PLURISUBHARMONIC FUNCTIONS AND THE KÄHLER-RICCI FLOW

By LEI NI and LUEN-FAI TAM

Abstract. In this paper, a sharp linear trace Li-Yau-Hamilton inequality for Kähler-Ricci flow is proved. The new inequality extends the previous trace Harnack inequality obtained by H.-D. Cao. We also establish sharp gradient estimates for the positive solution of the time-dependent heat equation for some cases. Finally, we apply this new linear trace Li-Yau-Hamilton inequality to study the Liouville properties of the plurisubharmonic functions on complete Kähler manifolds with bounded nonnegative holomorphic bisectional curvature.

0. Introduction. Consider the Kähler-Ricci flow on a Kähler manifold $(M, g_{\alpha\bar{\beta}}(x))$:

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}, \quad g_{\alpha\bar{\beta}}(x,0) = g_{\alpha\bar{\beta}}(x). \tag{0.1}$$

In this work, $(M, g_{\alpha\bar{\beta}}(x))$ will be assumed to be complete and noncompact with bounded nonnegative holomorphic bisectional curvature. Solutions of (0.1) on complete noncompact Kähler manifolds with bounded nonnegative bisectional curvature were extensively studied in a series of paper of Shi [Sh1–3].

Important properties and applications have also been obtained, see [Sh2–3], [C-Z], [C-T-Z]. In [N-T], the authors studied the Kähler-Ricci flow from another point of view. Namely, solutions of (0.1) are investigated by using the solution to the Poincaré-Lelong equation obtained in [M-S-Y], [N-S-T1]. More precisely, it was proved in [N-S-T1], under some mild average assumptions on scalar curvature $\mathcal{R}_0(x)$ of the initial metric $g_{\alpha\bar{\beta}}(x)$, one can solve the Poincaré-Lelong equation:

$$\sqrt{-1}\partial\bar{\partial} u_0 = \mathrm{Ric}_0, \tag{0.2}$$

where Ric_0 is the Ricci form of the initial metric. We should mention that (0.2) was solved by Mok-Siu-Yau [M-S-Y] and Mok [M1] in the case when M has maximal volume growth and the scalar curvature has quadratical pointwise decay. Using the solution of (0.2), one can easily find a function $u(x,t)$ so that

Manuscript received October 21, 2001.

Research of the first author supported in part by NSF grant DMS-0196405 and DMS-0203023; research of the second author supported in part by an Earmarked Grant of Hong Kong #CUHK4217/99P.

American Journal of Mathematics 125 (2003), 0–00.

$\sqrt{-1}\partial\bar{\partial}u(x,t) = \text{Ric}(x,t)$ where Ric is the Ricci form of the metric $g(t)$. Moreover $u(x,t)$ satisfies the time-dependent heat equation:

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) u = 0 \\ u(x,0) = u_0(x). \end{cases} \tag{0.3}$$

Hence one can study (0.1) by using (0.3). For example, we gave a simple proof of the long time existence of (0.1) under some growth conditions on $\mathcal{R}_0$ in [N-T]. Note that in this case u_0 and $u(\cdot,t)$ are both plurisubharmonic because $g(t)$ has nonnegative holomorphic bisectional curvature [Sh2–3].

This motivates us to study (0.3) for general plurisubharmonic initial data u_0. Under a rather mild assumption on the growth rate of u_0, we can prove that (0.3) has a long time solution in the sense that if the Kähler-Ricci flow (0.1) has a solution up to time T, then (0.3) also has a unique solution up to time T.

The next important question is whether or not under the flow (0.3) the plurisubharmonicity will be preserved. In order to study this problem, we have to study the complex Hessian $u_{\alpha\bar{\beta}}$ of u. One can show that $u_{\alpha\bar{\beta}}$ satisfies the complex Lichnerowicz-Laplacian heat equation (see (1.2) for details). We shall prove that if the $\widetilde{\Delta}u_0$ is of at most exponential growth, then plurisubharmonicity will be preserved. Here $\widetilde{\Delta}$ is the Laplacian of the initial metric. In fact, we shall prove the result for more general Hermitian symmetric (1,1) tensors which satisfy the Lichnerowicz-Laplacian heat equation. See Proposition 1.1. In case u_0 is the solution of (0.2), the assumption on the rate of growth of $\widetilde{\Delta}u_0$ is the same as the assumption on the rate of growth of the scalar curvature $\mathcal{R}_0$, which is assumed to be bounded in [Sh2–3].

There are many important differential Harnack type inequalities for Ricci flow and curvature flows obtained by various people, see [L-Y], [H4], [Cw1–2], [Co1–2], [A] for examples. Works in this area can be traced back to the fundamental works of Li-Yau [L-Y] and Hamilton [H4]. For this reason, in this paper we shall call this kind of inequalities to be Li-Yau-Hamilton type inequalities, or LYH inequalities for short. In [C-H], Chow and Hamilton obtained a linear trace LYH inequality for a symmetric two-tensor on a Riemannian manifold with a family of metric $g(t)$ satisfying the Ricci flow equation so that the initial metric has nonnegative curvature operator. The two-tensor is assumed to satisfy the real Lichnerowicz-Laplacian heat equation. In this paper, using the results of Cao [Co1–2], we shall prove a complex version of Chow-Hamilton's result. More precisely, suppose (0.1) has a solution on $M \times [0,T]$ so that the initial metric has nonnegative bounded holomorphic bisectional curvature. Let $h_{\alpha\bar{\beta}}(x,t)$ be a solution of the complex Lichnerowicz-Laplacian heat equation so that $h_{\alpha\bar{\beta}}(x,0) \geq 0$ and $h_{\alpha\bar{\beta}}(x,t)$ satisfies some growth conditions. Then on $M \times (0,T]$, we have

$$\begin{aligned} Z = & \frac{1}{2}[g^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}\operatorname{div}(h)_\alpha + g^{\gamma\bar{\delta}}\nabla_\gamma\operatorname{div}(h)_{\bar{\delta}}] \\ & + g^{\alpha\bar{\beta}}g^{\gamma\bar{\delta}}[R_{\alpha\bar{\delta}}h_{\gamma\bar{\beta}} + \nabla_\gamma h_{\alpha\bar{\delta}}V_{\bar{\beta}} + \nabla_{\bar{\beta}}h_{\alpha\bar{\delta}}V_\gamma + h_{\alpha\bar{\delta}}V_{\bar{\beta}}V_\gamma] + \frac{H}{t} \geq 0 \end{aligned}$$

where $\operatorname{div}(h)_\alpha = g^{\gamma\bar\delta}\nabla_\gamma h_{\alpha\bar\delta}$, $\operatorname{div}(h)_{\bar\beta} = g^{\gamma\bar\delta}\nabla_{\bar\delta} h_{\gamma\bar\beta}$, H is the trace of $h_{\alpha\bar\beta}$ with respect to $g_{\alpha\bar\beta}(x,t)$, and V is any vector field of type (1,0). See Theorem 1.1 for details. In case $h_{\alpha\bar\beta}$ is the complex Hessian of a plurisubharmonic solution of (0.3) so that u_0 is not harmonic, then our result implies that $w = u_t$ satisfies

$$w_t - \frac{|\nabla w|^2}{w} + \frac{w}{t} \geq 0$$

which extends Cao's trace LYH type inequality for the scalar curvature [Co1–2]. Unlike [C-H], which mainly considers compact manifolds and is not very specific on noncompact manifolds, we need the growth conditions on $h_{\alpha\bar\beta}$ so that one can apply the maximum principle in [N-T].

As an application of the above results on the study of (0.3) and the linear trace LYH type inequality, we shall study Liouville properties for plurisubharmonic functions on $(M, g_{\alpha\bar\beta}(x))$. Suppose (0.1) has long time solution. Then we have the following:

THEOREM 3.1. *Let $(M, g_{\alpha\bar\beta}(x))$ be a complete noncompact manifold with bounded nonnegative holomorphic bisectional curvature so that the (0.1) has long time solution. Suppose u_0 is a plurisubharmonic function such that (i) u is bounded; and (ii) $\widetilde\Delta u_0(x) \leq \exp(a(1 + r_0(x))$ for some constant $a > 0$. Then u_0 must be constant.*

In $\mathbb{C}^m$, a plurisubharmonic function with sub-logarithmic growth must be constant. It is conjectured that this is still true for complete noncompact Kähler manifolds with nonnegative Ricci curvature. In this paper, we shall also prove that in some cases, the condition that u_0 is bounded in the above theorem can be relaxed. For example, one can prove that if the scalar curvature has quadratic decay in the average sense, then Theorem 3.1 is still true if the condition (i) is replaced by the condition that u_0 has sub-logarithmic growth. This is a special case of a more general result, see Theorem 3.2. In particular, when $u_0(x)$ is the solution of (0.2), the Liouville result mentioned above implies the gap theorem proved in [C-Z] and [N-T]. For previous results of Liouville properties of plurisubharmonic functions, please see [N], [N-S-T1–2], [N-T].

As a by-product of our argument, we also prove a Li-Yau type differential inequality for the positive plurisubharmonic solution $u(x,t)$ of (0.3) (see Theorem 2.2). Namely, we have

$$\frac{u_t}{u} - \frac{|\nabla u|^2}{u^2} + \frac{m}{t} \geq 0$$

exactly as in [L-Y] for a fixed metric. Hopefully, this differential inequality will have applications to the study of the plurisubharmonic functions, the Kähler-Ricci flow and other problems.

Here is the organization of the paper. In §1, we shall prove the preservation of nonnegativity of (1,1) tensors and the linear trace LYH type inequality for

(1,1) tensors. In §2, we shall study the initial value problem (0.3) and prove the Li-Yau type inequality for the positive solution to (0.3). In §3, we shall study Liouville properties of plurisubharmonic functions.

Acknowledgments. The authors would like to thank Ben Chow for his interest and discussions and also Huai-Dong Cao for discussions.

1. A Li-Yau-Hamilton inequality. In this section we shall prove a linear trace Li-Yau-Hamilton inequality which is the Kähler version of the one obtained by Chow and Hamilton in [C-H]. Just as Chow-Hamilton's LYH inequality extends the trace Li-Yau-Hamilton inequality of Hamilton [H4] our differential inequality extends the trace LYH inequality of Cao [Co1] for the scalar curvature. Applications of this new inequality will be given in the following sections.

Let $(M^m, g_{\alpha\bar{\beta}}(x))$ be a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature. Because of the results in [Sh1], in this section we always assume that solution of the following Ricci-Kähler flow exists on $M \times [0, T]$

$$\begin{cases} \frac{\partial}{\partial t} g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}} \\ g_{\alpha\bar{\beta}}(x, 0) = g_{\alpha\bar{\beta}}(x) \end{cases} \tag{1.1}$$

such that on $M \times [0, T]$,

(i) $g_{\alpha\bar{\beta}}(x, t)$ is nonincreasing in t and is uniformly equivalent to $g_{\alpha\bar{\beta}}(x, 0)$;

(ii) the curvature tensors of $g_{\alpha\bar{\beta}}(x, t)$ are uniformly bounded;

(iii) there exists a constant C such that

$$|\nabla \operatorname{Rm}|(x, t) \le \frac{C}{t^{\frac{1}{2}}}; \text{and}$$

(iv) $g_{\alpha\bar{\beta}}(x, t)$ has nonnegative holomorphic bisectional curvature.

Sufficient conditions that (1.1) has long time existence are given in [Sh2–3], [N-T], see also [C-T-Z] for the surfaces case.

In this work, we will use the maximum principle of the authors [N-T, Theorem 1.2] from time to time. For the convenience of the readers, we include the statement of this maximum principle here.

THEOREM 1.1. *Let $g_{ij}(x, t)$ be a smooth family of complete Riemannian metrics defined on M with $0 \le t \le T$ for some $T > 0$ such that for any $T \ge t_2 \ge t_1 \ge 0$*

$$C g_{ij}(x, t_1) \le g_{ij}(x, t_2) \le g_{ij}(x, t_1)$$

for some constant $C > 0$ for all $x \in M$ and let $f(x, t)$ be a smooth function such *that* $(\Delta - \frac{\partial}{\partial t})f(x,t) \geq 0$ *whenever* $f(x,t) \geq 0$. *Assume that*

$$\int_0^T \int_M \exp\left(-ar_0^2(x)\right) f_+^2(x,s)\, dV_0\, ds < \infty$$

for some $a > 0$, *where* $r_0(x)$ *is the distance function to a fixed point* $o \in M$ *with respect to* $g_{i\bar{j}}(x,0)$. *Suppose* $f(x,0) \leq 0$ *for all* $x \in M$. *Then* $f(x,t) \leq 0$ *for all* $(x,t) \in M \times [0,T]$.

In the following let $h_{\alpha\bar{\beta}}(x,t)$ be a Hermitian symmetric tensor defined on $M \times [0,T]$, which is also deformed by the complex Lichnerowicz-Laplacian heat equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right) h_{\gamma\bar{\delta}} = R_{\beta\bar{\alpha}\gamma\bar{\delta}} h_{\alpha\bar{\beta}} - \frac{1}{2}(R_{\gamma\bar{p}} h_{p\bar{\delta}} + R_{p\bar{\delta}} h_{\gamma\bar{p}}). \tag{1.2}$$

We shall obtain a LYH inequality for $h_{\alpha\bar{\beta}}$ provided $h_{\alpha\bar{\beta}}$ is nonnegative and does not grow very fast on $M \times [0,T]$. In application, usually we only know that $h_{\alpha\bar{\beta}}$ is nonnegative initially. Hence we shall discuss conditions on $h_{\alpha\bar{\beta}}$ so that nonnegativity is preserved under the flow. The following lemma is basically from [H4, Lemma 5.1].

