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Abstract

Consider the problem of estimating a common mean of two independent normal distributions, each with unknown variances. Note that the problem of recovery of interblock information in balanced incomplete blocks designs is such a problem. Suppose a random sample of size m is drawn from the first population and a random sample of size n is drawn from the second population. We first show that the sample mean of the first population can be improved on (with an unbiased estimator having smaller variance), provided $m \geqq 2$ and $n \geqq 3$. The method of proof is applicable to the recovery of information problem. For that problem, it is shown that interblock information could be used provided $b \geqq 4$. Furthermore for the case $b = t = 3$, or in the common mean problem, where $n = 2$, it is shown that the prescribed estimator does not offer improvement. Some of the results for the common mean problem are extended to the case of K means. Results similar to some of those obtained for point estimation, are also obtained for confidence estimation.

Keywords

common mean, unbiased estimators, balanced incomplete blocks designs, inadmissibility, interblock information, confidence intervals

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POINT AND CONFIDENCE ESTIMATION OF A COMMON MEAN AND RECOVERY OF INTERBLOCK INFORMATION

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Consider the problem of estimating a common mean of two independent normal distributions, each with unknown variances. Note that the problem of recovery of interblock information in balanced incomplete blocks designs is such a problem. Suppose a random sample of size m is drawn from the first population and a random sample of size n is drawn from the second population. We first show that the sample mean of the first population can be improved on (with an unbiased estimator having smaller variance), provided $m \geqq 2$ and $n \geqq 3$. The method of proof is applicable to the recovery of information problem. For that problem, it is shown that interblock information could be used provided $b \geqq 4$. Furthermore for the case $b = t = 3$, or in the common mean problem, where $n = 2$, it is shown that the prescribed estimator does not offer improvement.

Some of the results for the common mean problem are extended to the case of K means. Results similar to some of those obtained for point estimation, are also obtained for confidence estimation.

1. Introduction and summary. The problem of estimating a common mean of two normal distributions and the related problem of recovery of interblock information has been studied in several papers. Yates [9] was apparently the first to suggest that information could be recovered in balanced incomplete block designs (BIBD). Graybill and Deal [2] showed, for the common mean problem, that an unbiased estimator with a smaller variance than either sample mean could be found, provided both sample sizes are greater than 10. For the recovery of information problem, Graybill and Deal claim that interblock information should always be used provided either

(i) $f = rt - b - t + 1 \geqq 18$, (where b = number of blocks, t = number of treatments, r = number of replications) and $b - t = 9$, or

(ii) $(b - t) \geqq 10$.

Zacks [11] offers Bayes and Fiducial equivariant estimators of the common mean and Zacks [10] studies this problem for small sample sizes. Seshadri [5] proved that interblock information should be used provided $t \geqq 9$. Shah [6] improved this result, in a sense, and showed interblock information should be recovered it $t \geqq 6$, and Stein [8] showed recovery should be made, provided

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$t \geqq 4$. Seshardri, Shah, and Stein all use estimators of the same type and these are studied numerically by Shah [7].

Recently Cohen and Sackrowitz [1] obtained results on the common mean problem. They proved for the equal sample size case, that the sample mean of the first population could be improved on provided the sample size is greater than 4. They also showed, either sample mean can be improved on, provided the sample size is at least 10. In addition, new estimators were obtained which not only had some optimality properties, but were such that an explicit non-trivial bound could be given for the variance of such estimators.

In this paper we first show that the sample mean of the first population can be improved on, provided the sample size in that population is greater than 1 and the sample size in the other population is greater than 2. The method of proof is applicable to the recovery of information problem. For that problem, it is shown that interblock information could be used provided $b \geqq 4$. In fact the result shows that the only practical BIBD, for which interblock information should not be recovered in the prescribed manner, is the case where $b = t = 3$. Furthermore, for the case $b = t = 3$, or in the common mean problem, where the sample size for the second population is 2, it is shown that the prescribed estimator does not offer improvement. This proof can be used to show other recommended estimators also do not offer uniform improvement.

Shah [7] raises the question of whether Yates type estimators are superior to the mean based on only one sample. We essentially answer this question in the affirmative for appropriate degrees of freedom.

Some of the results for the common mean problem are extended to the case of K means. That is, for example, suppose there are three normal populations with unknown common mean and different unknown variances. If the sample size for the second population is at least 6, then the data from that population can be used to improve on the sample mean of the first population (based on at least two observations). Further, if the sample size for the third population is at least 6, then the data from that population can be used to improve on the estimator based on the data from the first two populations.

Another result for the common mean problem (and recovery of information problem) concerns the confidence interval based on the t-statistic, computed from the sample on the first population. Suppose the sample size for the first population is at least 2, and for the second population, at least 6. Then another confidence interval is given, which has the same length as the interval based on "t", but which has greater probability of coverage, uniformly in the mean and the pair of unknown variances.

One may question the value, other than theoretical, of coming up with an unbiased estimator which is better than the sample mean based on a single population. The sample mean based on a single population does not appear to be a very good estimator to begin with, especially for large or moderate size samples. There are several rebuttals. First, one hopes to suggest a good estimator using both samples. In most cases it is difficult to obtain the variance of the recommended estimator. If it is demonstrated that the estimator is better than the sample mean, that implies its variance is bounded by the variance of the mean.

Furthermore, and of significant importance, the estimator which is better than both sample means is an appropriate candidate to be used for recovery of interblock information. (See aforementioned references.) Still further, for small sample sizes a sample mean *is* a contending estimator because of the difficulty in finding an estimator which beats it uniformly. In some situations the experimenter has a choice of taking a second sample or not, depending also perhaps on the costs of observations. The result here, stating when an improved combined estimator is available, provides guidance for such a choice.

The advantage of improving on a confidence interval based on the sample mean of the first population is as follows. The new interval will match the given interval in length and will have coverage probability at least equal to the coverage probability of the given interval. Since it is very difficult to compute coverage probabilities for all but the simplest type of confidence interval, the lower bound on the probability of coverage for the new confidence interval is a bonafide confidence coefficient. Although one can suggest many confidence intervals which appear sensible, it would be quite possible that many would have coverage probabilities that dipped below a desired level for some values in the parameter space.

In the next section we state the model for the two population common mean problem and present the point estimation results. In Section 3 we discuss applications to the BIBD model. In Section 4, results for K means are shown. Finally, in Section 5, the confidence interval result is given.

