing, smoothing and extending distance functions

We shall prove in this section the inverse maximum principles stated in subsection 1.3.

### 4.1. Staircase of distance functions

Let  $X$  be a Riemannian manifold let  $U_1 \supset U_2 \supset U_3 \supset \dots \supset U_i \supset \dots \subset X$  be closed domains with boundaries  $Y_i = \partial U_i$ , let  $d_i(x)$ ,  $x \in U_i$ , denote the minus distance function from  $x$  to the boundary  $Y_j = \partial U_j$  and let some numbers  $\delta_i > 0$  satisfy

$$\delta_i > \sup_{x \in Y_{i+1}} \text{dist}(x, Y_i).$$

Then there exists a negative proper continuous function  $h: X \rightarrow \mathbb{R}_-$ , such that locally, in a neighbourhood of every point  $x \in X$ , this  $h$  equals the maximum of the functions  $\beta_i \circ d_i$ , for  $k_x \leq i \leq l_x$ , where

- $l_x$  is the maximal  $l$  such that  $x \in U_l$ ;
- $k_x$  is the minimal  $k$  such that  $\text{dist}(x, Y_k) \leq \delta_k$ ;
- $\beta_j$  are smooth monotone increasing functions,  $\beta_k: \mathbb{R} \rightarrow \mathbb{R}$ , with strictly positive derivatives,  $\beta_j' > 0$ .

**Proof.** The required max-function  $h$  is determined by its sublevels, call them  $Y_{-\rho} = h^{-1}(-\infty, -\rho) \subset X$ , that come as intersections of certain sublevels of the functions  $d_i$  that are  $(U_i)_{-\rho_i} \subset U_i = d_i^{-1}(\infty, \rho_i]$  for some  $\rho_i$  that must be continuous strictly increasing functions in  $\rho$ . The essential point is to choose these  $\rho_i$  such that if the boundary of some  $(U_i)_{-\rho_i}$  passes through a point  $x$  contained in the boundary of the intersection  $\cap_i (U_i)_{-\rho_i}$ , then  $\text{dist}(x, Y_i) \leq \delta_i$ . Since  $\delta_i > \sup_{x \in Y_{i+1}} \text{dist}(x, Y_i)$ , this inequality can be obviously satisfied with some  $\rho_i$  and the proof follows.  $\square$

#### Corollary (Non-Smoothed Inverse Maximal Principle for compact manifolds).

Let  $X$  be a compact Riemannian  $C^2$ -smooth manifold with strictly mean convex boundary. Then either  $X$  contains a compact minimal hypersurface in its interior or it admits a continuous negative strictly mean convex function that vanishes on the boundary of  $X$ .

**Proof.** Shrinking the mean convex “ends” of  $X$  (see subsection 2.4) provides a finite descending sequences of  $\phi$ -convex bubbles  $U_i$  with a fixed (albeit very small) strictly positive  $\phi$  and with arbitrarily small  $\sup_{x \in Y_{i+1}} \text{dist}(x, Y_i) > 0$ . Then the above  $h$  is strictly mean convex being local maximum of distance functions that are strictly convex by the  $k$ -convexity lemma in subsection 3.3.  $\square$

#### Remark.

Bruce Kleiner explained to me how a version of this follows by an application of the mean curvature flow, but this does not seem to be simpler than our more pedestrian argument.

### 4.2. Proofs of Inverse Maximum Principles

What remains is to justify Step 4 in the proof of IMP in subsection 1.4. Let, for instance,  $X$  be a complete  $C^2$ -smooth Riemannian manifold that is connected and thick at infinity. We already know (see subsection 2.4) that if  $X$  contains no minimal hypersurface then it can be exhausted by compact strictly mean convex bubbles  $U_j$ . We also know that each  $U_j$  can be shrunk via smaller bubbles  $U_{ji} \subset U_j$ ,

$$U_{j0} = U_j \supset U_{j1} \supset U_{j2} \supset \dots \supset U_{ji} \supset \dots,$$

where the minus distance functions  $d_{ij}(x) = -\text{dist}(x, \partial U_{ji})$ ,  $x \in U_{ji}$ , can be “spliced” to continuous mean curvature convex functions  $h_j$  on  $U_j$ .

If the (positive!) mean curvatures of the boundary hypersurfaces  $Y_{ji} = \partial U_{ji}$  are bounded from above at all points  $x \in Y_{ji}$  by  $\alpha(x)$ , where  $\alpha(x)$  is a (possibly very fast growing) continuous function on  $X$ , then, by the usual compactness principle of the geometric measure theory, some subsequence of  $h_j$  converges on all compact subsets in  $X$  to the required  $h$ .

A transparent way to achieve the control over  $\sup \text{mn.curv}(\partial U')$  of a bubble  $U'$  inside a given bubble  $U$  is to see the construction of  $U'$  in terms of an obstacle (see subsection 1.4) that is a subdomain  $V \subset U$  that must be engulfed by  $U'$ . If the mean curvatures of the boundary of  $V$  at all boundary points are bounded by  $\alpha(x)$ , then the same bound will hold for  $\partial(U')$ . If, for instance,  $X$  has Ricci curvature bounded from below, one may take  $V = (U_{-\rho})_{+\rho/2}$ , where this  $V$  (pinched between  $U_{-\rho/2}$  and  $U_{-\rho}$ ) has its mean curvatures bounded by above roughly by  $\rho^{-(n-1)}$ . In general, one modifies this by replacing constant  $\rho$  by a positive function  $\rho(x)$  on  $X$ , that must decay, roughly, as  $(1 + |R(x)|^{(n-1)})^{-1}$ , for a negative function  $R(x)$  that serves as a lower bound for the Ricci curvature of  $X$ .

The curvature of the boundary of such  $V$ , that is obtained by pushing  $U$  inward by  $\rho(x)$  and then outward by  $\rho(x)/2$ , can be easily bounded by some (very fast growing)  $\alpha(x)$ . This argument, that extends to multi-ended manifolds with the preparations made in subsections 2.2–2.4, yields the following non-regular IMP stated in subsection 1.4.

### Trichotomy Theorem.

Let  $X$  be a complete Riemannian  $C^2$ -smooth  $n$ -manifold. Then (at least) one of the the following three conditions is satisfied:

- $X$  admits a proper strictly mean curvature convex function  $h: X \rightarrow \mathbb{R}_+$ .
- $X$  contains a minimal hypersurface  $H$  that is closed in  $X$  as a subset and such that  $\text{vol}_{n-1}(H) < \infty$ .
- $X$  admits a non-proper continuous strictly mean curvature convex function  $h: X \rightarrow \mathbb{R}$  and also there is a sequence of minimal hypersurfaces  $H_i \subset X$  with boundaries  $\partial H_i$  contained in a fixed compact subset  $X_0 \subset X$ , where these  $H_i$  are closed in  $X$  as subsets and such that  $\text{vol}_{n-1}(H_i) < \infty$ ,  $\text{diam}(H_i) \rightarrow \infty$ ,  $i \rightarrow \infty$ .