LEMMA 1.1. *For any* $a > 0$ *and* $C > 0$ *there exists a positive function* $\phi(x,t)$ *and* $b > 0$ *such that* $\exp(b(r_0(x)+1)) \geq \phi(x,t) \geq \exp(a(r_0(x)+1))$ *and*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi \geq C\phi$$

on $M \times [0,T]$, *where* $r_0(x)$ *is the distance from a fixed point* o *with respect to the initial metric* $g(0)$.

Proof. By Lemma 5.1 in [H4], there is a smooth function $f(x)$ and a constant $C_1 > 0$ on M such that

$$C_1^{-1}(1 + r_0(x)) \leq f(x) \leq C_1(1 + r_0(x)),$$
$$|\nabla f| + |\nabla^2 f| \leq C_1.$$

As in [H4], we can choose $\phi(x,t) = \exp(At + \alpha f(x))$ for suitable positive constants A and α, then ϕ will be the required function.

To simplify notations, in the rest of the section, let $\|h\|$ be the norm of h with respect to $g_{\alpha\bar{\beta}}(x,t)$,

$$\begin{cases} \Phi = \|h\|^2 \\ \Psi = \|\nabla h\|^2 = \sum_{\alpha\beta\gamma}(\|\nabla_\gamma h_{\alpha\bar{\beta}}\|^2 + \|\nabla_{\bar{\gamma}} h_{\alpha\bar{\beta}}\|^2) \\ \Lambda = \|\nabla\nabla h\|^2 = \sum_{\alpha\beta\gamma\delta}(\|\nabla_\delta\nabla_\gamma h_{\alpha\bar{\beta}}\|^2 + \|\nabla_\delta\nabla_{\bar{\gamma}} h_{\alpha\bar{\beta}}\|^2). \end{cases} \tag{1.3}$$

Suppose h satisfies (1.2), then direct computations show (see [H1] for example):

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Phi = -\Psi + A, \tag{1.4}$$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Psi = -\Lambda + B, \tag{1.5}$$

where A and B satisfy the following conditions: There exists a constant $C > 0$ such that $|A| \leq C\Phi$ and $t|B| \leq C(\Phi + \Psi)$ on $M \times [0, T]$. Here we have used properties (ii) and (iii) of $g_{\alpha\bar{\beta}}$.

Moreover, in normal coordinates

$$\begin{aligned} \|\nabla\Phi\|^2 &= \sum_{\alpha} \Phi_\alpha \Phi_{\bar{\alpha}} \\ &= \sum_{\alpha} \left(\sum_{\xi,\tau} h_{\xi\bar{\tau},\alpha} h_{\bar{\xi}\tau} + h_{\xi\bar{\tau}} h_{\bar{\xi}\tau,\alpha}\right) \left(\sum_{\xi,\tau} h_{\xi\bar{\tau},\bar{\alpha}} h_{\bar{\xi}\tau} + h_{\xi\bar{\tau}} h_{\bar{\xi}\tau,\bar{\alpha}}\right) \\ &\leq 4\|h\|^2 \sum_{\alpha} \left[\left(\sum_{\xi,\tau} |h_{\xi\bar{\tau},\alpha}|^2\right)^{\frac{1}{2}} \left(\sum_{\xi,\tau} |h_{\xi\bar{\tau},\bar{\alpha}}|^2\right)^{\frac{1}{2}}\right] \\ &\leq 2\|h\|^2 \sum_{\alpha,\xi\tau} (|h_{\xi\bar{\tau},\alpha}|^2 + |h_{\xi\bar{\tau},\bar{\alpha}}|^2). \end{aligned}$$

Hence

$$\|\nabla\Phi\|^2 \leq 2\Phi\Psi. \tag{1.6}$$

Similarly,

$$\|\nabla\Psi\|^2 \leq 2\Psi\Lambda. \tag{1.7}$$

LEMMA 1.2. *Let $h_{\alpha\bar{\beta}}$ be a tensor satisfying (1.2). Suppose*

$$\|h_{\alpha\bar{\beta}}(x, 0)\| \leq \exp\left(a(1 + r_0(x)\right) \tag{1.8}$$

and

$$\int_0^T \int_M \exp\left(-br_0^2(x)\right)\|h\|^2(x, t)\, dV_t\, dt < \infty \tag{1.9}$$

for some positive constants a and b. Then there exists a positive constant $c > 0$ such that

$$\|h_{\alpha\bar{\beta}}(x, t)\| \leq \exp\left(c(1 + r_0(x)\right) \tag{1.10}$$

on $M \times [0, T]$.

Proof. By (1.4) and (1.8), it is easy to see that

$$\left(\frac{\partial}{\partial t}-\Delta\right)[e^{-C_1 t}(1+\Phi)^{\frac{1}{2}}]\le 0 \tag{1.11}$$

for some constant $C_1>0$.

By Lemma 1.1, there exists a function $\phi(x,t)$ and constant $c>0$ such that $\exp(c(r_0(x)+1))\ge\phi(x,t)\ge\exp(a(r_0(x)+1))$ and

$$\left(\frac{\partial}{\partial t}-\Delta\right)\phi\ge C\phi$$

with $C>0$. By (1.9), (1.10) and (1.11), we have $\phi+1\ge(1+\Phi)^{\frac{1}{2}}$ by the maximum principle Theorem 1.1. The lemma follows by choice of an even larger c.

Next we shall prove that nonnegativity of h will be preserved by the flow under certain conditions.

PROPOSITION 1.1. *Suppose $h_{\alpha\bar\beta}$ satisfy (1.2) and the conditions (1.8) and (1.9) of Lemma 1.2. Suppose also that $h_{\alpha\bar\beta}(x,0)\ge 0$. Then $h_{\alpha\bar\beta}(x,t)\ge 0$ for $t>0$.*

Proof. By Lemma 1.2, there exists a constant $c>0$ such that

$$\|h\|(x,t)\le\exp(c(1+r_0(x))). \tag{1.12}$$

By Lemma 1.1, for any $C'>0$, there exists a function ϕ such that

$$\exp(c'(1+r_0(x)))\ge\phi\ge\exp(2c(1+r_0(x))) \tag{1.13}$$

and

$$\left(\frac{\partial}{\partial t}-\Delta\right)\phi>C'\phi. \tag{1.14}$$

It is enough to show that $h_{\alpha\bar\beta}(x,t)+\epsilon\phi g_{\alpha\bar\beta}(x,t)>0$, for any $\epsilon>0$. Now we calculate

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)(h_{\alpha\bar\beta}+\epsilon\phi g_{\alpha\bar\beta}) = {} & R_{\alpha\bar\beta\gamma\bar\delta}(h_{\bar\gamma\delta}+\epsilon\phi g_{\bar\gamma\delta})-\frac{1}{2}R_{\alpha\bar p}(h_{p\bar\beta}+\epsilon\phi g_{p\bar\beta})\\ & -\frac{1}{2}R_{p\bar\beta}(h_{\alpha\bar p}+\epsilon\phi g_{\alpha\bar p})\\ & +\epsilon(\phi_t-\Delta\phi)g_{\alpha\bar\beta}-\epsilon\phi R_{\alpha\bar\beta}.\end{aligned} \tag{1.15}$$

Here we have used (1.2) and the Ricci flow equation. By (1.12), (1.13) and the fact that at $t=0$, $h_{\alpha\bar\beta}+\epsilon\phi g_{\alpha\bar\beta}>0$, if $h_{\alpha\bar\beta}(x,t)+\epsilon\phi g_{\alpha\bar\beta}(x,t)>0$ fails to hold at some $t>0$, then there is (x_0,t_0) and unit (1,0) vector at x_0 with $t_0>0$ such that $(h_{\alpha\bar\beta}(x_0,t_0)+\epsilon\phi g_{\alpha\bar\beta}(x,t))v^\alpha\bar v^\beta=0$ and t_0 is the first time that happens. As in [Cw3], we can extend v in a neighborhood in space-time of (x_0,t_0) such that ∇v and $\Delta v=0$ at (x_0,t_0) with respect to the metric $g(t_0)$ and such that v is independent of time. Hence at (x_0,t_0) we have

$$\begin{aligned}
0 &\geq \left(\frac{\partial}{\partial t}-\Delta\right)[(h_{\alpha\bar\beta}+\epsilon\phi g_{\alpha\bar\beta})v^\alpha\bar v^\beta]\\
&= \left[\left(\frac{\partial}{\partial t}-\Delta\right)(h_{\alpha\bar\beta}+\epsilon\phi g_{\alpha\bar\beta})\right]v^\alpha\bar v^\beta\\
&= R_{\alpha\bar\beta\gamma\bar\delta}(h_{\bar\gamma\delta}+\epsilon\phi g_{\bar\gamma\delta})v^\alpha\bar v^\beta-\frac12 R_{\alpha\bar p}(h_{p\bar\beta}+\epsilon\phi g_{p\bar\beta})v^\alpha\bar v^\beta\\
&\quad -\frac12 R_{p\bar\beta}(h_{\alpha\bar p}+\epsilon\phi g_{\alpha\bar p})v^\alpha\bar v^\beta+\epsilon(\phi_t-\Delta\phi)g_{\alpha\bar\beta}v^\alpha\bar v^\beta-\epsilon\phi R_{\alpha\bar\beta}v^\alpha\bar v^\beta.
\end{aligned}$$

Since v minimizes $h_{\alpha\bar\beta}+\epsilon\phi g_{\alpha\bar\beta}$ among all (1,0) unit vectors at x_0, first variation gives

$$(h_{\alpha\bar\beta}+\epsilon\phi g_{\alpha\bar\beta})v^\alpha=(h_{\alpha\bar\beta}+\epsilon\phi g_{\alpha\bar\beta})\bar v^\beta=0.$$

Using also the fact that $M,g(t_0)$ has nonnegative holomorphic bisectional curvature, we conclude that

$$0\geq\epsilon(\phi_t-\Delta\phi)-\epsilon\phi R_{\alpha\bar\beta}v^a\bar v^\beta>0$$

for sufficiently large C', since $|Rm|$ is bounded. This is a contradiction.

We should remark that the result is still true if M is compact. In this case, there is no need to impose a growth condition on $h_{\alpha\bar\beta}$. Moreover if $h_{\alpha\bar\beta}(x,0)$ is positive at some point, then $h_{\alpha\bar\beta}(x,t)$ will be positive for all $t>0$.

In order to apply the maximum principle we also need the following estimates.

LEMMA 1.3. *Let $h_{\alpha\bar\beta}$ as in Lemma 1.2. Then for any $a>0$,*

$$\int_0^T\int_M e^{-ar_0^2(x)}\Psi(x,t)\,dV_t\,dt<\infty, \tag{1.16}$$

$$\int_0^T\int_M te^{-ar_0^2(x)}\Lambda(x,t)\,dV_t\,dt<\infty \tag{1.17}$$

and

$$\int_0^T\int_M te^{-ar_0^2(x)}\Psi^2(x,t)\,dV_t\,dt<\infty. \tag{1.18}$$

Proof. Let $f(x)$ be a smooth function such that $0\leq f\leq 1$, $f=1$ on $B_0(o,R)$, $f=0$ outside $B_0(o,2R)$ and $|\widetilde\nabla f|\leq C/R$ for some constant C independent of R.

Here $B_0(o,R)$ is the geodesic ball with center at o and radius R with respect to $g(0)$. Multiply (1.4) by f^2 and integrating by parts, we have:

$$\begin{aligned}\int_0^T\int_M f^2\Psi\,dV_t\,dt &\le -\int_0^T\int_M f^2\left(\frac{\partial}{\partial t}-\Delta\right)\Phi\,dV_t\,dt + C_1\int_0^T\int_M f^2\Phi\,dV_t\,dt\\ &\le \int_M f^2\Phi\,dV_0\,dt + 2\int_0^T\int_M f|\nabla f|\,|\nabla\Phi|\,dV_t\,dt\\ &\quad + C_1\int_0^T\int_M f^2\Phi\,dV_t\,dt\end{aligned}$$

for some constant C_1. Here we have used the fact that dV_t is nonincreasing. Using (1.6) and Schwarz inequality, we have

$$\int_0^T\int_M f^2\Psi\,dV_t\,dt \le C_2\left[\int_M f^2\Phi\,dV_0\,dt + \int_0^T\int_M (f^2+|\widetilde{\nabla} f|^2)\Phi\,dV_0\,dt\right],$$

for some constant C_2, where we have used the fact that $g(t)$ and $g(0)$ are equivalent in $[0,T]$. Hence

$$\int_0^T\int_{B_0(o,R)}\Psi\,dV_t\,dt \le C_3\left[\int_{B_0(o,2R)}\Phi\,dV_0\,dt + \int_0^T\int_{B_0(o,2R)}\Phi\,dV_0\,dt\right], \tag{1.19}$$

for some constant C_3, where we assume $R\ge 1$. By Lemma 1.1, $\|h\|$ is at most of exponential growth, hence it is easy to see (1.16) is true because $g(0)$ has nonnegative Ricci curvature.

