Plug-and-play decentralized model predictive control for linear systems

Stefano Riverso, *Graduate Student Member, IEEE*, Marcello Farina and Giancarlo Ferrari-Trecate, *Senior Member, IEEE*

***Abstract*—In this paper we consider a linear system structured into physically coupled subsystems and propose a decentralized control scheme capable to guarantee asymptotic stability and satisfaction of constraints on system inputs and states. The design procedure is totally decentralized, since the synthesis of a local controller uses only information on a subsystem and its neighbors, i.e. subsystems coupled to it. We show how to automatize the design of local controllers so that it can be carried out in parallel by smart actuators equipped with computational resources and capable to exchange information with neighboring subsystems. In particular, local controllers exploit tube-based Model Predictive Control (MPC) in order to guarantee robustness with respect to physical coupling among subsystems. Finally, an application of the proposed control design procedure to frequency control in power networks is presented.**

Key Words: Decentralized Control; Decentralized Synthesis; Large-scale Systems; Model Predictive Control; Plug-and-Play Control; Robust Control

I. INTRODUCTION

Decentralized regulators have been studied since the 70's as a viable solution to the control of large-scale systems composed by physically coupled subsystems [1]. The problem of guaranteeing stability and suitable performance levels for decentralized control system has been addressed in the 70's and 80's mainly for unconstrained systems [2], [1]. Distributed (also known as overlapping decentralized) control, where controllers can exchange pieces of information through a communication network, has also been studied (see, e.g. [3], and references therein).

Decentralized and distributed control schemes for constrained systems have been proposed only much more recently in the context of Model Predictive Control (MPC) [4], [5], [6], [7], [8], [9], [10]. These results are particularly appealing because they replace large-scale optimization problems stemming from centralized MPC with several smaller-scale problems that can be solved in parallel using computational resources collocated with sensors. While the main focus of decentralized and distributed control is on limiting the computational burden and communication cost associated to real-time operations of the control system, attention has also been paid to the complexity of the controller design procedure. In this respect, decentralized and distributed controllers can be designed either in a centralized fashion, i.e. relying on the knowledge of the collective model, or in a decentralized fashion, i.e. the whole model of the system is never used in any design step [1]. However, decentralized design does not prevent from using collective quantities based on pieces of information from all subsystems.

In this paper we propose a decentralized MPC (DeMPC) scheme with Plug-and-Play (PnP) capabilities. In line with [11], PnP means that

(i) the design of a single controller involves at most information about the subsystem under control and its neighbors, i.e. no step of the design procedure involves collective quantities;

(ii) when a subsystem joins/leaves an existing plant there is a procedure for
   (a) assessing if the operation does not spoil stability and constraint satisfaction for the overall plant;
   (b) automatically retuning at most the controllers of the subsystem and its successors, i.e. subsystems influenced by it.

Note that plugging in and unplugging of subsystems are here considered as off-line operations, i.e. they do not lead to switching between different dynamics in real time. PnP controllers are attractive because the complexity of designing a controller for a given subsystem scales with the number of neighboring subsystem only. Moreover, PnP eases the revamping of control systems by enabling the replacement of actuators with limited interaction of human operators. It is well known that, for general interconnection topologies, requirement (i) above implies the design of regulators for each subsystem that are robust to the coupling with neighboring subsystems [1]. Our design procedure is no exception and we will exploit tube-based MPC [12] for the design of robust local controllers. While this introduces an unavoidable degree of conservatism, we argue that PnP DeMPC can be successfully applied in a number of real world plants where coupling among subsystems is sufficiently weak. As an example, we will use PnP DeMPC for designing the Automatic Generation Control (AGC) layer for frequency control in a realistic power network and discuss the plugging in and unplugging of power generation areas.

The paper is structured as follows. The design of decentralized controllers is introduced in Section II. In Section III we discuss how to design the local controllers in a distributed fashion and in Section IV we describe PnP operations. In Section V we present the application of PnP DeMPC to frequency control in a power network and Section VI is devoted to concluding remarks.

Notation. We use $a:b$ for the set of integers $\{a, a+1, \dots, b\}$. The column vector with s components $v_1, \dots, v_s$ is $\mathbf{v} = (v_1, \dots, v_s)$. The function $\text{diag}(G_1, \dots, G_s)$ denotes the block-diagonal matrix composed by s block G_i, $i \in 1:s$. The pseudo-inverse of a matrix $A \in \mathbb{R}^{m\times n}$ is denoted with $A^\flat$. The symbol $\oplus$ denotes the Minkowski sum, i.e. $A = B \oplus C$ if and only if $A = \{a : a = b + c, \text{ for all } b \in B \text{ and } c \in C\}$. Moreover, $\bigoplus_{i=1}^s G_i = G_1 \oplus \dots \oplus G_s$. The symbols $\mathbf{1}_n$ and $\mathbf{0}_n$ denote column vectors with n elements equal to 1 and 0, respectively. The set $\mathbb{X} \subseteq \mathbb{R}^n$ is Robust Positively Invariant (RPI) [6] for $x(t+1) = f(x(t), w(t))$, $w(t) \in \mathbb{W} \subseteq \mathbb{R}^m$ if $x(t) \in \mathbb{X} \Rightarrow f(x(t), w(t)) \in \mathbb{X}$, $\forall w(t) \in \mathbb{W}$. The RPI set $\underline{\mathbb{X}}$ is minimal if every other RPI $\mathbb{X}$ verifies $\underline{\mathbb{X}} \subseteq \mathbb{X}$. The RPI set $\mathbb{X}(\delta)$ is a δ-outer approximation of the minimal RPI $\underline{\mathbb{X}}$ if

$$x \in \mathbb{X}(\delta) \Rightarrow \exists\ \underline{x} \in \underline{\mathbb{X}} \text{ and } \tilde{x} \in B_\delta(0) : x = \underline{x} + \tilde{x}$$

where, for $\delta > 0$, $B_\delta(v) = \{x \in \mathbb{R}^n | \|x - v\| < \delta\}$.

II. DECENTRALIZED TUBE-BASED MPC OF LINEAR SYSTEMS

We consider a discrete-time Linear Time-Invariant (LTI) system

$$\mathbf{x}^+ = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$ are the state and the input, respectively, at time t and $\mathbf{x}^+$ stands for $\mathbf{x}$ at time $t+1$. We will use the notation $\mathbf{x}(t)$, $\mathbf{u}(t)$ only when necessary. The state is partitioned into M state vectors $x_{[i]} \in \mathbb{R}^{n_i}$, $i \in \mathcal{M} = 1:M$ such that $\mathbf{x} = (x_{[1]}, \dots, x_{[M]})$, and $n = \sum_{i\in\mathcal{M}} n_i$. Similarly, the input is partitioned into M vectors $u_{[i]} \in \mathbb{R}^{m_i}$, $i \in \mathcal{M}$ such that $\mathbf{u} = (u_{[1]}, \dots, u_{[M]})$ and $m = \sum_{i\in\mathcal{M}} m_i$.

We assume the dynamics of the $i-th$ subsystem is given by

$$\Sigma_{[i]}: \quad {x^+}_{[i]} = A_{ii}x_{[i]} + B_i u_{[i]} + w_{[i]} \tag{2}$$

The research leading to these results has received funding from the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreement n° 257462 HYCON2 Network of excellence.