**Proof of the Regularized Maximum Principles.** The above functions  $h_i$  are approximated by  $C^2$ -Morse functions and the minimal  $H$  is approximated by slightly concave hypersurfaces according to the smoothing lemma. (See subsection 3.4). This accomplishes the proof of the IMPs stated in subsection 1.3.  $\square$

### Remark.

It seems, I did not check the details, the above theorem remains true with “convex” replaced everywhere by  $\phi$ -convex for a given continuous (not even necessarily positive) function  $\phi(x)$ , where the minimality condition on  $H$  must be replaced by  $\text{mn.curv}_x(H) = \phi(x)$ ,  $x \in H$ , and where the finiteness requirement for the  $(n-1)$ -volume of  $H$  must be replaced by a suitable finiteness condition for some  $\phi$ -area.

## 5. Generalized convexity

We look at the mean convexity from a broader perspective in this section and we prove the  $k$ -mean convexity lemma from subsection 3.4. All of what we say is known but dispersed in the literature.

### 5.1. Smooth and non-smooth convexity classes

A *coorientation* of a germ of hypersurface  $Y$  at a point  $x$  in a manifold  $X$  is expressed by calling the closure of one of the two “halves” in the complement  $B_x \setminus Y$ , for a small ball at  $x$ , being *inside*  $Y$  and the closure of the other half *outside*  $Y$ . Thus, cooriented germs at  $x \in X$  are partially ordered. We agree, thinking of  $Y_2$  being *more* convex than  $Y_1$ , that  $Y_2 \succeq_x Y_1$  stands for  $Y_2$  is *inside*  $Y_1$ . Formally, being inside a cooriented  $Y_1$  does not need any coorientation of  $Y_2$ . In fact,  $Y_2$  is *inside*  $Y_1$  implies that  $Y_1$  is *outside*  $Y_2$  *only for one* of the two coorientations of  $Y_2$ . So the above “ $Y_2$  is *inside*  $Y_1$ ” tacitly assumes that this does imply “ $Y_1$  is *outside*  $Y_2$ ”; moreover, if, geometrically, without coorientations,  $Y_1 = Y_2$ , then “inside” means that their coorientations are equal as well.

Assume  $X$  is smooth, let  $T(X)$  denote the tangent bundle of  $X$  and  $\mathcal{H}$  be the space of tangent cooriented hyperplanes  $H = H_x \subset T_x(X)$ ,  $x \in X$ , that are the tangent spaces to germs of smooth cooriented hypersurfaces in  $X$ . Observe that the relation  $Y_2 \geq_x Y_1$  between  $C^1$ -smooth cooriented hypersurfaces implies that they have equal *oriented* tangent spaces (hyperplanes) at  $x$ . Accordingly, we may write  $Y_2 \geq_H Y_1$  instead of  $Y_2 \geq_x Y_1$  for their common cooriented tangent hyperplane  $H \subset T_x(X)$ .

Given a cooriented hyperplane  $H \subset T_x(X)$ , denote by  $\mathcal{Q}_H = \mathcal{Q}_H(X)$  the space of quadratic functions (forms)  $H \rightarrow T_x(X)/H = \mathbb{R}$  and by  $\mathcal{Q} = \mathcal{Q}(X)$  the space of  $\mathcal{Q}_H$  over all  $H \subset T_x(X)$ ,  $x \in X$ .

The affine space  $\text{aff}(\mathcal{Q}_H)$  naturally represents the space of 2-jets  $J_x^2(Y)$  of germs of cooriented hypersurfaces  $Y$  at  $x$  that are tangent to  $H$  and one may speak of the difference  $J_x^2(Y_1) - J_x^2(Y_2) \in \mathcal{Q}_H$ . Obviously,

$$Y_2 \geq_H Y_1 \quad \implies \quad J_x^2(Y_1) - J_x^2(Y_2) \geq 0,$$

where we refer to the usual partial order on the space  $\mathcal{Q}_H$  regarded as a space of  $\mathbb{R}$ -valued functions on  $H$ , where this implication is reversible for

**Strict Order.** The above implication is not, in general, reversible but it is reversible in the strict form:

$$Y_2 >_H Y_1 \quad \iff \quad J_x^2(Y_1) - J_x^2(Y_2) > 0,$$

where the *strict inequality between germs* signifies that not only  $Y_2 \geq_H Y_1$ , but also that this non-strict inequality is *stable* under small  $C^2$ -perturbations of the germs that remain tangent to  $H$ .

If  $X$  is endowed with an affine connection, then one may identify  $\text{aff}(\mathcal{Q}_H)$  with  $\mathcal{Q}_H$ ; thus, every germ  $Y$  is assigned the quadratic form  $Q$  on  $H = T_x(Y) \subset T_x(X)$  with values in  $T_x(X)/H$ . If, moreover,  $X$  is a Riemannian manifold, then  $T_x(X)/H$  is *canonically* isomorphic to  $\mathbb{R}$ , and  $Q$  equals the *second fundamental form* of  $Y$  at  $x$ .

Denote by  $\mathcal{Q}_{\text{aff}}(X)$  the space of 2-jets of cooriented hypersurfaces  $Y \subset X$  and call a subset  $\mathcal{R} \subset \mathcal{Q}_{\text{aff}}(X)$  a *convexity relation* (of second order) if

$$J_x^2(Y_1) \in \mathcal{R} \quad \implies \quad J_x^2(Y_2) \in \mathcal{R} \quad \text{for all germs} \quad Y_2 \geq Y_1,$$

where  $Y_2 \geq Y_1$  signifies that both  $Y$  have the *same* underlying cooriented tangent space (hyperplane)  $H \subset T_x(X)$ , where this inequality makes sense. We say that a cooriented  $C^2$ -smooth hypersurface  $Y \subset X$  *satisfies*  $\mathcal{R}$ , or it is  $\mathcal{R}$ -convex, if the 2-jets of  $Y$  are contained in  $\mathcal{R}$  at all points  $y \in Y$ . If  $X$  is a Riemannian manifold, then  $\mathcal{Q}_{\text{aff}}(X) = \mathcal{Q}(X)$  and such a relation is expressed in terms of the second fundamental forms of hypersurfaces.

## 5.2. $k$ -convexity and $(n - k)$ -mean convexity

Let  $X$  be an  $n$ -manifold with an affine, e.g. Riemannian, connection and say that a cooriented  $C^2$ -hypersurface  $Y \subset X$  is  $([k_{\geq}] + [k_{>}])$ -convex if the second fundamental form of  $Y$  with values in  $T(X|Y)/T(Y)$ , when diagonalized, has at least  $k_{\geq}$  nonnegative terms and  $k_{>}$  positive terms. If only one of the two terms in the sum  $k = k_{\geq} + k_{>}$  is present, one speaks of  $k$ -convexity for  $k = k_{\geq}$  and of *strict  $k$ -convexity* for  $k = k_{>}$ . Accordingly, a domain  $V \subset X$  is called  $([k_{\geq}] + [k_{>}])$ -convex if its boundary is  $([k_{\geq}] + [k_{>}])$ -convex.

For instance, a small  $\varepsilon$ -neighbourhood of compact smooth submanifold  $P^{n-k-1}$  of codimension  $k + 1$  in a Riemannian  $X$  is strictly  $k$ -convex and it is easy to show that every curve-linear subpolyhedron in  $X$  of codimension  $k + 1$  also admits an arbitrarily small strictly  $k$ -convex neighbourhood.