To prove (1.17), multiplying (1.5) by tf^2 and integrating by parts we have for $R\ge 1$,

$$\begin{aligned}&\int_0^T\int_M tf^2\Lambda\,dV_t\,dt\\ &\quad\le -\int_0^T\int_M tf^2\left(\frac{\partial}{\partial t}-\Delta\right)\Psi\,dV_t\,dt + C_4\int_0^T\int_M f^2(\Phi+\Psi)\,dV_t\,dt\\ &\quad\le \int_0^T\int_M f^2\Psi\,dV_t\,dt + 2\int_0^T\int_M tf|\nabla f|\,|\nabla\Psi|\,dV_t\,dt\\ &\qquad + C_4\int_0^T\int_M f^2(\Phi+\Psi)\,dV_t\,dt\\ &\quad\le C_5\int_0^T\int_M (f^2+|\widetilde{\nabla} f|^2)(\Phi+\Psi)\,dV_t\,dt + \frac{1}{2}\int_0^T\int_M tf^2\Lambda\,dV_t\,dt\end{aligned}$$

for some constants C_4, C_5, where we have used (1.7). Hence if R is large

$$\int_0^T\int_{B_0(o,R)} t\Lambda\,dV_t\,dt \le 3C_5\int_0^T\int_{B_0(o,2R)}(\Phi+\Psi)\,dV_t\,dt. \tag{1.20}$$

Combining (1.19) and (1.20), we can conclude that (1.17) is true.

To prove (1.18), multiplying (1.4) by $tf^2\Psi$ and integrating by parts, we have

$$\begin{aligned}
&\int_0^T\int_M tf^2\Psi^2\,dV_t\,dt\\
&\quad\le -\int_0^T\int_M tf^2\Psi\left(\frac{\partial}{\partial t}-\Delta\right)\Phi\,dV_t\,dt+\int_0^T\int_M tf^2\Psi A\,dV_t\,dt\\
&\quad\le C_6\int_0^T\int_M f^2\Psi\Phi\,dV_t\,dt+\int_0^T\int_M tf^2 t\left(\Phi\frac{\partial}{\partial t}\Psi+\Psi\Delta\Phi\right)dV_t\,dt\\
&\quad\le C_6\int_0^T\int_M f^2\Psi\Phi\,dV_t\,dt+\int_0^T\int_M tf^2\left(\Phi\Delta\Psi+\Psi\Delta\Phi\right)dV_t\,dt\\
&\qquad+\int_0^T\int_M tf^2\Phi B\,dV_t\,dt\\
&\quad\le C_7\int_0^T\int_M f^2(\Psi+\Phi)\Phi\,dV_t\,dt+2\int_0^T\int_M tf^2|\nabla\Phi|\,|\nabla\Psi|\,dV_t\,dt\\
&\qquad+2\int_0^T\int_M tf|\nabla f|(\Phi|\nabla\Psi|+\Psi|\nabla\Phi|)\,dV_t\,dt
\end{aligned}$$

for some constants C_6, C_7, where we have used (1.5). Apply (1.6) and (1.7) to $|\nabla\Phi|$ and $|\Psi|$ respectively, and use Schwarz inequality, we have,

$$\int_0^T\int_M tf^2\Psi^2\,dV_t\,dt\le C_8\left(\int_0^T\int_M (f^2+|\widetilde{\nabla} f|^2)(\Phi+\Psi)\Phi\,dV_t\,dt+tf^2\Lambda\Phi\,dV_t\,dt\right)$$

for some constant C_8. Combining this with (1.16) and (1.17) and the fact that Φ grows at most exponentially, we can conclude that (1.18) is true.

Remark 1.1. (1.17) and (1.18) imply that for any $\epsilon>0$,

$$\int_\epsilon^T\int_M te^{-ar_0^2(x)}\Lambda(x,t)\,dV_t\,dt<\infty,$$

and

$$\int_\epsilon^T\int_M te^{-ar_0^2(x)}\Psi^2(x,t)\,dV_t\,dt<\infty.$$

Now we are ready to prove a LYH inequality. Let $\operatorname{div}(h)_\alpha=g^{\gamma\bar\delta}\nabla_\gamma h_{\alpha\bar\delta}$ and $\operatorname{div}(h)_{\bar\beta}=g^{\gamma\bar\delta}\nabla_{\bar\delta}h_{\gamma\bar\beta}$. Consider the quantity

$$Z=g^{\alpha\bar\beta}g^{\gamma\bar\delta}\left[\frac{1}{2}(\nabla_{\bar\beta}\nabla_\gamma+\nabla_\gamma\nabla_{\bar\beta})h_{\alpha\bar\delta}+R_{\alpha\bar\delta}h_{\gamma\bar\beta}\right. \tag{1.21}$$

$$+ (\nabla_\gamma h_{\alpha\bar\delta} V_{\bar\beta} + \nabla_{\bar\beta} h_{\alpha\bar\delta} V_\gamma) + h_{\alpha\bar\delta} V_{\bar\beta} V_\gamma\Big] + \frac{H}{t}$$

$$= \frac{1}{2}[g^{\alpha\bar\beta}\nabla_{\bar\beta}\operatorname{div}(h)_\alpha + g^{\gamma\bar\delta}\nabla_\gamma \operatorname{div}(h)_{\bar\delta}]$$

$$+ g^{\alpha\bar\beta} g^{\gamma\bar\delta}[R_{\alpha\bar\delta} h_{\gamma\bar\beta} + \nabla_\gamma h_{\alpha\bar\delta} V_{\bar\beta} + \nabla_{\bar\beta} h_{\alpha\bar\delta} V_\gamma + h_{\alpha\bar\delta} V_{\bar\beta} V_\gamma] + \frac{H}{t},$$

where H is the trace of $h_{\alpha\bar\beta}$ with respect to $g_{\alpha\bar\beta}(x,t)$.

THEOREM 1.2. *Let $h_{\alpha\bar\beta}$ be a Hermitian symmetric tensor satisfying (1.2) on $M\times[0,T]$. Suppose $h_{\alpha\bar\beta}(x,0)\ge 0$ and satisfies (1.8) and (1.9) in Lemma 1.2. Then $Z\ge 0$ on $M\times(0,T]$ for any smooth vector field V of type* $(1,0)$.

In order to prove the theorem, we need to compute $(\frac{\partial}{\partial t}-\Delta)Z$. As in [C-H], we need to calculate $(\frac{\partial}{\partial t}-\Delta)Z$. We break up the computations into several lemmas.

LEMMA 1.4. *Under normal coordinates at a point,*

$$\left(\frac{\partial}{\partial t}-\Delta\right)(\operatorname{div}(h)_\alpha) = R_{s\bar t}\nabla_t h_{\alpha\bar s} + \nabla_\alpha R_{s\bar t} h_{\bar s t} - \frac{1}{2} R_{\alpha\bar t}(\operatorname{div}(h)_t). \tag{1.22}$$

Proof. Direct calculation shows

$$\begin{aligned}
\frac{\partial}{\partial t}(g^{\gamma\bar\delta}\nabla_\gamma h_{\alpha\bar\delta}) &= \frac{\partial}{\partial t}[g^{\gamma\bar\delta}(\partial_\gamma h_{\alpha\bar\delta} - \Gamma^p_{\alpha\gamma} h_{p\bar\delta})] \\
&= g^{\gamma\bar t} R_{s\bar t} g^{s\bar\delta}\nabla_\gamma h_{\alpha\bar\delta} + g^{\gamma\bar\delta}\nabla_\gamma\left(\frac{\partial}{\partial t} h_{\alpha\bar\delta}\right) - g^{\gamma\bar\delta}\left(\frac{\partial}{\partial t}\Gamma^p_{\alpha\gamma}\right) h_{p\bar\delta} \\
&= R_{s\bar t}\nabla_t h_{\alpha\bar s} + \nabla_\gamma\left(\Delta h_{\alpha\bar\gamma} + R_{\alpha\bar\gamma s\bar t} h_{\bar s t} - \frac{1}{2}R_{\alpha\bar t} h_{t\bar\gamma} - \frac{1}{2} R_{t\bar\gamma} h_{\alpha\bar t}\right) \\
&\quad + \nabla_\gamma R_{\alpha\bar p} h_{p\bar\gamma} \\
&= R_{s\bar t}\nabla_t h_{\alpha\bar s} + \nabla_\alpha R_{s\bar t} h_{\bar s t} + R_{\alpha\bar\gamma s\bar t}\nabla_\gamma h_{\bar s t} + \frac{1}{2}\nabla_\gamma R_{\alpha\bar t} h_{t\bar\gamma} \\
&\quad - \frac{1}{2} R_{\alpha\bar t}\nabla_\gamma h_{t\bar\gamma} - \frac{1}{2}\nabla_t R h_{\alpha\bar t} - \frac{1}{2} R_{t\bar\gamma}\nabla_\gamma h_{\alpha\bar t} + \nabla_\gamma(\Delta h_{\alpha\bar\gamma}).
\end{aligned} \tag{1.23}$$

Now we calculate $\nabla_\gamma(\Delta h_{\alpha\bar\gamma})$. By definition,

$$\nabla_\gamma(\Delta h_{\alpha\bar\gamma}) = \frac{1}{2}\nabla_\gamma(\nabla_s\nabla_{\bar s} + \nabla_{\bar s}\nabla_s) h_{\alpha\bar\gamma}.$$

On the other hand,

$$\begin{aligned}
\nabla_\gamma\nabla_s\nabla_{\bar s} h_{\alpha\bar\gamma} &= \nabla_s\nabla_\gamma\nabla_{\bar s} h_{\alpha\bar\gamma} \\
&= \nabla_s[\nabla_{\bar s}\nabla_\gamma h_{\alpha\bar\gamma} - R_{\alpha\bar p\gamma\bar s} h_{p\bar\gamma} + R_{p\bar\gamma\gamma\bar s} h_{\alpha\bar p}] \\
&= \nabla_s\nabla_{\bar s}\nabla_\gamma h_{\alpha\bar\gamma} - \nabla_\gamma R_{\alpha\bar p} h_{p\bar\gamma} - R_{\alpha\bar p\gamma\bar s}\nabla_s h_{p\bar\gamma} + \nabla_p R h_{\alpha\bar p} + R_{p\bar s}\nabla_s h_{\alpha\bar p}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\nabla_\gamma\nabla_{\bar s}\nabla_s h_{\alpha\bar\gamma} &= \nabla_{\bar s}\nabla_\gamma\nabla_s h_{\alpha\bar\gamma} + R_{p\bar\gamma\gamma\bar s}\nabla_s h_{\alpha\bar p} - R_{s\bar p\gamma\bar s}\nabla_p h_{\alpha\bar\gamma} \\ &\quad - R_{\alpha\bar p\gamma\bar s}\nabla_s h_{p\bar\gamma} \\ &= \nabla_{\bar s}\nabla_s\nabla_\gamma h_{\alpha\bar\gamma} + R_{p\bar s}\nabla_s h_{\alpha\bar p} - R_{\gamma\bar p}\nabla_p h_{\alpha\bar\gamma} - R_{\alpha\bar p\gamma\bar s}\nabla_s h_{p\bar\gamma}.\end{aligned}$$

Combining the above three we have

$$\begin{aligned}\nabla_\gamma(\Delta h_{\alpha\bar\gamma}) &= \Delta(\nabla_\gamma h_{\alpha\bar\gamma}) - \frac{1}{2}\nabla_\gamma R_{\alpha\bar p}h_{p\bar\gamma} - R_{\alpha\bar p\gamma\bar s}\nabla_s h_{p\bar\gamma} \\ &\quad + \frac{1}{2}\nabla_p R h_{\alpha\bar p} + R_{p\bar s}\nabla_s h_{\alpha\bar p} - \frac{1}{2}R_{\gamma\bar p}\nabla_p h_{\alpha\bar\gamma}.\end{aligned}$$

Plugging the above into (1.23), the lemma is proved.