S. Riverso and G. Ferrari-Trecate are with the Dipartimento di Ingegneria Industriale e dell'Informazione, Università degli Studi di Pavia, via Ferrata 1, 27100 Pavia, Italy. {stefano.riverso,giancarlo.ferrari}@unipv.it

M. Farina is with Dipartimento di Elettronica e Informazione, Politecnico di Milano, via Ponzio 34/5, 20133 Milan, Italy. farina@elet.polimi.it

$$w_{[i]} = \sum_{j\in\mathcal{N}_i} A_{ij}x_{[j]} \tag{3}$$

where $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, $i,j \in \mathcal{M}$, $B_i \in \mathbb{R}^{n_i\times m_i}$ and $\mathcal{N}_i$ is the set of neighbors to subsystem i defined as $\mathcal{N}_i = \{j \in \mathcal{M} : A_{ij} \neq 0, i \neq j\}$. According to (2), the matrix $\mathbf{A}$ in (1) is decomposed into blocks A_{ij}, $i,j \in \mathcal{M}$. We also define $\mathbf{A_D} = \text{diag}(A_{11},\dots,A_{MM})$ and $\mathbf{A_C} = \mathbf{A} - \mathbf{A_D}$, i.e. $\mathbf{A_D}$ collects the state transition matrices of every subsystem and $\mathbf{A_C}$ collects coupling terms between subsystems. From (2) one also obtains $\mathbf{B} = \text{diag}(B_1,\dots,B_M)$ because submodels (2) are input decoupled.

In this Section we propose a decentralized controller for (1) guaranteeing asymptotic stability of the origin of the closed-loop system and constraint satisfaction.

In the spirit of tube-based control [12], we treat $w_{[i]}$ as a disturbance and we define the nominal system

$$\hat{\Sigma}_{[i]}: \quad \hat{x}^+_{[i]} = A_{ii}\hat{x}_{[i]} + B_i v_{[i]} \tag{4}$$

with input $v_{[i]}$, obtained from (2) by neglecting the disturbance term $w_{[i]}$. System $\Sigma_{[i]}$ will be equipped with controller $\mathcal{C}_{[i]}$ given by

$$u_{[i]} = v_{[i]} + K_i(x_{[i]} - \bar{x}_{[i]}). \tag{5}$$

where $K_i \in \mathbb{R}^{m_i\times n_i}$, $i \in \mathcal{M}$ and variables $v_{[i]}$ and $\bar{x}_{[i]}$ will be computed by a local state-feedback MPC scheme, i.e. there exist functions $\kappa_i : \mathbb{R}^{n_i} \to \mathbb{R}^{m_i}$ and $\eta_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ such that $v_{[i]} = \kappa_i(x_{[i]})$ and $\bar{x}_{[i]} = \eta_i(x_{[i]})$. Note that the controller $\mathcal{C}_{[i]}$ is completely decentralized, since it depends upon quantities of system $\Sigma_{[i]}$ only.

Remark 1. *In order to illustrate the meaning of* (5)*, assume that* $\bar{x}_{[i]}(t) = \hat{x}_{[i]}(t)\ \forall t \geq 0$*. Then, introducing the error* $z_{[i]} = x_{[i]} - \hat{x}_{[i]}$*, from* (2)*,* (4) *and* (5) *we obtain*

$$z^+{}_{[i]} = (A_{ii} + B_iK_i)z_{[i]} + w_{[i]}. \tag{6}$$

When $A_{ii}+B_iK_i$ *is Schur and* $w_{[i]}$ *is bounded* $\forall t \geq 0$*,* (6) *guarantees that* $x_{[i]} - \hat{x}_{[i]}$ *remains bounded regardless of the exerted control action* $v_{[i]}$*. Moreover, if* $w_{[i]}(t) = 0$*, the state* $x_{[i]}(t)$ *achieves perfect tracking of the nominal state* $\hat{x}_{[i]}(t)$ *in the asymptotic regime.*

Defining the collective variables $\bar{\mathbf{x}} = (\bar{x}_{[1]},\dots,\bar{x}_{[M]}) \in \mathbb{R}^n$, $\mathbf{v} = (v_{[1]},\dots,v_{[M]}) \in \mathbb{R}^m$ and the matrix $\mathbf{K} = \text{diag}(K_1,\dots,K_M) \in \mathbb{R}^{m\times n}$, from (2) and (5) one obtains the collective model

$$\mathbf{x}^+ = (\mathbf{A} + \mathbf{BK})\mathbf{x} + \mathbf{B}(\mathbf{v} - \mathbf{K}\bar{\mathbf{x}}). \tag{7}$$

The following assumptions will be needed for designing stabilizing controllers $\mathcal{C}_{[i]}$.

Assumption 1. *(i) The matrices* $F_i = A_{ii} + B_iK_i$*,* $i \in \mathcal{M}$ *are Schur.*

(ii) The matrix $\mathbf{F} = \mathbf{A} + \mathbf{BK}$ *is Schur.* ■

We discuss now constraint satisfaction, assuming subsystems $\Sigma_{[i]}$, $i \in \mathcal{M}$ are equipped with constraints $x_{[i]} \in \mathbb{X}_i$ and $u_{[i]} \in \mathbb{U}_i$. If we define the sets $\mathbb{X} = \prod_{i\in\mathcal{M}} \mathbb{X}_i$ and $\mathbb{U} = \prod_{i\in\mathcal{M}} \mathbb{U}_i$, then we obtain the following constraints for the collective system (1)

$$\mathbf{x} \in \mathbb{X},\ \mathbf{u} \in \mathbb{U}. \tag{8}$$

As in tube-based MPC [12], our goal is to compute tightened state constraints $\hat{\mathbb{X}}_i \subseteq \mathbb{X}_i$ and input constraints $\mathbb{V}_i \subseteq \mathbb{U}_i$ that, through (5), will allow us to fulfill (8) at time $k+1$ when $\bar{x}_{[i]}(k) \in \hat{\mathbb{X}}_i(k)$ and $v_{[i]}(k) \in \mathbb{V}_i(k)$. Next, we characterize the shape of sets $\mathbb{X}_i$, $\hat{\mathbb{X}}_i$, $\mathbb{U}_i$ and $\mathbb{V}_i$, $i \in \mathcal{M}$. Sets $\mathbb{X}_i$ and $\hat{\mathbb{X}}_i$ are zonotopes, i.e., centrally symmetric convex polytopes:

$$\begin{aligned}\mathbb{X}_i &= \{x_{[i]} \in \mathbb{R}^{n_i} | f^T_{i,r}x_{[i]} \leq 1, r \in 1:\bar{r}_i\} = \{x_{[i]} \in \mathbb{R}^{n_i} | \mathcal{F}_i x_{[i]} \leq \mathbf{1}_{\bar{r}_i}\} \\ &= \{x_{[i]} \in \mathbb{R}^{n_i} | x_{[i]} = \Xi_i d_i,\ \|d_i\|_\infty \leq 1\}\end{aligned} \tag{9}$$

$$\begin{aligned}\hat{\mathbb{X}}_i &= \{\hat{x}_{[i]} \in \mathbb{R}^{n_i} | \hat{f}^T_{i,r}\hat{x}_{[i]} \leq \hat{l}_i, r \in 1:\bar{\hat{r}}_i\} = \{\hat{x}_{[i]} \in \mathbb{R}^{n_i} | \hat{\mathcal{F}}_i\hat{x}_{[i]} \leq \hat{l}_i\mathbf{1}_{\bar{\hat{r}}_i}\} \\ &= \{\hat{x}_{[i]} \in \mathbb{R}^{n_i} | \hat{x}_{[i]} = \hat{\Xi}_i\hat{d}_i,\ \|\hat{d}_i\|_\infty \leq \hat{l}_i\}\end{aligned} \tag{10}$$

