Invent. math. (2020) 221:301–327
https://doi.org/10.1007/s00222-020-00950-1

Poincaré polynomials of moduli spaces of Higgs bundles and character varieties (no punctures)

Anton Mellit[1]

Received: 17 October 2018 / Accepted: 9 January 2020 / Published online: 29 January 2020

Abstract Using our earlier results on polynomiality properties of plethystic logarithms of generating series of certain type, we show that Schiffmann's formulas for various counts of Higgs bundles over finite fields can be reduced to much simpler formulas conjectured by Mozgovoy. In particular, our result implies the conjecture of Hausel and Rodriguez-Villegas on the Poincaré polynomials of twisted character varieties and the conjecture of Hausel and Thaddeus on independence of E-polynomials on the degree.

1 Introduction

Schiffmann [17] computed the number of absolutely indecomposable vector bundles of rank r and degree d over a compete curve C of genus g over $\mathbb{F}_q$. Suppose the eigenvalues of the Frobenius acting on the first cohomology of C are $\alpha_1, \dots, \alpha_{2g}$ with $\alpha_{i+g} = q\alpha_i^{-1}$ for $i = 1, \dots, g$. This means that for all $k \geq 1$ we have

$$\#C(\mathbb{F}_{q^k}) = 1 + q^k - \sum_{i=1}^{2g} \alpha_i^k.$$

✉ Anton Mellit
anton.mellit@univie.ac.at

1 Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

Schiffmann's result says that the number of absolutely indecomposable vector bundles of rank r and degree d on C is given by a Laurent polynomial independent of C

$$A_{g,r,d}(q, \alpha_1, \dots, \alpha_g) \in \mathbb{Z}[q, \alpha_1^{\pm 1}, \dots, \alpha_g^{\pm 1}],$$

symmetric in α_i and invariant under $\alpha_i \to q\alpha_i^{-1}$.

Suppose $(r, d) = 1$. Schiffmann showed that the number of stable Higgs bundles of rank r and degree d is given by $q^{1+(g-1)r^2} A_{g,r,d}$. Let C be a curve over $\mathbb{C}$. The moduli space of stable Higgs bundles $\mathcal{M}_{g,r,d}(C)$ is a quasi-projective variety, and by a theorem of Katz [9] its E-polynomial defined as

$$E_{g,r,d}(x, y) = \sum_{i,j,k} (-1)^k x^i y^j \dim \mathrm{Gr}_F^i \mathrm{Gr}_{i+j}^W H_c^k(\mathcal{M}_{g,r,d}(C), \mathbb{C})$$

is given by $(xy)^{1+(g-1)r^2} A_{g,r,d}(xy, x, \dots, x)$. It is known [7] that this moduli space has pure cohomology. In particular, the Poincaré polynomial

$$P_{q,r,d}(q) = \sum_i (-1)^i q^{\frac{i}{2}} \dim H_c^i(\mathcal{M}_{g,r,d}(C))$$

is the following specialization:

$$P_{q,r,d}(q) = E_{g,r,d}(q^{\frac{1}{2}}, q^{\frac{1}{2}}) = q^{1+(g-1)r^2} A_{g,r,d}(q, q^{\frac{1}{2}}, \dots, q^{\frac{1}{2}}).$$

Since twisted character varieties are diffeomorphic to the moduli spaces of stable Higgs bundles (see [9]), their Poincaré polynomials coincide.

The formula of Schiffmann was difficult to work with. In particular, it was not clear that his formula is equivalent to a much simpler formula conjectured earlier by Hausel and Rodriguez-Villegas for Poincaré polynomials [9], and then extended by Mozgovoy for the polynomials $A_{g,r,d}$ [15].

Here we study Schiffmann's formula from the combinatorial point of view and establish these conjectures. Our main result is:

Theorem 1.1 *Let $g \geq 1$. Let Ω_g denote the series*

$$\Omega_g = \sum_{\mu \in \mathcal{P}} T^{|\mu|} \prod_{\square \in \mu} \frac{\prod_{i=1}^g (z^{a(\square)+1} - \alpha_i q^{l(\square)})(z^{a(\square)} - \alpha_i^{-1} q^{l(\square)+1})}{(z^{a(\square)+1} - q^{l(\square)})(z^{a(\square)} - q^{l(\square)+1})},$$

and let

$$H_g = -(1-q)(1-z) \operatorname{Log} \Omega_g, \qquad H_g = \sum_{r=1}^{\infty} H_{g,r} T^r.$$

Then for all $r \geq 1$, $H_{g,r}$ is a Laurent polynomial in q, z and $\alpha_1, \ldots, \alpha_g$, and for all d, $A_{g,r,d}$ is obtained by setting $z = 1$ in $H_{g,r}$:

$$A_{g,r,d}(q, \alpha_1, \ldots, \alpha_g) = H_{g,r}(q, 1, \alpha_1, \ldots, \alpha_g).$$

As a corollary, we obtain the GL-version of the conjecture of Hausel and Thaddeus (see Conjecture 3.2 in [7]):

Corollary 1.2 *For r, d, d' satisfying $(r, d) = (r, d') = 1$, the E-polynomials of $\mathcal{M}(g, r, d)$ and $\mathcal{M}(g, r, d')$ coincide.*

Davesh Maulik and Aaron Pixton announced an independent proof of Theorem 1.1. Their approach is to make rigorous the physical considerations of [1]. They claim that their work will settle the more general conjectures about Higgs bundles with parabolic structures.

In the next paper [13], we extend Schiffmann's [17] and Schiffmann–Mozgovoy's [16] methods to the parabolic case. Combined with the results of the present work, we obtain a proof of the conjecture of Hausel et al. [8] on the Poincaré polynomials of character varieties with punctures.

For a more precise technical version of the main result the reader is referred to Theorem 5.2. We warn the reader that variables q, t in Sect. 5 correspond to z, q in the rest of the paper and apologize for the inconvenience.

In Sect. 6 we discuss motivic classes of moduli stacks and connect our results to the work of Fedorov, Soibelman and Soibelman [4].

2 Arms and legs

We begin by stating an elementary formula which relates the generating series of arms and legs and the generating series of weights of partitions, proved in [2] (we follow notations from [3]). For a partition λ and any cell $\square$ we denote by $a_\lambda(\square)$ and $l_\lambda(\square)$ the arm and leg lengths of $\square$ with respect to λ. These numbers are non-negative when $\square \in \lambda$ and negative otherwise. For partitions μ, ν define

$$E_{\mu,\nu} = \sum_{\square \in \mu} q^{-a_\nu(\square)} t^{l_\mu(\square)+1} + \sum_{\square \in \nu} q^{a_\mu(\square)+1} t^{-l_\nu(\square)}.$$

For any partition μ, let

$$B_\mu = \sum_{\square \in \mu} q^{c(\square)} t^{r(\square)},$$

where $c(\square), r(\square)$ denote the column and row indices. For any f let f^* be obtained from f by the substitution $q \to q^{-1}, t \to t^{-1}$.

Lemma 2.1 *For any partitions μ, ν, we have*

$$E_{\mu,\nu} = qtB_\mu + B^*_\nu - (q-1)(t-1)B_\mu B^*_\nu. \tag{1}$$

Proof We prove by induction on the largest part μ_1 of μ (defined to be 0 if $\mu = \varnothing$). If $\mu = \varnothing$, we have $a_\mu(\square) = -1 - c(\square)$. Therefore

$$E_{\varnothing,\nu} = \sum_{\square\in\mu} q^{-c(\square)} t^{-l_\nu(\square)}.$$

For each fixed value of $c(\square)$, the numbers $l_\nu(\square)$ go over the same range as the numbers $r(\square)$. Thus we obtain

$$E_{\mu,\varnothing} = B^*_\nu.$$

This establishes the case $\mu_1 = 0$.

For the induction step, let μ' be obtained from μ by removing the first column, i.e. $\mu' = (\mu_1 - 1, \mu_2 - 1, \ldots)$. Splitting the sum according to whether $\square$ is in the first column, we obtain

$$\sum_{\square\in\mu} q^{-a_\nu(\square)} t^{l_\mu(\square)+1} = q \sum_{\square\in\mu'} q^{-a_\nu(\square)} t^{l_\mu(\square)+1} + \sum_{i=1}^{l(\mu)} q^{1-\nu_i} t^{l(\mu)-i+1}.$$

For any cell $\square$, we have

$$a_\mu(\square) = \begin{cases} a_{\mu'}(\square) + 1 & \text{if } r(\square) < l(\mu), \\ -1 - c(\square) & \text{otherwise.} \end{cases}$$

This implies

$$\sum_{\square\in\nu} q^{a_\mu(\square)+1} t^{-l_\nu(\square)} = q \sum_{\square\in\nu} q^{a'_\mu(\square)+1} t^{-l_\nu(\square)} \\ + (1-q) \sum_{\square\in\nu\,:\, r(\square)\geq l(\mu)} q^{-c(\square)} t^{-l_\nu(\square)}.$$