2. Point estimation for two populations. Let $X_1, X_2, \cdots, X_m$ be a random sample from a normal population with unknown mean θ and unknown variance σ_x^2. Let $\bar{X}$, s_x^2, and $s_{\bar{x}}^2$ denote the sample mean, sample variance and sample variance of $\bar{X}$, respectively. That is, $\bar{X} = \sum X_i/m$, $s_x^2 = \sum (X_i - \bar{X})^2/(m-1)$, and $s_{\bar{x}}^2 = s_x^2/m$. Let $Y_1, Y_2, \cdots, Y_n$ be a random sample from a normal population with unknown mean θ and unknown variance σ_y^2. The Y sample is independent of the X sample. Define $\bar{Y}$, s_y^2, and $s_{\bar{y}}^2$ in analogy with their counterparts in the X sample. We assume $m \geqq 2$ and $n \geqq 2$. The problem is to estimate the common mean θ. We evaluate the merits of an estimator by its mean square error. We seek estimators which improve uniformly on $\bar{X}$.

To start, let $\tau = \sigma_{\bar{y}}^2/\sigma_{\bar{x}}^2$ and note that any translation equivariant estimate is of the form

$$\bar{X} + \gamma(\bar{Y} - \bar{X}, s_x^2, s_y^2)\,, \tag{2.1}$$

where γ is a real function of $\bar{Y} - \bar{X}, s_x^2, s_y^2$. Let $\gamma \equiv \gamma(\bar{Y} - \bar{X}, s_x^2, s_y^2)$. We prove

LEMMA 2.1. *The estimator* (2.1) *is better than* $\bar{X}$ *if and only if*

$$(1 + \tau)E\gamma^2 \leqq 2E[(\bar{Y} - \bar{X})\gamma] \tag{2.2}$$

with strict inequality for at least one point in the parameter space.

PROOF. The mean square error of (2.1) is

$$\sigma_{\bar{x}}^2 + E\gamma^2 + 2E(\bar{X} - \theta)\gamma \,. \tag{2.3}$$

But

$$E[(\bar{X} - \theta)\gamma \mid (\bar{Y} - \bar{X}), s_x^2, s_y^2] = -[\sigma_{\bar{x}}^2/(\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2)](\bar{Y} - \bar{X})\gamma \,. \tag{2.4}$$

Use (2.3), (2.4) and the definition of τ to complete the proof of the lemma.

Now let

$$\gamma = (\bar{Y} - \bar{X})r((\bar{Y} - \bar{X})^2, s_x^2, s_y^2) \,, \tag{2.5}$$

where r is a real function and let W^* be a χ^2 variable with 3 degrees of freedom, independent of (s_x^2, s_y^2). Then we prove

LEMMA 2.2. *The estimator in* (2.1) *with* γ *given in* (2.5) *is better than* $\bar{X}$ *if and only if*

$$(1 + \tau)Er^2([\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2]W^*, s_x^2, s_y^2) \leqq 2Er([\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2]W^*, s_x^2, s_y^2) \,, \tag{2.6}$$

with strict inequality for at least one point in the parameter space.

PROOF. Let $W = (\bar{Y} - \bar{X})^2/(\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2)$, so that W is a χ^2 variable with one degree of freedom, independent of (s_x^2, s_y^2). Use Lemma 2.1 and (2.2) becomes

$$(1 + \tau)E[Wr^2([\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2]W, s_x^2, s_y^2)] \leqq 2E[Wr([\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2]W, s_x^2, s_y^2)] \,. \tag{2.7}$$

Note that

$$E[Wr([\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2]W, s_x^2, s_y^2)] = CEr([\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2]W^*, s_x^2, s_y^2) \,, \tag{2.8}$$

where C is a positive constant and W^* is a χ^2 variable with 3 degrees of freedom, independent of (s_x^2, s_y^2). The term on the left-hand side of (2.7) can be treated as in (2.8) and thus using (2.8) we see that (2.7) is equivalent to (2.6). This completes the proof of the lemma.

When $n \geqq 3$, consider estimators of the form

$$\begin{aligned} T_a = \bar{X} + (\bar{Y} - \bar{X})\{as_{\bar{x}}^2/[s_{\bar{x}}^2 + (n - 1)(s_{\bar{y}}^2/(n + 2)) \\ + (\bar{Y} - \bar{X})^2/(n + 2)]\} \,. \end{aligned} \tag{2.9}$$

We prove

THEOREM 2.1. *For* $n \geqq 3$ *any estimator* T_a *is unbiased and better than* $\bar{X}$ *if* $0 < a \leqq a_{\max}$ *where*

$$a_{\max} = a_{\max}(m, n) = 2(n + 2)/\{nE(\max V^{-1}, V^{-2})\} \,, \tag{2.10}$$

where V *has an* F *distribution with* $(n + 2)$ *and* $(m - 1)$ *degrees of freedom.*

(Some values of $a_{\max}$ are given in Table 1. See also Remarks 2.1.)