where $\mathcal{F}_i = (f^T_{i,1},\dots,f^T_{i,\bar{r}_i}) \in \mathbb{R}^{\bar{r}_i\times n_i}$, $\text{rank}(\mathcal{F}_i) = n_i$, $d_i \in \mathbb{R}^{e_i}$, $\Xi_i \in \mathbb{R}^{n_i\times e_i}$, $\hat{l}_i \in \mathbb{R}_+$, $\hat{\mathcal{F}}_i = (\hat{f}^T_{i,1},\dots,\hat{f}^T_{i,\bar{\hat{r}}_i}) \in \mathbb{R}^{\bar{\hat{r}}_i\times n_i}$, $\hat{d}_i \in \mathbb{R}^{\bar{e}_i}$ and $\hat{\Xi}_i \in \mathbb{R}^{n_i\times\bar{e}_i}$. Moreover we assume that $\hat{\mathcal{F}}_i$ and $\hat{\Xi}_i$ are given, whereas $\hat{l}_i$ are free parameters that will be tuned in the control design procedure. Sets $\mathbb{U}_i$ and $\mathbb{V}_i$, $i \in \mathcal{M}$ are polytopes containing the origin in their interior, that, without loss of generality, are defined as follows

$$\begin{aligned}\mathbb{U}_i &= \{u_{[i]} \in \mathbb{R}^{m_i} | h^T_{i,r}u_{[i]} \leq 1, r \in 1:r_{u_i}\} \\ &= \{u_{[i]} \in \mathbb{R}^{m_i} | \mathcal{H}_i u_{[i]} \leq \mathbf{1}_{r_{u_i}}\},\end{aligned}$$

$$\begin{aligned}\mathbb{V}_i &= \{v_{[i]} \in \mathbb{R}^{m_i} | h^T_{i,r}v_{[i]} \leq 1 - l_{v_{i,r}}, r \in 1:r_{u_i}\} \\ &= \{v_{[i]} \in \mathbb{R}^{m_i} | \mathcal{H}_i v_{[i]} \leq \mathbf{1}_{r_{u_i}} - l_{v_i}\},\end{aligned} \tag{11}$$

where $\mathcal{H}_i = (h^T_{i,1},\dots,h^T_{i,r_{u_i}}) \in \mathbb{R}^{r_{u_i}\times m_i}$ are given, $l_{v_i} = (l_{v_{i,1}},\dots,l_{v_{i,r_{u_i}}})$ and $l_{v_{i,r}} \in \mathbb{R}_+$, $r \in 1:r_{u_i}$ are free parameters.

Similarly to [13], under Assumption 1-(i) and the definition of sets $\mathbb{X}_i$, $i \in \mathcal{M}$ there exist nonempty RPIs $\mathbb{Z}_i \subseteq \mathbb{R}^{n_i}$, $i \in \mathcal{M}$ for the dynamics (6) and

$$w_{[i]} \in \mathbb{W}_i = \bigoplus_{j\in\mathcal{N}_i} A_{ij}\mathbb{X}_j. \tag{12}$$

In particular, for $\delta_i > 0$, we denote with $\mathbb{Z}_i(\delta_i)$ an RPI set that is a δ_i-outer approximation of the minimal RPI for (6) and $w_{[i]} \in \mathbb{W}_i$. For guaranteeing (8) we introduce the following Assumption.

Assumption 2. *For all* $i \in \mathcal{M}$ *there exist* $\delta_i > 0$ *and nonempty constraint sets* $\hat{\mathbb{X}}_i$ *and* $\mathbb{V}_i$ *verifying*

$$\hat{\mathbb{X}}_i \oplus \mathbb{Z}_i(\delta_i) \subseteq \mathbb{X}_i \tag{13}$$

$$\mathbb{V}_i \oplus K_i\mathbb{Z}_i(\delta_i) \subseteq \mathbb{U}_i. \tag{14}$$

■

Under Assumptions 1-2, as in [12], we set in (5)

$$v_{[i]}(t) = \kappa_i(x_{[i]}(t)) = v_{[i]}(0|t), \qquad \bar{x}_{[i]}(t) = \eta_i(x_{[i]}(t)) = \hat{x}_{[i]}(0|t) \tag{15}$$

where $v_{[i]}(0|t)$ and $\hat{x}_{[i]}(0|t)$ are optimal values of variables $v_{[i]}(0)$ and $\hat{x}_{[i]}(0)$, respectively, obtained by solving the following MPC-i problem at time t

$$\mathbb{P}^N_i(x_{[i]}(t)) = \min_{\substack{\hat{x}_{[i]}(0)\\ v_{[i]}(0:N_i-1)}} \sum_{k=0}^{N_i-1} \ell_i(\hat{x}_{[i]}(k), v_{[i]}(k)) + V_{f_i}(\hat{x}_{[i]}(N_i)) \tag{16a}$$

$$x_{[i]}(t) - \hat{x}_{[i]}(0) \in \mathbb{Z}_i(\delta_i) \tag{16b}$$

$$\hat{x}_{[i]}(k+1) = A_{ii}\hat{x}_{[i]}(k) + B_i v_{[i]}(k) \qquad k \in 0:N_i-1 \tag{16c}$$

$$\hat{x}_{[i]}(k) \in \hat{\mathbb{X}}_i,\ v_{[i]}(k) \in \mathbb{V}_i \qquad k \in 0:N_i-1 \tag{16d}$$

$$\hat{x}_{[i]}(N_i) \in \hat{\mathbb{X}}_{f_i} \tag{16e}$$

In (16), $N_i \in \mathbb{N}$ is the prediction horizon, $\ell_i(\hat{x}_{[i]}(k), v_{[i]}(k)) : \mathbb{R}^{n_i\times m_i} \to \mathbb{R}_+$ is the stage cost and $V_{f_i}(\hat{x}_{[i]}(N_i)) : \mathbb{R}^{n_i} \to \mathbb{R}_+$ is the final cost, fulfilling the following assumption.

Assumption 3. *For all* $i \in \mathcal{M}$*, there exist an auxiliary control law* $\kappa^{aux}_i(\hat{x}_{[i]})$ *and a* $\mathcal{K}_\infty$ *function* $\mathcal{B}_i$ *such that:*

(i) $\ell_i(\hat{x}_{[i]}, v_{[i]}) \geq \mathcal{B}_i(\|(\hat{x}_{[i]}, v_{[i]})\|)$*, for all* $\hat{x}_{[i]} \in \mathbb{R}^{n_i}$*,* $v_{[i]} \in \mathbb{R}^{m_i}$ *and* $\ell_i(0,0) = 0$*;*

(ii) $\hat{\mathbb{X}}_{f_i} \subseteq \hat{\mathbb{X}}_i$ *is an invariant set for* $\hat{x}^+_{[i]} = A_{ii}\hat{x}_{[i]} + B_i\kappa^{aux}_i(\hat{x}_{[i]})$*;*
(iii) $\forall \hat{x}_{[i]} \in \hat{\mathbb{X}}_{f_i}$, $\kappa^{aux}_i(\hat{x}_{[i]}) \in \mathbb{V}_i$*;*
(iv) $\forall \hat{x}_{[i]} \in \hat{\mathbb{X}}_{f_i}$, $V_{f_i}(\hat{x}^+_{[i]}) - V_{f_i}(\hat{x}_{[i]}) \leq -\ell_i(\hat{x}_{[i]}, \kappa^{aux}_i(\hat{x}_{[i]}))$. ■

We highlight that there are several methods, discussed e.g. in [6], for computing $\ell_i(\cdot)$, $V_{f_i}(\cdot)$ and $\mathbb{X}_{f_i}$ verifying Assumption 3.
The next Theorem provides the main results on stability of the closed-loop system (7) and (15) equipped with constraints (8).