In the last sum for each fixed value of $c(\square)$ the numbers $l_\nu(\square)$ go over the same range as the numbers $r(\square) - l(\mu)$, so we have

$$\sum_{\square\in\nu\,:\,r(\square)\geq l(\mu)} q^{-c(\square)}t^{-l_\nu(\square)} = \sum_{\square\in\nu\,:\,r(\square)\geq l(\mu)} q^{-c(\square)}t^{l(\mu)-r(\square)}$$
$$= \sum_{i=l(\mu)+1}^{\infty} t^{l(\mu)-i+1}\frac{1-q^{-\nu_i}}{1-q^{-1}}.$$

Putting things together, we have

$$E_{\mu,\nu} - qE_{\mu',\nu} = \sum_{i=1}^{\infty} t^{l(\mu)-i+1}(q^{1-\nu_i}-q) + q\sum_{i=1}^{l(\mu)} t^{l(\mu)-i+1}.$$

The first sum reduces to

$$\sum_{i=1}^{\infty} t^{l(\mu)-i+1}(q^{1-\nu_i}-q) = qt^{l(\mu)}(q^{-1}-1)B_\nu^* = (1-q)t^{l(\mu)}B_\nu^*.$$

The second sum becomes

$$q\sum_{i=1}^{l(\mu)} t^{l(\mu)-i+1} = qt\frac{t^{l(\mu)}-1}{t-1}.$$

This implies

$$E_{\mu,\nu} - qE_{\mu',\nu} = (1-q)t^{l(\mu)}B_\nu^* + qt\frac{t^{l(\mu)}-1}{t-1}$$

On the other hand we have

$$B_\mu - qB_{\mu'} = \sum_{i=1}^{l(\mu)} t^{i-1} = \frac{t^{l(\mu)}-1}{t-1}.$$

Therefore if we denote the right hand side of (1) by $E'_{\mu,\nu}$, we obtain

$$\begin{aligned} E'_{\mu,\nu} - qE'_{\mu',\nu} =& qt\frac{t^{l(\mu)}-1}{t-1} + (1-q)B_\nu^* - (q-1)(t-1)B_\nu^*\frac{t^{l(\mu)}-1}{t-1} \\ =& qt\frac{t^{l(\mu)}-1}{t-1} + (1-q)t^{l(\mu)}B_\nu^*. \end{aligned}$$

So $E_{\mu',\nu} = E'_{\mu',\nu}$ implies $E_{\mu,\nu} = E'_{\mu,\nu}$ and the induction step is established. □

For a partition μ, we define $z_i(\mu)$ to match z_i in [17]:

$$z_i(\mu) = t^{-l(\mu)+i} q^{\mu_i} \qquad (i = 1, 2, \ldots, l(\mu)).$$

Our notations match after the substitution $(q, z) \to (t, q)$. Note the following generating series identity:

$$\sum_{i=1}^{l(\mu)} z_i(\mu) = t^{-l(\mu)+1} \left((q-1) B_\mu + \frac{t^{l(\mu)} - 1}{t - 1} \right). \tag{2}$$

What we will actually need is the following generating series:

$$K_\mu := (1 - t) \sum_{i<j} \frac{z_i(\mu)}{z_j(\mu)}.$$

It can be obtained as follows. Note that the sum K_μ contains only terms with non-positive powers of t. So we can start with

$$\tilde{K}_\mu := (1 - t) \sum_{i=1}^{l(\mu)} z_i(\mu) \sum_{i=1}^{l(\mu)} z_i(\mu)^{-1} = (1 - t) \sum_{i,j=1}^{l(\mu)} \frac{z_i(\mu)}{z_j(\mu)},$$

and take only non-positive powers of t. Let L be the operator

$$L(t^i q^j) = \begin{cases} t^i q^j & (i \le 0) \\ 0 & (i > 0) \end{cases}.$$

Then

$$K_\mu = L(\tilde{K}_\mu) - l(\mu).$$

Note that we had to subtract $l(\mu)$ to cancel the contribution from the terms $i = j$ appearing in $\tilde{K}_\mu$. We can calculate $\tilde{K}_\mu$ using Lemma 2.1 and (2):

$$\begin{aligned} \tilde{K}_\mu &= (1 - t) \left((q-1) B_\mu + \frac{t^{l(\mu)} - 1}{t - 1} \right) \left((q^{-1} - 1) B_\mu^* + \frac{t^{-l(\mu)} - 1}{t^{-1} - 1} \right) \\ &= (q^{-1} - 1) E_{\mu,\mu} - t^{l(\mu)} (q^{-1} - 1) B_\mu^* + t^{1-l(\mu)} (q - 1) B_\mu \\ &\quad - (t^{l(\mu)} - 1) \sum_{i=0}^{l(\mu)-1} t^{-i}, \end{aligned}$$

from which it is clear that

$$
\begin{aligned}
L(\tilde{K}_\mu) &= (q^{-1}-1)L(E_{\mu,\mu}) + t^{1-l(\mu)}(q-1)B_\mu + \sum_{i=0}^{l(\mu)-1} t^{-i}. \\
&= (q^{-1}-1)\sum_{\square\in\mu} q^{a_\mu(\square)+1}t^{-l_\mu(\square)} + \sum_{i=1}^{l(\mu)} z_i(\mu).
\end{aligned}
$$

The conclusion is the following

Proposition 2.2 *For any partition μ we have*

$$
(1-t)\sum_{i<j}\frac{z_i(\mu)}{z_j(\mu)} = (q^{-1}-1)\sum_{\square\in\mu} q^{a_\mu(\square)+1}t^{-l_\mu(\square)} + \sum_{i=1}^{l(\mu)}(z_i(\mu)-1).
$$

Converting additive generating functions to multiplicative with an extra variable u we obtain

Corollary 2.3 *For any partition μ we have*

$$
\prod_{i<j}\frac{1-tu\frac{z_i(\mu)}{z_j(\mu)}}{1-u\frac{z_i(\mu)}{z_j(\mu)}} = \prod_{\square\in\mu}\frac{1-uq^{a_\mu(\square)+1}t^{-l_\mu(\square)}}{1-uq^{a_\mu(\square)}t^{-l_\mu(\square)}}\prod_{i=1}^{l(\mu)}\frac{1-u}{1-uz_i(\mu)}.
$$

Note that the left hand side contains "non-symmetric" ratios $\frac{z_i(\mu)}{z_j(\mu)}$ for $i<j$, while the right hand side contains "simple terms" $z_i(\mu)$ and 1, "correct arm-leg terms" $q^{a_\mu(\square)+1}t^{-l_\mu(\square)}$ and "incorrect arm-leg terms" $q^{a_\mu(\square)}t^{-l_\mu(\square)}$. Our strategy is to trade incorrect arm-leg terms in Schiffmann's formula for non-symmetric ratios, which will complement or cancel other non-symmetric ratios so that the result contains only correct arm-leg terms and something symmetric.

3 Schiffmann's terms

Let X be a smooth projective curve over $\mathbb{F}_q$ of genus g with zeta function

$$
\zeta_X(x) = \frac{\prod_{i=1}^{2g}(1-\alpha_i x)}{(1-x)(1-qx)}.
$$

Let us order α_i in such a way that $\alpha_{i+g}=\frac{q}{\alpha_i}$ holds. We will treat $\alpha_1,\alpha_2,\ldots,\alpha_g$ as formal variables and set $\alpha_{i+g}=\frac{q}{\alpha_i}$. An alternative way to think of the parameters α_i is to view them as the exponentials of the chern roots of the

Hodge bundle on the moduli space of curves times $q^{\frac{1}{2}}$. The expressions we will be writing will depend on $q, z, \alpha_1, \ldots, \alpha_g$. There is a correspondence between these variables and the variables from [14] given as follows:

$$q, z, \alpha_1, \ldots, \alpha_g \to t, q, u_1^{-1}, \ldots, u_g^{-1}. \tag{3}$$

The formula of Schiffmann (see [16,17]) involves a sum over partitions

$$\Omega^{\mathrm{Sch}} := \sum_\mu \Omega_\mu T^{|\mu|}.$$

For each partition μ the corresponding coefficient is

$$\Omega_\mu := q^{(g-1)\langle\mu,\mu\rangle} J_\mu H_\mu.$$

Here $\langle\mu,\mu\rangle = \sum_i \mu_i'^2$ where μ' is the conjugate partition of μ. We will proceed defining J_μ and H_μ and taking them apart in the process. We have

$$J_\mu = \prod_{\square\in\mu} \frac{\prod_{i=1}^{2g}(1-\alpha_i q^{-1-l(\square)} z^{a(\square)})}{(1-q^{-1-l(\square)} z^{a(\square)})(1-q^{-l(\square)} z^{a(\square)})_{\neq 0}}.$$

The notation $(-)_{\neq 0}$ means we omit the corresponding factor if it happens to be zero. This naturally splits as follows:

$$J_\mu = \prod_{\square\in\mu} \frac{\prod_{i=1}^{g}(1-\alpha_i q^{-1-l(\square)} z^{a(\square)})}{1-q^{-1-l(\square)} z^{a(\square)}} \prod_{\square\in\mu} \frac{\prod_{i=1}^{g}(1-\alpha_i^{-1} q^{-l(\square)} z^{a(\square)})}{(1-q^{-l(\square)} z^{a(\square)})_{\neq 0}}.$$