TABLE 1
Values of $a_{\max}(m, n)$

n	m = 2	3	4	5	6	7	8	9	10	12	14	16	18	20	30	40	50	100	∞
3	.1329	.1989	.2383	.2646	.2834	.2975	.3805	.3172	.3244	.3354	.3436	.3498	.3547	.3586	.3708	.3770	.3808	.3900	.3975
4	.2208	.3298	.3949	.4382	.4691	.4923	.5103	.5248	.5366	.5548	.5682	.5784	.5865	.5931	.6132	.6236	.6299	.6500	.6660
5	.2833	.4224	.5052	.5602	.5995	.6291	.6520	.6704	.6855	.7088	.7259	.7389	.7493	.7577	.7835	.7967	.8048	.8400	.8570
6	.3299	.4911	.5869	.6506	.6960	.7302	.7568	.7781	.7955	.8225	.8423	.8575	.8696	.8793	.9094	.9248	.9343	.9800	1.0000
7	.3660	.5441	.6498	.7200	.7701	.8078	.8372	.8607	.8800	.9098	.9317	.9486	.9619	.9728	1.0061	1.0233	1.0338	1.0800	1.1110
8	.3947	.5862	.6997	.7750	.8288	.8692	.9008	.9261	.9468	.9789	1.0025	1.0207	1.0351	1.0467	1.0828	1.1014	1.1128	1.1600	1.2000
9	.4181	.6205	.7402	.8196	.8764	.9191	.9523	.9790	1.0010	1.0349	1.0599	1.0791	1.0944	1.1067	1.1450	1.1649	1.1770	1.2400	1.3726
10	.4376	.6489	.7737	.8565	.9157	.9602	.9949	1.0228	1.0457	1.0811	1.1073	1.1274	1.1434	1.1564	1.1966	1.2174	1.2302	1.3000	1.3334
12	.4681	.6932	.8260	.9140	.9769	1.0242	1.0612	1.0909	1.1153	1.1531	1.1811	1.2026	1.2198	1.2337	1.2769	1.2994	1.3133	1.4000	1.4284
14	.4909	.7262	.8648	.9567	1.0223	1.0717	1.1103	1.1414	1.1669	1.2065	1.2358	1.2585	1.2765	1.2911	1.3367	1.3605	1.3752	1.4600	1.5000
16	.5086	.7517	.8948	.9895	1.0573	1.1083	1.1482	1.1803	1.2067	1.2477	1.2781	1.3015	1.3202	1.3355	1.3830	1.4079	1.4232	1.5200	1.5556
18	.5227	.7720	.9186	1.0157	1.0851	1.1374	1.1782	1.2111	1.2283	1.2803	1.3116	1.3357	1.3550	1.3707	1.4198	1.4456	1.4616	1.5600	1.6000
20	.5342	.7886	.9380	1.0369	1.1077	1.1610	1.2027	1.2362	1.2639	1.3069	1.3389	1.3636	1.3833	1.3994	1.4499	1.4765	1.4929	1.6000	1.6364
30	.5100	.8399	.9980	1.1026	1.1775	1.2339	1.2781	1.3137	1.3431	1.3890	1.4232	1.4497	1.4710	1.4884	1.5434	1.5726	1.5909	1.7000	1.7500
40	.5886	.8666	1.0291	1.1366	1.2135	1.2715	1.3170	1.3537	1.3840	1.4314	1.4668	1.4943	1.5164	1.5346	1.5922	1.6231	1.6426	1.7600	1.8094
50	.6001	.8829	1.0481	1.1573	1.2355	1.2944	1.3407	1.3781	1.4090	1.4573	1.4934	1.5216	1.5442	1.5629	1.6222	1.6543	1.6745	1.8000	1.8460
100	.6236	.9163	1.0869	1.1996	1.2802	1.3412	1.3890	1.4277	1.4598	1.5100	1.6420	1.6950	1.7192	1.7385	1.7975	1.8278	1.8462	1.8000	1.9214
∞	.6494	.9708	1.1383	1.3000	1.4000	1.5000	1.5556	1.6000	1.6364	1.6924	1.7334	1.7648	1.7894	1.8094	1.8708	1.9024	1.9214	1.9604	2

PROOF. By symmetry T_a is unbiased. From Lemma 2.2 we must show

$$(2.11)\quad a^{-1} \geqq \frac{[(1+\tau)/2]E\{(s_{\bar{x}}^2/[s_{\bar{x}}^2+(n-1)(s_{\bar{y}}^2/(n+2))+(\sigma_{\bar{x}}^2+\sigma_{\bar{y}}^2)W^*/(n+2)])^2\}}{E\{s_{\bar{x}}^2/[s_{\bar{x}}^2 + (n-1)(s_{\bar{y}}^2/(n+2)) + (\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2)W^*/(n+2)]\}} .$$

Let

$$V = \sigma_{\bar{x}}^2[(n-1)s_{\bar{y}}^2 + \sigma_{\bar{y}}^2 W^*]/(n+2)\sigma_{\bar{y}}^2 s_{\bar{x}}^2 ,$$
$$U = \sigma_{\bar{y}}^2 W^*/[(n-1)s_{\bar{y}}^2 + \sigma_{\bar{y}}^2 W^*] .$$

V has the desired F distribution and is independent of U which has a beta distribution. The above inequality may be rewritten as

$$(2.12)\quad a^{-1} \geqq [(1+\tau)/2]E\{1/[1+\tau V + UV]^2\}/E\{1/[1+\tau V + UV]\} .$$

Note that

$$(2.13)\quad \begin{aligned}[(1+\tau)/(1+\tau V + UV)] &\leqq \max[1/(1+UV), 1/V] \\ &\leqq \max[1, 1/V] = h(V) .\end{aligned}$$

Hence it suffices to have

$$(2.14)\quad a^{-1} \geqq E(h(V)/[1+\tau V + UV])/2E(1/[1+\tau V + UV]) .$$

Now,

$$(2.15)\quad \frac{E(h(V)/[1+\tau V + UV])}{E(1/[1+\tau V + UV])} \leqq \sup_{\tau,u} \left\{ \frac{E\left(\frac{h(V)}{1+\tau V + UV} \middle| U = u\right)}{E\left(\frac{1}{1+\tau V + UV} \middle| U = u\right)} \right\} .$$

Compute the derivative of the bracketed term on the r.h.s. of (2.15) and find that the derivative with respect to τ is greater than or equal to 0 since $V/(1 + \tau V + UV)$ is increasing in V for given U and $h(V)$ is non-increasing in V. Hence for each fixed $U = u$ the supremum on the right side of (2.15) occurs when $\tau \to \infty$. This supremum is easily seen to be $E(h(V)V^{-1} | U)/E(V^{-1} | U)$, which is independent of U. Combining (2.15) and (2.14) it follows that it suffices to choose a as in the statement of the theorem.

REMARK 2.1. In many applications it is reasonable to feel that $\tau = \sigma_{\bar{y}}^2/\sigma_{\bar{x}}^2$ is large. This is usually the case in the application to the recovery of interblock information discussed in Section 3. When this is so it is most reasonable to use the estimator T_a for which $a \leqq a_{\max}$ and for which

$$(2.16)\quad \lim_{\tau\to\infty} [\tau(\operatorname{Var}_\tau \bar{X} - \operatorname{Var}_\tau T_a)/(\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2)]$$

is maximized. Computations like the preceding show that (2.16) is maximized for the value $a_0 = a_0(m, n) = E(V^{-1})/E(V^{-2}) = (m-1)(n-2)/(m+1)(n+2)$. Note that $a_0 < a_{\max}$; and so, except for small values of m, n, T_{a_0} is a reasonable estimator of the form T_a to use for most situations.