Theorem 1. *Let Assumptions 1-3 hold. Define the feasibility region for the MPC-i problem as*

$$\mathbb{X}^N_i = \{s_{[i]} \in \mathbb{X}_i : \quad (16) \text{ is feasible for } x_{[i]}(t) = s_{[i]}\}$$

and the collective feasibility region as $\mathbb{X}^N = \prod_{i\in\mathcal{M}} \mathbb{X}^N_i$.
Then

(i) if $\mathbf{x}(0) \in \mathbb{X}^N$*, i.e.* $x_{[i]}(0) \in \mathbb{X}^N_i$ *for all* $i \in \mathcal{M}$*, constraints* (8) *are fulfilled at all time instants;*
(ii) the origin of the closed-loop system (7) *and* (15) *is asymptotically stable and* $\mathbb{X}^N$ *is a region of attraction.*

■

The proof of Theorem 1 uses arguments similar to the ones adopted in [8]. For a complete proof and details we refer the reader to [14].

III. Decentralized synthesis of DeMPC

In this Section we discuss the decentralized design of the DeMPC scheme given by (5) and (16). Our method hinges on the following proposition, that is proved in the Appendix.

Proposition 1. *For given matrices* K_i, $i \in \mathcal{M}$*, verifying Assumption 1-(i), if the following conditions are fulfilled*

$$\alpha_i = \sum_{j\in\mathcal{N}_i}\sum_{k=0}^{\infty} ||\mathcal{F}_i F_i^k A_{ij}\mathcal{F}_j^\flat||_\infty < 1, \ \forall i \in \mathcal{M} \tag{17}$$

then

(I) Assumption 1-(ii) holds.
(II) For all $i \in \mathcal{M}$*, defining for* $r \in 1 : \bar{r}_i$

$$\underline{\hat{L}}_{i,r} = \frac{1 - \sum_{j\in\mathcal{N}_i}\sum_{k=0}^{\infty} ||f^T_{i,r}F^k_i A_{ij}\Xi_j||_\infty}{||f^T_{i,r}\hat{\Xi}_i||_\infty}, \tag{18}$$

there is $\delta_i > 0$ *such that*

$$\hat{L}_i = \min_{r\in 1:\bar{r}_i} \ \underline{\hat{L}}_{i,r} - \frac{||f^T_{i,r}||_\infty\delta_i}{||f^T_{i,r}\hat{\Xi}_i||_\infty} > 0. \tag{19}$$

Furthermore, choosing $\hat{l}_i \in (0, \hat{L}_i]$ *and the set* $\hat{\mathbb{X}}_i$ *as in* (10)*, the inclusion* (13) *holds.*
(III) For $\delta_i > 0$ *verifying* (19) *assume the following condition is fulfilled*

$$\beta_i(\delta_i) = \max_{r\in 1:r_{u_i}} \ \hat{l}_{v_{i,r}}(\delta_i) < 1 \tag{20}$$

with

$$\hat{l}_{v_{i,r}}(\delta_i) = \sup_{z_i\in\mathbb{Z}_i(\delta_i)} \ h^T_{i,r}K_i z_i, \ r \in 1 : r_{u_i}. \tag{21}$$

Then, choosing $\mathbb{V}_i$ *as in* (11) *for* $l_{v_{i,r}} = \hat{l}_{v_{i,r}}(\delta_i)$*, the inclusion* (14) *holds.* ■

We highlight that, for a given $i \in \mathcal{M}$, the quantities α_i in (17), $\hat{L}_i$ in (19), and $\beta_i(\delta_i)$ in (20) depend only upon local fixed parameters $\{A_{ii}, B_i, \mathcal{F}_i, \mathcal{H}_i\}$, neighbors' fixed parameters $\{A_{ij}, \Xi_j\}_{j\in\mathcal{N}_i}$ (or equivalently $\{A_{ij}, \mathcal{F}_j\}_{j\in\mathcal{N}_i}$) and local tunable parameters $\{K_i, \delta_i\}$ but not on neighbors' tunable parameters. Moreover, also the computation of sets $\mathbb{Z}_i(\delta_i)$ depends upon the same parameters. This implies that the choice of $\{K_i, \delta_i\}$ does not influence the choice of $\{K_j, \delta_j\}_{j\neq i}$ and therefore, in order to verify Assumptions 1 and 2, we need to solve the following independent problems for $i \in \mathcal{M}$.

Problem $\mathcal{P}_i$*:* Check if there exist K_i and $\delta_i > 0$ such that $\alpha_i < 1$, $\hat{L}_i > 0$ and $\beta_i(\delta_i) < 1$. ■

The solution to Problems $\mathcal{P}_i$ also allows for the decentralized design of controllers MPC-i that, using the synthesis methods reviewed in [6], can also satisfy Assumption 3. The overall procedure for the decentralized synthesis of local controllers $\mathcal{C}_{[i]}$, $i \in \mathcal{M}$ is summarized in Algorithm 1.

Algorithm 1 Design of controller $\mathcal{C}_{[i]}$ for system $\Sigma_{[i]}$

Input: A_{ii}, B_i, $\mathbb{X}_i$, $\mathbb{U}_i$, $\mathcal{N}_i$, $\{A_{ij}\}_{j\in\mathcal{N}_i}$, $\{\mathbb{X}_j\}_{j\in\mathcal{N}_i}$
Output: controller $\mathcal{C}_{[i]}$ in (5)

1) Find K_i and $\delta_i > 0$ such that Assumption 1-(i) is fulfilled, $\alpha_i < 1$, (19) holds and $\beta_i(\delta_i) < 1$. If they do not exist **stop** (the controller $\mathcal{C}_{[i]}$ cannot be designed).
2) Compute sets $\mathbb{W}_i = \bigoplus_{j\in\mathcal{N}_i} A_{ij}\mathbb{X}_j$ and $\mathbb{Z}_i(\delta_i)$.
3) Compute $\hat{L}_i$ as in (19), choose $\hat{l}_i = \hat{L}_i$ and define $\hat{\mathbb{X}}_i$ as in (10).
4) Compute $\hat{l}_{v_{i,r}}(\delta_i)$ as in (21), set $l_{v_{i,r}} = \hat{l}_{v_{i,r}}(\delta_i)$ and define $\mathbb{V}_i$ as in (11).
5) Compute $\ell_i(\cdot)$, $V_{f_i}(\cdot)$ and $\mathbb{X}_{f_i}$ verifying Assumption 3.

Next, we propose an automatic method for computing the matrix K_i and $\delta_i > 0$ in step 1 of Algorithm 1. We design K_i as the LQ control gain associated to matrices $Q_i \geq 0$ and $R_i > 0$, i.e.

$$K_i = (R_i + B^T_i\bar{P}_iB_i)^{-1}B^T_i\bar{P}_iA_{ii}$$

where $\bar{P}_i$ is the solution to the algebraic Riccati equation

$$A^T_{ii}\bar{P}_iA_{ii} + Q_i - A^T_{ii}\bar{P}_iB_i(R_i + B^T_i\bar{P}_iB_i)^{-1}B^T_i\bar{P}_iA_{ii} = \bar{P}_i.$$

We then solve the following nonlinear optimization problem

$$\min_{\delta_i,\ Q_i,\ R_i} \ \mu_{\alpha_i}\alpha_i + \mu_{\beta_i}\beta_i(\delta_i) \tag{22a}$$
$$Q_i \geq 0, \ R_i > 0 \tag{22b}$$
$$\alpha_i < 1, \ \hat{L}_i > 0 \tag{22c}$$
$$\delta_i > 0, \ \beta_i(\delta_i) < 1 \tag{22d}$$

where $\mu_{\alpha_i} \geq 0$ and $\mu_{\beta_i} \geq 0$. In order to simplify the optimization problem (22) one can assume $Q_i = \text{diag}(q_{i,1}, \ldots, q_{i,n_i})$, $R_i = \text{diag}(r_{i,1}, \ldots, r_{i,m_i})$ and replace the matrix inequalities in (22b) with the scalar inequalities $q_{i,k} \geq 0$, $k \in 1 : n_i$ and $r_{i,k} > 0$, $k \in 1 : m_i$. The feasibility of problem (22) guarantees that the controller $\mathcal{C}_{[i]}$ can be successfully designed. Moreover, in (22a), weights μ_{α_i} and μ_{β_i}, that can be chosen by the user, establish a trade-off between the maximization of sets $\hat{\mathbb{X}}_i$ and $\mathbb{V}_i$, respectively. Note also that the series in (17) and (18) involve only positive terms and they can be easily truncated either if (22c) is violated or if summands fall below the machine precision. Additional computational remarks on the calculation of sets $\mathbb{Z}_i(\delta_i)$ are discussed in [14].