LEMMA 1.5. *Under normal coordinates at a point,*

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)(g^{\alpha\bar\beta}\nabla_{\bar\beta}\operatorname{div}(h)_\alpha) &= R_{s\bar\alpha}\nabla_{\bar s}\operatorname{div}(h)_\alpha + \nabla_{\bar\alpha}R_{s\bar t}\nabla_t h_{\alpha\bar s} + \nabla_\alpha R_{s\bar t}\nabla_{\bar\alpha}h_{\bar s t} \\ &\quad + R_{s\bar t}\nabla_{\bar\alpha}\nabla_t h_{\alpha\bar s} + (\nabla_{\bar\alpha}\nabla_\alpha R_{s\bar t})h_{\bar s t}.\end{aligned}\tag{1.24}$$

Proof. Direct calculation shows that

$$\begin{aligned}\frac{\partial}{\partial t}(g^{\alpha\bar\beta}\nabla_{\bar\beta}\operatorname{div}(h)_\alpha) &= g^{\alpha\bar t}R_{s\bar t}g^{s\bar\beta}\nabla_{\bar\beta}\operatorname{div}(h)_\alpha + \nabla_{\bar\alpha}\left(\frac{\partial}{\partial t}\operatorname{div}(h)_\alpha\right) \\ &= R_{s\bar\alpha}\nabla_{\bar s}(\operatorname{div}(h)_\alpha) \\ &\quad + \nabla_{\bar\alpha}\left[\Delta\operatorname{div}(h)_\alpha + R_{s\bar t}\nabla_t h_{\alpha\bar s} + \nabla_\alpha R_{s\bar t}h_{\bar s t} - \frac{1}{2}R_{\alpha\bar t}\operatorname{div}(h)_t\right],\end{aligned}$$

by Lemma 1.4. Therefore we have that

$$\begin{aligned}\frac{\partial}{\partial t}(g^{\alpha\bar\beta}\nabla_{\bar\beta}\operatorname{div}(h)_\alpha) &= \nabla_{\bar\alpha}(\Delta\operatorname{div}(h)_\alpha) + \frac{1}{2}R_{s\bar\alpha}\nabla_{\bar s}(\operatorname{div}(h)_\alpha) \\ &\quad + \nabla_{\bar\alpha}R_{s\bar t}\nabla_t h_{\alpha\bar s} + \nabla_\alpha R_{s\bar t}\nabla_{\bar\alpha}h_{\bar s t} \\ &\quad + R_{s\bar t}\nabla_{\bar\alpha}\nabla_t h_{\alpha\bar s} + \nabla_{\bar\alpha}\nabla_\alpha R_{s\bar t}h_{\bar s t} - \frac{1}{2}\nabla_{\bar t}R(\operatorname{div}(h)_t).\end{aligned}\tag{1.25}$$

Now we calculate $\nabla_{\bar\alpha}(\Delta\operatorname{div}(h)_\alpha)$. By definition

$$\nabla_{\bar\alpha}(\Delta\operatorname{div}(h)_\alpha) = \frac{1}{2}\nabla_{\bar\alpha}\nabla_s\nabla_{\bar s}\operatorname{div}(h)_\alpha + \frac{1}{2}\nabla_{\bar\alpha}\nabla_{\bar s}\nabla_s\operatorname{div}(h)_\alpha.$$

On the other hand

$$\begin{aligned}\nabla_{\bar\alpha}\nabla_{\bar s}\nabla_s \operatorname{div}(h)_\alpha &= \nabla_{\bar s}\nabla_{\bar\alpha}\nabla_s \operatorname{div}(h)_\alpha\\ &= \nabla_{\bar s}[\nabla_s\nabla_{\bar\alpha}\operatorname{div}(h)_\alpha + R_{\alpha\bar p s\bar\alpha}\operatorname{div}(h)_p]\\ &= \nabla_{\bar s}\nabla_s\nabla_{\bar\alpha}\operatorname{div}(h)_\alpha + (\nabla_{\bar s}R)(\operatorname{div}(h)_s) + R_{s\bar p}\nabla_{\bar s}\operatorname{div}(h)_p\end{aligned}$$

and

$$\begin{aligned}\nabla_{\bar\alpha}\nabla_s\nabla_{\bar s}\operatorname{div}(h)_\alpha &= \nabla_s\nabla_{\bar\alpha}\nabla_{\bar s}\operatorname{div}(h)_\alpha + R_{\bar p s}\nabla_{\bar s}\operatorname{div}(h)_p - R_{p\bar\alpha}\nabla_{\bar p}\operatorname{div}(h)_\alpha\\ &= \nabla_s\nabla_{\bar s}\nabla_{\bar\alpha}\operatorname{div}(h)_\alpha + R_{\bar p s}\nabla_{\bar s}\operatorname{div}(h)_p - R_{p\bar\alpha}\nabla_{\bar p}\operatorname{div}(h)_\alpha.\end{aligned}$$

Combining the above three we have that

$$\nabla_{\bar\alpha}(\Delta\operatorname{div}(h)_\alpha) = \Delta(\nabla_{\bar\alpha}\operatorname{div}(h)_\alpha) + \frac{1}{2}\nabla_{\bar s}R(\operatorname{div}(h)_s) + \frac{1}{2}R_{s\bar p}\nabla_{\bar s}\operatorname{div}(h)_p.$$

Plugging into (1.25), this completes the proof of Lemma 1.3.

Taking the conjugation we will have the following lemmas.

LEMMA 1.4′. *Under normal coordinates at a point,*

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\operatorname{div}(h)_{\bar\beta}) = R_{s\bar p}\nabla_{\bar s}h_{p\bar\beta} + \nabla_{\bar\beta}R_{p\bar\gamma}h_{\gamma\bar p} - \frac{1}{2}R_{t\bar\beta}\operatorname{div}(h)_{\bar t}. \tag{1.26}$$

LEMMA 1.5′. *Under normal coordinates at a point,*

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(g^{\beta\bar\alpha}\nabla_\beta\operatorname{div}(h)_{\bar\alpha}) &= R_{\alpha\bar p}\nabla_p\operatorname{div}(h)_{\bar\alpha} + \nabla_\alpha R_{s\bar p}\nabla_{\bar s}h_{p\bar\alpha}\\ &\quad + \nabla_{\bar\alpha}R_{p\bar\gamma}\nabla_\alpha h_{\gamma\bar p}\\ &\quad + R_{s\bar p}\nabla_\alpha\nabla_{\bar s}h_{p\bar\alpha} + (\nabla_\alpha\nabla_{\bar\alpha}R_{p\bar\gamma})h_{\gamma\bar p}.\end{aligned} \tag{1.27}$$

Now we are ready to calculate $(\frac{\partial}{\partial t} - \Delta)Z$. By Proposition 1.1, h is nonnegative. However h may be zero somewhere, we consider $\widehat{Z}$ instead, where:

$$\widehat{Z} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V},$$

where

$$\begin{aligned}\mathrm{I} &= \frac{1}{2}[g^{\alpha\bar\beta}\nabla_{\bar\beta}\operatorname{div}(h)_\alpha) + g^{\gamma\bar\delta}\nabla_\gamma\operatorname{div}(h)_{\bar\delta}],\\ \mathrm{II} &= g^{\alpha\bar\beta}g^{\gamma\bar\delta}[R_{\alpha\bar\delta}h_{\gamma\bar\beta} + \epsilon\mathcal{R}],\end{aligned}$$

$$\begin{aligned}
\mathrm{III} &= g^{\alpha\bar{\beta}} \operatorname{div}(h)_\alpha V_{\bar{\beta}} + g^{\gamma\bar{\delta}} \operatorname{div}(h)_{\bar{\delta}} V_\gamma, \\
\mathrm{IV} &= g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} \widetilde{h}_{\alpha\bar{\delta}} V_{\bar{\beta}} V_\alpha, \\
\mathrm{V} &= \frac{H + \epsilon m}{t},
\end{aligned}$$

and where $\epsilon > 0$ is a fixed constant, $\mathcal{R}$ is the scalar curvature, and $\widetilde{h}_{\alpha\bar{\beta}} = h_{\alpha\bar{\beta}} + \epsilon g_{\alpha\bar{\beta}}$. We calculate them one by one. In the following, we always do computations in normal coordinates at a point because the final result will not depend on the choice of coordinates.

From Lemma 1.5, Lemma 1.5′, and the second Bianchi identity we have that

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{I} &= \frac{1}{2}[R_{\alpha\bar{p}} \nabla_p \operatorname{div}(h)_{\bar{\alpha}} + R_{p\bar{\alpha}} \nabla_{\bar{p}} \operatorname{div}(h)_\alpha] \\
&\quad + \frac{1}{2}[R_{s\bar{p}} \nabla_\alpha \nabla_{\bar{s}} h_{p\bar{\alpha}} + R_{s\bar{p}} \nabla_{\bar{\alpha}} \nabla_p h_{\alpha\bar{s}}] \\
&\quad + \Delta R_{s\bar{t}} h_{\bar{s}t} + \nabla_{\bar{t}} R_{s\bar{\alpha}} \nabla_t h_{\alpha\bar{s}} + \nabla_t R_{s\bar{\alpha}} \nabla_{\bar{t}} h_{\bar{s}\alpha}.
\end{aligned} \tag{1.28}$$

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{II} &= \left(\frac{\partial}{\partial t} - \Delta\right) (g^{\alpha\bar{\delta}} g^{\gamma\bar{\beta}} R_{\alpha\bar{\beta}} h_{\gamma\bar{\delta}} + \epsilon\mathcal{R}) \\
&= 2R_{\beta\bar{\gamma}} R_{\alpha\bar{\beta}} h_{\gamma\bar{\alpha}} + \left(\left(\frac{\partial}{\partial t} - \Delta\right) R_{\alpha\bar{\beta}}\right) h_{\beta\bar{\alpha}} \\
&\quad + R_{\alpha\bar{\beta}} \left(\left(\frac{\partial}{\partial t} - \Delta\right) h_{\beta\bar{\alpha}}\right) \\
&\quad - \nabla_s R_{\alpha\bar{\beta}} \nabla_{\bar{s}} h_{\beta\bar{\alpha}} - \nabla_{\bar{s}} R_{\alpha\bar{\beta}} \nabla_s h_{\beta\bar{\alpha}} + \epsilon |R_{\alpha\bar{\beta}}|^2 \\
&= 2R_{\alpha\bar{\beta}s\bar{t}} R_{\bar{s}t} h_{\beta\bar{\alpha}} - \nabla_s R_{\alpha\bar{\beta}} \nabla_{\bar{s}} h_{\beta\bar{\alpha}} \\
&\quad - \nabla_{\bar{s}} R_{\alpha\bar{\beta}} \nabla_s h_{\beta\bar{\alpha}} + \epsilon |R_{\alpha\bar{\beta}}|^2.
\end{aligned} \tag{1.29}$$

Here we have used (1.2) and the equation satisfies by the Ricci form [Sh3]:

$$\left(\frac{\partial}{\partial t} - \Delta\right) R_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}\gamma\bar{\delta}} R_{\bar{\gamma}\delta} - R_{\alpha\bar{s}} R_{s\bar{\beta}}. \tag{1.30}$$

Using Lemma 1.4, Lemma 1.4′ and the second Bianchi identity we have that

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) \mathrm{III} &= \left(\frac{\partial}{\partial t} - \Delta\right) [g^{\alpha\bar{\beta}} (\operatorname{div}(h)_\alpha V_{\bar{\beta}} + \operatorname{div}(h)_{\bar{\beta}} V_\alpha)] \\
&= R_{\bar{\alpha}\beta} \operatorname{div}(h)_\alpha V_{\bar{\beta}} + R_{\bar{\alpha}\beta} \operatorname{div}(h)_{\bar{\beta}} V_\alpha + \operatorname{div}(h)_\alpha \\
&\quad \times \left(\left(\frac{\partial}{\partial t} - \Delta\right) V_{\bar{\alpha}}\right) + \operatorname{div}(h)_{\bar{\alpha}} \left(\left(\frac{\partial}{\partial t} - \Delta\right) V_\alpha\right)
\end{aligned} \tag{1.31}$$

$$
\begin{aligned}
&+\left(R_{s\bar t}\nabla_t h_{\alpha\bar s}+\nabla_s R_{\alpha\bar t}h_{\bar s t}-\frac{1}{2}R_{\alpha\bar t}\,\mathrm{div}\,(h)_t\right)V_{\bar\alpha}\\
&+\left(R_{s\bar t}\nabla_{\bar s}h_{t\bar\alpha}+\nabla_{\bar s}R_{t\bar\alpha}h_{s\bar t}-\frac{1}{2}R_{t\bar\alpha}\,\mathrm{div}\,(h)_{\bar t}\right)V_{\alpha}\\
&-\nabla_s\,\mathrm{div}\,(h)_\alpha\nabla_{\bar s}V_{\bar\alpha}-\nabla_{\bar s}\,\mathrm{div}\,(h)_\alpha\nabla_s V_{\bar\alpha}\\
&-\nabla_s\,\mathrm{div}\,(h)_{\bar\alpha}\nabla_{\bar s}V_{\alpha}-\nabla_{\bar s}\,\mathrm{div}\,(h)_{\bar\alpha}\nabla_s V_{\alpha}.
\end{aligned}
$$

Using (1.2) we have

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{IV} \;=\; &\left(\frac{\partial}{\partial t}-\Delta\right)(g^{\alpha\bar\beta}g^{\gamma\bar\delta}\widetilde{h}_{\alpha\bar\delta}V_\gamma V_{\bar\beta})\\
=\; &R_{\alpha\bar\beta s\bar t}h_{\bar s t}V_\beta V_{\bar\alpha}+\frac{1}{2}R_{\alpha\bar s}h_{s\bar\gamma}V_\gamma V_{\bar\alpha}+\frac{1}{2}h_{\alpha\bar s}R_{s\bar\gamma}V_\gamma V_{\bar\alpha}\\
&+\widetilde{h}_{\alpha\bar\gamma}\left(\left(\frac{\partial}{\partial t}-\Delta\right)V_\gamma\right)V_{\bar\alpha}+\widetilde{h}_{\alpha\bar\gamma}V_\gamma\left(\left(\frac{\partial}{\partial t}-\Delta\right)V_{\bar\alpha}\right)\\
&-\nabla_s h_{\alpha\bar\gamma}\nabla_{\bar s}(V_\gamma V_{\bar\alpha})-\nabla_{\bar s}h_{\alpha\bar\gamma}\nabla_s(V_\gamma V_{\bar\alpha})\\
&-\widetilde{h}_{\alpha\bar\gamma}[\nabla_s V_\gamma\nabla_{\bar s}V_{\bar\alpha}+\nabla_{\bar s}V_\gamma\nabla_s V_{\bar\alpha}+\epsilon R_{\alpha\bar\gamma}V_{\bar\alpha}V_\gamma].
\end{aligned}
\tag{1.32}
$$

Taking trace on (1.2) one can have

$$
\left(\frac{\partial}{\partial t}-\Delta\right)\mathrm{V}=\frac{R_{\alpha\bar s}h_{s\bar\alpha}}{t}-\frac{H+\epsilon m}{t^2}.
\tag{1.33}
$$

