

Plebański–Demiański solution of general relativity and its expressions quadratic and cubic in curvature: analogies to electromagnetism

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Motivation & Outline

Formal analogies between general relativity and **electrodynamics**: Matte 1953, Bel 1962

Physical analogies in gravitoelectromagnetism (GEM): Mashhoon et al. 1984

Question: Are there physical analogies that go beyond the linearized level?

- The Plebański–Demiański solution in brief
- Conventions, exterior calculus
- Curvature invariants and their similarities to electromagnetism
- The Bel–Robinson tensor and its 3-form
- The Kummer tensor
- Conclusions & Summary

The Plebański–Demiański solution in brief

Found by Plebański & Demiański in 1976.

It has **seven** free parameters and is of **Petrov type D** (Szekeres: “Coulomb-like”).

It can describe a massive,
rotating,
electrically & magnetically charged,
uniformly accelerating

black hole in a de Sitter background
with an additional NUT parameter.

Various subclasses: Schwarzschild, **Kerr**, C-metric, Taub–NUT, ...

Physical interpretation: Griffiths & Podolský (2006)

Conventions, Exterior Calculus

We use exterior calculus. We have a **frame** e_μ and a dual **coframe** ϑ^ν :

$$e_\mu = e^b{}_\mu \partial_b, \quad \vartheta^\nu = e_a{}^\nu dx^a, \quad e_\mu \lrcorner \vartheta^\nu = \delta^\nu{}_\mu$$

By means of the **metric**, we choose a pseudo-orthogonal frame and coframe:

$$\left(g(e_\mu, e_\nu) \right) = \left(g_{\mu\nu} \right) = \text{diag}(-1, 1, 1, 1)$$

Expand **exterior forms** — say, the curvature 2-form — in terms of their components:

$$\text{Riem}^\mu{}_\nu = \frac{1}{2!} \text{Riem}_{\alpha\beta}{}^\mu{}_\nu \vartheta^\alpha \wedge \vartheta^\beta = \frac{1}{2!} \text{Riem}_{ij}{}^\mu{}_\nu dx^i \wedge dx^j$$

anholonomic
coframe

holonomic
coordinate cobasis

Curvature invariants (for any type D spacetime)

1. **Kretschmann** scalar:

$$K := - \star \left[\text{Weyl}_{\alpha\beta} \wedge (\star \text{Weyl}^{\alpha\beta}) \right] = \frac{1}{2} \text{Weyl}_{\alpha\beta\gamma\delta} \text{Weyl}^{\alpha\beta\gamma\delta} = -24 (\mathbb{B}^2 - \mathbb{E}^2)$$

2. **Chern–Pontryagin** pseudo-scalar:

$$\mathcal{P} := \star (\text{Weyl}_{\alpha\beta} \wedge \text{Weyl}^{\alpha\beta}) = \frac{1}{2} (\star \text{Weyl}_{\alpha\beta\gamma\delta}) \text{Weyl}^{\alpha\beta\gamma\delta} = -48 \mathbb{E} \mathbb{B}$$

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2. Chern–Pontryagin pseudo-scalar:

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Physical analogy between general relativity and **electrodynamics** (for **Kerr** solution):

$\mathbb{E} \propto$ mass (gravitational charge), $\mathbb{B} \propto$ angular momentum (gravitational current)
 $\mathbf{E} \propto$ electric charge, $\mathbf{B} \propto$ electric current

The Bel–Robinson tensor

Related to **super-energy**. Traditionally defined as a $\binom{0}{4}$ tensor:

$$\tilde{\mathbb{B}}_{\mu\nu\rho\sigma} := \text{Weyl}_{\mu\alpha\beta\rho} \text{Weyl}_{\nu}{}^{\alpha\beta}{}_{\sigma} + (*\text{Weyl}_{\mu\alpha\beta\rho}) (*\text{Weyl}_{\nu}{}^{\alpha\beta}{}_{\sigma})$$

Physical dimension:

$$[\tilde{\mathbb{B}}_{\mu\nu\rho\sigma}] = \left(\frac{\text{energy}}{\text{3-volume}} \right)^2$$

Algebraic properties:

$$\tilde{\mathbb{B}}_{\mu\nu\rho\sigma} = \tilde{\mathbb{B}}_{(\mu\nu\rho\sigma)}, \quad \tilde{\mathbb{B}}^{\alpha}{}_{\nu\alpha\sigma} = 0$$

Similar to the **electromagnetic energy momentum** (tracefree and symmetric).

The Bel–Robinson 3-form

Geometrically, energy-momentum is a current 3-form.