IV. Plug-and-play operations

Consider a plant composed by subsystems $\Sigma_{[i]}$, $i \in \mathcal{M}$ equipped with local controllers $\mathcal{C}_{[i]}$, $i \in \mathcal{M}$ produced by Algorithm 1. In case subsystems are added or removed, existing controllers have to be redesigned. In this Section we propose a plug-and-play distributed solution, which requires the redesign of a limited number of controllers. As mentioned in Section I, we assume subsystems get plugged in and unplugged off-line.

A. Plugging in operation

We start considering the plugging in of subsystem $\Sigma_{[M+1]}$, characterized by parameters $A_{M+1,M+1}$, B_{M+1}, $\mathbb{X}_{M+1}$, $\mathbb{U}_{M+1}$, $\mathcal{N}_{M+1}$ and $\{A_{M+1,j}\}_{j\in\mathcal{N}_{M+1}}$, into an existing plant. In particular $\mathcal{N}_{M+1}$ identifies the subsystems that will be physically coupled to $\Sigma_{[M+1]}$ and $\{A_{M+1,j}\}_{j\in\mathcal{N}_{M+1}}$ are the corresponding coupling terms. For designing the controller $\mathcal{C}_{[M+1]}$ we execute Algorithm 1 that needs information only from systems $\Sigma_{[j]}$, $j\in\mathcal{N}_{M+1}$. If Algorithm 1 stops before the last step we declare that $\Sigma_{[M+1]}$ cannot be plugged in. Let

$$\mathcal{S}_i = \{j : i \in \mathcal{N}_j\} \tag{23}$$

be the set of successors to system i. Since $\Sigma_{[M+1]}$ is now a new neighbor to systems $\Sigma_{[j]}$, $j\in\mathcal{S}_{M+1}$, existing matrices K_j, $j\in\mathcal{S}_{M+1}$ may now result in $\alpha_j \geq 1$ or $\hat{L}_j \leq 0$ or $\beta_i(\delta_i)\geq 1$. Indeed, when $\mathcal{N}_j$ gets larger, the quantity α_j in (17) (respectively $\hat{L}_j$ in (19)) can only increase (respectively decrease). Furthermore, the size of the set $\mathbb{Z}_j(\delta_j)$ increases (because the set $\mathbb{W}_i$ in (12), gets bigger) and therefore the condition in (20) could be violated. This means that for each $j\in\mathcal{S}_{M+1}$ the controllers $\mathcal{C}_{[j]}$ must be redesigned according to Algorithm 1. Again, if Algorithm 1 stops before completion for some $j\in\mathcal{S}_{M+1}$, we declare that $\Sigma_{[M+1]}$ cannot be plugged in.
In conclusion, the addition of system $\Sigma_{[M+1]}$ triggers the design of controller $\mathcal{C}_{[M+1]}$ and the redesign of controllers $\mathcal{C}_{[j]}$, $j\in\mathcal{S}_{M+1}$ according to Algorithm 1. Note that controller redesign does not propagate further in the network, i.e. even without changing controllers $\mathcal{C}_{[i]}$, $i\notin\{M+1\}\bigcup\mathcal{S}_{M+1}$ stability of the origin and constraint satisfaction are guaranteed for the new closed-loop system.

B. Unplugging operation

We consider the unplugging of system $\Sigma_{[k]}$, $k\in\mathcal{M}$ and define $\mathcal{S}_k$ as in (23). Since for each $i\in\mathcal{S}_k$ the set $\mathcal{N}_i$ gets smaller, we have that α_i in (17) (respectively $\hat{L}_i$ in (19)) cannot increase (respectively decrease). Furthermore, the size of the set $\mathbb{Z}_i(\delta_i)$ cannot increase and therefore the inequality (20) cannot be violated. This means that for each $i\in\mathcal{S}_k$ the controller $\mathcal{C}_{[i]}$ does not have to be redesigned. Moreover, since for each system $\Sigma_{[j]}$, $j\notin\{k\}\bigcup\mathcal{S}_k$, the set $\mathcal{N}_j$ does not change, the redesign of controller $\mathcal{C}_{[j]}$ is not required.
In conclusion, the removal of system $\Sigma_{[k]}$ does not require the redesign of any controller, in order to guarantee stability of the origin and constraint satisfaction for the new closed-loop system. However systems $\Sigma_{[i]}$, $i\in\mathcal{S}_k$, have one neighbor less and redesign of controllers $\mathcal{C}_{[i]}$ through Algorithm 1 could improve the performance. Furthermore, as discussed in [14], redesign is mandatory when matrices A_{ii} and B_i in (2) contain parameters that depend upon neighboring subsystems (see Section V for an example).

V. Example: Power Network System

In this Section we apply the proposed DeMPC scheme to a power network system composed by several power generation areas coupled through tie-lines. We aim at designing the AGC layer with the goals of keeping the frequency at the nominal value and to reduce power exchanges between areas. We show the advantages of PnP DeMPC when generation areas are connected/disconnected to/from an existing network.
The dynamics of an area equipped with primary control and linearized around the equilibrium value for all variables can be described by the following model [15]

$$\Sigma^C_{[i]}: \quad \dot{x}_{[i]} = A_{ii}x_{[i]} + B_i u_{[i]} + L_i \Delta P_{L_i} + \sum_{j\in\mathcal{N}_i} A_{ij}x_{[j]} \tag{24}$$

where $x_{[i]}=(\Delta\theta_i,\ \Delta\omega_i,\ \Delta P_{m_i},\ \Delta P_{v_i})$ is the state, $u_{[i]}=\Delta P_{ref_i}$ is the control input of each area, ΔP_L is the local power load and $\mathcal{N}_i$ is the sets of neighboring areas, i.e. areas directly connected to $\Sigma^C_{[i]}$ through tie-lines. The matrices of system (24) are

$$A_{ii}(\{P_{ij}\}_{j\in\mathcal{N}_i}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{\sum_{j\in\mathcal{N}_i}P_{ij}}{2H_i} & -\frac{D_i}{2H_i} & \frac{1}{2H_i} & 0 \\ 0 & 0 & -\frac{1}{T_{t_i}} & \frac{1}{T_{t_i}} \\ 0 & -\frac{1}{R_iT_{g_i}} & 0 & -\frac{1}{T_{g_i}} \end{bmatrix}$$

$$B_i = \begin{bmatrix} 0\\0\\0\\\frac{1}{T_{g_i}}\end{bmatrix},\ A_{ij} = \begin{bmatrix} 0&0&0&0\\ \frac{P_{ij}}{2H_i}&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\ L_i = \begin{bmatrix}0\\-\frac{1}{2H_i}\\0\\0\end{bmatrix}$$

For the meaning of constants as well as parameter values we defer the reader to [14]. We highlight that all parameter values are within the range of those used in Chapter 12 of [15]. Model (24) is input decoupled since both ΔP_{ref_i} and ΔP_{L_i} act only on subsystem $\Sigma^C_{[i]}$. Moreover, subsystems $\Sigma^C_{[i]}$ are parameter dependent since the local dynamics depends on the quantities $-\frac{\sum_{j\in\mathcal{N}_i}P_{ij}}{2H_i}$. Each subsystem $\Sigma^C_{[i]}$ is subject to constraints on $\Delta\theta_i$ and on ΔP_{ref_i} specified in [14]. We obtain models $\Sigma_{[i]}$ by discretizing models $\Sigma^C_{[i]}$ with 1 s sampling time, using exact discretization and treating $u_{[i]}$, ΔP_{L_i}, $x_{[j]}$, $j\in\mathcal{N}_i$ as exogenous signals.
In the following we first design the AGC layer for a power network composed by four areas (Scenario 1) and we show how that when a new area is connected (Scenario 2) the AGC can be redesigned via a plugging in operation. A further example considering disconnection of an area is provided in [14].