Now combining them together we have that

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)Z \;=\; &Y_1+\frac{1}{2}[R_{\alpha\bar p}\nabla_p\,\mathrm{div}\,(h)_{\bar\alpha}+R_{p\bar\alpha}\nabla_{\bar p}\,\mathrm{div}\,(h)_\alpha]\\
&+\frac{1}{2}[R_{s\bar p}\nabla_\alpha\nabla_{\bar s}h_{p\bar\alpha}+R_{s\bar p}\nabla_{\bar\alpha}\nabla_p h_{\alpha\bar s}]\\
&+R_{\alpha\bar\beta s\bar t}R_{\bar\alpha\beta}h_{t\bar s}+R_{\bar\alpha\beta}\,\mathrm{div}\,(h)_\alpha V_{\bar\beta}+R_{\bar\alpha\beta}\,\mathrm{div}\,(h)_{\bar\beta}V_\alpha\\
&+\mathrm{div}\,(h)_\alpha\left(\left(\frac{\partial}{\partial t}-\Delta\right)V_{\bar\alpha}\right)+\mathrm{div}\,(h)_{\bar\alpha}\left(\left(\frac{\partial}{\partial t}-\Delta\right)V_\alpha\right)\\
&-\nabla_s\,\mathrm{div}\,(h)_\alpha\nabla_{\bar s}V_{\bar\alpha}-\nabla_{\bar s}\,\mathrm{div}\,(h)_\alpha\nabla_s V_{\bar\alpha}\\
&-\nabla_s\,\mathrm{div}\,(h)_{\bar\alpha}\nabla_{\bar s}V_{\alpha}-\nabla_{\bar s}\,\mathrm{div}\,(h)_{\bar\alpha}\nabla_s V_{\alpha}\\
&+R_{s\bar t}\nabla_t h_{\alpha\bar s}V_{\bar\alpha}-\frac{1}{2}R_{\alpha\bar t}\,\mathrm{div}\,(h)_t V_{\bar\alpha}+R_{s\bar t}\nabla_{\bar s}h_{t\bar\alpha}V_\alpha\\
&-\frac{1}{2}R_{t\bar\alpha}\,\mathrm{div}\,(h)_{\bar t}V_\alpha\\
&+\frac{1}{2}R_{\alpha\bar s}h_{s\bar\gamma}V_\gamma V_{\bar\alpha}+\frac{1}{2}h_{\alpha\bar s}R_{s\bar\gamma}V_\gamma V_{\bar\alpha}
\end{aligned}
\tag{1.34}
$$

$$
\begin{aligned}
&+\widetilde{h}_{\alpha\bar{\gamma}}\left(\left(\left(\frac{\partial}{\partial t}-\Delta\right)V_{\gamma}\right)V_{\bar{\alpha}}+\widetilde{h}_{\alpha\bar{\gamma}}V_{\gamma}\left(\left(\frac{\partial}{\partial t}-\Delta\right)V_{\bar{\alpha}}\right)\right.\\
&-\nabla_{s}h_{\alpha\bar{\gamma}}\nabla_{\bar{s}}(V_{\gamma}V_{\bar{\alpha}})-\nabla_{\bar{s}}h_{\alpha\bar{\gamma}}\nabla_{s}(V_{\gamma}V_{\bar{\alpha}})\\
&-\widetilde{h}_{\alpha\bar{\gamma}}[\nabla_{s}V_{\gamma}\nabla_{\bar{s}}V_{\bar{\alpha}}+\nabla_{\bar{s}}V_{\gamma}\nabla_{s}V_{\bar{\alpha}}]-\frac{H+\epsilon m}{t^{2}}\\
&+\epsilon|R_{\alpha\bar{\beta}}|^{2}+\epsilon R_{\alpha\bar{\gamma}}V_{\bar{\alpha}}V_{\gamma},
\end{aligned}
$$

where

$$
Y_{1}=\left[\Delta R_{s\bar{t}}+R_{s\bar{t}\alpha\bar{\beta}}R_{\bar{\alpha}\beta}+\nabla_{\alpha}R_{s\bar{t}}V_{\bar{\alpha}}+\nabla_{\bar{\alpha}}R_{s\bar{t}}V_{\alpha}+R_{s\bar{t}\alpha\bar{\beta}}V_{\bar{\alpha}}V_{\beta}+\frac{R_{s\bar{t}}}{t}\right]h_{\bar{s}t}. \tag{1.35}
$$

By Proposition 1.1, $h_{\alpha\bar{\beta}}\geq 0$ on $M\times[0,T]$. Hence by Cao's LYH inequality [Co1–2] and the fact that $(M,g_{\alpha\bar{\beta}}(x,t))$ has nonnegative holomorphic bisectional curvature, the two factors in Y_1 are all nonnegative tensors. Therefore $Y_1\geq 0$. Since $\widetilde{h}_{\alpha\bar{\beta}}\geq \epsilon g_{\alpha\bar{\beta}}$, for each (x,t), $\widehat{Z}$ attains minimum for some V. Then by the first variation we have

$$
\operatorname{div}(h)_{\alpha}+\widetilde{h}_{\alpha\bar{\gamma}}V_{\gamma}=0 \quad\text{and}\quad \operatorname{div}(h)_{\bar{\alpha}}+\widetilde{h}_{\gamma\bar{\alpha}}V_{\bar{\gamma}}=0. \tag{1.36}
$$

Direct calculation also shows that

$$
\begin{aligned}
R_{p\bar{\alpha}}\nabla_{\bar{p}}\operatorname{div}(h)_{\alpha}+R_{\alpha\bar{p}}\nabla_{p}\operatorname{div}(h)_{\bar{\alpha}} \;=\;& R_{s\bar{p}}\nabla_{\alpha}\nabla_{\bar{s}}h_{p\bar{\alpha}}+R_{s\bar{p}}\nabla_{\bar{\alpha}}\nabla_{p}h_{\alpha\bar{s}}\\
&+2R_{\alpha\bar{p}}R_{p\bar{\alpha}s\bar{\gamma}}h_{\gamma\bar{s}}-2R_{\alpha\bar{p}}R_{p\bar{s}}h_{s\bar{\alpha}}.
\end{aligned} \tag{1.37}
$$

Combining (1.34)–(1.37) we have that

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\widehat{Z} \;=\;& Y_{1}+[R_{\alpha\bar{p}}\nabla_{p}\operatorname{div}(h)_{\bar{\alpha}}+R_{p\bar{\alpha}}\nabla_{\bar{p}}\operatorname{div}(h)_{\alpha}]+R_{\alpha\bar{p}}R_{p\bar{s}}h_{s\bar{\alpha}}\\
&-\nabla_{s}\operatorname{div}(h)_{\alpha}\nabla_{\bar{s}}V_{\bar{\alpha}}-\nabla_{\bar{s}}\operatorname{div}(h)_{\alpha}\nabla_{s}V_{\bar{\alpha}}\\
&-\nabla_{s}\operatorname{div}(h)_{\bar{\alpha}}\nabla_{\bar{s}}V_{\alpha}-\nabla_{\bar{s}}\operatorname{div}(h)_{\bar{\alpha}}\nabla_{s}V_{\alpha}\\
&+R_{s\bar{t}}\nabla_{t}h_{\alpha\bar{s}}V_{\bar{\alpha}}+R_{s\bar{t}}\nabla_{\bar{s}}h_{t\bar{\alpha}}V_{\alpha}\\
&-\nabla_{s}h_{\alpha\bar{\gamma}}\nabla_{\bar{s}}(V_{\gamma}V_{\bar{\alpha}})-\nabla_{\bar{s}}h_{\alpha\bar{\gamma}}\nabla_{s}(V_{\gamma}V_{\bar{\alpha}})\\
&-\widetilde{h}_{\alpha\bar{\gamma}}[\nabla_{s}V_{\gamma}\nabla_{\bar{s}}V_{\bar{\alpha}}+\nabla_{\bar{s}}V_{\gamma}\nabla_{s}V_{\bar{\alpha}}]-\frac{H+\epsilon m}{t^{2}}.
\end{aligned} \tag{1.38}
$$

Differentiate (1.36) and we have

$$
\begin{aligned}
\nabla_{s}\operatorname{div}(h)_{\alpha}+(\nabla_{s}h_{\alpha\bar{\gamma}})V_{\gamma}+\widetilde{h}_{\alpha\bar{\gamma}}\nabla_{s}V_{\gamma} \;&=\; 0,\\
\nabla_{s}\operatorname{div}(h)_{\bar{\alpha}}+(\nabla_{s}h_{\gamma\bar{\alpha}})V_{\bar{\gamma}}+\widetilde{h}_{\gamma\bar{\alpha}}\nabla_{s}V_{\bar{\gamma}} \;&=\; 0,
\end{aligned} \tag{1.39}
$$

$$\nabla_{\bar{s}} \operatorname{div}(h)_\alpha + (\nabla_{\bar{s}} h_{\alpha\bar{\gamma}}) V_\gamma + \widetilde{h}_{\alpha\bar{\gamma}} \nabla_{\bar{s}} V_\gamma = 0,$$
$$\nabla_{\bar{s}} \operatorname{div}(h)_{\bar{\alpha}} + (\nabla_{\bar{s}} h_{\gamma\bar{\alpha}}) V_{\bar{\gamma}} + \widetilde{h}_{\gamma\bar{\alpha}} \nabla_{\bar{s}} V_{\bar{\gamma}} = 0.$$

Plugging the above into (1.34) we have that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \widehat{Z} = Y_1 + R_{\alpha\bar{p}} R_{p\bar{s}} h_{s\bar{\alpha}} - R_{\alpha\bar{p}} h_{\gamma\bar{\alpha}} \nabla_p V_{\bar{\gamma}} - R_{p\bar{\alpha}} h_{\alpha\bar{\gamma}} \nabla_{\bar{p}} V_\gamma + \widetilde{h}_{\gamma\bar{\alpha}} \nabla_s V_{\bar{\gamma}} \nabla_{\bar{s}} V_\alpha + \widetilde{h}_{\gamma\bar{\alpha}} \nabla_{\bar{s}} V_{\bar{\gamma}} \nabla_s V_\alpha - \frac{H + \epsilon m}{t^2} + \epsilon |R_{\alpha\bar{\beta}}|^2. \tag{1.40}$$

Let

$$Y_2 = \widetilde{h}_{\gamma\bar{\alpha}} \left[\nabla_p V_{\bar{\gamma}} - R_{p\bar{\gamma}} - \frac{1}{t} g_{p\bar{\gamma}}\right] \left[\nabla_{\bar{p}} V_\alpha - R_{\alpha\bar{p}} - \frac{1}{t} g_{\bar{p}\alpha}\right] + \widetilde{h}_{\gamma\bar{\alpha}} \nabla_{\bar{p}} V_{\bar{\gamma}} \nabla_p V_\alpha. \tag{1.41}$$

By Proposition 1.1 again, $Y_2 \geq 0$.

$$\left(\frac{\partial}{\partial t} - \Delta\right) \widehat{Z} = Y_1 + Y_2 - \frac{1}{t}\left[- \widetilde{h}_{\gamma\bar{\alpha}} \nabla_\alpha V_{\bar{\gamma}} - \widetilde{h}_{\gamma\bar{\alpha}} \nabla_{\bar{\alpha}} V_\gamma + 2R_{\alpha\bar{\gamma}} h_{\gamma\bar{\alpha}} + \frac{2(H + \epsilon m)}{t} + 2\epsilon \mathcal{R}\right]. \tag{1.42}$$

Using (1.36) we also know that

$$\widehat{Z} = R_{\alpha\bar{\beta}} h_{\bar{\alpha}\beta} - \frac{1}{2} \widetilde{h}_{\alpha\bar{\beta}} \nabla_{\bar{\alpha}} V_\beta - \frac{1}{2} \widetilde{h}_{\beta\bar{\alpha}} \nabla_\alpha V_{\bar{\beta}} + \frac{H + \epsilon m}{t} + \epsilon \mathcal{R}. \tag{1.43}$$

Plugging into (1.42) and using the fact that $Y_1 \geq 0$ and $Y_2 \geq 0$, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) \widehat{Z} \geq -\frac{2\widehat{Z}}{t}, \tag{1.44}$$

where V is the smooth vector field given by (1.36). Note that both sides of (1.44) do not depend on the choice of coordinates.

Proof of Theorem 1.2. Since $\widetilde{h}_{\alpha\bar{\beta}} \geq \epsilon g_{\alpha\bar{\beta}}$ on $M \times [0, T]$, by (1.36) and (1.39), we have

$$\|V\| \leq C_1 \|\nabla h\|,$$

and

$$\|\nabla V\| \leq C_2(\|\nabla\nabla h\| + \|\nabla h\|^2),$$

for some constants C_1 and C_2. Combining this with (1.44), we have

$$|t^2\widehat{Z}|^2 \leq C_3(\Phi + \Phi(\Psi^2 + \Lambda) + 1) \tag{1.45}$$

for some constant C_3. By (1.43), the corresponding $\widehat{Z}$ satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right)(t^2\widehat{Z}) \geq 0 \tag{1.46}$$

for the vector field which minimizes $\widehat{Z}$. By Lemma 1.2, Lemma 1.3, and (1.45), we have

$$\int_0^T \int_M \exp\left(-ar_0^2(x)\right)(t^2\widehat{Z})^2\, dV_t\, dt < \infty$$

for any $a > 0$. By the maximum principle Theorem 1.1, we have $t^2\widehat{Z} \geq 0$ because it is obvious that $t^2\widetilde{Z} = 0$ at $t = 0$. Since this is true for the vector field V minimizing $\widehat{Z}$, we have $\widehat{Z} \geq 0$ for any (1,0) vector field. Let $\epsilon \to 0$ and the proof of the theorem is completed.