Before proceeding we note also that $a_{\max}/2 \leqq a_0 \leqq 1$, and that $a_{\max} \geqq 1$ for many values of m and n. This last remark essentially answers a question raised in Shah [7], page 1562.

REMARK 2.2. Although $a_{\max} > 1$ for many values of m and n, we do not recommend choosing values greater than 1 for a in T_a.

REMARK 2.3. If $\sigma_{\bar{y}}^2 > \sigma_{\bar{x}}^2$ (that is, $\tau > 1$) and (2.6) holds then the estimator in (2.1) with γ given in (2.5) is better than $\bar{X}$ and $\bar{Y}$ simultaneously.

REMARK 2.4. It can be shown that there exist values of $a > 0$ such that estimators of the form

$$T_a(p) = \bar{X} + (\bar{Y} - \bar{X})\{as_{\bar{x}}^2/[p(s_{\bar{x}}^2 + s_{\bar{y}}^2) + (1-p)(\bar{Y} - \bar{X})^2]\}, \tag{2.17}$$

$0 < p < 1$, are better than $\bar{X}$ for $m \geqq 2$, $n \geqq 3$.

When $p = 1$, we prove

THEOREM 2.2. *The estimator* $T_a(1) = \bar{X} + (\bar{Y} - \bar{X})\{as_{\bar{x}}^2/[s_{\bar{x}}^2 + s_{\bar{y}}^2]\}$ *is better than* $\bar{X}$ *for* $m \geqq 2$, *and* $n \geqq 6$ *whenever* $0 < a < a_{\max}(m, n-3)$.

PROOF. The proof is similar to the proof of Theorem 2.1.

REMARK 2.5. As in Remark 2.1 a satisfactory value of a to use in an estimator of the form $T_a(1)$ in some circumstances is $a_0' = a_0(m, n-3)$, as long as $a_0' \leqq a_{\max}(m, n-3)$.

Concerning the relative merits of estimators of the form T_a and of the form $T_a(1)$ note the following: $T_a(1)$ can improve on $\bar{X}$ only if $n \geqq 6$ whereas T_a can improve on $\bar{X}$ for $n \geqq 3$. Also, even for $n \geqq 6$

$$\begin{aligned}\lim_{\tau\to\infty} [\tau(\mathrm{Var}_\tau T_{a_0'}(1) - \mathrm{Var}_\tau T_{a_0})/(\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2)] \\ = C\sigma_{\bar{x}}^2(a_0^2(m, n) - a_0^2(m, n-3)) > 0 .\end{aligned} \tag{2.18}$$

However, the difference on the right of (2.18) is small even for moderately large values of n. On the other hand, for fixed $\sigma_{\bar{x}}^2$, $\lim_{\tau\to 0} \mathrm{Var}_0 T_{a_0'}(1) = (1 - a_0')^2\sigma_{\bar{x}}^2$. Since a_0' is near 1 for moderately large values of n, this quantity may be a small multiple of $\sigma_{\bar{x}}^2$. Meanwhile

$$\lim_{\tau\to 0} \mathrm{Var}_\tau T_{a_0} = E\{(\bar{X} - \theta)(1 - a_0)/(1 + [(\bar{X} - \theta)^2/s_{\bar{x}}^2])\}^2 .$$

No matter what a_0 is, this latter quantity is bigger than a positive multiple of $\sigma_{\bar{x}}^2$. Hence for moderately large values of n $T_{a_0'}(1)$ is noticeably better than T_{a_0} for small values of τ.

In summary, for general purposes it appears usually preferable to use T_{a_0} for smaller values of n and $T_{a_0'}(1)$ for larger values of n.

We now demonstrate the following

THEOREM 2.3. *Let* $n = 2$, $m \geqq 2$. *Then the estimator* T_a, *given in* (2.9) *is not better than* $\bar{X}$ *for any value of* a.

PROOF. We appeal to Lemma 2.2 and rewrite (2.11) as

$$\begin{aligned}a(1+\tau)E\{(s_{\bar{x}}^2/[s_{\bar{x}}^2 + (n-1)(s_{\bar{y}}^2/(n+2)) + (\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2)W^*/(n+2)])^2\} \\ \leqq 2E\{s_{\bar{x}}^2/[s_{\bar{x}}^2 + (n-1)(s_{\bar{y}}^2/(n+2)) + (\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2)W^*/(n+2)]\} .\end{aligned} \tag{2.19}$$

It is easy to show that the product of $(1 + \tau)$ and the r.h.s. of (2.19) is uniformly bounded, while the limit as $\tau \to \infty$ of the product of $(1 + \tau)$ and the l.h.s. of (2.19) is infinite. This proves the theorem.

We conclude this section with the following remarks:

REMARK 2.6. The proof of Theorem 2.2 can be used to show other estimators do not improve on $\bar{X}$ when $n = 2$. For example, the estimator $T_a(p)$, as given in (2.17).

REMARK 2.7. Suppose $\delta(\bar{X}, s_{\bar{x}}^2)$ is any estimator of θ, which is admissible with respect to squared error loss when there are no observations from the Y population. An interesting question is, suppose there are now n observations on the Y population; then does $\delta(\bar{X}, s_{\bar{x}}^2)$ become inadmissible? We can only answer this question for some $\delta(\bar{X}, s_{\bar{x}}^2)$. In particular, the class of estimators $\alpha\bar{X}$, $0 < \alpha \leqq 1$, can all be improved on by αT_a, where T_a is given in (2.9) under the conditions of Theorem 2.1. This follows by examining the bias and variance of $\alpha\bar{X}$ and αT_a, and then applying Theorem 2.1.

REMARK 2.8. The methods of this section can be used to show that $\bar{X}$ is inadmissible for other convex loss functions, under appropriate conditions on sample sizes and existence of moments.