Applying Corollary 2.3 we obtain

$$\begin{aligned} J_\mu = {} & \prod_{\square\in\mu} \frac{\prod_{i=1}^{g}(1-\alpha_i q^{-1-l(\square)} z^{a(\square)})}{1-q^{-1-l(\square)} z^{a(\square)}} \times \prod_{\square\in\mu} \frac{\prod_{i=1}^{g}(1-\alpha_i^{-1} q^{-l(\square)} z^{a(\square)+1})}{1-q^{-l(\square)} z^{a(\square)+1}} \\ & \times \prod_{i<j} \prod_{k=1}^{g} \frac{1-\alpha_k^{-1}\frac{z_i(\mu)}{z_j(\mu)}}{1-q\alpha_k^{-1}\frac{z_i(\mu)}{z_j(\mu)}} \times \prod_{i=1}^{l(\mu)} \prod_{k=1}^{g} \frac{1-\alpha_k^{-1}}{1-\alpha_k^{-1} z_i(\mu)} \\ & \times \prod_{i<j} \frac{\left(1-q\frac{z_i(\mu)}{z_j(\mu)}\right)_{\neq 0}}{1-\frac{z_i(\mu)}{z_j(\mu)}} \times \prod_{i=1}^{l(\mu)} (1-z_i(\mu)), \end{aligned}$$

where $z_i(\mu) = q^{-l(\mu)+i} z^{\mu_i}$ coincides with Schiffmann's z_{n-i+1}. Denote the last four products in the above right hand side by A, B, C, D. Note that

$\sum l(\square)+\sum(l(\square)+1) = \langle\mu,\mu\rangle$, so $q^{\langle\mu,\mu\rangle}$ together with the first two products produce

$$\frac{\prod_{i=1}^{g} N_\mu(\alpha_i^{-1})}{N_\mu(1)},$$

where N_μ is the arm-leg product as in [14]:

$$N_\mu(u) = \prod_{\square\in\mu}(z^{a(\square)} - uq^{1+l(\square)})(z^{a(\square)+1} - u^{-1}q^{l(\square)}). \tag{4}$$

So we have

$$q^{\langle\mu,\mu\rangle} J_\mu = \frac{\prod_{i=1}^{g} N_\mu(\alpha_i^{-1})}{N_\mu(1)} ABCD,$$

where

$$A = \prod_{i<j}\prod_{k=1}^{g}\frac{1-\alpha_k^{-1}\frac{z_i(\mu)}{z_j(\mu)}}{1-q\alpha_k^{-1}\frac{z_i(\mu)}{z_j(\mu)}}, \qquad B = \prod_{i=1}^{l(\mu)}\prod_{k=1}^{g}\frac{1-\alpha_k^{-1}}{1-\alpha_k^{-1}z_i(\mu)}$$

$$C = \prod_{i<j}\frac{\left(1-q\frac{z_i(\mu)}{z_j(\mu)}\right)_{\neq 0}}{1-\frac{z_i(\mu)}{z_j(\mu)}} \qquad D = \prod_{i=1}^{l(\mu)}(1-z_i(\mu)).$$

We proceed by defining H_μ. Let[1]

$$\tilde{\zeta}'(x) = x^{1-g}\zeta(x) = \frac{\prod_{k=1}^{g} x^{-1}(1-\alpha_k x)(1-q\alpha_k^{-1}x)}{x^{-1}(1-x)(1-qx)}.$$

Let $L(z_1,\dots,z_{l(\mu)})$ be the rational function (note that we reversed the order of z_i)

$$L(z_1,\dots,z_{l(\mu)}) = \frac{1}{\prod_{i>j}\tilde{\zeta}'\left(\frac{z_i}{z_j}\right)}\sum_{\sigma\in S_{l(\mu)}}\sigma\left\{\prod_{i<j}\tilde{\zeta}'\left(\frac{z_i}{z_j}\right)\frac{1}{\prod_{i<l(\mu)}\left(1-q\frac{z_{i+1}}{z_i}\right)}\frac{1}{1-z_1}\right\}.$$

[1] What we call $\tilde{\zeta}'$ is usually denoted $\tilde{\zeta}$, but we prefer to modify it slightly and use $\tilde{\zeta}$ for the modified version.

Note that $\tilde{\zeta}'$ appears in the numerator as many times as in the denominator, so it can be multiplied by a constant without changing L. So we replace $\tilde{\zeta}'$ with something more resembling the other products we have seen:

$$\tilde{\zeta}(x) = \frac{\prod_{k=1}^{g}(1-\alpha_k^{-1}x^{-1})(1-q\alpha_k^{-1}x)}{(1-x^{-1})(1-qx)}.$$

H_μ is defined as the iterated residue (remember that our ordering of z_i is the opposite of Schiffman's)

$$H_\mu = \operatorname{res}_{z_i=z_i(\mu)} L(z_1,\ldots,z_{l(\mu)}) \prod_{i\,:\,\mu_i=\mu_{i+1}} \frac{dz_{i+1}}{z_{i+1}}.$$

Note that the only poles L can have at $z_i = z_i(\mu)$ are coming from factors of the form $1-q\frac{z_i}{z_{i+1}}$ for i such that $\mu_i=\mu_{i+1}$. Each such factor can appear at most once in the denominator of L. We have

$$\operatorname{res}_{z_{i+1}=qz_i} \frac{1}{1-q\frac{z_i}{z_{i+1}}}\frac{dz_{i+1}}{z_{i+1}} = 1.$$

Thus we will obtain the same result if we multiply L by the product of these factors and then evaluate at $z_i = z_i(\mu)$. Note that C has precisely the same factors removed. Therefore we have

$$CH_\mu = \left(\prod_{i<j}\frac{1-q\frac{z_i}{z_j}}{1-\frac{z_i}{z_j}}L\right)(z_1(\mu),\ldots,z_{l(\mu)}(\mu)).$$

Putting in A as well we obtain a nice expression:

$$ACH_\mu = \left(\prod_{i\neq j}\frac{1-q\frac{z_i}{z_j}}{\prod_{k=1}^{g}1-q\alpha_k^{-1}\frac{z_i}{z_j}}\sum_{\sigma\in S_{l(\mu)}}\sigma\{\cdots\}\right)(z_1(\mu),\ldots,z_{l(\mu)}(\mu)).$$

We see that the product is symmetric in z_i, so it can be moved inside the summation. Since B and D are symmetric, they can also be moved inside the summation. After some cancellations we arrive at the following. Define for any n

$$\begin{aligned} f(z_1,\ldots,z_n) = {} & \prod_i\prod_{k=1}^{g}\frac{1-\alpha_k^{-1}}{1-\alpha_k^{-1}z_i} \\ & \times \sum_{\sigma\in S_n}\sigma\left\{\prod_{i>j}\left(\frac{1}{1-\frac{z_i}{z_j}}\prod_{k=1}^{g}\frac{1-\alpha_k^{-1}\frac{z_i}{z_j}}{1-q\alpha_k^{-1}\frac{z_i}{z_j}}\right)\prod_{i>j+1}\left(1-q\frac{z_i}{z_j}\right)\prod_{i\geq 2}(1-z_i)\right\}. \end{aligned} \tag{5}$$

Then

$$ABCDH_\mu = f(z_1(\mu), \ldots, z_{l(\mu)}(\mu)).$$

Summarizing we obtain

Proposition 3.1 *For any partition μ the term Ω_μ is given by*

$$\Omega_\mu = \frac{f_\mu \prod_{i=1}^{g} N_\mu(\alpha_i^{-1})}{N_\mu(1)}, \qquad f_\mu = f(z_1(\mu), \ldots, z_{l(\mu)}(\mu)),$$

where $z_i(\mu) = q^{-l(\mu)+i} z^{\mu_i}$, and N, f are defined in (4), (5).

Example 3.1 Let us calculate f in a few cases. It is convenient to set

$$P(x) = \prod_{i=1}^{g} (1 - \alpha_i^{-1} x).$$

We have

$$f(z_1) = \frac{P(1)}{P(z_1)}$$

$$f(z_1, z_2) = \frac{P(1)^2}{P(z_1)P(z_2)(z_1 - z_2)} \left(z_1(1 - z_2) \frac{P(\frac{z_2}{z_1})}{P(q\frac{z_2}{z_1})} - z_2(1 - z_1) \frac{P(\frac{z_1}{z_2})}{P(q\frac{z_1}{z_2})} \right)$$

Note that the denominator of this expression is $P(z_1)P(z_2)P(q\frac{z_1}{z_2})P(q\frac{z_2}{z_1})$ if no cancellations happen. If $z_2 = qz_1$, the denominator reduces to $P(z_1)P(z_2)P(q^2)$, so it has only 3 P-factors instead of 4.