If  $X = \mathbb{R}^n$ , these  $[k_{\geq}] + [k_{>}]$  are *the only* convexity relations that are invariant under *affine* transformations of  $\mathbb{R}^n$ , where  $k = n - 1$  corresponds to the ordinary local convexity, while 1-convex hypersurfaces are nowhere concave.

The distinction between “ $\leq$ ” and “ $<$ ” is nonessential for *compact*  $Y \subset \mathbb{R}^n$ , since, (almost) obviously (see [2, § 1/2]) every smooth compact (possibly with a boundary and with a self-intersection)  $k$ -convex hypersurface  $Y$  in  $\mathbb{R}^n$  can be  $C^2$ -approximated by *strictly*  $k$ -convex hypersurfaces  $Y'$  that may be positioned, depending on what you wish, inside or outside  $Y$ .

**Remark/Question.**

If  $Y$  is non-compact, then a “strict” approximation of  $Y$  by  $Y'$  may be possible in one topology, e.g. for  $Y'$  being obtained from  $Y$ , by a map  $f$  with  $\text{dist}(y, f(y)) \leq \varepsilon$  but not in a finer topology where  $\varepsilon = \varepsilon(y) \rightarrow 0$  for  $y \rightarrow \infty$ . Besides, an approximation of  $([k_{\geq}] + [k_{>}])$ -convex hypersurfaces by  $([k'_{\geq} - l] + [k'_{>} + l])$ -convex ones may depend on  $l$  and on your positioning  $Y'$  inside/outside  $Y$ . Is there a comprehensive description of what may happen?

Since a generic linear function  $f$  on a  $k$ -convex domain  $V \subset \mathbb{R}^n$  bounded by  $k$ -convex hypersurface adds no  $l$ -handles to sublevels of  $f$  at the critical points of  $f$  on  $Y$ , a compact  $k$ -convex domain  $V \subset \mathbb{R}^n$  is diffeotopic to a regular neighbourhood of  $(n - k - 1)$ -dimensional subpolyhedron  $P^{n-k-1} \subset \mathbb{R}^n$ .

**Questions.**

Does there exist such a diffeotopy  $f_t: V \rightarrow \mathbb{R}^n$ , (that eventually “shrinks”  $V$  to  $P^{n-k-1}$ ) where all intermediate domains  $f_t(V)$ ,  $t > 0$ , (for  $f_0(V) = V$ ) are  $k$ -convex?

What are topological possibilities of  $k$ -convex domains in the Euclidean  $n$ -sphere?

Observe that the complement to a disjoint union of  $\varepsilon$ -neighbourhoods of two or more equatorial spheres of dimension  $k$ ,  $k < n/2$ , is strictly  $k$ -convex; it is contractible to some  $P^{n-1}$  but not to any  $P^{n-k-1}$ . (A more traditional problem concerns  $k$ -convex domains  $V \subset S^n$ , such that, moreover, the complementary domains  $S^n \setminus V$  are  $(n - 1 - k)$ -convex.)

Recall that a  $C^2$ -smooth cooriented hypersurface  $Y \subset X$  is called  $(n - k)$ -mean convex if the traces of the second fundamental form of  $Y$  restricted to the tangent  $(n - k)$ -planes  $H^{n-k} \subset T(Y)$  are non-negative. In other words, the principal curvatures of  $Y \subset X$ , say  $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_{n-1}$ , satisfy

$$\kappa_1 + \kappa_2 + \dots + \kappa_{n-k} \geq 0 \quad \text{at all points } y \in Y.$$

(If  $k = 1$  this means that  $Y$  is convex and if  $k = n - 1$  this says that the mean curvature of  $Y$  is non-negative.) Accordingly, strict  $(n - k)$ -mean convexity requires this inequality to be strict, i.e. all traces to be positive.

**Question.**

Can every closed  $(n - k)$ -mean convex hypersurface in  $\mathbb{R}^n$  be approximated by strictly  $(n - k)$ -mean convex ones? (This is easy for  $k = 1$  and  $k = n - 1$ , but I see no immediate proof it for other  $k$ . Am I missing something obvious?)<sup>2</sup>

Clearly, (strictly)  $(n - k)$ -mean convex hypersurfaces are (strictly)  $k$ -convex, and every embedded closed  $k$ -convex hypersurface in the Euclidean space  $\mathbb{R}^n$  is isotopic to a strictly  $(n - k)$ -mean convex one (since it can be brought to a neighbourhood of  $k$ -subpolyhedron  $P^k \mathbb{R}^n$ ). But this is far from being true, even on the homotopy level, in non-Euclidean spaces.

**Mean Convex Surgery.** Let  $V \subset X$  be a smooth  $(n - k)$ -mean convex domain and let  $B^l \subset X$  be a smooth disk that all lies outside  $V$  except for its boundary sphere  $S^{l-1} = \partial B^l \subset Y = \partial V$ , where we assume (just for the civility sake) that  $B^l$  meets  $Y$  normally, i.e. under the angle  $\pi/2$  along  $S^{l-1} = B^l \cap Y$ . Let us slightly thicken  $B^l$  by taking its  $\varepsilon$ -neighbourhood, denoted  ${}_{\varepsilon}B^l \subset X$ , and observe, assuming  $\varepsilon > 0$  is sufficiently small, that

- the union  $V \cup {}_{\varepsilon}B^l$  has smooth boundary except for a  $\sim \pi/2$  corner along the boundary of a small tubular neighbourhood of  $S^{l-1} \subset Y$ ;
- the new smooth part of the boundary of  $V \cup {}_{\varepsilon}B^l$ , that is  $\partial(V \cup {}_{\varepsilon}B^l) \setminus \partial V = \partial({}_{\varepsilon}B^l) \cap (X \setminus V)$ , is  $(l + 1)$ -mean convex.

<sup>2</sup> Bruce Kleiner pointed out to me that such approximation is possible with the mean curvature flow.

If  $l \leq n - k - 1$ , then the corner in  $V \cup_\varepsilon B^l$  can be  $(n - k)$ -mean convexly smoothed.

*About the proof.* The boundary of the union  $V \cup_\varepsilon B^l$  is concave along the corner and the obvious smoothing of  $V \cup_\varepsilon B^l$  does not give us an  $(n - k)$ -mean convex domain. However, the  $(n - k)$ -mean curvature of the boundary of the  $\varepsilon$ -tube around  $B^l$  for  $l \leq n - k - 1$  tends to  $+\infty$  for  $\varepsilon \rightarrow 0$ . This “infinite excess of positivity” allows one to construct strictly  $(n - k)$ -mean smoothing similarly but easier than how it was done in [4] for scalar curvature.

### 5.3. Convergence stability

The limit behavior of *embedded*  $\mathcal{R}$ -convex hypersurfaces is opposite to what is demanded by the  $C^0$ -dense  $h$ -principle: the spaces of such hypersurfaces are *closed* rather than dense in the  $C^0$ -topology for closed subsets  $\mathcal{R} \subset \Omega$ .