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Analogous expressions for vacuum electrodynamics and general relativity:

$$2\Sigma_\mu := F \wedge (e_\mu \lrcorner *F) - (*F) \wedge (e_\mu \lrcorner F)$$

The Bel–Robinson 3-form

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Analogous expressions for **vacuum electrodynamics** and general relativity:

$$2\Sigma_\mu := F \wedge (e_\mu \lrcorner \star F) - (\star F) \wedge (e_\mu \lrcorner F)$$

$$\tilde{\Sigma}_{\nu\rho\sigma} := \text{Weyl}_{\rho\alpha} \wedge (e_\nu \lrcorner \star \text{Weyl}^\alpha{}_\sigma) - (\star \text{Weyl}_{\rho\alpha}) \wedge (e_\nu \lrcorner \text{Weyl}^\alpha{}_\sigma)$$

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The symmetric energy-momentum tensors are derived analogously:

$$T_{\mu\nu} := e_\mu \lrcorner (\star \Sigma_\nu)$$

$$\tilde{B}_{\mu\nu\rho\sigma} := e_\mu \lrcorner (\star \tilde{\Sigma}_{\nu\rho\sigma})$$

Both yield “energy density” squared:

$$\star [\Sigma_\alpha \wedge (\star \Sigma^\alpha)] = (\mathbf{E}^2 + \mathbf{B}^2)^2$$

$$\star [\tilde{\Sigma}_{\alpha\beta\gamma} \wedge (\star \tilde{\Sigma}^{\alpha\beta\gamma})] = 24^2 (\mathbf{E}^2 + \mathbf{B}^2)^2$$

Both are traceless:

$$\vartheta^\alpha \wedge \Sigma_\alpha = 0$$

$$\vartheta^\alpha \wedge \tilde{\Sigma}_{\alpha\rho\sigma} = 0$$

Kummer tensor

Cubic in curvature, can be defined for Riemann as well as Weyl.

Systematic introduction to electrodynamics and gravity by Baekler et al. (2014)

$$K^{\mu\nu\rho\sigma} := \text{Weyl}^{\alpha\mu\beta\nu} * \text{Weyl}^*_{\alpha\gamma\beta\delta} \text{Weyl}^{\gamma\rho\delta\sigma}$$

It is related to so-called Kummer surfaces (propagation of waves), and principal null directions of curvature.

Can be irreducibly decomposed into six pieces. There are two invariants:

$$\begin{aligned} S &:= {}^{(1)}K^{\alpha}_{\alpha}{}^{\beta}_{\beta} = 24\mathbb{E} (3\mathbb{B}^2 - \mathbb{E}^2) \\ \mathcal{A} &:= \eta_{\alpha\beta\gamma\delta} {}^{(6)}K^{\alpha\beta\gamma\delta} = 24\mathbb{B} (3\mathbb{E}^2 - \mathbb{B}^2) \end{aligned}$$

They are Bel's fundamental vacuum scalars.

Conclusions & Summary

Physical analogy between general relativity and **electrodynamics** (Kerr solution):

$\mathbb{E} \propto$ mass (gravitational charge), $\mathbb{B} \propto$ angular momentum (gravitational current)

$\mathbf{E} \propto$ electric charge, $\mathbf{B} \propto$ electric current

Result 1) can be generalized to the Plebański–Demiański solution.

Analogous expressions for **vacuum electrodynamics** and general relativity:

$$2\Sigma_\mu := F \wedge (e_\mu \lrcorner \star F) - (\star F) \wedge (e_\mu \lrcorner F)$$

$$\tilde{\Sigma}_{\nu\rho\sigma} := \text{Weyl}_{\rho\alpha} \wedge (e_\nu \lrcorner \star \text{Weyl}^\alpha_\sigma) - (\star \text{Weyl}_{\rho\alpha}) \wedge (e_\nu \lrcorner \text{Weyl}^\alpha_\sigma)$$

Bel-Robinson 3-form $\tilde{\Sigma}_{\nu\rho\sigma}$: needs more attention.

More details in J.B., “Plebański–Demiański solution of general relativity and its expressions quadratic and cubic in curvature: analogies to electromagnetism”, arXiv:1412.1958 [gr-qc], submitted to Gen. Rel. Grav.

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“Plebański–Demiański solution of general relativity and its expressions quadratic and cubic in curvature: analogies to electromagnetism”

arXiv:1412.1958

Abstract:

Analogies between gravitation and electromagnetism have been known since the 1950s. Here, we examine the exact seven parameter solution of Plebański–Demiański (PD) to demonstrate these analogies for a physically meaningful spacetime:

The two quadratic curvature invariants are shown to take the form $\mathbf{B}^2 - \mathbf{E}^2$ and $\mathbf{E} \cdot \mathbf{B}$, and we show that \mathbf{E} and \mathbf{B} are related to the mass and angular momentum parameters of the PD solution, respectively.

Furthermore, the square of the Bel–Robinson tensor reads $(\mathbf{B}^2 + \mathbf{E}^2)^2$ for the PD solution, reminiscent of the square of the energy density in electrodynamics. Analogously to the energy-momentum 3-form of the electromagnetic field, we introduce the Bel–Robinson 3-form, from which the Bel–Robinson tensor can be derived.

We also determine the Kummer tensor, a tensor cubic in curvature, for the PD solution for the first time, and calculate the pieces of its irreducible decomposition.

The calculations are carried out in two coordinate systems: in the original polynomial PD coordinates, and in a modified Boyer–Lindquist-like version introduced by Griffiths and Podolský (GP) allowing for a more straightforward physical interpretation of the free parameters.