4 Combinatorics of the function *f*

4.1 Bounding denominators

First we analyse denominators of f defined in (5). For generic values of z_i, the denominator of f can be as bad as the full product

$$\prod_i P(z_i) \prod_{i \neq j} P(q\frac{z_i}{z_j}),$$

where $P(x) = \prod_{k=1}^{g} (1 - \alpha_k^{-1} x)$. Pick numbers $r_1, r_2, \ldots$ such that $\sum_m r_m = n$. Split $z_1, z_2, \ldots, z_n$ into a union of subsequences of sizes $r_1, r_2, \ldots$. Let $j_m = 1 + \sum_{i<m} r_i$. For each m the m-th subsequence looks like

$z_{j_m}, z_{j_m+1}, \dots, z_{j_m+r_m-1}$. Suppose each subsequence forms a geometric progression with quotient q:

$$z_{j_m+i} = q^i z_{j_m} \qquad (i < r_m).$$

Then f can be viewed as a function of variables z_{j_m}. The denominator can be bounded as follows

Proposition 4.1 *The following expression is a Laurent polynomial:*

$$f \prod_{i=1}^{n} \left(P(z_i) \prod_{m:\, j_m+r_m>i} P\left(q^{r_m} \frac{z_{j_m}}{z_i}\right) \prod_{m:\, j_m>i} P\left(q \frac{z_i}{z_{j_m}}\right)\right)$$

Proof First write the definition of f as follows:

$$f = \prod_i \frac{P(1)}{P(z_i)} \sum_{\sigma \in S_n} \prod_{\sigma(i)>\sigma(j)} \frac{P\left(\frac{z_i}{z_j}\right)}{\left(1-\frac{z_i}{z_j}\right) P\left(q\frac{z_i}{z_j}\right)} \prod_{\sigma(i)>\sigma(j)+1} \left(1 - q\frac{z_i}{z_j}\right) \prod_{\sigma(i)\geq 2} (1-z_i).$$

Note that $1-\frac{z_i}{z_j}$ does not contribute to the denominator because of symmetrization. Next note that if $j = i+1$ and j, i belong to the same subsequence, then $1 - q\frac{z_i}{z_j} = 0$. So all summands with $\sigma(i) > \sigma(j)+1$ vanish. So it is enough to sum only over those σ which satisfy the condition

$$\sigma(i+1) \geq \sigma(i) - 1 \quad \text{whenever } i, i+1 \text{ are in the same subsequence.} \tag{6}$$

So in each sequence $\sigma(j_m), \dots, \sigma(j_m + r_m - 1)$ if there is a drop, the size of the drop is 1. Now for each such σ we look at the product

$$\prod_{\sigma(i)>\sigma(j)} \frac{P\left(\frac{z_i}{z_j}\right)}{P\left(q\frac{z_i}{z_j}\right)} = \prod_{i<j,\, \sigma(i)>\sigma(j)} \frac{P\left(\frac{z_i}{z_j}\right)}{P\left(q\frac{z_i}{z_j}\right)} \prod_{i<j,\, \sigma(i)<\sigma(j)} \frac{P\left(\frac{z_j}{z_i}\right)}{P\left(q\frac{z_j}{z_i}\right)}.$$

It is enough to show that for each value of i and each σ the following expressions are Laurent polynomials:

$$P(1) \prod_{m:\, j_m>i} P\left(q\frac{z_i}{z_{j_m}}\right) \times \prod_{i<j,\, \sigma(i)>\sigma(j)} \frac{P\left(\frac{z_i}{z_j}\right)}{P\left(q\frac{z_i}{z_j}\right)},$$

$$\prod_{m:\, j_m+r_m>i} P\left(q^{r_m}\frac{z_{j_m}}{z_i}\right) \times \prod_{i<j,\, \sigma(i)<\sigma(j)} \frac{P\left(\frac{z_j}{z_i}\right)}{P\left(q\frac{z_j}{z_i}\right)}.$$

Further, let us split the product over all $j > i$ into products over our subsequences. We only need to consider values of m such that $j_m > i$ (when j and i are in different subsequences) or $j_m \le i < j_m + r_m$ (when they are in the same subsequence). So it is enough to show that the following products are Laurent polynomials:

$$P\left(q\frac{z_i}{z_{j_m}}\right) \prod_{k<r_m,\ \sigma(i)>\sigma(j_m+k)} \frac{P\left(\frac{z_i}{z_{j_m+k}}\right)}{P\left(q\frac{z_i}{z_{j_m+k}}\right)} \qquad (j_m > i), \tag{7}$$

$$P\left(q^{r_m}\frac{z_{j_m}}{z_i}\right) \prod_{k<r_m,\ \sigma(i)<\sigma(j_m+k)} \frac{P\left(\frac{z_{j_m+k}}{z_i}\right)}{P(q\frac{z_{j_m+k}}{z_i})} \qquad (j_m > i), \tag{8}$$

$$P(1) \prod_{i-j_m<k<r_m,\ \sigma(i)>\sigma(j_m+k)} \frac{P\left(\frac{z_i}{z_{j_m+k}}\right)}{P\left(q\frac{z_i}{z_{j_m+k}}\right)} \qquad (j_m > i), \tag{9}$$

$$P\left(q^{r_m}\frac{z_{j_m}}{z_i}\right) \prod_{i-j_m<k<r_m,\ \sigma(i)<\sigma(j_m+k)} \frac{P\left(\frac{z_{j_m+k}}{z_i}\right)}{P\left(q\frac{z_{j_m+k}}{z_i}\right)} \qquad (j_m > i). \tag{10}$$

Observe that because of the condition (6) in each of the cases (7)–(10) the values of k from a contiguous set $k_{min}, \dots, k_{max}$ (if non-empty). So the arguments to P from a geometric progression with ratio q or q^{-1}. Hence the product collapses and the only remaining denominator is $P(q\frac{z_i}{z_{j_m+k_{min}}})$ in cases (7) and (9), and $P(q\frac{z_i}{z_{j_m+k_{max}}})$ in cases (8) and (10). Further analysis leads to $k_{min} = 0$ in (7), $k_{max} = r_m - 1$ in (8), $k_{min} = i - j_m + 1$ in (9) and $k_{max} = r_m - 1$ in (10). □

Example 4.1 In the situation of $n = 1$ we obtain that $f P(z_1)P(q)$ is a Laurent polynomial. For $n = 2$ and $z_2 = qz_1$ we obtain $f P(z_1)P(z_2)P(q^2)P(q)$ is a Laurent polynomial. Comparing with Example 3.1 one can notice that our denominator bound is not optimal.

For the case when $z_i = z_i(\mu) = z^{\mu_i} q^{i-l(\mu)}$ for a partition μ we obtain

Proposition 4.2 *The following product is a Laurent polynomial for any partition μ:*

$$f_\mu \prod_{\square\in\mu} P(z^{a(\square)+1} q^{-l(\square)}) P(z^{-a(\square)} q^{l(\square)+1}). \tag{11}$$

Proof Recall that f_μ is a shorthand for $f(z_1(\mu), \dots, z_{l(\mu)}(\mu))$ where $z_i(\mu) = z^{\mu_i} q^{i-l(\mu)}$. In view of Proposition 4.1 it is enough to show that for each i the product

$$P(z_i) \prod_{m:\, j_m+r_m>i} P\left(q^{r_m} \frac{z_{j_m}}{z_i}\right) \prod_{m:\, j_m>i} P\left(q \frac{z_i}{z_{j_m}}\right) \tag{12}$$

divides the arm-leg product in (11) for cells $\square \in \mu$ occurring in the row i. Note that our subsequences of geometric progressions in z_i simply correspond to repeated parts of μ. Let $\square$ be the cell in row i and column μ_{j_m} ($j_m + r_m > i$). Then we have $a(\square) = \mu_i - \mu_{j_m}$, $l(\square) = j_m + r_m - 1 - i$. Therefore

$$z^{-a(\square)} q^{l(\square)+1} = q^{r_m} \frac{z_{j_m}}{z_i}.$$

Let $\square$ be the cell in row i and column $\mu_{j_m} + 1$ ($j_m > i$). Then $a(\square) = \mu_i - \mu_{j_m} - 1$, $l(\square) = j_m - 1 - i$. Therefore

$$z^{a(\square)+1} q^{-l(\square)} = q \frac{z_i}{z_{j_m}}.$$

For the cell in column 1 we have $a(\square) = \mu_i - 1$, $l(\square) = l(\mu) - i$, so

$$z^{a(\square)+1} q^{-l(\square)} = z_i.$$

Thus the factors of (12) form a sub-multiset of the factors of the arm-leg product (11), and the claim follows. $\square$

Corollary 4.3 *For any partition μ, the product $N_\mu(1)\Omega_\mu$ is a Laurent polynomial.*

Proof We have

$$N_\mu(1)\Omega_\mu = \prod_{i=1}^{g} N_\mu(\alpha_i^{-1}) f_\mu$$

and

$$\prod_{i=1}^{g} N_\mu(\alpha_i^{-1}) = \prod_{\square\in\mu} P(z^{a(\square)+1} q^{-l(\square)}) P(z^{-a(\square)} q^{l(\square)+1}) \times \pm \text{ a monomial.}$$

$\square$

4.2 Interpolation

We remind the reader that f is not a function in fixed number of variables, but a sequence of functions: a function in n variables for each n. A nice property of f is that the substitution $z_1 = 1$ into the function in $n + 1$ variables leads to essentially the same function in n variables:

Proposition 4.4 *For any n we have*

$$f(1, z_1, \ldots, z_n) = f(qz_1, \ldots, qz_n).$$

Proof Note that because of the product $\prod_{i=2}^{n}(1 - z_i)$ in the definition of $f(1, z_1, \ldots, z_n)$, only the terms with $\sigma(1) = 1$ survive. So we can reduce the summation over S_{n+1} to a summation over S_n. After cancellation of $\prod_i(1-z_i)$ we obtain

$$f(1, z_1, \ldots, z_n) = \prod_i \frac{P(1)}{P(z_i)}$$
$$\times \sum_{\sigma \in S_n} \sigma \left\{ \prod_{i>j} \frac{P\left(\frac{z_i}{z_j}\right)}{\left(1 - \frac{z_i}{z_j}\right) P\left(q\frac{z_i}{z_j}\right)} \prod_{i>j+1} \left(1 - q\frac{z_i}{z_j}\right) \prod_{i>1}(1 - qz_i) \prod_i \frac{P(z_i)}{P(qz_i)} \right\},$$

which coincides with $f(qz_1, \ldots, qz_n)$. □

Corollary 4.5 *Let μ be a partition and let $n \geq l(\mu)$. Define $z_{n,i}(\mu) = z^{\mu_i} q^{i-n}$ for $i = 1, \ldots, n$. Then*

$$f(z_{n,1}(\mu), \ldots, z_{n,n}(\mu)) = f(z_1(\mu), \ldots, z_{l(\mu)}(\mu)).$$

Thus, instead of having a separate function for each value of $l(\mu)$ we can use the same function of n arguments if n is big enough.