Moreover, let  $\mathcal{R} \subset \Omega(X)$  be a closed convexity relation and let  $Y \subset X$  be a  $C^2$ -smooth cooriented hypersurface that is closed in  $X$  as a subset. Let  $U_i \supset Y$ ,  $i = 1, 2, \dots$ , be a sequence of neighbourhoods such that  $\bigcap_i U_i = Y$  and let  $Y_i \subset U_i$  be smooth cooriented hypersurfaces closed in  $U_i$  as subsets, the closures of which do not intersect the boundaries of  $U_i$  and that separate the components of the boundaries  $\partial U_i$  in the same way as  $Y$  does. In other words,  $Y_i$  are homologous to  $Y$  in  $U_i$  (in the sense of homology with infinite supports if  $Y$  is non-compact). If all  $Y_i$  satisfy a closed convexity relation  $\mathcal{R}$  then  $Y$  also satisfies  $\mathcal{R}$ .

**Proof.** In fact let  $Q_0$  be the jet of  $Y$  at some point  $Y_0$  and  $\Omega_0 \subset \Omega$  be an arbitrarily small neighbourhood of  $Q_0$ . Then, by the weak (and obvious) *maximal principle*, every  $Y_i$  for all  $i \geq i_0 = i_0(\Omega_0)$  contains a point  $y_i^-$  such that the 2-jet  $J_{y_i^-}^2(Y_i) \in \Omega$  (or a germ at this point, if you wish) satisfies

$$J_{y_i^-}^2(Y_i) \leq \omega_i \quad \text{for some } \omega_i \in \Omega_0$$

(as well as a point  $y_i^+$ , where  $J_{y_i^+}^2(Y_i) \geq \omega'_i$  for some  $\omega'_i \in \Omega_0$ ). □

#### (Counter)examples.

- (a) Every curve  $Y$  in the plane can be (obviously)  $C^0$ -approximated by locally convex *immersed* curves  $Y_i$ . By the above, these  $Y_i$  must have lots of double points as  $Y_i$  come close to the region where  $Y$  is concave.
- (b) Similarly, according to Lawson and Michelson [6] every co-orientable immersion  $f$  of an  $(n - 1)$ -manifold  $Y$  to a Riemannian  $n$ -manifold  $X$  can be  $C^0$ -approximated by immersions  $f_i: Y \rightarrow X$  with positive mean curvatures. (The building blocks of  $f_i$  are finite coverings maps onto the boundaries of  $\varepsilon$ -neighbourhoods of  $(n - 2)$ -submanifolds in  $X$ , where, observe, such boundaries have mean curvatures  $\sim \varepsilon^{-1}$  for  $\varepsilon \rightarrow 0$ .)
- (c) In contrast with the above, if  $k > n/2$ , then every closed cooriented strictly  $(n - k)$ -mean convex hypersurface  $Y$  in a complete Riemannian manifold  $X$  with non-negative sectional curvatures bounds a compact Riemannian manifold  $U$ , i.e.  $\partial U = Y$ , such that the immersion  $Y \rightarrow X$  extends to an isometric immersion  $U \rightarrow X$ . (This  $U$  is contractible to its  $k$ -skeleton, since the minus distance function  $u \mapsto -\text{dist}_U(u, \partial U = Y)$  admits an approximation by an  $(n - k)$ -mean convex Morse function on  $U$  that provides an isotopy of  $U$  to a regular neighbourhood of a  $k$ -dimensional subpolyhedron  $P^k \subset U$  [10].)

On the other hand,  $k$ -convex hypersurfaces for  $k < n - 1$  in general *non-flat*  $n$ -manifolds  $X$  with non-negative curvatures do not necessarily bound immersed  $n$ -manifolds in  $X$ . But this is true in the presence of many “movable” totally geodesic submanifolds in  $X$  by the Euclidean argument from [2, § 1/2], where the standard examples of such manifolds are Riemannian cylinders  $X = X_0 \times \mathbf{R}$  and complete simply connected  $n$ -spaces  $X$  of constant negative curvature.

#### Questions.

What are possible topologies of (embedded and immersed)  $k$ -convex hypersurfaces in the Euclidean  $n$ -sphere? Are there any constraints on immersed  $(n - k)$ -mean convex hypersurfaces in the Euclidean  $n$ -space for  $n \geq 2k$ ?

The convergence stability suggests that the notion of  $\mathcal{R}$ -convexity can be extended to non-smooth subsets. The cheapest way to produce non-smooth examples starting with the class  $\mathcal{U}$  of smooth  $\mathcal{R}$ -convex domains  $U \subset X$ , i.e. having smooth  $\mathcal{R}$ -convex boundaries, is enlarging/completing  $\mathcal{U}$  by some/all of the following four operations over subsets.

$[\cap]_{<\infty}$  Locally finite intersections of smooth domains  $U_i \in \mathcal{U}$  with mutually transversal boundaries.

$[\cap \downarrow]_{\infty}$  Intersections of infinite *decreasing* families of subsets.

$[\cap]_{\leq\infty}$  Finite and infinite intersections of smooth domains, that is essentially the same as  $[\cap]_{<\infty} + [\cap \downarrow]_{\infty}$ .

$[\cup \uparrow]_{-\infty}$  Union of infinite *increasing* families.

### Question.

Given, say an open, convexity relation  $\mathcal{R} \subset \mathcal{Q}(X)$ , let  $\mathcal{C}(\mathcal{R})$  denote the class of all compact subsets in  $X$  obtained from compact smooth  $\mathcal{R}$ -convex domains  $U \subset X$  by some of the above operations, e.g. by  $[\cap]_{\leq\infty}$ , i.e. by taking infinite intersections of compact smooth  $\mathcal{R}$ -convex domains  $U$ . Is there any, not necessarily exhaustive, characterization of subsets in  $\mathcal{C}(\mathcal{R})$  in terms of  $\mathcal{R}$ ? Is every  $\mathcal{R}$  uniquely determined by  $\mathcal{C}(\mathcal{R})$ ?

For instance, which *Cantor sets*  $C \subset \mathbb{R}^n$  are representable as infinite intersections of disjoint finite unions of compact convex subsets? Clearly, this is possible if the Hausdorff dimension of  $C$  satisfies  $\dim_{\text{Hau}} C < 1$ , but “generic” subsets  $C$  with  $\dim_{\text{Hau}} C > 1$  admit no such representation.

In fact, the geometry of a Cantor set  $C \subset \mathbb{R}^n$  at a point  $x \in C$  may be characterized by the minimal possible “oscillatory complexity”,  $\text{osc}_\varepsilon = \text{osc}_\varepsilon(C, x)$ ,  $\varepsilon > 0$ , e.g. the *total curvature*  $\text{curv}_\varepsilon$  (that is the  $(n-1)$ -volume of the tangential Gauss map counted with multiplicity) of the boundaries of smooth neighborhoods  $U_\varepsilon \subset \mathbb{R}^n$  of  $x$  such that  $\text{diam}(U_\varepsilon) \leq \varepsilon$  and where the boundaries  $\partial U_\varepsilon$  do not intersect  $C$ . It seems “most” Cantor sets in  $\mathbb{R}^n$ ,  $n \geq 2$ , (I checked this only for a few particular classes of sets) have  $\text{osc}_\varepsilon \rightarrow \infty$ , e.g.  $\text{curv}_\varepsilon \rightarrow \infty$  for  $\varepsilon \rightarrow 0$ , and they do not belong to any convexity class  $\mathcal{R} \subset \mathcal{Q}$ , unless  $\mathcal{R}$  equals  $\mathcal{Q}$  minus a “very thin” subset.