3. Recovery of interblock information. Graybill and Weeks [3] give, in a sense, a canonical form for a BIBD with blocks and errors random. The canonical form is as follows: Let t = number of treatments, b = number of blocks, r = number of replications, k = number of cells per block, and λ = number of times any pair of treatments appears in the same block. Recall $f = rt - b - t + 1$. The $(t-1) \times 1$ vector $U = (u_i)$ is distributed normally with mean $\mu = (\mu_i)$ and covariance matrix $(k/\lambda t)\sigma^2 I$. The $(t-1) \times 1$ vector $Z = (z_i)$ is distributed normally with mean μ and covariance matrix $[k/(r-\lambda)](\sigma^2 + k\sigma_\beta^2)I$. The scalar S^2/σ^2 is distributed as a chi-square variate with f degrees of freedom. The scalar $S^{*2}/(\sigma^2 + k\sigma_\beta^2)$ is distributed as a chi-square variate with $(b-t)$ degrees of freedom. The scalar $y\cdots$ is distributed normally with mean ν and variance $(\sigma^2 + k\sigma_\beta^2)/bk$. The statistics $u_1, u_2, \cdots, u_{t-1}, z_1, z_2, \cdots, z_{t-1}, S^2, S^{*2}, y\cdots$ are mutually independent.

Consider the problem of estimating μ_i (μ_i represents a treatment contrast) and let us make the following match ups ($\sim$) with terms used in Section 2:

$$\bar{X} \sim u_i\,, \qquad \sigma_x^2 \sim k\sigma^2\,, \qquad m \sim \lambda t\,, \qquad \sigma_{\bar{x}}^2 \sim \sigma_{u_i}^2\,, \qquad \bar{Y} \sim z_i\,,$$
$$\sigma_y^2 \sim k(\sigma^2 + k\sigma_\beta^2)\,, \qquad n \sim (r-\lambda)\,, \qquad \sigma_{\bar{y}}^2 \sim \sigma_{z_i}^2\,.$$

Note $\tau = \lambda t(\sigma^2 + k\sigma_\beta^2)/(r-\lambda)\sigma^2$. Since $k \geqq 2$, it will follow that $\tau \geqq 1$. To see this, use $\lambda = r(k-1)/(t-1)$ and note $\lambda t \geqq (r-\lambda)$ is equivalent to $k(t+1) \geqq 2t$. To apply the results of Section 2 we note that the derivation of a, and hence the theorems, are in terms of the degrees of freedom. In Section 2 sample sizes determine degrees of freedom. However, here the degrees of freedom related to S^2 and S^{*2} are f and $(b - t)$ respectively.

Using Theorem 2.1 and Remark 2.3 it follows that for $0 < a \leqq a_{\max}(f + 1, b - t + 1)$ the estimator

$$\hat{\mu}_a = u_i + \frac{(z_i - u_i)a(k/\lambda t)(S^2/f)}{[(k/\lambda t)(S^2/f) + (k/(r-\lambda))(S^{*2}/(b-t+3)) + (z_i - u_i)^2/(b-t+3)]} \tag{3.1}$$

is better than both u_i or z_i for $(b - t) \geqq 2$. As in Section 2, $a = a_0(f + 1, b - t + 1)$ is a reasonable general choice for a whenever $a_0 \leqq a_{\max}$. (Actually, since it is known a priori that $\tau \geqq 1$, a slightly larger value may be substituted for $a_{\max}$ in Theorem 2.1. Namely

$$a''_{\max}(f + 1, b - t + 1) = 2E(V^{-1})/E\{\max([2/V(1 + U)], 1/V^2)\},$$

where V has an F distribution with $b - t + 3$, and f degrees of freedom. This can be seen by improving the bound in (2.15) in the proof of Theorem 2.1.)

Another interesting possibility is the estimator

$$\begin{aligned}\hat{\mu}_a(1) = u_i + (z_i - u_i)\{a(k/\lambda t)(S^2/f)/[(k/\lambda t)(S^2/f) \\ + (k/(r - \lambda))S^{*2}/(b - t)]\}\end{aligned} \tag{3.2}$$

which is better than both u_i and z_i for $a \leqq a_{\max}(f + 1, b - t - 2)$ as long as $b - t \geqq 5$. Again a reasonable general choice for a here is $a = a_0(f + 1, b - t - 2)$ when $a_0 \leqq a''_{\max}$.

Perhaps more interesting is consideration of the estimator

$$\begin{aligned}\hat{\mu}_a^* = u_i + (z_i - u_i)\{a(k/\lambda t)(S^2/f)/[(k/\lambda t)(S^2/f) \\ + (k/(r - \lambda))(S^{*2}/(b + 1)) + (U - Z)'(U - Z)/(b + 1)]\}.\end{aligned} \tag{3.3}$$

We state the formal result, whose proof is analogous to the preceding results.

THEOREM 3.1. *Let $b \geqq 4$. Then the estimator $\hat{\mu}_a^*$ is better than u_i or z_i for*

$$a \leqq a''_{\max}(f + 1, b - 1). \tag{3.4}$$

(Since $a''_{\max} > a_{\max}$, conservative maximum values of a may be found by referring to Table 1. Again, a reasonable choice for a is $a = a_0(f + 1, b - 1)$ as long as $a_0 \leqq a''_{\max}$.)

Note that $\hat{\mu}_a^*$ is a minor modification of the estimate considered by Graybill and Weeks [3] page 804, Equation (6) and Scheffé [4] page 175, using formula (5.2.45). The modification consists in alterations of the constants which are here given as a, $k/\lambda t$, $k/(r - \lambda)(b + 1)$, and $1/(b + 1)$. Modifications of our methods can be applied to the estimators given by Graybill and Weeks and by Scheffé, but the form $\hat{\mu}_a^*$, which we have chosen, is both more convenient and more natural from our point of view.

Using the proof of Theorem 2.3, one can establish that the estimate of the form $\hat{\mu}_a^*$ (with any $a > 0$, including $a = 1$), is not better than u_i if $b < 4$. Hence the variation of Yates's estimator which appears in Graybill and Weeks [3] and Scheffé [4] is not better than u_i when $b = t = 3$.

REMARK 3.1. Theorem 3.1 is interesting in light of a commentary by Stein [8], page 352. His method and that of Shah [6] seem to be more appropriate when the number of treatments is large. The result here is more in line with that of Yates which appears relevant when the number of blocks is large.