5 Polynomiality and the main result

In this section we return to variables q, t which correspond to Schiffmann's variables z, q respectively. First we prove the following statement. The proof is straightforward using methods of [14], but tedious. Let R be a lambda ring containing $\mathbb{Q}(t)[q^{\pm 1}]$. We denote by R^* the tensor product $R \otimes_{\mathbb{Q}(t)[q^{\pm 1}]} \mathbb{Q}(q, t)$ and assume $R \subset R^*$.

Definition 5.1 A regular function of z_i is a sequence of Laurent polynomials

$$f_n \in R[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \qquad (n \geq 0)$$

such that

(i) f_n is symmetric in $z_1, \ldots, z_n$,
(ii) $f_{n+1}(1, z_1, \ldots, z_n) = f_n(tz_1, \ldots, tz_n)$.

For a regular function f and a partition μ, we set

$$f_\mu = f_{l(\mu)}(z_1(\mu), \ldots, z_{l(\mu)}(\mu)), \qquad z_i(\mu) = q^{\mu_i} t^{i-l(\mu)}.$$

We will use the modified Macdonald polynomials $\tilde{H}_\mu[X;q,t]$, see [6,14].

Lemma 5.1 *Let $f(u) = 1 + f^{(1)}u + f^{(2)}u^2 + \cdots$ be a power series whose coefficients $f^{(i)}$ are regular functions in the above sense. Let*

$$\Omega[X] = \sum_{\mu\in\mathcal{P}} c_\mu \tilde{H}_\mu[X;q,t]$$

be a series with $c_\mu \in R^*$, $c_\varnothing = 1$ such that all coefficients of

$$\mathbb{H}[X] = (q-1)\operatorname{Log}\Omega[X]$$

are in R. Let

$$\Omega_f[X,u] = \sum_\mu c_\mu \tilde{H}_\mu[X;q,t] f_\mu(u), \qquad \mathbb{H}_f[X,u] = (q-1)\operatorname{Log}\Omega_f[X,u].$$

Consider the expansion

$$\mathbb{H}_f[X,u] = \mathbb{H}[X] + u\mathbb{H}_{f,1}[X] + u^2\mathbb{H}_{f,2}[X] + \cdots .$$

Then all coefficients of $\mathbb{H}_{f,i}[X]$ for $i \geq 1$ are in $(q-1)R$. In other words, the specialization $q = 1$ of $\mathbb{H}_f[X,u]$ is independent of u.

Proof Let $S = -(q-1)(t-1)$. Recall the notation $\int_X^S F[X, X^*]$ (see [14]). This is a linear operation such that

$$\int_X^S G[X]F[X^*] = (G[X], F[X])_X^S = (G[X], F[SX])_X,$$

and $(-,-)_X$ is the standard Hall scalar product,

$$(s_\mu[X], s_\lambda[X])_X = \delta_{\mu,\lambda}.$$

Recall that modified Macdonald polynomials are orthogonal with respect to $(-,-)_X^S$. In this proof we call an expression F admissible if $(q-1)\operatorname{Log} F$ has all of its coefficients in R. It was proved in [14] that $\int_X^S$ preserves admissibility. By the assumption $\Omega[X]$ is admissible. We will "construct" $\mathbb{H}_f[X,u]$ from admissible building parts.

Let $R[Z, Z^*]$ be the free lambda ring over R with two generators Z and Z^*. Fix a large integer N. For each $i \geq 1$ let $\tilde{f}^{(i)} \in R[Z, Z^*]$ be any element such that

$$\tilde{f}^{(i)}\left[\sum_{i=1}^N z_i, \sum_{i=1}^N z_i^{-1}\right] = f_N^{(i)}(z_1, \ldots, z_N).$$

One way to construct such an element is to find $m \geq 0$ such that $(z_1 \cdots z_N)^m f_N^{(i)}(z_1, \ldots, z_N) = p(z_1, \ldots, z_n)$ does not contain negative powers of z_i, then lift p to a symmetric function $\tilde{p} \in R[Z]$ and set

$$\tilde{f}^{(i)}[Z, Z^*] = \tilde{p}[Z] e_N [Z^*]^m.$$

Then set

$$\tilde{f}(u) = 1 + \tilde{f}^{(1)} u + \tilde{f}^{(2)} u^2 + \cdots \in R[Z, Z^*][[u]].$$

We can take plethystic logarithm:

$$\operatorname{Log} \tilde{f}(u) = g(u) = g^{(1)} u + g^{(2)} u^2 + \cdots \in u R[Z, Z^*][[u]].$$

For any partition μ satisfying $l(\mu) \leq N$ by regularity of f we have

$$f_\mu = f_{l(\mu)}(q^{\mu_1} t^{1-l(\mu)}, q^{\mu_2} t^{2-l(\mu)}, \ldots, q^{\mu_{l(\mu)}}) = f_N(q^{\mu_1} t^{1-N}, q^{\mu_2} t^{2-N}, \ldots, q^{\mu_N}).$$

Thus we can obtain f_μ from $\tilde{f}$ by specializing at

$$\begin{aligned} Z = Z_\mu &= \sum_{i=1}^{N} q^{\mu_i} t^{i-N} = t^{1-N}(q-1) B_\mu + \sum_{i=1}^{N} t^{i-N} \\ &= \frac{t^{1-N}}{1-t} S B_\mu + \frac{t^{-N}-1}{t^{-1}-1}, \end{aligned}$$

and similarly for Z^*. Hence there exists a series

$$g'(u) \in u R[Z, Z^*][[u]]$$

such that for any partition μ with $l(\mu) \leq N$ we have

$$f_\mu = \operatorname{Exp}[g'(u)[S B_\mu, S B_\mu^*]].$$

This g' is obtained from g by the lambda ring homomorphism which sends Z to $\frac{t^{1-N}}{1-t} Z + \frac{t^{-N}-1}{t^{-1}-1}$ and similarly for Z^*. Specialization can be replaced by scalar product using the identity

$$F[SY] = (F[X], \operatorname{Exp}[SXY])_X = (F[X], \operatorname{Exp}[XY])_X^*,$$

and we obtain

$$f_\mu = \int_{Z,V}^{S} \operatorname{Exp}[g'(u)[Z, V]] \operatorname{Exp}[Z^* B_\mu + V^* B_\mu^*] \qquad (l(\mu) \leq N). \tag{13}$$

Let us show that the sum

$$\tilde{\Omega}[X, Z, V] = \sum_{\mu \in \mathcal{P}} c_\mu \tilde{H}_\mu[X; q, t] \operatorname{Exp}[Z B_\mu + V B_\mu^*] \tag{14}$$

is admissible. Begin with the series

$$\sum_{\mu \in \mathcal{P}} \frac{\tilde{H}_\mu[X]\tilde{H}_\mu[Y]\tilde{H}_\mu[Z]\tilde{H}_\mu[V]}{(\tilde{H}_\mu, \tilde{H}_\mu)^S}, \tag{15}$$

which is admissible by the main theorem of [14]. Recall the nabla operator ∇, the shift operator τ and the multiplication by $\operatorname{Exp}\left[\frac{X}{S}\right]$ operator τ^*, and Tesler's identity

$$\nabla \tau \tau^* \tilde{H}_\mu[X] = \operatorname{Exp}\left[\frac{D_\mu X}{S}\right],$$

where $D_\mu = -1 - S B_\mu$. This implies

$$\tau^* \nabla \tau \tau^* \tilde{H}_\mu[X] = \operatorname{Exp}[-X B_\mu].$$

All of the operators involved preserve admissibility (Corollary 6.3 from [14]). In particular, we see that the operator that sends $\tilde{H}_\mu[X]$ to $\operatorname{Exp}[X B_\mu]$ preserves admissibility. Let ω be the operator that sends q, t, X to $q^{-1}, t^{-1}, -X$. Then using $\omega\nabla = \nabla^{-1}\omega$, $\omega\tilde{H}_\mu[X] = \frac{\tilde{H}_\mu[X]}{\tilde{H}_\mu[-1]}$ and the fact that ∇^{-1} preserves admissibility (Corollary 6.4 from [14]) we see that the operator that sends $\tilde{H}_\mu[X]$ to $\operatorname{Exp}[X B_\mu^*]$ preserves admissibility too. Applying these operators to (15) in the variables Z, V we obtain that the following series is admissible:

$$\sum_{\mu \in \mathcal{P}} \frac{\tilde{H}_\mu[X]\tilde{H}_\mu[Y] \operatorname{Exp}[Z B_\mu + V B_\mu^*]}{(\tilde{H}_\mu, \tilde{H}_\mu)^S}.$$

Finally, pairing this series with $\Omega[X]$ we obtain admissibility of (14).