**Convergence Stability for  $k$ -Mean Convexity for Functions.** Since this convexity is defined by *linear* inequalities on the (second) derivatives of functions  $f$ , it is stable under all kinds of weak limits and it non ambiguously extends to continuous functions as we saw in subsection 3.3.

## 5.4. Riemannian curvature digression

The above is a baby version of the following Riemannian problems. Given two 2-jets, or germs  $g_1$  and  $g_2$  of Riemannian metrics at a point  $x$  in a smooth manifold  $X$ , write  $g_1 \preceq g_2$ , if the two have equal 1-jets and their sectional curvatures satisfy

$$\text{curv}_\tau(g_1) \geq \text{curv}_\tau(g_2) \quad \text{for all tangent 2-planes } \tau \subset T_x(X).$$

For example, metrics with “large amount” of positive curvature are regarded as small.

A *lower curvature relation/bound*  $\mathcal{B}$  is a subset of 2-jets  $g$  of Riemannian metrics at the origin in  $\mathbb{R}^n$  such that

- 1-jets of  $g$  equal the 1-jet of the Euclidean metric;
- if  $g_2 \in \mathcal{B}$  and  $g_1 \preceq g_2$  then  $g_1 \in \mathcal{B}$ ;
- the subset  $\mathcal{B}$  in the space of jets is invariant under orthogonal transformations of  $\mathbb{R}^n$ .

The latter condition allows one to invariantly speak of  $\mathcal{B}$ -positive metrics on *all* smooth  $n$ -manifolds  $X$  that are, in other words, *Riemannian  $n$  manifolds that satisfy  $\mathcal{B}$*  (compare with [2, § 7]).

The fundamental questions are as follows.

- A. Given  $\mathcal{B}$  what is the weakest topology/convergence  $\mathcal{T} = \mathcal{T}(\mathcal{B})$  in the space of Riemannian manifolds, such that the limits of  $\mathcal{B}$ -positive manifolds are  $\mathcal{B}$ -positive?

B. What are singular  $\mathcal{B}$ -positive metrics spaces?

C. What are  $\mathcal{B}$  for which the above two questions have satisfactory answers?

If  $\mathcal{B}$  consists of the metrics with a given bound *all* sectional curvatures, then the (best known) answer to A. is the *Hausdorff convergence* of metric spaces and B. is essentially resolved by the theory of Alexandrov spaces.

The starting point of the theory for spaces with a lower bound on the *Ricci* curvature is the (almost obvious) stability of the inequality  $\text{Ricci}(g) \geq \text{const} \cdot g$  under  $C^0$ -limits of Riemannian metrics on a given underlying (and unchangeable) smooth manifold  $X$  while the general theory, albeit not fully established, is well underway, see [1, 7, 9] and references therein.

The most tantalizing relation  $\mathcal{B}$  is expressed with the *scalar curvature* by  $\text{scal}(g) \geq \text{const}$ , where even the  $C^0$ -limit stability is not fully established and where some possibilities are suggested by the *intrinsic flat distance* [11].

Nothing seems to be known about other  $\mathcal{B}$ , e.g. those encoding some positivity of the curvature operator, e.g. positivity of the *complexified sectional curvature*, see [8] and [2, §7].

## 5.5. Cut locus, focality and curvature blow-up

Let us see what happens to convexity under equidistant deformations of a hypersurface  $Y \subset X$ , where an attention must be paid to singularities on the *cut locus* that may be aggravated by the presence of *focal points*. Recall that the cut locus  $\text{cut}(U) \subset X$  of an open subset  $U \subset X$  (or of a closed domain  $U$ ) with respect to  $Y = \partial U$  is defined as the closure of the set of points  $u \in U$  that have more than one *ancestor* in  $Y$ , where a point  $x$  in the closure of  $U$  is called a *d-ancestor*, for  $d = \text{dist}(x, u)$ , or just “ancestor” of a point  $u \in U$ , with  $u$  being called a *d-descender*, or “descender” of  $x$ , if  $\text{dist}(x, u) = \text{dist}(u, Y) - \text{dist}(x, Y)$ . Assume that  $X$  is a complete  $C^2$ -smooth Riemannian manifold and recall a few obvious properties of  $\text{cut}(U)$ .

If  $Y = \partial U$  is a  $C^2$ -hypersurface, then the cut locus of  $U$  does not intersect  $Y$  and the  $\rho$ -equidistant hypersurfaces denoted  $Y_{-\rho} \subset U$  are  $C^2$ -smooth away from  $\text{cut}(U)$ , i.e. the complements  $Y_{-\rho} \setminus \text{cut}(U)$  are  $C^2$ -smooth (locally closed) hypersurfaces in  $U$ .

If the boundary  $Y = \partial U$  is  $C^1$ -smooth, then  $x \in Y$  is  $\rho$ -accessible from  $U$  if and only if the geodesic segment of length  $\rho$  issuing from  $x$  normally to  $Y$  inward  $U$  either *does not* intersect  $\text{cut}(U)$ , or, if it meets  $\text{cut}(U)$ , then only at its terminal in  $U$ .

All open  $U \subset X$  satisfy (by a simple Čech homology argument)

$$\rho \leq \text{dist}(y, \text{cut}(U)) \quad \implies \quad y \text{ is } \rho\text{-accessible from } U \text{ for all } y \in Y = \partial U.$$

Consequently, if  $U$  is  $C^2$ -quasi-regular, then  $Y \setminus \text{cut}(U)$  is  $C^2$ -smooth.

**Focal Points.** Let  $y_0 \in Y$  be an ancestor of  $u_0 \in U$ , i.e. a (*global*) *minimum* point of the function  $y \mapsto \text{dist}(y, u_0)$  on  $Y$ . Assume  $X$  is complete and let  $\gamma = \gamma(s)$  in  $X$  be a geodesic ray issuing from  $x_0$  inward  $U$  such that

$$\gamma(s_0) = u_0 \quad \text{for } s_0 = \text{dist}(u_0, y_0),$$

where  $s \geq 0$  denotes the geodesic length parameter. (If  $Y$  is  $C^1$ -smooth hypersurface at  $y_0$  then  $\gamma$  is unique being normal to  $Y$ .)

The point  $u_0$  is called *non-focal* for  $y_0$  along  $\gamma$  if  $y_0$  remains a *local* minimum of the function  $y \mapsto \text{dist}(y, u)$  on  $Y$  as we slightly move along  $\gamma$  inward, i.e. for  $u = \gamma(s_0 + \varepsilon)$  and all sufficiently small  $\varepsilon > 0$ . In other words, the  $(s_0 + \varepsilon)$ -ball in  $X$  around  $u_\varepsilon \in U$ , say

$$B_{u_\varepsilon}(s_0 + \varepsilon) \supset B_{u_0}(s_0) \subset U,$$

is “contained in  $U$  at  $y_0$ ”, i.e. the intersection of  $B_{u_\varepsilon}(s_0 + \varepsilon)$  with a small neighbourhood of  $y_0$  in  $X$  is contained in  $U$ .



Notice that focal/non-focal for  $y_0 \in Y$  depends only on the geometry of  $Y$  in a small neighbourhood of  $Y_0$  plus on how one defines "inward". Thus, one can extend the above definition by taking an arbitrarily small neighbourhood  $B_0 \subset X$  of  $y_0$ , (e.g. a small  $\varepsilon$ -ball around  $y_0$ ), letting

$$U_0 = X \setminus (B_0 \cap (X \setminus U)) \supset U$$

and defining focal/non-focal along geodesic segments in  $U_0$  that starts at  $y_0$  and may go beyond  $U$ .