Remark 1.1. (i) The theorem is still true for the case that M is compact with positive holomorphic bisectional curvature because of the result in [Co1]. (ii) When $h_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}}$, it is known that the Ricci tensor satisfies (1.2). Therefore we can apply Theorem 1.1 to this case. Since

$$\operatorname{div}(h)_\alpha = \nabla_\gamma R_{\alpha\bar{\gamma}} = \nabla_\alpha \mathcal{R} \quad \text{and} \quad \operatorname{div}(h)_{\bar{\delta}} = \nabla_{\bar{\alpha}} R_{\alpha\bar{\delta}} = \nabla_{\bar{\delta}}\mathcal{R}$$

we have

$$Z = \Delta\mathcal{R} + R_{\alpha\bar{\beta}}R_{\bar{\alpha}\beta} + \nabla_\alpha\mathcal{R}V_{\bar{\alpha}} + \nabla_{\bar{\alpha}}\mathcal{R}V_\alpha + R_{\alpha\bar{\beta}}V_{\bar{\alpha}}V_\beta + \frac{\mathcal{R}}{t} \geq 0. \tag{1.47}$$

It is the trace of the LYH inequality proved by Cao in [Co1–2]. Hence Theorem 1.1 can be considered as a generalization of the LYH inequality of Cao for the scalar curvature. However, we should emphasize that Cao's result has been used in the proof of Theorem 1.2.

2. Deforming plurisubharmonic functions. Let $(M^m, g_{\alpha\bar{\beta}}(x,t))$ be a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature deformed by the Kähler-Ricci flow (1.1). As in the previous section we assume that (1.1) has solution on $M \times [0,T]$ which satisfies conditions (i)–(iv) in that section. In this section we shall study the plurisubharmonic functions deformed by the time-dependent heat equation:

$$\begin{cases} \left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{2.1}$$

where $\Delta = g^{\alpha\bar{\beta}}(x,t)\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta}$ and $u_0(x)$ is a plurisubharmonic function on M.

First, we shall consider the more general case and drop the assumption that u_0 is plurisubharmonic. We have the following existence result.

PROPOSITION 2.1. *Let u_0 be a continuous function such that $|u_0(x)| \leq \exp(a(r_0(x)+1))$ for all x for some positive constant $a > 0$. Then there is a unique solution of (2.1) on $M \times [0,T]$ such that*

$$|u(x,t)| \leq \exp(b(r_0(x)+1)) \tag{2.2}$$

on $M \times [0,T]$ for some positive constant b.

Proof By Lemma 1.1, there exists a function $\varphi(x)$ such that

$$\exp(b(r_0(x)+1)) \geq \varphi(x) \geq \exp(a(r_0(x)+1))$$

for some positive constant and b for all $(x,t) \in M \times [0,T]$, and

$$\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \geq 0.$$

Using φ and $-\varphi$ as barriers, the existence part of the proposition follows. Uniqueness follows from the maximum principle Theorem 1.1.

Next we shall study properties of the solution u obtained in the proposition. We need the following lemma.

LEMMA 2.1. *Let $u(x,t)$ be a solution of (2.1). Then $u_{\alpha\bar{\beta}}$ satisfies the complex Lichnerowicz heat equation (1.2).*

Proof. Differentiate (2.1) and we have

$$(u_t)_{\gamma\bar{\delta}} = R_{\beta\bar{\alpha}\gamma\bar{\delta}}u_{\alpha\bar{\beta}} + g^{\alpha\bar{\beta}}u_{\alpha\bar{\beta}\gamma\bar{\delta}}. \tag{2.3}$$

By definition $\Delta u_{\alpha\bar{\beta}} = \frac{1}{2}(u_{\alpha\bar{\beta},\gamma\bar{\gamma}} + u_{\alpha\bar{\beta},\bar{\gamma}\gamma})$, in normal coordinates at a point. We need to calculate the difference between the partial derivative $u_{\alpha\bar{\beta}\gamma\bar{\delta}}$ and the covariant derivative $u_{\alpha\bar{\beta},\gamma\bar{\delta}}$. Direct computations show that, for normal coordinates at a point,

$$u_{\gamma\bar{\delta},\alpha\bar{\beta}} = u_{\gamma\bar{\delta}\alpha\bar{\beta}} + u_{s\bar{\delta}}R_{\alpha\bar{\beta}\gamma\bar{s}}. \tag{2.4}$$

Using the fact that

$$u_{\gamma\bar{\delta},\alpha\bar{\alpha}} = u_{\gamma\bar{\delta},\bar{\alpha}\alpha} + R_{\gamma\bar{p}}u_{p\bar{\delta}} - R_{p\bar{\delta}}u_{\gamma\bar{p}} \tag{2.5}$$

we have

$$\begin{aligned} \Delta u_{\gamma\bar{\delta}} &= \frac{1}{2}(u_{\gamma\bar{\delta},\alpha\bar{\alpha}} + u_{\gamma\bar{\delta},\bar{\alpha}\alpha}) \\ &= u_{\gamma\bar{\delta}\alpha\bar{\alpha}} + \frac{1}{2}(R_{\gamma\bar{p}}u_{p\bar{\delta}} + R_{p\bar{\delta}}u_{\gamma\bar{p}}) \end{aligned} \tag{2.6}$$

Combining with (2.3), we conclude that $u_{\alpha\bar{\beta}}$ satisfies (1.2).

In the following, $\widetilde{\nabla}$ and $\widetilde{\Delta}$ denote the covariant derivative and the Laplacian with respect to the initial metric.

PROPOSITION 2.2. *Let u_0 be a smooth function such that $|u_0(x)| \le \exp(a(r_0(x)+1))$ for all x for some positive constant $a > 0$. Let $u(x,t)$ be the solution of (2.1) obtained in Proposition 2.1. We have the following:*

(i) *For any $b > 0$*

$$\int_0^T \int_M \exp(-br_0^2(x))(|\nabla u|^2(x,t) + t\|u_{\alpha\bar{\beta}}\|^2(x,t))\,dV_t\,dt < \infty, \tag{2.7}$$

where $\|u_{\alpha\bar{\beta}}\|^2 = g^{\alpha\bar{\delta}}g^{\gamma\bar{\beta}}u_{\alpha\bar{\beta}}u_{\gamma\bar{\delta}}$.

(ii) *If in addition, $\int_{B_0(o,r)} |\widetilde{\nabla} u_0|^2\,dV_0 \le \exp(a'(1+r))$ for some $a' > 0$, where $B_0(o,r)$ is the geodesic ball with center at o and radius r with respect to the initial metric $g(0)$, then*

$$\int_0^T \int_M \exp(-br_0^2(x))\|u_{\alpha\bar{\beta}}\|^2(x,t)\,dV_t\,dt < \infty, \tag{2.8}$$

(iii) *If in addition $|\widetilde{\nabla} u_0|^2 \le C_1$ on M then*

$$|\nabla u|^2 \le C_1 \tag{2.9}$$

and

$$\|u_{\alpha\bar{\beta}}\|^2(x,t) \le \frac{C_2}{t} \tag{2.10}$$

for some constant C_2 on $M \times [0,T]$.

Proof. By Proposition 2.1, there exist a positive constant and c such that

$$|u(x,t)| \le \exp(c(r_0(x)+1)) \tag{2.11}$$

on $M \times [0, T]$. Since

$$\left(\frac{\partial}{\partial t} - \Delta\right) u^2 = -|\nabla u|^2,$$

we can proceed as in the proof of Lemma 1.1 to conclude that for any $b > 0$

$$\int_0^T \int_M \exp\left(-br_0^2(x)\right)|\nabla u|^2\, dV_t\, dt < \infty. \tag{2.12}$$

Direct computations show (see [N-T, Lemma 1.1] for example)

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\nabla u|^2 = -\|u_{\alpha\beta}\|^2 - \|u_{\alpha\bar\beta}\|^2. \tag{2.13}$$

Combining with (2.12), one can also proceed as in the proof of Lemma 1.1 and conclude that (2.7) is true. In case $|\widetilde\nabla u_0|^2$ satisfies the condition in (ii), then one can prove (2.8) similarly.

By (2.13), it is easy to see that $(\frac{\partial}{\partial t} - \Delta)(\sqrt{|\nabla u|^2 + 1}) \le 0$. Suppose $|\widetilde\nabla u|^2 \le C_1$ on M, then by (2.7) we can apply Theorem 1.1 to conclude that (2.9) is true.

Since $u_{\alpha\bar\beta}$ satisfies (1.2), as in the proof of (1.11) we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) (1 + t\Phi)^{\frac{1}{2}} \le C_3\Phi$$

on $M \times [0, T]$ for some constant $C_3 > 0$, where $\Phi = \|u_{\alpha\bar\beta}\|^2$. Hence on $M \times [0, T]$,

$$\left(\frac{\partial}{\partial t} - \Delta\right) (C_3|\nabla u|^2 + (1 + t\Phi)^{\frac{1}{2}}) \le 0$$

where we have used (2.13). By (2.8) and (2.9), we can apply the maximum principle in [N-T] and conclude that

$$\sup_{M\times[0,T]} (C_3|\nabla u|^2 + (1 + t\Phi)^{\frac{1}{2}}) \le C_3C_1 + 1$$

where we have used the fact that $|\widetilde\nabla u|^2 \le C_1$. From this (2.10) follows.

Next, we shall study the properties of $u(x, t)$ in case the initial value u_0 is plurisubharmonic.

THEOREM 2.1. *Let $u_0(x)$ be a smooth function on M such that (a) u_0 is plurisubharmonic; and (b) there exists $a > 0$ such that $|u_0(x)| \le \exp(a(1 + r_0(x))$ and $\widetilde\Delta u_0 \le \exp(a(1 + r_0(x))$. Let u be the solution of (2.1) obtained in Proposition 2.1. We have the following:*

(i) *$u(x, t)$ is plurisubharmonic for $t > 0$.*

(ii) *If u_0 is not harmonic, then $w = u_t > 0$ for $t > 0$, and we have the following differential inequality:*

$$w_t - \frac{|\nabla w|^2}{w} + \frac{w}{t} \geq 0 \tag{2.14}$$

for $t > 0$.

If in addition, $\sup_M |\widetilde{\nabla} u_0|^2 \leq C_1 < \infty$ for some constant C_1, then $u_{\alpha\bar{\beta}}$ satisfies (2.10) for some constant C_2.

Proof. Let $f = \widetilde{\Delta} u_0 \geq 0$. By assumptions, $|u_0(x)| \leq \exp(a(1 + r_0(x))$ and $f(x) \leq \exp(a(1 + r_0(x))$. It is easy to see that

$$\int_{B_0(o,r)} |\widetilde{\nabla} u_0|^2 \, dV_0 \leq \exp(a'(1 + r_0(x))$$

for some $a' > 0$. Hence $u_{\alpha\bar{\beta}}$ satisfies (2.8) by Proposition 2.2. Since u_0 is plurisubharmonic, we also have $\|u_{\alpha\bar{\beta}}\|^2(x, 0) \leq \exp(a''(1 + r_0(x))$ for some $a'' > 0$. By (i), Proposition 1.1 and Lemma 2.1, we conclude that u is plurisubharmonic for $t > 0$.

Since $u_{\alpha\bar{\beta}}$ satisfies (1.2) by Lemma 2.1 and $w = u_t = \Delta u$, taking trace of (1.2), we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) w = R_{\alpha\bar{\beta}} u_{\beta\bar{\alpha}} \geq 0.$$

If $w(x, t) = 0$ for some x and $t > 0$, then by the strong maximum principle (see [Cw3, Proposition 3.6]), we have $\widetilde{\Delta} u_0 = 0$ on M. Hence if u_0 is not harmonic, then $w > 0$ for $t > 0$.

Since $u_{\alpha\bar{\beta}}$ satisfies the conditions in Theorem 1.2, if we let $h_{\alpha\bar{\beta}}$, in Theorem 1.2 to be $u_{\alpha\bar{\beta}}$, then in normal coordinates

$$\operatorname{div}(h)_\alpha = \nabla_\gamma u_{\alpha\bar{\gamma}} = \nabla_\alpha(u_t) \quad \text{and} \quad \operatorname{div}(h)_{\bar{\delta}} = \nabla_{\bar{\alpha}} u_{\alpha\bar{\delta}} = \nabla_{\bar{\delta}}(u_t),$$

and

$$Z = \Delta(u_t) + R_{\alpha\bar{\beta}} u_{\bar{\alpha}\beta} + \nabla_{\bar{\alpha}}(u_t) V_\alpha + \nabla_\alpha(u_t) V_{\bar{\alpha}} + u_{\alpha\bar{\beta}} V_{\bar{\alpha}} V_\beta + \frac{u_t}{t} \geq 0$$

for any $(1, 0)$ vector field V. Combining this with (2.15), we have

$$w_t + \nabla_{\bar{\alpha}} w V_\alpha + \nabla_\alpha w V_{\bar{\alpha}} + u_{\alpha\bar{\beta}} V_{\bar{\alpha}} V_\beta + \frac{w}{t} \geq 0.$$

Choosing $V_\alpha = -\frac{\nabla_\alpha w}{w}$ we conclude that (2.14) is true.

The last assertion follows from Proposition 2.2 immediately.