Because of (13) we have

$$\Omega_f(u) = \int_{Z,V}^S \operatorname{Exp}[g'(u)[Z, V]] \tilde{\Omega}[X, Z^*, V^*] \qquad \text{up to terms of degree } > N \text{ in } X.$$

In what follows we ignore the terms of degree $> N$ in X. Since N can be chosen as large as possible, this is enough. Notice that $\operatorname{Exp}[g'(u)[Z, V]]$ is "more" than admissible in the following sense. Introduce a new free (in the lambda ring sense) variable W. Then $\operatorname{Exp}[\frac{W}{S}g'(u)[Z,V]]$ is admissible. Therefore the following is admissible:

$$\Omega_f[X,W,u] := \int_{Z,V}^{S} \operatorname{Exp}[\tfrac{W}{S}g'(u)[Z,V]]\tilde{\Omega}[X,Z^*,V^*].$$

So we have

$$\mathbb{H}_f[X,W,u] = (q-1)\operatorname{Log}\Omega_f[X,W,u] = \sum_{i\geq 0}\mathbb{H}_{f,i}[X,W]u^i$$

with $\mathbb{H}_{f,i}[X,W] \in R[X,W]$. Finally notice that

$$\mathbb{H}_{f,i}[X] = \mathbb{H}_{f,i}[X,S] \equiv \mathbb{H}_{f,i}[X,0] \pmod{(q-1)R[X]},$$

and

$$\begin{aligned}
&\mathbb{H}_f[X,0,u] = (q-1)\operatorname{Log}\Omega_f[X,0,u],\\
&\Omega_f[X,0,u] = \int_{Z,V}^{S}\tilde{\Omega}[X,Z^*,V^*] = \tilde{\Omega}[X,0,0] = \Omega[X].
\end{aligned}$$

This implies

$$\mathbb{H}_{f,i}[X] \equiv 0 \pmod{(q-1)R[X]} \qquad (i \geq 1).$$

□

Then our main result is

Theorem 5.2 *For any $g \geq 0$ let*

$$\Omega(T,q,t,\alpha_1,\ldots,\alpha_g) = \sum_{\mu\in\mathcal{P}} \frac{\prod_{i=1}^{g} N_\mu(\alpha_i^{-1})}{N_\mu(1)} T^{|\mu|},$$

where

$$N_\mu(u) = \prod_{\square\in\mu}(q^{a(\square)} - ut^{1+l(\square)})(q^{a(\square)+1} - u^{-1}t^{l(\square)}).$$

Let

$$\Omega^{\mathrm{Sch}}(T,q,t,\alpha_1,\ldots,\alpha_g) = \sum_{\mu\in\mathcal{P}} \Omega_\mu(q,t,\alpha_1,\ldots,\alpha_g)T^{|\mu|},$$

where Ω_μ are the Schiffmann's terms defined in Sect. 3. Let

$$\mathbb{H}(T, q, t, \alpha_1, \ldots, \alpha_g) = -(q-1)(t-1)\operatorname{Log}\Omega(T, q, t, \alpha_1, \ldots, \alpha_g),$$
$$\mathbb{H}^{\mathrm{Sch}}(T, q, t, \alpha_1, \ldots, \alpha_g) = -(q-1)(t-1)\operatorname{Log}\Omega^{\mathrm{Sch}}(T, q, t, \alpha_1, \ldots, \alpha_g),$$

and let $\mathbb{H}(q, t, \alpha_1, \ldots, \alpha_g)_k$ denote the k-th coefficient of $\mathbb{H}(T, q, t, \alpha_1, \ldots, \alpha_g)$ viewed as a power series in T, and similarly for $\mathbb{H}^{\mathrm{Sch}}$. Then we have

(i) $\mathbb{H}^{\mathrm{Sch}}(q, t, \alpha_1, \ldots, \alpha_g)_k \in \mathbb{Q}(t)[q^{\pm 1}, \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1}]$,
(ii) $\mathbb{H}^{\mathrm{Sch}}(1, t, \alpha_1, \ldots, \alpha_g)_k = \mathbb{H}(1, t, \alpha_1, \ldots, \alpha_g)_k$.

Proof By the main result of [14] we have

$$\mathbb{H}(q, t, \alpha_1, \ldots, \alpha_g)_k \in \mathbb{Q}[t, q, \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1}].$$

By Corollary 4.3 we have

$$\mathbb{H}^{\mathrm{Sch}}(q, t, \alpha_1, \ldots, \alpha_g)_k \in \mathbb{Q}(t, q)[\alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1}].$$

So we can pass to the ring of Laurent series in $\alpha_1^{-1}, \ldots, \alpha_g^{-1}$ and it is enough to prove the corresponding statements (i) and (ii) for the coefficients in front of monomials of the form $\prod_{i=1}^g \alpha_i^{m_i}$. Let us apply Lemma 5.1 for the ring $R = \mathbb{Q}(t)[q^{\pm 1}, \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1}]$, series

$$\Omega[X, q, t, \alpha_1, \ldots, \alpha_g] = \sum_{\mu \in \mathcal{P}} \frac{\prod_{i=1}^g N_\mu(\alpha_i^{-1}) \tilde{H}_\mu[X, q, t]}{N_\mu(1)},$$

and the regular function $f(u)$ obtained from f (see (5) and Proposition 4.4) by setting $u\alpha_i^{-1}$ in place of α_i^{-1}, so that $f(u)$ becomes a power series in u with coefficients in R.

To be able to apply Lemma 5.1, we need to show that the constant coefficient of $f(u)$ is 1, in other words we need to check that

$$\sum_{\sigma \in S_n} \sigma \left\{ \prod_{i>j} \left(\frac{1}{1 - \frac{z_i}{z_j}} \right) \prod_{i>j+1} \left(1 - q\frac{z_i}{z_j}\right) \prod_{i \geq 2} (1 - z_i) \right\} = 1.$$

We do this by induction. Denote the left hand side by L_n. Notice that L_n is a polynomial. Suppose we know that $L_{n-1} = 1$. Then by Proposition 4.4 we know that $L_n - 1$ is divisible by $z_1 - 1$. Since it is a symmetric polynomial, it must be divisible by $\prod_{i=1}^n (z_i - 1)$. On the other hand, the degree of L_n is at most $n-1$, so necessarily $L_n - 1 = 0$.

After applying Lemma 5.1 we can set $X = T$, where T is the variable from the statement of the Theorem. In particular, T is assumed to satisfy $p_k[T] = T^k$ and we can use the identity $\tilde{H}_\mu[T; q, t] = T^{|\mu|}$. Let

$$\mathbb{H}^{\mathrm{Sch}}(T, q, t, u) = -(q-1)(t-1)\,\mathrm{Log}\left[\sum_{\mu\in\mathcal{P}} \frac{\prod_{i=1}^{g} N_\mu(\alpha_i^{-1})}{N_\mu(1)} T^{|\mu|} f_\mu(u)\right].$$

Lemma 5.1 says that

$$\mathbb{H}^{\mathrm{Sch}}(T, q, t, u) - \mathbb{H}(T, q, t) \in (q-1)\mathbb{Q}(t)[q^{\pm 1}, \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1}][[T, u]].$$

On the other hand, the coefficient in front of any monomial in $\alpha_1, \ldots, \alpha_g, T$ has bounded degree in u, wo we can set $u = 1$ and obtain a statement about Laurent series in α_i^{-1}:

$$\mathbb{H}^{\mathrm{Sch}}(T, q, t, 1) - \mathbb{H}(T, q, t) \in (q-1)\mathbb{Q}(t)[q^{\pm 1}]((\alpha_1^{-1}, \ldots, \alpha_g^{-1}))[[T]].$$

Finally we remember that $\mathbb{H}^{\mathrm{Sch}}(T, q, t) = \mathbb{H}^{\mathrm{Sch}}(T, q, t, 1)$ and remember that the coefficients of $\mathbb{H}^{\mathrm{Sch}}(T, q, t)$ are Laurent *polynomials* in α_i to obtain

$$\mathbb{H}^{\mathrm{Sch}}(T, q, t) - \mathbb{H}(T, q, t) \in (q-1)\mathbb{Q}(t)[q^{\pm 1}, \alpha_1^{\pm 1}, \ldots, \alpha_g^{\pm 1}][[T]].$$

□

Remark 5.1 We have been using the substitution $(q, z) \to (t, q)$ to relate Schiffmann's variables to the HLV variables. Note that the Hausel–Villegas functions Ω and $\mathbb{H}$ are symmetric in q, t because replacing μ by the conjugate partition interchanges arms and legs. So we could have worked with the substitution $(q, z) \to (q, t)$, but then we would need to replace a partition by the conjugate partition somewhere in the argument.