If  $Y$  is a  $C^1$ -smooth hypersurface and  $y$  is an ancestor of  $u$  with  $\text{dist}(u, y) = \text{dist}(u, Y) = \rho$  then the  $\rho$ -sphere around  $u$ , say  $S_u(\rho) = \partial B_u(\rho) \subset U$ , that contains  $y$  is  $C^2$ -smooth at  $y$ , provided our Riemannian metric is  $C^2$ -smooth. If, moreover,  $Y$  is a  $C^2$ -smooth hypersurface, then the second fundamental form  $Q_Y$  of  $Y$  at  $y$  is minorized by the form  $Q_S$  at  $y$ , i.e.  $Q_Y - Q_S$  is *negative semidefinite* since  $B_u(\rho) \subset U$ . (Our sign convention for  $Q$ 's is the one for which the boundaries of convex subsets  $U \subset X$  have *positive* definite second fundamental forms  $Q$ .)

Obviously,  $u \in U$  is *non-focal* for  $Y = \partial U$  (along the minimal geodesic segment between the two points) if and only if the quadratic form  $Q_Y - Q_S$  is *negative definite*.

Denote by  $\text{foc}(U) \subset U \cap \text{cut}(U)$  the subset of *the focal points* where  $u$  is called focal if it is focal for some ancestor of  $u$  in  $Y = \partial U$  and observe that if  $Y$  is  $C^2$ -quasi-regular, e.g.  $C^2$ -smooth, then the subset  $\text{foc}(U) \subset U$ , is closed in  $U$ . (This is not, in general, true for  $C^1$ -hypersurfaces  $Y$ .)

The appearance of focal points can be seen in terms of the hypersurfaces  $Y_{-\rho} \subset U$  equidistant to  $Y = \partial U$  as follows. Join a point  $u_0 \in Y_{-\rho}$  with one of its ancestors, say  $y_0 \in Y$ , by a minimal geodesic segment  $\gamma$  in the closure of  $U$ , where  $\text{length}(\gamma) = \rho$ , and observe that the hypersurfaces  $Y_{-\rho+\varepsilon}$ ,  $0 < \varepsilon \leq \rho$ , are  $C^2$ -smooth at the points  $u_{+\varepsilon} = Y_{-\rho+\varepsilon} \cap \gamma$ , provided  $Y$  is  $C^2$ -smooth at  $y_0$ . Then, *the second fundamental forms  $Q_\varepsilon$  of  $Y_{-\rho+\varepsilon}$  at the points  $u_{+\varepsilon}$  are uniformly bounded from below*.

If  $u_0$  is a *non-focal* for  $y_0$  then the forms  $Q_\varepsilon$  are also bounded from *above*; moreover, the hypersurfaces  $Y_{-\rho+\varepsilon}$  can be locally, around  $\gamma$ , included into a  $C^2$ -smooth family of local equidistant hypersurfaces to a small neighbourhood of  $y_0 \in Y$ . But if  $u_0$  is *focal* for  $y_0$  then these forms "blow up" for  $\varepsilon \rightarrow 0$  as follows.

The  $((n-1)$ -dimensional) spaces  $T(\varepsilon)$  normal to  $\gamma$  at the points  $u_{+\varepsilon}$ , that serve as tangent spaces to  $Y$  for  $\varepsilon > 0$ , admit orthogonal splittings  $T(\varepsilon) = T_0(\varepsilon) \oplus T_1(\varepsilon)$ , where these  $T_0(\varepsilon)$  and  $T_1(\varepsilon)$  continuously depend on  $\varepsilon \in [0, \rho]$  and are such that

- The subspace  $T_0(\varepsilon = \rho) \subset T_{y_0}(Y)$  equals the kernel of the above difference form  $Q_Y - Q_S$  at  $y_0$ .
- The forms  $Q_\varepsilon$  restricted to  $T_1(\varepsilon)$  are continuous for all  $0 < \varepsilon \leq \rho$  and they continuously extend to the space  $T_1(\varepsilon = 0)$ .
- The forms  $Q_\varepsilon$  on the subspaces  $T_0(\varepsilon)$  tend to  $+\infty$  for  $\varepsilon \rightarrow 0$ . In fact, the values of  $Q_\varepsilon$  on the unit vectors in  $T_0(\varepsilon)$  is of order  $1/\varepsilon$ .

## 5.6. $C^2$ -approximation with corners

We show here how equidistant hypersurfaces to a quasiregular  $Y$  can be approximated by piecewise smooth hypersurfaces with one sided controls on their curvatures. Let  $U \subset X$  be a  $C^2$ -quasiregular open subset (domain) with boundary  $Y = \partial U$  in a complete Riemannian manifold  $X$ , let  $Y_{-\rho} = \partial U_{-\rho} = U_{-\rho}^\leq \subset U$ ,  $\rho > 0$ , be the equidistant hypersurface, where, as earlier,  $U_{-\rho}^\leq$  denotes the set of  $u \in U$ , where  $\text{dist}(u, Y) \leq \rho$ .

### $\sim$ -Approximation Lemma.

Given  $\varepsilon > 0$  and  $0 < \rho' < \rho$ , there exists a domain  $U' = U_{\rho'}^{\sim \varepsilon}$  in  $X$  such that

$$U_{-\rho} \supset U' \supset U_{-(\rho+\varepsilon)},$$

and such that the boundary  $Y' = \partial U'$  is piecewise  $C^2$ -smooth.

In fact, there are  $C^2$ -diffeomorphisms  $D_i: X \rightarrow X$  such that  $D_i(U_{-\rho})$  do not intersect the singular locus of  $Y_{-\rho'}$  and  $Y'$  equals the union of the  $D_i$ -pullbacks of  $Y_{-\rho'}$

$$Y' = \bigcup_i D_i^{-1}(Y_{-\rho'}).$$

Moreover, if  $\rho - \rho'$  is small, then these  $D_i$  are  $C^2$ -close to the identity map  $X \rightarrow X$ ; consequently, the curvatures of the smooth pieces  $D_i^{-1}(Y_{-\rho'})$  are close to the curvatures of their  $D_i$  images in  $Y_{-\rho'}$ .

**Proof.** Let  $\gamma \subset X$  be a minimal geodesic segment. Then, obviously, there exists smooth vector field  $V_\gamma(x)$  on  $X$  that is tangent to  $\gamma$ , where it equals the unit field directed from  $x_0$  to  $x_1$  and such that the norm of  $V_\gamma$  satisfies

$$\|V_\gamma(x)\| < 1, \quad x \notin \gamma, \quad \limsup_{x \rightarrow \infty} \|V_\gamma(x)\| < 1.$$

Integrate  $V_\gamma$  for the flow time equal  $\text{length}(\gamma)$ ; thus, obtain a  $C^2$ -diffeomorphism  $D_\gamma: X \rightarrow X$  such that  $D_\gamma$  sends one end of  $\gamma$ , say  $x_0$ , to the other one, called  $x_1 = D_\gamma(x_0)$ , where this diffeomorphism is sharp at  $\gamma$  in the sense that  $\text{dist}(x, D_\gamma(x)) < \text{length}(\gamma)$  for all  $x \notin \gamma$ , and where one can achieve a map  $\gamma \mapsto D_\gamma$  to be continuous for the  $C^2$ -topology in the space of diffeomorphisms.  $\square$

### Remark.

It is easy to arrange the maps  $D_\gamma: X \rightarrow X$  such that their differentials  $T_{x_0}(X) \rightarrow T_{x_1}(X)$  are isometries for all  $\gamma$ . Moreover, if  $X$  has positive sectional curvatures, one can make  $D_\gamma$  second order isometries at these points, i.e. such that every geodesic through  $x_0$  goes to a curve with zero curvature at  $x_1$ . However, this is impossible for manifolds of negative curvature.