Remark 2.1. If $u_0(x)$ is a solution to the Poincaré-Lelong equation $\sqrt{-1}\partial\bar\partial u_0 = \mathrm{Ric}(x,0)$, by Theorem 1.3 of [N-T] we know that we have a solution $u(x,t)$ to (2.1) in this case with $\sqrt{-1}\partial\bar\partial u(x,t) = \mathrm{Ric}(x,t)$. Then (2.14) in Theorem 2.1 is nothing but the differential LYH inequality of Cao on the scalar curvature since $w(x,t) = R(x,t)$.

Next we shall prove a Li-Yau type differential inequality for the positive plurisubharmonic solution of (2.1). The result will not be needed in the next section.

THEOREM 2.2. *Let $u(x,t)$ be a positive solution to (2.1) such that $u(x,t)$ is plurisubharmonic for all t. Then we have the following differential inequality:*

$$\frac{u_t}{u} - \frac{|\nabla u|^2}{u^2} + \frac{m}{t} \geq 0. \tag{2.15}$$

Proof. As in Li-Yau [L-Y], we let $v = \log u$. Then

$$\Delta v - v_t = -|\nabla v|^2. \tag{2.16}$$

Let

$$G(x,t) = t(|\nabla v|^2 - \eta v_t),$$

where $\eta > 1$ is a constant. Direct calculation shows that in normal coordinates at a point:

$$\Delta|\nabla v|^2 = R_{\bar\alpha\beta} v_\alpha v_{\bar\beta} + v_{\alpha\gamma} v_{\bar\alpha\bar\gamma} + v_{\alpha\bar\gamma} v_{\bar\alpha\gamma} + (\Delta v)_\alpha v_{\bar\alpha} + v_\alpha (\Delta v)_{\bar\alpha}, \tag{2.17}$$

$$\frac{\partial}{\partial t}|\nabla v|^2 = R_{\bar\alpha\beta} v_\alpha v_{\bar\beta} + (v_t)_\alpha v_{\bar\alpha} + v_\alpha (v_t)_{\bar\alpha} \tag{2.18}$$

and

$$v_{tt} - \Delta(v_t) = R_{\bar\alpha\beta} v_{\alpha\bar\beta} + R_{\bar\alpha\beta} v_\alpha v_{\bar\beta} + (v_t)_\alpha v_{\bar\alpha} + v_\alpha (v_t)_{\bar\alpha}. \tag{2.19}$$

Combining (2.16)–(2.19) we have that

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)(|\nabla v|^2 - \eta v_t) \;=\; & v_{\alpha\gamma} v_{\bar\alpha\bar\gamma} + v_{\alpha\bar\gamma} v_{\bar\alpha\gamma} - (|\nabla v|^2 - \eta v_t)_\alpha v_{\bar\alpha} \\ & - v_\alpha (|\nabla v|^2 - \eta v_t)_{\bar\alpha} + \eta R_{\bar\alpha\beta}(v_{\alpha\bar\beta} + v_\alpha v_{\bar\beta}).\end{aligned}$$

Using the fact that

$$R_{\bar\alpha\beta}(v_{\alpha\bar\beta} + v_\alpha v_{\bar\beta}) = \frac{1}{u} R_{\bar\alpha\beta} u_{\alpha\bar\beta} \geq 0$$

we then have

$$
\begin{aligned}
\left(\Delta - \frac{\partial}{\partial t}\right) G \geq\ & t v_{\alpha\bar{\gamma}} v_{\bar{\alpha}\gamma} - 2\langle \nabla G, \nabla v\rangle - \frac{G}{t} \\
\geq\ & \frac{t}{m}(\Delta v)^2 - 2\langle \nabla G, \nabla v\rangle - \frac{G}{t} \\
=\ & \frac{t}{m}(|\nabla v|^2 - v_t)^2 - 2\langle \nabla G, \nabla v\rangle - \frac{G}{t}.
\end{aligned} \tag{2.20}
$$

Once we have (2.20), we can use the cut-off function argument as in [L-Y] to carry the interior estimates. For the sake of the completeness we include the argument here. Let $\psi(s)$ be a cut-off function such that $0 \leq \psi \leq 1$, $\psi(s) \equiv 1$ for $s \in [0, 1]$ and $\psi(s) \equiv 0$ for $s \geq 2$. We also require that

$$
\psi' \leq 0, \quad \psi'' \geq -C_1 \quad \text{and} \quad \frac{|\psi'|^2}{\psi} \leq C_1 \tag{2.21}
$$

for some positive constant C_1. Now we let $\phi(x) = \psi(r_t(x)/R)$. Let $\Phi = \phi G$. Suppose Φ attains a positive maximum at (x_0, t_0). Then we have at (x_0, t_0):

$$
0 \geq \left(\Delta - \frac{\partial}{\partial t}\right)\Phi \quad \text{and} \quad \nabla \Phi = 0.
$$

Note that ϕ may not be smooth at x_0 in the space variable, but we can always use the trick of Calabi as in [L-Y]. ϕ may not be smooth in the t variable at t_0, but we can use the difference quotient so that the final result of the following computations is correct. Hence the above differential inequality together with (2.20) implies that at (x_0, t_0)

$$
\begin{aligned}
0 \geq\ & t_0\phi\left(\Delta - \frac{\partial}{\partial t}\right)\Phi \\
\geq\ & \frac{1}{m}(t_0\phi|\nabla v|^2 - t_0\phi v_t)^2 - 2Gt_0\frac{|\nabla\phi|^2}{\phi} + t\left(\Delta\phi - \frac{\partial}{\partial t}\phi\right)G \\
& - G\phi^2 + 2\langle\nabla\phi, \nabla v\rangle G\phi t_0 \\
\geq\ & \frac{1}{m}(t_0\phi|\nabla v|^2 - t_0\phi v_t)^2 - G\left[2t_0\frac{|\nabla\phi|^2}{\phi} - t_0\left(\Delta\phi - \frac{\partial}{\partial t}\phi\right) + 1\right] \\
& - 2\frac{|\nabla\phi|}{\phi^{1/2}}G(|\nabla v|\phi^{1/2}t_0^{1/2})t_0^{1/2}.
\end{aligned} \tag{2.22}
$$

Using (2.21) we have that

$$
\frac{|\nabla\phi|^2}{\phi} \leq \frac{C_2}{R^2} \quad \text{and} \quad -\Delta\phi \geq \frac{C_2}{R^2}. \tag{2.23}
$$

Also Theorem 17.2 of [H4] implies that

$$\left|\frac{\partial}{\partial t}\phi\right| \leq \frac{C_2}{R}. \tag{2.24}$$

Here C_2 is a constant dependent of C_1, m and the upper bound of $|Rm|(x,t)$. Combining (2.22)–(2.24) we have, at the maximum of Φ over $M \times [0,T]$, that

$$0 \geq \frac{1}{m}(y-z)^2 - \frac{C_2}{R}(y-\eta z)y^{1/2}t_0^{1/2} - (y-\eta z)\left(\frac{C_2 t_0}{R^2} + \frac{C_2 t_0}{R} + 1\right). \tag{2.25}$$

Here $y = t_0\phi|\nabla v|^2(x_0,t_0)$, $z = t_0\phi v_t(x_0,t_0)$. Using the trick of [L-Y], we write

$$(y-z)^2 = \frac{1}{\eta^2}(y-\eta z)^2 + 2\frac{\eta-1}{\eta}(y-\eta z)y + \left(\frac{\eta-1}{\eta}\right)^2 y^2.$$

Using the $ax^2 + bx \geq -\frac{b^2}{4a}$, for $R \gg 1$ we have that

$$0 \geq \frac{1}{m\eta^2}(y-\eta z)^2 - (y-\eta z)\left(\frac{C_3 t_0}{R} + 1\right)$$

for some constant C_3 independent of R. From which we have that

$$\sup_{B_0(o,R)\times[0,T]} t(|\nabla v|^2 - \eta v_t) \leq m\eta^2\left(1 + \frac{C(m,\eta,T,\sup_{M\times[0,T]}|Rm|)}{R}\right).$$

Here we have used the fact that $g(t)$ is nonincreasing so that $B_t(o,R) \supset B_0(o,R)$. Letting $R \to \infty$ and then $\eta \to 1$ we have (2.15).

3. Liouville properties of plurisubharmonic functions. In this section, we shall discuss Liouville properties of plurisubharmonic functions using the LYH type inequality in §1 and the results of §2. In this section, we always assume that $(M, g_{\alpha\bar\beta}(x))$ is a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature. We also assume that for all $x \in M$ and $r > 0$, $k(x,r) \leq \epsilon(r)$ for some nonincreasing function $\epsilon(r)$ with $\lim_{r\to\infty}\epsilon(r) = 0$, where

$$k(x,r) = \fint_{B_0(x,r)} \mathcal{R}_0\, dV_0 \tag{3.1}$$

and $\mathcal{R}_0$ is the scalar curvature of $M, g_{\alpha\bar\beta}(x)$. By [N-T], we know that (1.1) has a solution $g_{\alpha\bar\beta}(x,t)$ on $M \times [0,\infty)$ such that for any $0 < T < \infty$, $g_{\alpha\bar\beta}$ satisfies (i)–(iv) in §1.

Define

$$F(x,t)=\log\left(\frac{\det(g_{\alpha\bar\beta}(x,t))}{\det(g_{\alpha\bar\beta}(x,0))}\right). \tag{3.2}$$

To illustrate the idea of the proof to a more general result, let us begin with the following particular case. In this case, what we need is to assume that (1.1) has long time solution $g_{\alpha\bar\beta}$ so that for any $T<\infty$, conditions (i)–(iv) in §1 are satisfied by $g_{\alpha\bar\beta}$ on $M\times[0,T]$.

THEOREM 3.1. *With the above assumptions, suppose u_0 is a plurisubharmonic function such that (i) u is bounded; and (ii) $\widetilde{\Delta}u_0(x)\le \exp(a(1+r_0(x))$ for some constant $a>0$. Then u_0 must be constant.*

Proof. Let $\widetilde{\Delta}u_0=f$, then $f\ge 0$. Since u_0 is bounded, by [N-S-T1, Corollary 2.1] we have

$$\int_0^\infty s\left(⨍_{B_0(x,s)} f\,dV_0\right)ds\le C_1$$

for some constant C_1 independent of x. By [N-S-T1, Corollary 1.2], we know that

$$\sup_M|\widetilde{\nabla}u_0|\le C_2.$$

By Proposition 2.1, there is a unique solution $u(x,t)$ with initial data u_0. Moreover, by Proposition 2.1 and the maximum principle in [N-T, Theorem 1.2], we conclude that u is uniformly bounded.

Since $\widetilde{\Delta}u_0(x)\le\exp(a(1+r_0(x)))$, by Theorem 2.1(i) we conclude that $u(x,t)$ is plurisubharmonic for all $t>0$. Moreover, suppose u_0 is not harmonic, then by Theorem 2.1(ii) $w=u_t>0$ for $t>0$ and tw is nondecreasing in t. Hence

$$\begin{aligned} u(x,t)-u_0(x) &= \int_0^t w(x,s)\,ds\\ &\ge w(x,1)\int_1^t\frac{1}{s}\,ds\\ &= w(x,1)\log t. \end{aligned}$$

Since $w(x,1)>0$, let $t\to\infty$, the above inequality contradicts the fact that u is uniformly bounded. Hence u_0 must be harmonic and is constant by [Y].

Next we shall generalize Theorem 3.1 by relaxing the condition that u_0 is bounded. In the following, we always assume that u_0 is a plurisubharmonic function on M such that there exists a constant $a > 0$ such that

$$\begin{cases} |\widetilde{\nabla} u_0(x)| \le a, \\ \widetilde{\Delta} u_0(x) \le \exp\left(a(1 + r_0(x)\right) \end{cases} \tag{3.3}$$

for all $x \in M$. Note that in the proof of Theorem 3.1, we know that if u_0 is bounded, then u_0 will satisfy the first inequality of (3.3).

Because of (3.3), let u be the solution of (2.1) with initial data u_0 constructed in Proposition 2.1. By Proposition 2.2, u is plurisubharmonic for all $t \ge 0$. Let $v(x,t) = u(x,t) - u_0(x)$. Also, let $\mathfrak{m}(t) = \inf_{x \in M} F(x,t)$. Then $\mathfrak{m}(t) \le 0$, nonincreasing, and is finite for fixed t by properties (ii) and (iv) in §1 and the fact that $F(x,t) = -\int_0^t \mathcal{R}(\S, \int)\, ds$, where $\mathcal{R}(\S, \int)$ is the scalar curvature at time s.

Lemma 3.1. *With the above assumptions and notations, we have*

$$\widetilde{\Delta} v - e^F v_t \ge -\widetilde{\Delta} u_0. \tag{3.4}$$

Proof. As in [Shi, p. 156], using the fact that $g_{\alpha\bar{\beta}}$ is nonincreasing, we have

$$\widetilde{\Delta} u \ge e^F \Delta u = e^F u_t = e^F v_t.$$

Hence

$$\widetilde{\Delta} v = \widetilde{\Delta} u - \widetilde{\Delta} u_0 \ge e^F v_t - \widetilde{\Delta} u_0.$$

The result follows.