A. Scenario 1

We consider four areas interconnected as in Figure 1. For each

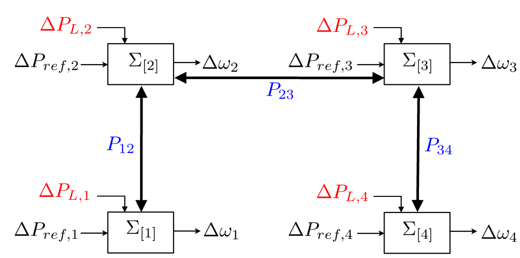

Fig. 1. Power network system of Scenario 1

system $\Sigma_{[i]}$ we design controller K_i, $i\in\mathcal{M}$ solving the optimization problem (22) for the nominal system, as shown in Section III, with $\mu_{\alpha_i}=1$, $\mu_{\beta_i}=1$, $\forall i\in\mathcal{M}$. This allows inequalities (17) to be fulfilled. Hence $\mathbf{K}$ verifies Assumption 1. Setting $\delta_i=10^{-4}$, $\forall i\in\mathcal{M}$ and applying steps 2-5 of Algorithm 1, we can compute sets $\mathbb{Z}_i(\delta_i)$, $\hat{\mathbb{X}}_i$ and $\mathbb{V}_i$ such that inclusions (13) and (14) hold $\forall i\in\mathcal{M}$. Control variables $u_{[i]}$ are obtained through (5) where $v_{[i]}$ and $\bar{x}_{[i]}$ solve the optimization problem (16): we replace the cost funciton in (16a) with the following one depending upon $x^O_{[i]}=(0,\ 0,\ \Delta P_{L_i},\ \Delta P_{L_i})$ and $u^O_{[i]}=\Delta P_{L_i}$

$$J^{N_i}_i = \sum_{k=t}^{t+N_i-1}(||\hat{x}_{[i]}(k)-x^O_{[i]}||_{\hat{Q}_i}+ \\ +||v_{[i]}(k)-u^O_{[i]}||_{\hat{R}_i}) + ||\hat{x}_{[i]}(t+N_i)-x^O_{[i]}||_{\hat{S}_i}.$$

Note that, except for the above modification of the cost function, needed for counteracting load disturbances, we followed exactly the

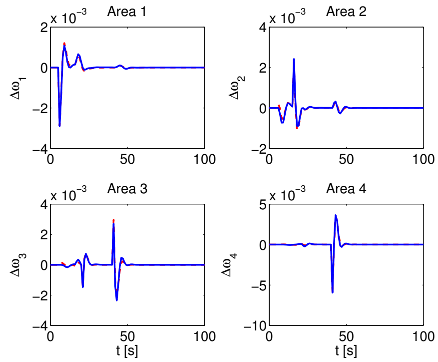

Fig. 2. Frequency deviation in each area controlled by the proposed DeMPC (bold line) and centralized MPC (dashed line).

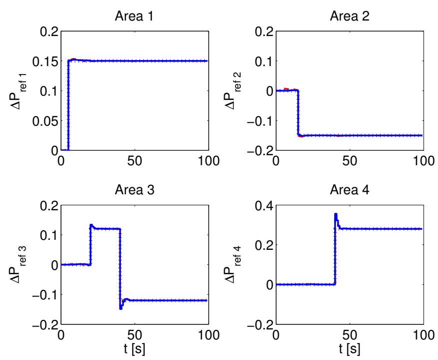

Fig. 3. Load reference set-point in each area controlled by the proposed DeMPC (bold line) and centralized MPC (dashed line).

design procedure described in Section II. Moreover, we highlight that each area can locally absorb load changes.

In Figures 2 and 3 we compare the performance of proposed DeMPC with the performance of centralized MPC. For centralized MPC we consider the overall system composed by the four areas, use the cost function $\sum_{i\in\mathcal{M}} J_i^N$ and impose the collective constraints (8). The prediction horizon is $N_i = 20$, $i \in \mathcal{M}$ for MPC-i controllers and $N = 20$ for centralized MPC. In the simulations step power loads ΔP_{L_i}, specified in [14], have been used and they account for the step-like changes of the control variables in Figure 3. We highlight that the performance of decentralized and centralized MPC are totally comparable, in terms of frequency deviation (Figure 2) and control variables (Figure 3).

B. Scenario 2

We consider the power network proposed in Scenario 1 and we add a fifth area connected as in Figure 4 with values of parameters and constraints provided in [14]. The set of successors to system 5 is $\mathcal{S}_5 = \{2,4\}$. Since systems $\Sigma_{[j]}$, $j \in \mathcal{S}_5$ depend on parameters

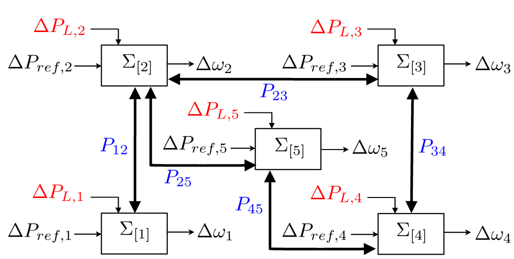

Fig. 4. Power network system of Scenario 2

related to $\mathcal{S}_5$, a re-tuning of their controllers is needed. As described in Section IV-A, only systems $\Sigma_{[j]}$, $j \in \mathcal{S}_5$ update their controller $\mathcal{C}_{[j]}$. For systems $\Sigma_{[j]}$, $j \in \mathcal{S}_5$, since the set $\mathcal{N}_j$ changes, we retune controllers $\mathcal{C}_{[j]}$ using Algorithm 1. In particular, we compute K_j, $j \in \mathcal{S}_5$ and K_5 using the procedure described in Section III with $\mu_{\alpha_k} = 1$ and $\mu_{\beta_k} = 1$, $k \in \{5\}\bigcup\mathcal{S}_5$ that allow one to verify inequalities (17) for systems $\Sigma_{[j]}$, $j \in \mathcal{S}_5$ and $\Sigma_{[5]}$. Therefore $\mathbf{K}$ fulfills Assumption 1-(ii). Setting $\delta_j = 10^{-4}$, $j \in \mathcal{S}_5$ and $\delta_5 = 10^{-4}$, the execution of Algorithm 1 does not stop before completion and hence we compute the new sets $\mathbb{Z}_j(\delta_j)$, $\hat{\mathbb{X}}_j$ and $\mathbb{V}_j$, $j \in \mathcal{S}_5$ and $\mathbb{Z}_5(\delta_5)$, $\hat{\mathbb{X}}_5$ and $\mathbb{V}_5$. We highlight that no retuning of controllers $\mathcal{C}_{[1]}$ and $\mathcal{C}_{[3]}$ is needed since systems $\Sigma_{[1]}$ and $\Sigma_{[3]}$ are not neighbors to system $\Sigma_{[5]}$. Simulation experiments, described in [14], show that also in this case the performance of centralized MPC and the proposed DeMPC scheme are almost identical.