Now, let  $\delta > 0$  be very small (depending, in particular, on  $\varepsilon$ ), take all minimal segments  $\gamma$  between the points  $y \in Y_{-\rho}$  and its  $(\rho' + \delta)$ -ancestors in  $U$  and let

$$U' = U_{-\rho} \setminus \bigcup_\gamma D_\gamma^{-1}(X \setminus U_{-\rho'}).$$

Finally, take a sufficiently dense locally finite set of geodesic segments, say  $\{\gamma_i\}$ , and take  $D_{\gamma_i}$  for the required diffeomorphisms  $D_i$ .

This  $\sim$ -approximation implies, in particular, that the distance function  $d$  to the boundary  $Y$  of  $U$  can be approximated by the maximum of smooth distance functions with their second partial derivatives close to those of  $h$  at nearby points. It follows that all  $k$ -convexity bounds extend from smooth to non-smooth points of  $d$ . In particular, the  $k$ -convexity lemma from subsection 3.3 follows from this  $\sim$ -approximation since one, obviously, has a uniform bound on the “bending”  $\beta$  in this case.

**On External Approximation.** The above piecewise smooth hypersurfaces  $Y_{-\rho}^{\delta-\Delta}$  that approximate the boundary  $\partial U_{-\rho}^<$  are positioned inside  $U_{-\rho}^<$ . Probably, there is no similar approximation by hypersurfaces lying outside but this is obviously possible if  $Y_{-\rho}$  is compact: just apply the inside approximation to  $Y_{-\rho'}$  for  $\rho' < \rho$  and let  $\rho' \rightarrow \rho$ .

## 5.7. Cornered domains and smoothing the corners

Let us indicate here a geometric alternative to the smoothing operators we used in subsection 3.4. A cornered domain of class  $C^k$  in a  $C^k$ -smooth  $n$ -manifold  $X$  is a closed subset  $V$  such that every boundary point  $v$  in  $V$  admits a neighbourhood  $U(v)$  in  $V$  that is  $C^k$ -diffeomorphic to the intersection of  $k \leq n$  mutually orthogonal halfspaces in  $\mathbb{R}^n$ .

The regular part of the boundary of  $V$ , denoted  $\text{reg}_{\partial V} \subset \partial V$ , consists of those  $v$ , where  $U(v) \subset V$  is diffeomorphic to a half space, i.e.  $k = 1$ . The  $(n-1)$ -faces  $W_i$ ,  $i \in I$ , of  $V$  are the closures of the connected components of  $\text{reg}_{\partial V} \subset V$  where, obviously,

$$\bigcup_{i \in I} W_i = \partial V.$$

(Sometimes, one takes finite unions of disjoint connected components for faces.) *Corners or  $(n-2)$ -faces of  $V$*  are, by definition, non-empty pairwise intersections of  $(n-1)$ -faces,

$$W_{i_1 i_2} = W_{i_1} \cap W_{i_2}.$$

(Since we assume the corner structure being “simple”, there is no non-empty intersection  $W_{i_1} \cap W_{i_2}$  of dimension  $< n-2$ . On the other hand, corners may be disconnected.)

It is easy to see that a cornered  $V$  equals an intersection  $V = \bigcap_{j \in J} V_j$ , where

- $V_j \subset X$  are  $n$ -submanifolds with *smooth* boundaries  $\partial V_j$ ,
- all intersections between  $(k$ -tuples of)  $\partial V_j$  are *transversal*,
- there are at most *finitely many* boundaries that intersect a given *compact* subset in  $X$ .

These three imply that the intersections of  $\partial V_j$  with  $V$  equal finite union of disjoint  $(n-1)$ -faces of  $V$ , where these  $V \cap \partial V_j \subset \partial V_j$  are cornered domains in  $\partial V_j$ . If one wishes, one may let  $J = I$  and choose  $V_i$  such that  $V \cap \partial V_i = W_i$ .

If  $X$  is a Riemannian manifold than one may speak of the dihedral angles between pairs of  $(n-1)$ -faces along  $(n-2)$ -faces. Clearly all these angles  $\angle(W_{i_1}, W_{i_2})$ , that are *continuous functions* on  $W_{i_1 i_2} = W_{i_1} \cap W_{i_2}$ , are bounded by  $\angle(W_{i_1}, W_{i_2}) < \pi$ .

### Essential Example.

A generic  $C^\infty$ -perturbation of the smooth pieces of  $U'$  that approximate  $U_{-p}$  in the previous section turn  $U'$  into a cornered domain.

Since the corners of  $V$  are convex for the dihedral angles  $< \pi$  one expects that cornered domains  $V$  admit approximations by *smooth* domains that are, up to an arbitrary small error, “as convex” as the faces  $W_i$  of  $V$ . Indeed, this is possible for quite a few, classes of *convexity relations* (see next section) including strict  $k$ -mean convexity where the picture is most transparent for the mean curvature convexity.

### Corner Smoothing Lemma.

Let  $X$  be a  $C^2$ -smooth Riemannian  $n$ -manifold,  $\phi: X \rightarrow \mathbb{R}$  a continuous function and  $V$  a cornered domain of class  $C^2$  such that the mean curvatures of the regular part of the boundary  $\partial V \subset V \subset X$  satisfy

$$\text{mn.curv}_x(\partial V) > \phi(x) \quad \text{for all } x \in \text{reg}_{\partial V} \subset \partial V.$$

Then, for an arbitrary neighbourhood  $\Delta \subset V$  of the boundary  $\partial V \subset V$ , there exists a domain  $V' \subset V$  with  $C^2$ -smooth boundary such that

$$\partial V' \subset \Delta, \quad \text{mn.curv}_x(\partial V') > \phi(x) \quad \text{for all } x \in \partial V';$$

moreover, the normal projection  $\partial V \rightarrow \partial V'$  is a  $C^2$ -diffeomorphism on every  $(n-1)$ -face of  $V$ .

**Proof.** Let  $\partial V$  be compact and let  $\partial_{+\delta} V \subset X$  be the boundary of the  $\delta$ -neighbourhood of  $V$  for a small  $\delta > 0$ . Clearly,  $\partial_{+\delta} V$  is a  $C^1$ -smooth hypersurface, that is, moreover, piecewise  $C^2$ . This is seen with the normal projection  $\partial_{+\delta} V \rightarrow \partial V$  that sends every  $C^2$ -piece of  $\partial_{+\delta} V$  onto an  $m$ -face of  $V$ , for some  $m = 1, 2, \dots, n-1$ . If  $m = n-1$ , then the mean curvature of this piece is  $\delta$ -close to that of the corresponding  $(n-1)$ -face and if  $m < n-1$  then the mean curvature is  $\sim \delta^{-1}$ .