The Plebański–Demiański solution (1/2)

Found by Plebański & Demiański in 1976, expressed in the coordinates $\{\tau, q, p, \sigma\}$. It has **seven** free parameters and is of **Petrov type D** (Szekeres: “Coulomb-like”).

The pseudo-orthogonal coframe 1-forms read:

$$\vartheta^{\hat{0}} := \frac{1}{1 - pq} \sqrt{\frac{\mathcal{Q}(q)}{p^2 + q^2}} (d\tau - p^2 d\sigma),$$

$$\vartheta^{\hat{1}} := \frac{1}{1 - pq} \sqrt{\frac{p^2 + q^2}{\mathcal{Q}(q)}} dq,$$

$$\vartheta^{\hat{2}} := \ominus \frac{1}{1 - pq} \sqrt{\frac{p^2 + q^2}{\mathcal{P}(p)}} dp,$$

$$\vartheta^{\hat{3}} := \ominus \frac{1}{1 - pq} \sqrt{\frac{\mathcal{P}(p)}{p^2 + q^2}} (d\tau + q^2 d\sigma)$$

The metric is given by $g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$, with $(g_{\mu\nu}) = \text{diag}(-1, 1, 1, 1)$.

$\mathcal{Q}(q)$ and $\mathcal{P}(p)$ are fourth-order polynomials, prescribed by the Einstein–Maxwell equations: $\mathcal{Q}(q)'''' = \mathcal{P}(p)'''' = -8\Lambda$, where Λ is the cosmological constant.

The Plebański–Demiański solution (2/2)

The polynomials are given by

$$\mathcal{P}(p) := \hat{k} + 2\hat{n}p - \hat{e}p^2 + 2\hat{m}p^3 + \left(\hat{k} + \hat{e}^2 + \hat{g}^2 - \frac{\Lambda}{3} \right) p^4,$$
$$\mathcal{Q}(q) := \hat{k} + \hat{e}^2 + \hat{g}^2 - 2\hat{m}q + \hat{e}q^2 - 2\hat{n}q^3 + \left(\hat{k} - \frac{\Lambda}{3} \right) q^4.$$

The vector potential 1-form is

$$A := \frac{1 - pq}{\sqrt{p^2 + q^2}} \left(\frac{\hat{e}q}{\sqrt{\mathcal{Q}(q)}} \vartheta^{\hat{0}} + \frac{\hat{g}p}{\sqrt{\mathcal{P}(p)}} \vartheta^{\hat{3}} \right).$$

The free parameters are thus $\{\hat{m}, \hat{n}, \hat{e}, \hat{g}, \hat{e}, \hat{k}, \Lambda\}$.

Griffiths & Podolský (1/2)

Physical interpretation of polynomial coordinates problematic.
New coordinates $\{t, r, \theta, \phi\}$ replace the polynomial $\{\tau, \rho, \sigma\}$.

$$\vartheta^{\hat{0}} := \frac{\sqrt{\Delta}}{\Omega\rho} \left[dt - \left(a \sin^2 \theta + 4\ell^2 \sin^2 \frac{\theta}{2} \right) d\phi \right]$$

$$\vartheta^{\hat{1}} := \frac{\rho}{\Omega\sqrt{\Delta}} dr$$

$$\vartheta^{\hat{2}} := \frac{\rho}{\Omega\sqrt{\chi}} d\theta$$

$$\vartheta^{\hat{3}} := \frac{\sqrt{\chi} \sin \theta}{\Omega\rho} \left\{ a dt - [r^2 + (a + \ell)^2] d\phi \right\}$$

The metric is given by $g = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$, with $(g_{\mu\nu}) = \text{diag}(-1, 1, 1, 1)$.

The new parameters are $\{m, \ell, a, \alpha, e, g, \Lambda\}$.

Griffiths & Podolský (2/2)

The auxiliary functions are given by

$$\Delta := \omega^2 k + e^2 + g^2 - 2mr + \epsilon r^2 - 2\frac{\alpha}{\omega} nr^3 - \left(\alpha^2 k - \frac{\Lambda}{3}\right) r^4 = \left(\frac{\omega}{\alpha}\right)^2 \mathcal{Q},$$

$$\chi := 1 - \alpha_3 \cos \theta - \alpha_4 \cos^2 \theta = \frac{\omega^2}{\alpha^2 a^2 \sin^2 \theta} \mathcal{P},$$

$$\Omega := 1 - \frac{\alpha}{\omega} r(\ell + a \cos \theta) = 1 - pq,$$

$$\rho^2 := r^2 + (\ell + a \cos \theta)^2 = \frac{\omega}{\alpha} (p^2 + q^2).$$

The vector potential 1-form is

$$A := \frac{\Omega}{\rho} \left[\frac{er}{\sqrt{\Delta}} \vartheta^{\hat{0}} + \frac{g(\ell/a + \cos \theta)}{\sin \theta \sqrt{\chi}} \vartheta^{\hat{3}} \right].$$