Theorem 1.1 is a direct corollary of Theorem 5.2 and [17].

6 Motivic interpretation

To answer some of the questions asked by Yan Soibelman, Davesh Maulik and an anonymous referee, we sketch an approach connecting the present work to the work of Fedorov et al. [4]. There they explain how to recast Schiffmann's formula to compute motivic classes of moduli spaces in the Grothendieck group of stacks over a field of characteristic zero. It is natural to ask if results of the present paper can be used to improve our understanding of motivic classes, and if Hausel–Rodriguez-Villegas formula can be recast in a similar way.

The Grothendieck group of stacks is denoted by Mot and consists of formal linear combinations of Artin stacks of finite type modulo cut-and-paste relations. For a stack X the corresponding element of Mot is denoted by $[X]$ and is called the *motivic class* of X. The dimensional completion of Mot is denoted by $\overline{\mathrm{Mot}}$. Fix a curve C and assume it has a divisor of degree 1. The Grothendieck ring of stacks Mot contains natural elements: the class of $\mathbb{A}^1$ denoted by $\mathbb{L}$, and the class of the curve $[C]$. There is a natural operation of symmetric power which for a stack X is defined by

$$S^n[X] = [X^n/S_n].$$

This operation satisfies the condition

$$S^n[X+Y] = \sum_{i=0}^{n} S^i[X]S^{n-i}[Y],$$

which means that Mot is a *pre-λ-ring*. S^n extends to $\overline{\mathrm{Mot}}$ and makes it into a pre-λ-ring too. It is not clear[2] whether Mot or $\overline{\mathrm{Mot}}$ are λ-rings, which would mean that $S^n(xy)$ and $S^n(S^m(x))$ can be expressed as certain prescribed polynomials in x, $S^2(x)$, $S^3(x)$, ... and y, $S^2(y)$, $S^3(y)$, ... Nevertheless, Totaro's lemma [5] tells us that $S^n(\mathbb{L}X) = \mathbb{L}^n S^n(X)$ holds for any X and n. The formal sum of the form

$$1 + zS^1[X] + z^2S^2[X] + \cdots = \mathrm{Exp}[zX]$$

is called *plethystic exponential* (a.k.a. motivic zeta function). The inverse operation is called the *plethystic logarithm* Log.

Denote by R_g the ring of polynomials in $\alpha_1^{\pm 1}$, ..., $\alpha_g^{\pm 1}$ and $q^{\pm 1}$ invariant under permutations of α_i and substitutions of the form $\alpha_i \to q\alpha_i^{-1}$. As a ring R_g is the polynomial ring in the first g elementary symmetric functions in $\alpha_1, \ldots, \alpha_g, q\alpha_1^{-1}, \ldots, q\alpha_g^{-1}$ over $\mathbb{Z}[q, q^{-1}]$. Using Kapranov's results [10] on the motivic zeta function of a curve one can show that there is a ring homomorphism

$$\mathrm{ev}_C : R_g \to \mathrm{Mot}$$

which sends q to $\mathbb{L}$ and when extended to $R_g[[z]] \to \mathrm{Mot}[[z]]$ sends the formal zeta function to the motivic zeta function:

[2] For some interesting counter-examples in this direction see [11,12].

$$\mathrm{ev}_C\left(\frac{\prod_{i=1}^{g}(1-z\alpha_i)(1-zq\alpha_i^{-1})}{(1-z)(1-qz)}\right)=\sum_{n=0}^{\infty}z^nS^n[C]=\zeta_C(z)\in\mathrm{Mot}[[z]].$$

The ring R_g is a λ-ring. R_g has a filtration by the total degree in $q,\alpha_1,\ldots,\alpha_g$. Denote the corresponding completion by $\overline{R}_g$. An infinite sum converges in the completion if degrees of the summands tend to $-\infty$. The homomorphism ev_C extends to the completions.

Consider Schiffmann's generating function (see Sect. 3 for details)

$$\Omega^{\mathrm{Sch}}(T,z,q,\alpha_1,\ldots,\alpha_g)=\sum_{\mu}\Omega_{\mu}(z,q,\alpha_1,\ldots,\alpha_g)T^{|\mu|}.$$

In [4] it is explained how the definition of Ω^{Sch} can be recast to produce an element $\Omega^{\mathrm{Mot}}\in\overline{\mathrm{Mot}}[[T,z]]$. Analysing their construction it is easy to see that in fact we have

$$\Omega^{\mathrm{Mot}}=\mathrm{ev}_C(\Omega^{\mathrm{Sch}}),$$

where we extend ev_C to formal power series in z and T. To obtain the motivic class of the moduli stack of semistable Higgs bundles of rank r and degree d one needs to first compute plethystic logarithm:

$$\sum_{r,d}B_{r,d}T^rz^d=\mathbb{L}\,\mathrm{Log}\,\Omega^{\mathrm{Mot}}(T,z),$$

then for any rational slope τ compute plethystic exponential

$$\sum_{d/r=\tau}\mathbb{L}^{(1-g)r^2}H^{\mathrm{Mot}}_{r,d}T^rz^d=\mathrm{Exp}\sum_{d/r=\tau}B_{r,d}T^rz^d,$$

and finally recover the motivic class of the moduli stack as

$$[\mathcal{M}^{\mathrm{ss}}_{r,d}]=H^{\mathrm{Mot}}_{r,d+er}\qquad(e\gg 0)$$

for sufficiently large e.

This recipe can be reformulated as follows. Let us decompose Ω^{Mot} into an infinite product according to slopes first:

$$\Omega^{\mathrm{Mot}}=\prod_{\tau\in\mathbb{Q}_{\geq 0}}\sum_{d/r=\tau}\widetilde{H}^{\mathrm{Mot}}_{r,d}T^rz^d.$$

Then we have

$$\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2} H^{\mathrm{Mot}}_{r,d} T^r z^d = \operatorname{Exp} \mathbb{L} \operatorname{Log} \sum_{d/r=\tau} \widetilde{H}^{\mathrm{Mot}}_{r,d} T^r z^d.$$

Analogously, we can decompose

$$\Omega^{\mathrm{Sch}}(T, z, q, \alpha_1, \ldots, \alpha_g) = \prod_{\tau\in\mathbb{Q}_{\geq 0}} \sum_{d/r=\tau} \widetilde{H}^{\mathrm{Sch}}_{r,d}(q, \alpha_1, \ldots, \alpha_g) T^r z^d,$$

and define $H^{\mathrm{Sch}}_{r,d}$ by

$$\sum_{d/r=\tau} q^{(1-g)r^2} H^{\mathrm{Sch}}_{r,d} T^r z^d = \operatorname{Exp} q \operatorname{Log} \sum_{d/r=\tau} \widetilde{H}^{\mathrm{Sch}}_{r,d} T^r z^d.$$

We make two observations. If $\overline{\mathrm{Mot}}$ was a λ-ring, it would follow that ev_C commutes with Exp and Log, and we would have $H^{\mathrm{Mot}}_{r,d} = \mathrm{ev}_C(H^{\mathrm{Sch}}_{r,d})$. So any result about Ω^{Sch} could have been directly translated to $H^{\mathrm{Mot}}_{r,d}$. For instance, we would have

$$\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2}[\mathcal{M}^{\mathrm{ss}}_{r,d}] T^r z^d = \left(\operatorname{Exp} \frac{\mathbb{L}\sum_{d/r=\tau} \mathrm{ev}_C(A_{g,r}) T^r z^d}{\mathbb{L}-1}\right) \quad (\text{in a } \lambda\text{-ring}). \tag{16}$$

In particular, in the universal λ-ring quotient of $\overline{\mathrm{Mot}}$ we conclude that the above formula holds.

On the other hand, if we are only interested in the case of coprime r, d, then we do not need the λ-ring property because in all the expansions of Exp and Log above we use only the first term. So we directly obtain

$$\begin{aligned} H^{\mathrm{Mot}}_{r,d} &= \mathbb{L}^{(g-1)r^2+1} \widetilde{H}^{\mathrm{Mot}}_{r,d} = \mathbb{L}^{(g-1)r^2} B_{r,d} = \mathrm{ev}_C(H^{\mathrm{Sch}}_{r,d}), \\ [\mathcal{M}^{\mathrm{s}}_{r,d}] &= \frac{\mathbb{L}^{(g-1)r^2+1}\mathrm{ev}_C(A_{g,r})}{\mathbb{L}-1} \qquad ((r,d)=1). \end{aligned} \tag{17}$$

So in the above two situations no information is lost when passing from motivic invariants to functions in $q, \alpha_1, \ldots, \alpha_g$.