VI. Conclusions

In this paper we proposed a tube-based DeMPC scheme for linear constrained systems, with the goal of stabilizing the origin of the closed-loop system and guaranteeing constraints satisfaction. The key feature of our approach is that the design procedure does not require any centralized computation. This enables PnP operations, i.e. when a subsystem is plugged-in or unplugged at most the synthesis of its controller and the redesign of successors' controllers are needed. In future we will generalize our approach to embrace decentralized output-feedback control and tracking problems.

VII. Acknowledgment

The authors are indebted with R. Scattolini for insightful discussions.

Appendix

A. Proof of Proposition 1

1) Proof of (I): Define a matrix $\mathbb{M}$ such that its ij-th entry μ_{ij} is

$$\begin{array}{ll} \mu_{ij} = -1 & \text{if} \quad i=j \\ \mu_{ij} = \sum_{k=0}^{\infty} \|\mathcal{F}_i F_i^k A_{ij} \mathcal{F}_j^{\flat}\|_\infty & \text{if} \quad i \neq j. \end{array}$$

Note that all the off-diagonal entries of matrix $\mathbb{M}$ are non-negative, i.e., $\mathbb{M}$ is Metzler [16]. We recall the following results.

Lemma 1 (see [17])**.** *Let matrix $\mathbb{M} \in \mathbb{R}^{M\times M}$ be Metzler. Then $\mathbb{M}$ is Hurwitz if and only if there is a vector $\nu \in \mathbb{R}_+^M$ such that $\mathbb{M}\nu < \mathbf{0}$.*

Lemma 2. *Define the matrix $\Gamma = \mathbb{M} + \mathbb{I}_M$ where $\mathbb{M} \in \mathbb{R}^{M\times M}$, $\mathbb{I}_M$ is the $M \times M$ identity matrix and Γ is non negative. Then the Metzler matrix $\mathbb{M}$ is Hurwitz if and only if Γ is Schur.*

The proof of Lemma 2 easily follows from Theorem 13 in [16]. Inequalities (17) are equivalent to $\mathbb{M}\nu < \mathbf{0}_M$ where $\nu = \mathbf{1}_M$. Then, from Lemma 1, $\mathbb{M}$ is Hurwitz. From Lemma 2, (17) implies that matrix $\Gamma = \mathbb{M} + \mathbb{I}_M$ is Schur.

For system $\Sigma_{[i]}$ in (2)-(3), when $u_{[i]}$ is defined as in (5), $v_{[i]} = 0$ and $\bar{x}_{[i]} = 0$, we have

$$x_{[i]}(t) = F_i^t x_{[i]}(0) + \sum_{k=0}^{t-1} F_i^k \sum_{j\in\mathcal{N}_i} A_{ij} x_{[j]}(t-k-1) \qquad (25)$$

In view of (25) we can write

$$\begin{aligned} ||\mathcal{F}_i x_{[i]}(t)||_\infty \leq{} & ||\mathcal{F}_i F_i^t \mathcal{F}_i^{\flat}||_\infty ||\mathcal{F}_i x_{[i]}(0)||_\infty + \\ & + \sum_{j\in\mathcal{N}_i} \gamma_{ij} \max_{k\leq t} ||\mathcal{F}_j x_{[j]}(k)||_\infty . \end{aligned}$$

where γ_{ij} are the entries of Γ. Denoting $\tilde{x}_{[i]} = \mathcal{F}_i x_{[i]}$, we can collectively define $\tilde{\mathbf{x}} = \tilde{\mathcal{F}}\mathbf{x}$, where $\tilde{\mathcal{F}} = \text{diag}(\mathcal{F}_1, \ldots, \mathcal{F}_M)$. From the definition of sets $\mathbb{X}_i$, we have $\text{rank}(\tilde{\mathcal{F}}) = n$. We define the system

$$\tilde{\mathbf{x}}^+ = (\tilde{\mathbf{A}} + \tilde{\mathbf{B}}\tilde{\mathbf{K}})\tilde{\mathbf{x}} \tag{26}$$

where $\tilde{\mathbf{A}} = \tilde{\mathcal{F}}\mathbf{A}\tilde{\mathcal{F}}^\flat$, $\tilde{\mathbf{B}} = \tilde{\mathcal{F}}\mathbf{B}$ and $\tilde{\mathbf{K}} = \mathbf{K}\tilde{\mathcal{F}}^\flat$. In order to analyze the stability of the origin of (26), we consider the method proposed in [18]. In view of Corollary 16 in [18], the overall system (26) is asymptotically stable if the gain matrix Γ is Schur. As shown above this property is implied by (17).

Moreover, system (26) is an expansion of the original system (see Chapter 3.4 in [1]). In view of the inclusion principle (see Theorem 3.3 in [1] and [19] for a discrete-time version), the asymptotic stability of (26) implies the asymptotic stability of the original system.

2) Proof of (II): First note that, for $i \in \mathcal{M}$, in view of (9) $||f_{i,r}^T \Xi_i||_\infty = 1$ for all $r \in 1:\bar{r}_i$ and therefore $||\mathcal{F}_i \Xi_i||_\infty = 1$. This implies that $||f_{i,r}^T F_i^k A_{ij} \Xi_j||_\infty \leq ||f_{i,r}^T F_i^k A_{ij} \mathcal{F}_j^\flat||_\infty ||\mathcal{F}_j \Xi_j||_\infty = ||f_{i,r}^T F_i^k A_{ij} \mathcal{F}_j^\flat||_\infty \leq ||\mathcal{F}_i F_i^k A_{ij} \mathcal{F}_j^\flat||_\infty$. Therefore, in view of (17), for all $r \in 1:\bar{r}_i$

$$\sum_{k=0}^{\infty} \sum_{j \in \mathcal{N}_i} ||f_{i,r}^T F_i^k A_{ij} \Xi_j||_\infty \leq \sum_{k=0}^{\infty} \sum_{j \in \mathcal{N}_i} ||\mathcal{F}_i F_i^k A_{ij} \mathcal{F}_j^\flat||_\infty < 1 \tag{27}$$

Now we want to find $\hat{l}_i > 0$ such that, simultaneously, the inclusion (13) holds and $\mathbb{Z}_i(\delta_i)$ is a δ_i-outer approximation of the mRPI $\underline{\mathbb{Z}}_i$. The mRPI for (6) is given by [20]

$$\underline{\mathbb{Z}}_i = \bigoplus_{k=0}^{\infty} F_i^k \bigoplus_{j \in \mathcal{N}_i} A_{ij} \mathbb{X}_j. \tag{28}$$

From [20], for given $\delta_i > 0$ there exist $\alpha_i \in \mathbb{R}$ and $s_i \in \mathbb{N}_+$ such that the set

$$\mathbb{Z}_i(\delta_i) = (1 - \alpha_i)^{-1} \bigoplus_{k=0}^{s_i - 1} F_i^k \bigoplus_{j \in \mathcal{N}_i} A_{ij} \mathbb{X}_j \tag{29}$$

is a δ_i-outer approximation of the mRPI $\underline{\mathbb{Z}}_i$.