Lemma 3.2. *With the same assumptions and notations as in Lemma 3.1, there is a constant C such that for all $(x,t) \in M \times [0,\infty)$, we have*

$$0 \le v(x,t) \le C t^{\frac{1}{2}} \left(-\mathfrak{m}(2t) + 1\right). \tag{3.5}$$

Proof. First note that $v(x,0) = 0$ and $v_t = u_t = \Delta u \ge 0$. Hence $v \ge 0$. We need a more refined estimate of (2.10). More precisely, the Bochner formula on $\|u_{\alpha\bar{\beta}}\|^2$ says that

$$\left(\Delta - \frac{\partial}{\partial t}\right) \|u_{\alpha\bar{\beta}}\|^2 \ge \|u_{\alpha\bar{\beta}\gamma}\|^2 + \|u_{\alpha\bar{\beta}\bar{\gamma}}\|^2 - \mathcal{R}(x,t)\|u_{\alpha\bar{\beta}}\|^2.$$

Using the LYH type inequality of H.-D. Cao as in [N-T] we have that

$$t\mathcal{R}(x,t) \le -2\mathfrak{m}(2t).$$

Combining them we have that

$$\left(\Delta - \frac{\partial}{\partial t}\right)(1+t\Phi)^{\frac{1}{2}} \geq -(-\mathfrak{m}(2t)+1)\Phi.$$

Here $\Phi = \|u_{\alpha\bar{\beta}}\|^2$. Now we can proceed as in the proof of Proposition 2.2 (iii) to conclude that

$$\sup_{M\times[0,T]} ((-\mathfrak{m}(2T)+1)|\nabla u|^2 + (1+t\Phi)^{\frac{1}{2}}) \leq a^2(-\mathfrak{m}(2T)+1)+1,$$

which then implies

$$\|u_{\alpha\bar{\beta}}\|(x,t) \leq C_1 t^{-\frac{1}{2}}(-\mathfrak{m}(2t)+1)$$

for some constant C_1 depending only on m and $\sup_M |\widetilde{\nabla} u_0|$. Hence

$$\begin{aligned} v(x,T) &= \int_0^T v_t(x,t)\,dt \\ &= \int_0^T u_t(x,t)\,dt \\ &= \int_0^T \Delta u(x,t)\,dt \\ &\leq C_1 \int_0^T t^{-\frac{1}{2}}(-\mathfrak{m}(2t)+1)\,dt \\ &\leq C_2 T^{\frac{1}{2}}(-\mathfrak{m}(2T)+1) \end{aligned}$$

for some constant C_2 independent of x and t. The proof of the lemma is completed.

Using the method of proof of Theorem 2.1 in [N-T], we have:

THEOREM 3.2. *Let u_0 be a plurisubharmonic function on M satisfying (3.3). Suppose*

(3.6) $$\limsup_{R\to\infty} \frac{\sup_{\partial B_0(o,R)} u_0}{\log t} \leq 0$$

where $R^2 = t^{\frac{3}{2}} e^{-\mathfrak{m}(2t)}(-\mathfrak{m}(2t)+1)$, then u_0 must be constant.

Remarks.

(a) It is easy to see that $R \to \infty$ if and only if $t \to \infty$.

(b) By [Sh2–3] and [N-T, Remark 2.2] if $\epsilon(r)$ at the beginning of this section satisfies $\epsilon(r) \leq Cr^{-2}$ or more generally if $\int_0^r s\epsilon(s)\,ds \leq C\log(r+2)$, then $-\mathfrak{m}(t) \leq$

$C' \log (t+1)$, and the assumption (3.6) can be replaced by

$$\limsup_{x\to\infty} \frac{u_0(x)}{\log r_0(x)} \le 0.$$

(c) Similarly, if $\epsilon(r) \le r^{-\theta}$ for some $\theta > 0$, then the assumption (3.6) can be replaced by

$$\limsup_{x\to\infty} \frac{u_0(x)}{\log\log r_0(x)} \le 0.$$

If $\int_0^r s\epsilon(s)\,ds \le Cr^2/\log(2+r)$, then the assumption (3.6) can be replaced by

$$\limsup_{x\to\infty} \frac{u_0(x)}{\log\log\log r_0(x)} \le 0,$$

and so on.

(d) It is easy to see that Theorem 3.1 is a particular case of Theorem 3.2.

Proof of Theorem 3.2. Let u and v as in Lemmas 3.1 and 3.2. Let $(x_0, T) \in M \times (0, \infty)$. For any $R > 0$, let G_R be the positive Green's function with zero boundary value on $B_0(x_0, R)$ with respect to the initial metric. By (3.4)

$$\begin{aligned}
(3.7)\qquad \int_0^T \int_{B_0(x_0,R)} & G_R(x_0,y)\widetilde{\Delta} v(y,t)\,dV_0\,dt \\
&\ge -T\int_{B_0(x_0,R)} G_R(x_0,y)\widetilde{\Delta} u_0(y)\,dV_0 \\
&\quad + \int_0^T \int_{B_0(x_0,R)} G_R(x_0,y)e^{F(y,t)} v_t(y,t)\,dV_0\,dt \\
&\ge -T\int_{B_0(x_0,R)} G_R(x_0,y)\widetilde{\Delta} u_0(y)\,dV_0 \\
&\quad + e^{\mathfrak{m}(T)} \int_{B_0(x_0,R)} G_R(x_0,y) v(y,T)\,dV_0 \\
&\ge C_1(m)\left[-T\left(-u_0(x_0) + \sup_{B_0(x_0,R)} u_0\right) + e^{\mathfrak{m}(T)}R^2 \fint_{B_0(x_0,\frac{R}{5})} v(y,T)\,dV_0\right],
\end{aligned}$$

for some positive constant C_1 depending only on m, where we have used Theorem 2.1 in [N-S-T1] and Lemma 2.2 in [N-T] and the fact that $v \ge 0$, $F_t \le 0$. On the other hand, by Green's formula and Lemma 3.2, we have that, for any $0 < t < T$,

$$\begin{aligned}
(3.8)\qquad \int_{B_0(x_0,R)} G_R(x_0,y)\widetilde{\Delta} v(y)\,dV_0 &= -v(x_0,t) - \int_{\partial B_0(x_0,R)} v\frac{\partial G_R}{\partial \nu} \\
&\le C_2 t^{\frac{1}{2}}(-\mathfrak{m}(2t)+1)
\end{aligned}$$

for some constant C_2 independent of (x,t). By (3.4) and the fact that $v_t \geq 0$, we have $\widetilde{\Delta} v \geq -\widetilde{\Delta} u_0$. Since $v \geq 0$, by the generalized mean value inequality [N-T, Lemma 2.1], (3.7) and (3.8), we have

$$\begin{aligned} v(x_0,T) \leq\ & C_3 \fint_{B_0(x_0,\frac{R}{5})} v(y,T)\, dV_0 + \int_{B_0(x_0,\frac{R}{5})} G_{\frac{R}{5}}(x_0,y)\widetilde{\Delta} u_0(y)\, dV_0 \\ \leq\ & C_4 \left[R^{-2} T e^{-\mathfrak{m}(2T)} \left(-u_0(x_0) + \sup_{B_0(o,2R)} u_0 + T^{\frac{1}{2}}(-\mathfrak{m}(2T)+1) \right) \right. \\ & \left. \quad - u_0(x_0) + \sup_{B_0(o,2R)} u_0 \right] \end{aligned}$$

if R is large, for some constants C_3 and C_4 independent of (x_0,T) and R. Let R be such that $(2R)^2 = T^{\frac{3}{2}}(1+T)e^{-\mathfrak{m}(2T)}(-\mathfrak{m}(2T)+1)$, then by (3.6), we can conclude that

$$\limsup_{t\to\infty} \frac{v(x_0,t)}{\log t} = 0. \tag{3.9}$$

We claim that u_0 is harmonic. Suppose not, then as in the proof of Theorem 3.1, we have $u(x_0,t) \geq C \log t$ for some constant $C > 0$ for all $t \geq 1$. This is impossible.

By the definition of R in (3.6), it is easy to see that $\log R \geq \log t$ when t is large. Hence (3.6) implies that

$$\limsup_{R\to\infty} \frac{\sup_{\partial B_0(o,R)} u_0}{\log R} = 0.$$

Since u_0 is harmonic, it must be constant by [C-Y].

Since one can solve the Poincaré-Lelong equation for a (1,1) form on a complete noncompact manifold with nonnegative holomorphic bisectional curvature under rather weak assumptions on the (1,1) form (see [N-S-T1]), one can apply Theorem 3.2 (or Theorem 3.1) to obtain results on the flatness of the holomorphic line bundles. As an example, we have the following:

COROLLARY 3.1. *Let $(M, g_{\alpha\bar{\beta}}(x))$ be a complete nocompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature satisfying the conditions in Theorem 3.1. Let $(L, \mathfrak{h}_0)$ be a holomorphic line bundle on M with the Hermitian metric $\mathfrak{h}_0$. Suppose $\Omega(\mathfrak{h}_0) \geq 0$ and suppose its trace $\mathcal{S}_0 = g^{\alpha\bar{\beta}}(x)\Omega_{\alpha\bar{\beta}}(\mathfrak{h}_0)(x)$ is bounded and*

$$\int_0^\infty s \fint_{B_0(x,s)} \mathcal{S}_0(y)\, dy\, ds \leq C \tag{3.10}$$

for some constant $C > 0$ for all $x \in M$. Then $(L, \mathfrak{h}_0)$ is flat.

Proof. Using the fact that $\mathcal{S}_0$ is bounded and (3.10), one can find bounded function u_0 such that $\sqrt{-1}\partial\bar{\partial}u_0 = \Omega(\mathfrak{h}_0)$ by [N-S-T1, Theorem 5.1]. Since $\Omega(\mathfrak{h}_0)$ is nonnegative, u_0 is plurisubharmonic. By Theorem 3.1, u_0 is constant and hence $(L, \mathfrak{h}_0)$ is flat.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LA JOLLA, CA 92093
E-mail: lni@math.ucsd.edu

DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG, CHINA
E-mail: lftam@math.cuhk.edu.hk

REFERENCES

[A] B. Andrews, Harnack inequalities for evolving hypersurfaces, *Math. Z.* **217** (1994), 179–197.
[Co1] H.-D. Cao, On Harnack inequalities for the Kähler-Ricci flow, *Invent. Math.* **109** (1992), 247–263.
[Co2] ———, Limits of solutions to the Kähler-Ricci flow, *J. Differential Geom.* **45** (1997), 257–272.
[C-Z] B. L. Chen and X. P. Zhu, On complete noncompact Kähler manifolds with positive bisectional curvature, preprint.
[C-T-Z] B. L. Chen, S. H. Tang and X. P. Zhu, A uniformization theorem of complete noncompact Kähler surfaces with positive bisectional curvature, preprint.
[C-Y] S. Y. Cheng and S.-T. Yau, Differential equations on Riemannian manifolds and their geometric applications, *Comm. Pure Appl. Math.* **28** (1975), 333–354.
[Cw1] B. Chow, On Harnack's inequality and entropy for the Gaussian curvature flow, *Comm. Pure Appl. Math.* **44** (1991), 469–483.
[Cw2] ———, The Yamabe flow on locally conformally flat manifolds with positive Ricci curvature, *Comm. Pure Appl. Math.* **45** (1992), 1003–1014.
[Cw3] ———, Ricci flow and Einstein metrics in low dimensions, *Surv. Differ. Geom.*, vol. 6, Int. Press, Boston, MA, 1999, pp. 187–220.
[C-H] B. Chow and R. Hamilton, Constrained and linear Harnack inequalities for parabolic equations, *Invent. Math.* **129** (1997), 213–238.
[H1] R. S. Hamilton, Three-manifolds with positive Ricci curvature, *J. Differential Geom.* **17** (1982), 255–306.
[H2] ———, Four-manifolds with positive curvature operator, *J. Differential Geom.* **24** (1986), 153–179.
[H3] ———, The Harnack estimate for the Ricci flow, *J. Differential Geom.* **37** (1993), 225–243.
[H4] ———, Formation of singularities in the Ricci flow, *Surv. Differ. Geom.*, vol. 2, Int. Press, Boston, MA, 1995, pp. 7–136.
[L-Y] P. Li and S.-T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.* **156** (1986), 139–168.
[M1] N. Mok, An embedding theorem of complete Kähler manifolds of positive bisectional curvature onto affine algebraic varieties, *Bull. Soc. Math. France* **112** (1984), 197–250.
[M2] ———, The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature, *J. Differential Geom.* **27** (1988), 179–214.
[M-S-Y] N. Mok, Y.-T. Siu and S.-T. Yau, The Poincaré-Lelong equation on complete Kähler manifolds, *Compositio Math.* **44** (1981), 183–218.
[N-S-T1] L. Ni, Y.-G. Shi and L.-F. Tam, Poisson equation, Poincaré-Lelong equation and curvature decay on complete Kähler manifolds, *J. Differential Geom.* **57** (2001), 339–388.

[N-S-T2] ———, Ricci flatness of asymptotically locally flat Euclidean metrics, *Trans. Amer. Math. Soc.* (to appear).

[N-T] L. Ni and L.-F. Tam, Kähler-Ricci flow and Poinaré-Lelong equation, *Comm. Anal. Geom* (to appear).

[Sh1] W. X. Shi, Deforming the metric on complete Riemannian manifolds, *J. Differential Geom.* **30** (1989), 223–301.

[Sh2] ———, Ricci deformation of metric on complete noncompact Kähler manifolds, Ph.D. thesis, Harvard University, 1990.

[Sh3] ———, Ricci flow and the uniformization on complete noncompact Kähler manifolds, *J. Differential Geom.* **45** (1997), 94–220.

[Y] S.-T. Yau, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.* **28** (1975), 201–228.