4. Point estimation for K populations. In this section we consider the problem of estimating the common mean of K normal populations. That is, let X_{ij}, $i = 1, 2, \cdots, K$, $j = 1, 2, \cdots, n_i$ be independent normal variates with unknown mean θ and unknown variances σ_i^2. Let $\bar{X}_i$, s_i^2, $i = 1, 2, \cdots, K$ denote the sample means and variances respectively. From Theorem 2.1 and equation (2.11) we know that for $n_1 \geqq 2$, $n_2 \geqq 3$,

$$T_{a_{12}} = \bar{X}_1 + (\bar{X}_2 - \bar{X}_1)\{as_{\bar{x}_1}^2/[s_{\bar{x}_1}^2 + (n-1)(s_{\bar{x}_1}^2/(n+2)) + (\bar{X}_2 - \bar{X}_1)^2/(n+2)]\}\,, \tag{4.1}$$

where a_{12} is to be determined from (2.12) (with m replaced by n_1 and n replaced by n_2) has a smaller variance than $\bar{X}_1$. From Theorem 2.2 we also know that for $n_1 \geqq 2$, $n_2 \geqq 6$,

$$T_{a_{12}}(1) = \bar{X}_1 + (\bar{X}_2 - \bar{X}_1)\{a_{12}s_{\bar{x}_1}^2/(s_{\bar{x}_1}^2 + s_{\bar{x}_2}^2)\}\,, \tag{4.2}$$

where a_{12} is determined from $a_{\max}(n_1, n_2 - 3)$, has a smaller variance than $\bar{X}_1$. The question is can we use X_{ij}, $i = 3, \cdots, K$, somehow, to improve on $T_{a_{12}}$ or $T_{a_{12}}(1)$. We answer the question for $T_{a_{12}}(1)$. We assume for now that $K = 3$. The development for $K > 3$ will follow after we prove

THEOREM 4.1. *Let $n_1 \geqq 2$, $n_2 \geqq 6$, $n_3 \geqq 6$, and let $a_{12} < 1$. Then the estimator*

$$T_{a_{12},a_{13}}(1) = T_{a_{12}}(1) + (\bar{X}_3 - \bar{X}_1)\{a_{13}s_{\bar{x}_1}^2/(s_{\bar{x}_1}^2 + s_{\bar{x}_3}^2)\}\,, \tag{4.3}$$

where a_{13} is determined in the same manner as a_{12} save that it must be multiplied by $(1 - a_{12})$, is unbiased and has a smaller variance than $T_{a_{12}}(1)$.

PROOF. Unbiasedness is obvious. Let $\tau_{12} = (\sigma_{\bar{x}_2}^2/\sigma_{\bar{x}_1}^2)$ and $\tau_{13} = (\sigma_{\bar{x}_3}^2/\sigma_{\bar{x}_1}^2)$. Also let $S_i = (n_i - 1)(s_{\bar{x}_i}^2/\sigma_{\bar{x}_i}^2)$, $i = 1, 2, 3$. Express the variance of $T_{a_{12},a_{13}}(1)$ in terms of Var $T_{a_{12}}(1)$ plus three other terms. Divide the other terms by $a_{13}\sigma_{\bar{x}_1}^2$ and note that the proof of the theorem reduces to finding an a_{13} for which

$$\begin{aligned} a_{13}(1 + \tau_{13})E\{1/[1 + \tau_{13}(n_1 - 1)S_3/(n_3 - 1)S_1]^2\} \\ \leqq 2E\{1/[1 + \tau_{13}(n_1 - 1)S_3/(n_3 - 1)S_1]\} \\ - 2a_{12}E\{1/[1 + \tau_{13}(n_1 - 1)S_3/(n_3 - 1)S_1]\} \\ \times \{1/[1 + \tau_{12}(n_1 - 1)S_2/(n_2 - 1)S_1]\}\,. \end{aligned} \tag{4.4}$$

But since $0 < a_{12} < 1$, the r.h.s. of (4.4) exceeds

$$2E\{1/[1 + \tau_{13}(n_1 - 1)S_3/(n_3 - 1)S_1]\}\{(1 - a_{12})\}\,. \tag{4.5}$$

Now we may proceed as in the proof of Theorem 2.1 to establish that a_{13} is determined in the same manner as a_{12} (n_3 must be used instead of n_2) except that we must multiply the resulting number derived from $a_{\max}(n_1, n_3 - 1)$ call it b_{13} by $(1 - a_{12})$, That is $a_{13} = b_{13}(1 - a_{12})$. This completes the proof of the theorem.

For $K = 4$, we can improve on $T_{a_{12},a_{13}}(1)$, provided $b_{13} < 1$, by using the approach of the previous theorem. The constant $a_{14} = c_{14}(1 - a_{12} - a_{13})$, where c_{14} would be determined from $a_{\max}(n_1, n_4 - 1)$. We note that a_{14} is well defined, i.e., $(1 - a_{12} - a_{13}) > 0$. This follows since $a_{13} = b_{13}(1 - a_{12})$. Therefore $1 - a_{12} - b_{13}(1 - a_{12}) = (1 - b_{13})(1 - a_{12}) > 0$, since $b_{13} < 1$. For arbitrary K we would make successive improvements in the above manner. We establish the validity of the constants a_{1j}, $j = 1, 2, \cdots, K$ by appealing to the following

LEMMA 4.1. *Let* p_i, $i = 1, 2, \cdots, K$ *be* K *numbers. Let* $q_1 = p_1$, $q_i = p_i(1 - \sum_{j=1}^{i-1} q_j)$, $i = 2, \cdots, K$. *Then*

$$(1 - \sum_{i=1}^{K} q_i) = \prod_{i=1}^{K} (1 - p_i) . \tag{4.6}$$

PROOF. The proof is immediate by induction.

REMARK 4.1. The approach used in Theorem 4.1 does not appear to work for estimators of the type in (2.9). In fact the other methods cited in the references (see introduction) of obtaining improvement over $\bar{X}_1$, in the two population case, do not seem easily adaptable in demonstrating further improvement when there are three populations.

5. Improved confidence interval. Let us return to the model of Section 2. Then clearly a confidence interval for θ, with confidence coefficient $(1 - \alpha)$, is

$$\bar{X} \pm s_x t_{m-1}(\alpha)/m^{\frac{1}{2}} , \tag{5.1}$$

where $t_{m-1}(\alpha)$ is the two-tailed α critical value determined from Student's t-distribution with $(m - 1)$ degrees of freedom. In the theorem below, for $m \geqq 2$, $n \geqq 6$, we will find a confidence interval for θ, whose length matches the length of the interval in (5.1) and whose probability of coverage is uniformly (in θ, σ_x^2, σ_y^2) greater than $(1 - \alpha)$.