Define $\bar{\mathbb{X}}_i = \hat{\mathbb{X}}_i \oplus \mathbb{Z}_i(\delta_i)$. Following the proof of Proposition 2 in [8] and using arguments from Section 3 of [13], we can then guarantee (13) if $\bar{\mathbb{X}}_i \subseteq \mathbb{X}_i$, which holds if, for all $r \in 1:\bar{r}_i$

$$\sup_{\substack{z_{[i]} \in \mathbb{Z}_i(\delta_i) \\ \hat{x}_{[i]} \in \hat{\mathbb{X}}_i}} f_{i,r}^T (z_{[i]} + \hat{x}_{[i]}) \leq 1. \tag{30}$$

Using (I) and (28), the inequalities (30) are verified if

$$\sup_{\substack{\{x_{[j]}(k) \in \mathbb{X}_j\}_{j \in \mathcal{N}_i}^{k=0,\ldots,\infty} \\ \hat{x}_{[i]} \in \hat{\mathbb{X}}_i \\ \sigma_i \in B_{\delta_i}(0)}} h_{i,r}^x(\{x_{[j]}(k)\}_{j \in \mathcal{N}_i}^{k=0,\ldots,\infty}, \hat{x}_{[i]}) + ||f_{i,r}^T \sigma_i||_\infty \leq 1 \tag{31}$$

where $h_{i,r}^x(\cdot) = f_{i,r}^T(\sum_{k=0}^\infty F_i^k \sum_{j \in \mathcal{N}_i} A_{ij} x_{[j]}(k) + \hat{x}_{[i]})$.

Since $||f_{i,r}^T \sigma_i||_\infty \leq ||f_{i,r}^T||_\infty \delta_i$, conditions (31) are satisfied if

$$\sup_{\substack{\{x_{[j]}(k) \in \mathbb{X}_j\}_{j \in \mathcal{N}_i}^{k=0,\ldots,\infty} \\ \hat{x}_{[i]} \in \hat{\mathbb{X}}_i}} h_{i,r}^x(\{x_{[j]}(k)\}_{j \in \mathcal{N}_i}^{k=0,\ldots,\infty}, \hat{x}_{[i]}) \leq 1 - ||f_{i,r}^T||_\infty \delta_i. \tag{32}$$

Using (9) and (10) we can rewrite (32) as

$$\sup_{\substack{\{||d_j(k)||_\infty \leq 1\}_{j \in \mathcal{N}_i}^{k=0,\ldots,\infty} \\ ||\hat{d}_i||_\infty \leq \hat{l}_i}} h_{i,r}^d(\{d_j(k)\}_{j \in \mathcal{N}_i}^{k=0,\ldots,\infty}, \hat{d}_i) \leq 1 - ||f_{i,r}^T||_\infty \delta_i \tag{33}$$

where $h_{i,r}^d(\cdot) = f_{i,r}^T(\sum_{k=0}^\infty F_i^k \sum_{j \in \mathcal{N}_i} A_{ij} \Xi_j d_j(k) + \hat{\Xi}_i \hat{d}_i)$.

The inequalities (33) are satisfied if

$$\sum_{k=0}^{\infty} \sum_{j \in \mathcal{N}_i} ||f_{i,r}^T F_i^k A_{ij} \Xi_j||_\infty + ||f_{i,r}^T \hat{\Xi}_i||_\infty \hat{l}_i \leq 1 - ||f_{i,r}^T||_\infty \delta_i \tag{34}$$

for all $r \in 1:\bar{r}_i$.

In view of (27) there exist sufficiently small $\delta_i > 0$ and $\hat{l}_i > 0$ satisfying (34) (and therefore verifying (13)), e.g. choosing $\hat{l}_i \in (0, \hat{L}_i]$.

3) Proof of (III): For each $i \in \mathcal{M}$, we want to find tightened input constraint $\mathbb{V}_i$ such that (14) holds. Following the rational used in Section 3 of [13], from definition of sets $\mathbb{U}_i$ and $\mathbb{V}_i$, (14) holds if (20) is satisfied. Hence, choosing $\mathbb{V}_i$ as in (11), for $l_{v_{i,r}} = \hat{l}_{v_{i,r}}(\delta_i)$ the inclusion (14) holds. ■

References

[1] J. Lunze, *Feedback control of large scale systems*. Upper Saddle River, NJ, USA: Prentice Hall, Systems and Control Engineering, 1992.

[2] S. Wang and E. J. Davison, "On the stabilization of decentralized control systems," *IEEE Transactions on Automatic Control*, vol. 18, no. 5, pp. 473–478, 1973.

[3] J. Lavaei and A. G. Aghdam, "Control of continuous-time LTI systems by means of structurally constrained controllers," *Automatica*, vol. 44, no. 1, pp. 141–148, 2008.

[4] E. Camponogara, D. Jia, B. H. Krogh, and S. Talukdar, "Distributed model predictive control," *IEEE Control Systems Magazine*, vol. 22, no. 1, pp. 44–52, 2002.

[5] D. M. Raimondo, L. Magni, and R. Scattolini, "Decentralized MPC of nonlinear systems: An input-to-state stability approach," *International Journal of Robust and Nonlinear Control*, vol. 17, pp. 1651–1667, 2007.

[6] J. B. Rawlings and D. Q. Mayne, *Model predictive control: theory and design*. Madison, WI, USA: Nob Hill Pub., 2009.

[7] P. Trodden and A. Richards, "Distributed model predictive control of linear systems with persistent disturbances," *International Journal of Control*, vol. 83, no. 8, pp. 1653–1663, 2010.

[8] M. Farina and R. Scattolini, "Distributed predictive control: A non-cooperative algorithm with neighbor-to-neighbor communication for linear systems," *Automatica*, vol. 48, no. 6, pp. 1088–1096, 2012.

[9] L. Grune and K. Worthmann, "A distributed NMPC scheme without stabilizing terminal constraints," in *Distributed Decision Making and Control*, R. Johansson and A. Rantzer, Eds. Springer, Lecture Notes in Control and Information Sciences vol. 417, 2012, ch. 12, pp. 261–287.

[10] S. Riverso and G. Ferrari-Trecate, "Tube-based distributed control of linear constrained systems," *Automatica*, vol. 48, no. 11, pp. 2860–2865, 2012.

[11] J. Stoustrup, "Plug & Play Control: Control Technology towards new Challenges," in *Proceedings of the 10th European Control Conference*, Budapest, Hungary, August 23-26, 2009, pp. 1668–1683.

[12] D. Q. Mayne, M. M. Seron, and S. V. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219–224, 2005.

[13] I. Kolmanovsky and E. G. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical Problems in Engineering*, vol. 4, no. 4, pp. 317–363, 1998.

[14] S. Riverso, M. Farina, and G. Ferrari-Trecate, "Plug-and-Play Decentralized Model Predictive Control," RIDIS 157/12, Dipartimento di Informatica e Sistemistica, Università degli Studi di Pavia, Pavia, Italy, Tech. Rep., 2012. [Online]. Available: arXiv:1302.0226

[15] H. Saadat, *Power System Analysis*, 2nd ed. New York. NY, USA: McGraw-Hill Series in Electrical and Computer Engineering, 2002.

[16] L. Farina and S. Rinaldi, *Positive Linear Systems*. New York. NY, USA: John Wiley & Sons, 2000.

[17] O. Mason and R. Shorten, "On Linear Copositive Lyapunov Functions and the Stability of Switched Positive Linear Systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 7, pp. 1346–1349, 2007.

[18] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "An ISS small gain theorem for general networks," *Mathematics of Control, Signals, and Systems*, vol. 19, no. 2, pp. 93–122, 2007.

[19] S. S. Stankovic, "Inclusion Principle for Discrete-Time Time-Varying Systems," *Dynamics of Continuous, Discrete and Impulsive Systems*, vol. 11, no. Series A: Mathematical Analysis, pp. 321–338, 2004.

[20] S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, "Invariant approximations of the minimal robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 406–410, 2005.