Next we would like to connect the motivic formula to the Hausel–Rodriguez-Villegas generating function Ω_g. Unfortunately, the function is not invariant under the map $\alpha_i \to q\alpha_i^{-1}$, so the function is not in R_g. Following Mozgovoy's approach [15], we apply change of variables $q \to qz^{-1}$:

$$\begin{aligned}\Omega_g^{\mathrm{Moz}} &= \sum_{\mu\in\mathcal{P}} T^{|\mu|} z^{(g-1)\langle\mu',\mu'\rangle} \prod_{\square\in\mu} \frac{\prod_{i=1}^g (1-\alpha_i q^{l(\square)} z^{-h(\square)})(1-\alpha_i^{-1} q^{l(\square)+1} z^{-h(\square)})}{(1-q^{l(\square)} z^{-h(\square)})(1-q^{l(\square)+1} z^{-h(\square)})}\\ &= \sum_{\mu\in\mathcal{P}} T^{|\mu|} (q/z)^{(g-1)\langle\mu,\mu\rangle} \prod_{\square\in\mu} \frac{\prod_{i=1}^g (1-\alpha_i q^{-l(\square)-1} z^{h(\square)})(1-\alpha_i^{-1} q^{-l(\square)} z^{h(\square)})}{(1-q^{-l(\square)-1} z^{h(\square)})(1-q^{-l(\square)} z^{h(\square)})},\end{aligned}$$

where $h(\square) = a(\square) + l(\square) + 1$ and $\langle\mu',\mu'\rangle = \sum_i \mu_i^2$. In this way we obtain $\Omega_g^{\mathrm{Moz}} \in R_g[[T, z]]$. We apply ev_C to define the motivic Mozgovoy function:

$$\Omega_g^{\mathrm{MotMoz}} = \mathrm{ev}_C(\Omega_g^{\mathrm{Moz}}) = \sum_{\mu\in\mathcal{P}} T^{|\mu|} (q/z)^{(g-1)\langle\mu,\mu\rangle} \prod_{\square\in\mu} \zeta_C(q^{-l(\square)-1} z^{h(\square)}),$$

where $\zeta_C(z) = \mathrm{Exp}[Cz]$ is the motivic zeta function of C. Theorem 1.1 implies

$$\Omega_g^{\mathrm{Moz}} = \mathrm{Exp}\left[\frac{z/q \sum_{r=1}^{\infty} H_{g,r}(q/z, z, \alpha_1, \ldots, \alpha_g) T^r}{(1-z/q)(1-z)}\right].$$

Using the evaluations $H_{g,r}(q, 1, \ldots) = H_{g,r}(1, q, \ldots) = A_{g,r}(q, \ldots)$, we obtain

$$\begin{aligned}\frac{z/q H_{g,r}(q/z, z, \alpha_1, \ldots, \alpha_g)}{(1-z/q)(1-z)} &= \frac{1}{q-1}\left(\frac{1}{1-z} - \frac{1}{1-z/q}\right) H_{g,r}(q/z, z, \alpha_1, \ldots, \alpha_g)\\ &= \sum_{d=1}^{\infty} \frac{A_{g,r}(q, \alpha_1, \ldots, \alpha_g)}{q-1}(1-q^{-d}) z^d\\ &\quad + \text{Laurent polynomial in } z.\end{aligned}$$

So the coefficients do not stabilize like they do for $\mathrm{Log}\,\Omega^{\mathrm{Sch}}$, but nevertheless tend to $\frac{A_{g,r}}{q-1}$ in $\overline{R_g}$ as d goes to ∞. Therefore in the product expansion of the ratio

$$\frac{\Omega^{\mathrm{Sch}}(T, z, q, \alpha_1, \ldots, \alpha_g)}{\Omega^{\mathrm{Moz}}(T, z, q, \alpha_1, \ldots, \alpha_g)} = \prod_{\tau\in\mathbb{Q}} \sum_{d/r=\tau} D_{r,d} T^r z^d$$

the coefficients $D_{r,d} \in \overline{R_g}$ tend to 0 when r is fixed and d goes to ∞. Applying ev_C to both sides we conclude that

$$\frac{\Omega^{\mathrm{Mot}}(T, z)}{\Omega^{\mathrm{MotMoz}}(T, z)} = \prod_{\tau\in\mathbb{Q}} \sum_{d/r=\tau} D_{r,d}^{\mathrm{Mot}} T^r z^d,$$

where $D_{r,d}^{\mathrm{Mot}}$ tend to 0 in $\overline{\mathrm{Mot}}$ when r is fixed and d goes to ∞. So constructing $H_{r,d}^{\mathrm{MotMoz}}$ from Ω^{MotMoz} in the same way as $H_{r,d}^{\mathrm{Mot}}$ was constructed from Ω^{Mot}

in [4], we obtain for $\tau \in \mathbb{Q}$

$$\frac{\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2} H^{\mathrm{Mot}}_{r,d+er} T^r}{\sum_{d/r=\tau} \mathbb{L}^{(1-g)r^2} H^{\mathrm{MotMoz}}_{r,d+er} T^r} \to 1 \qquad (e \to \infty)$$

coefficientwise. This immediately implies a version of [4] for Mozgovoy's function:

Corollary 6.1 *For any r, d we have*

$$[\mathcal{M}^{ss}_{r,d}] = \lim_{e\to\infty} H^{\mathrm{MotMoz}}_{r,d+er} \in \overline{\mathrm{Mot}}.$$

Acknowledgements Open access funding provided by University of Vienna. Olivier Schiffmann drew my attention to his paper [17], suggested to try to deduce the conjectures of Hausel–Rodriguez-Villegas and pointed out that the conjecture of Hausel–Thaddeus also follows. He also made several corrections in a preliminary version of this paper. Fernando Rodriguez-Villegas experimented with Schiffmann's formula on a computer, and I had interesting conversations with him on this subject. I would like to thank them and Tamas Hausel for encouragement. I thank Sergey Galkin for answering my questions about the Grothendieck ring of varieties, Yan Soibelman and Roman Fedorov for bringing my attention to their work and interesting discussions. I would like to thank the organizers of the workshops on Higgs bundles at EPFL Lausanne in 2016 and SISSA Trieste in 2017. I discovered some of the ideas leading to the result of this work during these workshops. The main body of this work was performed during my stay at IST Austria, where I was supported by the Advanced Grant "Arithmetic and Physics of Higgs moduli spaces" No. 320593 of the European Research Council. This work was partially supported by the Austrian Science Fund (FWF) Projects Y963-N35 and P-31705.

References

1. Chuang, W., Diaconescu, D.-E., Donagi, R., Pantev, T.: Parabolic refined invariants and Macdonald polynomials. Commun. Math. Phys. **335**(3), 1323–1379 (2015)
2. Carlsson, E., Okounkov, A.: Exts and vertex operators. Duke Math. J. **161**(9), 1797–1815 (2012)
3. Carlsson, E., Rodriguez-Villegas, F.: Vertex operators and character varieties. Adv. Math. **330**, 38–60 (2018)
4. Fedorov, R., Soibelman, A., Soibelman, Y.: Motivic classes of moduli of Higgs bundles and moduli of bundles with connections. Commun. Number Theory Phys. **12**(4), 687–766 (2018)

5. Göttsche, L.: On the motive of the Hilbert scheme of points on a surface. Math. Res. Lett. **8**(5–6), 613–627 (2001)
6. Garsia, A.M., Haiman, M., Tesler, G.: Explicit plethystic formulas for Macdonald q, t-Kostka coefficients. Sém. Lothar. Combin. **42**, Art. B42m, 45 (1999). The Andrews Festschrift (Maratea, 1998)
7. Hausel, T.: Mirror symmetry and Langlands duality in the non-abelian Hodge theory of a curve. In: Bogomolov, F., Tschinkel, Y. (eds.) Geometric methods in algebra and number theory. Progress in Mathematics, vol. 235, pp. 193–217. Birkhäuser, Boston (2005)
8. Hausel, T., Letellier, E., Rodriguez-Villegas, F.: Arithmetic harmonic analysis on character and quiver varieties. Duke Math. J. **160**(2), 323–400 (2011)
9. Hausel, T., Rodriguez-Villegas, F.: Mixed Hodge polynomials of character varieties. Invent. Math. **174**(3), 555–624 (2008)
10. Kapranov, M.: The elliptic curve in the S-duality theory and Eisenstein series for Kac–Moody groups (2000). arXiv preprint arXiv:math/0001005
11. Larsen, M., Lunts, V.A.: Rationality criteria for motivic zeta functions. Compos. Math. **140**(6), 1537–1560 (2004)
12. Larsen, M., Lunts, V.: Irrationality of motivic zeta functions (2018). arXiv preprint arXiv:1802.03661
13. Mellit, A.: Poincare polynomials of character varieties, Macdonald polynomials and affine Springer fibers (2017). arXiv preprint arXiv:1710.04513
14. Mellit, A.: Integrality of Hausel–Letellier–Villegas kernels. Duke Math. J. **167**(17), 3171–3205 (2018)
15. Mozgovoy, S.: Solutions of the motivic ADHM recursion formula. Int. Math. Res. Not. **18**, 4218–4244 (2012)
16. Mozgovoy, Sergey, Schiffmann, Olivier: Counting Higgs bundles, (2014) arXiv preprint arXiv:1411.2101
17. Schiffmann, O.: Indecomposable vector bundles and stable Higgs bundles over smooth projective curves. Ann. Math. (2) **183**(1), 297–362 (2016)