Recall $\tau = \sigma_{\bar{y}}^2/\sigma_{\bar{x}}^2$. Let $z = s_{\bar{y}}^2/s_{\bar{x}}^2$, $v = s_y^2\sigma_x^2/s_x^2\sigma_y^2$, $u = (m - 1)^{\frac{1}{2}}s_x/\sigma_x$, $w = u^2$, $t_{m-1}(\alpha) \equiv t(\alpha) \equiv t$, $t^* = t(\alpha)/(m - 1)^{\frac{1}{2}}$, $\Phi(x)$ denote the cdf of a standard normal, $G_\nu(x)$ denote the cdf of a chi-square variate with ν degrees of freedom. Note that w is distributed as a chi-square variate with $(m - 1)$ degrees of freedom, v is distributed as an F variate with $(n - 1)$ and $(m - 1)$ degrees of freedom and the joint density of w and v is

$$h(w, v) = Ce^{-w/2}e^{-wv/2}w^{[(m+n-2)/2]-1}v^{[(n-1)/2]-1} , \tag{5.2}$$

where $C = (1/2^{(m+n-2)/2})\Gamma((m - 1)/2)\Gamma((n - 1)/2)$.

Now let $r(z) = a/(1 + z)$, where a is a number such that $0 < a < 1 - \varepsilon$, for $0 < \varepsilon < 1$, and a satisfies relation

$$\begin{aligned} &[\Gamma((n - 5)/2)/2^{(m+3)/2}\Gamma((m - 1)/2)\Gamma((n - 1)/2)]a \\ &\quad \times \textstyle\int_0^\infty w^{(m+2)/2}e^{-w/2}G_{(n-5)}(aw)\, dw \\ &\qquad \leqq ((1 - a)/2^{\frac{3}{2}}) \textstyle\int_0^\infty \int_1^\infty w^{\frac{1}{2}}e^{-2t^{*2}w}(1/(1 + v)^3)h(w, v)\, dv\, dw \end{aligned} \tag{5.3}$$

where

$$G_{(n-5)}(aw) = \textstyle\int_0^{wa} e^{-t/2}t^{[(n-1)/2]-3}\, dt .$$

Consider the confidence interval

$$[\bar{X} + (\bar{Y} - \bar{X})r(z)] \pm s_x t_{m-1}(\alpha)/m^{\frac{1}{2}} . \tag{5.4}$$

We prove

THEOREM 5.1. *Let $m \geqq 2$, $n \geqq 6$. Then the confidence interval in* (5.4) *is better than the confidence interval in* (5.1), *in the sense that the intervals have the same length and the probability of coverage for* (5.4) *is uniformly greater than the probability of coverage for* (5.1).

PROOF. Clearly the lengths of the two intervals match. We must show

$$P_{\theta,\sigma_x^2,\sigma_y^2}\{|\bar{X} + (\bar{Y} - \bar{X})r(z) - \theta| \leqq s_{\bar{x}} t(\alpha)\} \geqq P_{\theta,\sigma_x^2,\sigma_y^2}\{|\bar{X} - \theta| \leqq s_{\bar{x}} t(\alpha)\} , \tag{5.5}$$

which is equivalent to

$$P_{0,\sigma_x^2,\sigma_y^2}\{|\bar{X} + (\bar{Y} - \bar{X})r(z)| \leqq s_{\bar{x}} t(\alpha)\} \geqq P_{0,\sigma_x^2,\sigma_y^2}\{|\bar{X}| \leqq s_{\bar{x}} t(\alpha)\} . \tag{5.6}$$

By conditioning on s_x^2, z, (5.6) becomes

$$\begin{aligned} E_{\sigma_x^2,\sigma_y^2}[\Phi\{ts_{\bar{x}}/(\sigma_{\bar{x}}^2 - 2r(z)\sigma_{\bar{x}}^2 + r^2(z)(\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2))^{\frac{1}{2}}\} \\ - \Phi\{-ts_{\bar{x}}/(\sigma_{\bar{x}}^2 - 2r(z)\sigma_{\bar{x}}^2 + r^2(z)(\sigma_{\bar{x}}^2 + \sigma_{\bar{y}}^2))^{\frac{1}{2}}\}] \\ \geqq E_{\sigma_x^2,\sigma_y^2}[\Phi\{ts_x/\sigma_x\} - \Phi\{-ts_x/\sigma_x\}] . \end{aligned} \tag{5.7}$$

Because of the symmetry of the normal distribution, and our definitions, it suffices to show

$$E_\tau \, \Phi\{t(s_x/\sigma_x)/[1 - (2a/(1 + z)) + a^2(1 + \tau)/(1 + z)^2]^{\frac{1}{2}}\} \geqq E_\tau \Phi(ts_x/\sigma_x) = \alpha/2 . \tag{5.8}$$

Call the l.h.s. of (5.8), $\rho(\tau)$, and rewrite the l.h.s. as

$$\rho(\tau) = E_\tau \, \Phi\{t(s_x/\sigma_x)/[1 - (2a/(1 + v\tau)) + a^2(1 + \tau)/(1 + v\tau)^2]^{\frac{1}{2}}\} . \tag{5.9}$$

We will choose an a such that, $0 < a < 1$. Therefore note that for $0 \leqq \tau \leqq 1$,

$$[1 - (2a/(1 + v\tau)) + a^2(1 + \tau)/(1 + v\tau)^2] < 1 . \tag{5.10}$$

This follows since $[a^2(1 + \tau)/(1 + v\tau)^2] \leqq 2a^2/(1 + v\tau) \leqq 2a/(1 + v\tau)$. The inequality (5.10) implies that (5.8) holds for τ such that, $0 \leqq \tau \leqq 1$. Clearly (5.8) will hold for all τ now if we can show that $\lim_{\tau\to\infty} \rho(\tau) = \alpha/2$ and if the derivative of $\rho(\tau)$, denoted by $\rho'(\tau) \leqq 0$, for $\tau > 1$.