Thus, the mean curvature of  $\partial_{+\delta} V$  is a piecewise continuous function on  $\partial_{+\delta} V$  that satisfies

$$\text{mn.curv}_x(\partial_{+\delta} V) > \phi(x) \quad \text{for all sufficiently small } \delta > 0 \text{ and all } x \in \partial_{+\delta} V.$$

Now, observe that  $\partial_{+\delta} V$  equals the  $\delta$ -level of the distance function  $d(x) = \text{dist}(x, V)$  and let  $d_\varepsilon(x)$  be the average of  $d(x)$  over the  $\varepsilon$ -ball  $B_x(\varepsilon) \subset X$  for a small  $\varepsilon > 0$ .

Since the second differential of  $d(x)$  is a bounded measurable function and  $\|\text{grad } d\| = 1$ , the  $\delta$ -level set  $\partial_{+\delta,\varepsilon}V \subset X$  of  $d_\varepsilon$  is a  $C^2$ -smooth hypersurface that  $C^1$ -converges to  $\partial_{+\delta}$  for  $\varepsilon \rightarrow 0$ . Since the mean curvature of a level of a function is *linear* in the second derivatives of the function the mean curvatures of  $\partial_{+\delta,\varepsilon}$  are, up to an  $\varepsilon$ -error, equal to the  $B_x(\varepsilon)$ -averages of these of  $\partial_{+\delta}$ ; hence, the mean curvatures of  $\partial V' = \partial_{+\delta,\varepsilon}$  are  $> \phi(x)$  for sufficiently small  $\varepsilon > 0$ .

Finally, in order to have  $\partial V'$  inside rather than outside  $V$ , we apply the above to an interior equidistant hypersurface  $\partial V_{-\delta}$  instead of  $\partial V = \partial V_0$ , where a minor readjustment of this argument is needed to take care of non-compact  $\partial V$ .  $\square$

This, together with [Essential Example](#) allows an alternative proof of

### Smoothing of Quasiregular Hypersurfaces.

Let  $U$  be an open domain in  $X$  with quasiregular boundary and let the mean curvatures at all regular points of  $\partial U$  be strictly minorized by a continuous function  $\psi$  on  $X$ , i.e.

$$\text{mn.curv}_x(\partial U) > \psi(x) \quad \text{for all regular points } x \in \partial U.$$

Then  $U$  can be exhausted by closed subsets  $U_i \subset U$  with smooth boundaries  $Y_i = \partial U_i$ , where the mean curvatures of these are strictly minorized by  $\psi(x)$  at all  $x \in \partial U_i$  and all  $U_i$ .

The two basic examples where this smooth approximation is used in the present paper are

- (i) strictly mean convex bubbles  $U \subset X$  with compact boundaries  $Y$ ,
- (ii) minimal hypersurfaces  $H$ .

In both cases the Almgren–Allard quasiregularity theorem applies and, in case (i), allows a *smooth strictly mean convex approximation of  $Y$* , while in case (ii) one approximates the boundary  $Y_\varepsilon$  of the  $\varepsilon$ -neighbourhood  $U_\varepsilon(H) \subset X$  of  $H$  by a *smooth  $c_\varepsilon$ -concave hypersurface*, i.e. with  $\text{mn.curv}(Y_\varepsilon) \leq c_\varepsilon$ , where  $c_\varepsilon \rightarrow 0$  for  $\varepsilon \rightarrow 0$ .

### Remarks.

- (a) When we discussed smoothing minimal hypersurfaces  $H$  with Joachim Lohkamp a few years ago he, on one hand, said he was well aware of possibility of such smoothing, but, on the other hand, he expressed a concern about singularities at the focal points.

Focal points are invisible in the argument with bending and standard linear smoothing (see subsection 3.4) but the above makes it clear why singularities at these points cause no additional complication.

- (b) The corner smoothing lemma remains valid for the  $(n - k)$ -mean convexity for all  $k$  but it fails, in general, for  $k$ -convexity, probably for all  $k \neq 1, n - 1$ . To see this for even  $n - 1 \geq 4$  and  $k = (n - 1)/2 \geq 2$ , let  $V \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$  be a compact domain with smooth boundary. Then there obviously exist  $C^\infty$ -small perturbations  $V', V_+$  and  $V_-$  of  $V$  in  $\mathbb{R}^n$  such that  $V_+$  and  $V_-$  transversally meet along the boundary  $\partial V'$  and bound together a domain  $U' \subset \mathbb{R}^n$  that is  $k$ -convex away from the corner along  $\partial V'$ . This  $U' \supset V'$  can be seen as a small thickening of  $V'$  that is homeomorphic to  $V' \times [0, 1] = V \times [0, 1]$ . Therefore, if the homology group  $H_{n-2}(V) \neq 0$ , then  $H_{n-2}(U) \neq 0$  as well; hence,  $U$  cannot be approximated by smooth  $k$ -convex domains if  $k < n - 2$ .

## Acknowledgements

The author thanks the anonymous referee who has indicated several errors in the original manuscript and has made useful suggestions.

---

## References

---

- [1] Cheeger J., Naber A., Lower bounds on Ricci curvature and quantitative behavior of singular sets, *Invent. Math.*, 2013, 191(2), 321–339
- [2] Gromov M., Sign and geometric meaning of curvature, *Rend. Sem. Mat. Fis. Milano*, 1991, 61, 9–123
- [3] Gromov M., Hilbert volume in metric spaces. Part 1, *Cent. Eur. J. Math.*, 2012, 10(2), 371–400
- [4] Gromov M., Lawson H.B. Jr., The classification of simply connected manifolds of positive scalar curvature, *Ann. of Math.*, 1980, 111(3), 423–434
- [5] Lawson H.B. Jr., Michelsohn M.-L., Embedding and surrounding with positive mean curvature, *Invent. Math.*, 1984, 77(3), 399–419
- [6] Lawson H.B. Jr., Michelsohn M.-L., Approximation by positive mean curvature immersions: frizzing, *Invent. Math.*, 1984, 77(3), 421–426
- [7] Lott J., Villani C., Ricci curvature for metric–measure spaces via optimal transport, *Ann. of Math.*, 2009, 169(3), 903–991
- [8] Micallef M.J., Moore J.D., Minimal two–spheres and the topology of manifolds with positive curvature on totally isotropic two–planes, *Ann. of Math.*, 1988, 127(1), 199–227
- [9] Ollivier Y., Ricci curvature of Markov chains on metric spaces, *J. Funct. Anal.*, 2009, 256(3), 810–864
- [10] Sha J.-P., Handlebodies and  $p$ -convexity, *J. Differential Geom.*, 1987, 25(3), 353–361
- [11] Sormani C., Wenger S., The intrinsic flat distance between Riemannian manifolds and other integral current spaces, *J. Differential Geom.*, 2011, 87(1), 117–199
- [12] Wenger S., Isoperimetric inequalities of Euclidean type in metric spaces, *Geom. Funct. Anal.*, 2005, 15(2), 534–554
- [13] White B., A local regularity theorem for mean curvature flow, *Ann. of Math.*, 2005, 161(3), 1487–1519