From (5.9) and the dominated convergence theorem it follows that $\lim_{\tau\to\infty} \rho(\tau) = E_\tau \, \Phi\{t_\alpha(s_x/\sigma_x)\} = \alpha/2$. Also from (5.9) we find

$$\begin{aligned} \rho'(\tau) = (-t^*/2(2\pi)^{\frac{1}{2}}) \textstyle\int_0^\infty w^{\frac{1}{2}} \int_0^\infty [\exp(-t^{*2}w/2) \\ \times [1 - (2a/(1 + v\tau)) + a^2(1 + \tau)/(1 + v\tau)^2]] \\ \times \{[1 - (2a/(1 + v\tau)) + a^2(1 + \tau)/(1 + v\tau)^2]^{-\frac{3}{2}}\} \\ \times \{(2av/(1 + v\tau)^2) + (a^2/(1 + v\tau)^2) - 2a^2v(1 + \tau)/(1 + v\tau)^3\} \\ \times h(w, v) \, dv \, dw . \end{aligned} \tag{5.11}$$

Since our goal is to show that $\rho'(\tau) \leqq 0$ for $\tau \geqq 1$, from (5.11) it suffices to show for $\tau \geqq 1$, that

$$\begin{aligned} &\textstyle\int_0^\infty w^{\frac{1}{2}} \int_0^\infty [\exp - t^{*2}w(1+v\tau)^2/2[((1+v\tau)-a)^2+a^2\tau]] \\ (5.12) \qquad &\quad \times v\{(1+v\tau)-a(1+\tau)\}/[((1+v\tau)-a)^2+a^2\tau]^{\frac{3}{2}} \\ &\quad \times h(w,v)\,dv\,dw \geqq 0\,. \end{aligned}$$

Note that to get (5.12), the positive term $(a^2/(1+v\tau)^2)$ in the last bracket in (5.11) is thrown out. To show (5.12) our plan is as follows: multiply (5.12) by τ^2. Then split the integral into two pieces. For the first piece let $a < v < \infty$, $0 < w < \infty$ while for the second piece let $0 < v < a$, $0 < w < \infty$. We will show that the first piece is positive and bounded away from zero uniformly in a and τ. The second piece, on the other hand, after multiplication by -1, will be bounded above by a function of a, which will converge to 0 as $a \to 0$. Clearly this plan proves the existence of an $a > 0$, for which (5.12) holds. The bounds for the two pieces can be used to determine the a specified in (5.3).

Now for (5.12), and for any ε, $0 < \varepsilon < 1$, given that we require $0 < a < 1 - \varepsilon$, the first piece is

$$\begin{aligned} &\tau^2 \textstyle\int_0^\infty w^{\frac{1}{2}} \int_a^\infty [\exp - t^{*2}w(1+v\tau)^2/2[((1+v\tau)-a)^2+a^2\tau]] \\ &\quad \times v\{(1+v\tau)-a(1+\tau)\}/[((1+v\tau)-a)^2+a^2\tau]^{\frac{3}{2}}h(w,v)\,dv\,dw \\ (5.13) \qquad &\quad \geqq ((1-a)/2^{\frac{3}{2}})\tau^2 \textstyle\int_0^\infty w^{\frac{1}{2}} \int_1^\infty [\exp - 2t^{*2}w] \\ &\qquad \times (\tau/(1+v\tau)^3)h(w,v)\,dv\,dw \\ &\quad \geqq ((1-a)/2^{\frac{3}{2}}) \textstyle\int_0^\infty w^{\frac{1}{2}} \int_1^\infty [\exp - 2t^{*2}w] \\ &\qquad \times (1/(1+v)^3)h(w,v)\,dv\,dw\,. \end{aligned}$$

The second piece, after multiplication by -1, is

$$\begin{aligned} &\tau^2 \textstyle\int_0^\infty w^{\frac{1}{2}} \int_0^a [\exp - t^{*2}w(1+v\tau)^2/2[((1+v\tau)-a)^2+a^2\tau]] \\ (5.14) \qquad &\quad \times v[a(1+\tau)-(1+v\tau)]/\{((1+v\tau)-a)^2+a^2\tau\}^{\frac{3}{2}}h(w,v)\,dv\,dw \\ &\quad \leqq a\tau^2 \textstyle\int_0^\infty w^{\frac{1}{2}} \int_0^a (v\tau/v^3\tau^3)h(w,v)\,dv\,dw \\ &\quad = a \textstyle\int_0^\infty w^{\frac{1}{2}} \int_0^a (1/v^2)h(w,v)\,dv\,dw\,. \end{aligned}$$

Using (5.2), we see that the r.h.s. of (5.14) is

$$\begin{aligned} &Ca \textstyle\int_0^\infty w^{[(m+n-3)/2]}e^{-w/2} \int_0^a e^{-wv/2}v^{[(n-1)/2]-3}\,dv\,dw \\ (5.15) \qquad &\quad = Ca \textstyle\int_0^\infty w^{(m+4)/2}e^{-w/2} \int_0^a e^{-wv/2}(wv)^{[(n-1)/2]-3}\,dv\,dw \\ &\quad = Ca \textstyle\int_0^\infty w^{(m+2)/2}e^{-w/2}(\int_0^{wa} e^{-t/2}t^{[(n-1)/2]-3}\,dt)\,dw\,. \end{aligned}$$

Note for $[(n-1)/2]-3 > -1$, or $n \geqq 6$, that $e^{-t/2}t^{[(n-1)/2]-3}$ is proportional to a gamma density. It follows then from $n \geqq 6$, that the last expression of (5.15) has a limit of 0 as $a \to 0$. Thus the product of the second piece and minus 1, is bounded above by zero as $a \to 0$. Furthermore, for $n \geqq 6$, the r.h.s. of (5.15) is the same as the l.h.s. of (5.3). This fact, plus the positive lower bound for the first piece in (5.13) implies we can choose an a according to (5.3) for which (5.12) holds. This completes the proof of the theorem.

REMARK 5.1. Suppose that we seek a confidence interval for a treatment contrast (or effect) in the BIBD model of Section 3. We initially would have a choice of the interval (5.1), whose $\bar{X}$ represents the intra-block estimate, or of

$$\bar{Y} \pm s_y t_{n-1}(\alpha)/n^{\frac{1}{2}}, \tag{5.16}$$

where $\bar{Y}$ represents the inter-block estimate. We might choose the interval with the smallest expected length. Clearly then, the improved interval in (5.4) (if we chose (5.1)), or the analogous interval to (5.4) (if we chose (5.16)) would be better than both (5.4) and (5.6) if the criterion was probability of coverage and expected length. As such, for the appropriate sample sizes ($m \geqq 2$, $n \geqq 6$ and/or $m \geqq 6$, $n \geqq 2$) we could recover inter-block information for confidence estimation.

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