IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received December 18, 2020, accepted January 2, 2021, date of publication January 5, 2021, date of current version January 13, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3049438

Point and Interval Estimators of Reliability Indices for Repairable Systems Using the Weibull Generalized Renewal Process

ZHIMING WANG[1,2] AND RONG PAN[2], (Senior Member, IEEE)
[1]School of Mechanical and Electronic Engineering, Lanzhou University of Technology, Lanzhou 730050, China
[2]School of Computing, Informatics, and Decision Systems Engineering, Arizona State University, Tempe, AZ 85282, USA

Corresponding author: Zhiming Wang (wangzhiming301@sohu.com)

This work was supported in part by the National Natural Science Foundation of China under Grant 51565032, and in part by the China Scholarship Council under Grant 201708625073.

ABSTRACT The generalized renewal process considers repair efficiency of imperfect repair in reliability assessment of repairable systems; therefore, its evaluation results are close to real repair circumstance than the ordinary renewal process or the non-homogeneous Poisson process. Based on the asymptotic distribution of maximum likelihood estimation, a calculating method of reliability indices for repairable systems with imperfect repair is proposed. The point and approximate interval estimators of model parameters for Kijima type Weibull generalized renewal processes Models I and II, as well as reliability indices of repairable systems, such as reliability, cumulative failure number and failure intensity, etc., are all presented. Two real cases are studied using generalized renewal processes Models I and II respectively to show the validity of our method. The results show that imperfect repair makes the instantaneous failure intensity of a repairable system discretely jump either up or down at the time of each failure, and the method proposed in this paper agrees well with the other exiting methods, and can also reduce the complexity of calculation.

INDEX TERMS Reliability assessment, repairable system, Weibull generalized renewal process, interval estimation, failure intensity.

I. INTRODUCTION

Repairable systems are these which can be restored to a normal operating state by some repair actions other than replacement of the entire systems after experiencing a failure. Two most widely used models for analyzing failure process of repairable systems are the ordinary renewal process (ORP) model and the non-homogeneous Poisson process (NHPP) model. The former assumes that a repair action brings repairable systems to perfect repair with an as good-as-new condition, and the latter, on the contrary, considers that the repair brings the systems to minimal repair with an as-bad-as-old condition. In the other words, in the case of perfect repair, each repair renovates the system as if it was new; whereas in the case of minimal repair, each repair leaves the system in the same state as it was just before failure. However, in reality most repair activities may result in an intermediate state where the system is better than old but worse than new. This imperfect repair restores the system's operating state to somewhere between as-good-as new and as-bad-as-old. Kijima and Sumita [1] and Kijima [2] proposed two virtual age models, called the generalized renewal process (GRP) Model I and Model II, to describe this imperfect repair. In Kijima's GRP, repairs are classified in accordance with a repair effectiveness parameter, which can be used as an index of repair efficiency. Based on a reduction of virtual age or failure intensity, Doyen and Gaudoin [3]–[6] proposed two classes of imperfect repair models called the ARA and ARI models. Because there is no closed-form mathematical solution for Kijima's virtual age models, Kaminskiy and Krivtsov [7] presented an approximate solution to GRP Model I by using Monte Carlo (MC) method. Gasmi *et al.* [8] analyzed the effect of minor and major repairs to failure intensity of repairable systems following the virtual age process by using the GAUSS package. For practical purposes, model parameter estimation can also be found via a probabilistic optimization algorithm, such as the simulated annealing algorithm, or some other numerical methods, including the methods of Nelder-Mead simplex and nonlinear constrained programming [9]–[11], etc. This requires an

The associate editor coordinating the review of this manuscript and approving it for publication was Cristian Zambelli.

explicit formula for the likelihood function of each model and an optimization algorithm to maximize the corresponding log-likelihood function [12], [13]. Veber *et al.* [14] used the expectation-maximization (EM) algorithm to estimate the parameters in Kijima's Model I with a Weibull mixture distribution for the time to first failure. To estimate the model parameters of mixed effect Kijima type, Si and Yang [15] developed a stochastic approximation expectation maximization algorithm. Nguyen *et al.* [16] assumed the failure rate of a new system follows a Weibull distribution and characterized the repair efficiency by Kijima's Model II. Gasmi [17] developed the estimation of the parameters of the Weibull intensity using the likelihood ratio statistic method. Pan and Rigdon [18] studied imperfect repair systems using a hierarchical Bayesian method and applied the Markov chain Monto Carlo (MCMC) algorithm to approximate the properties of parameter posterior distributions. Maximum likelihood estimation (MLE) is one of the most used and most effective methods in reliability analysis of repairable systems [19]–[21]. For these virtual age models, empirical studies on MLE have also been presented in the literature [22]–[27]. Imperfect repair is also considered in the strategy research of preventive maintenance. Aimed at minimizing operational costs, Finkelstein [28] studied the expected cost of repair using the asymptotic approach to the Kijima II type virtual age model. Zhang and Jardine [29] proposed a system improvement model due to overhaul. Wang *et al.* [30] analyzed the preventive maintenance policy of wind turbines using general renewal processes model. On the other hand, imperfect repair actions may be ineffective and could cause repair error, therefore, Qiu *et al.* [31] investigated the maintenance policy of inspected systems using the Kijima virtual age model.

However, previous studies mostly focused on parameter estimation of imperfect repair models from failure data, only a few articles deal with statistical inferences of reliability indices of repairable systems. More recently, to obtain the confidence intervals for model parameters to predict the failure process behavior of a repairable system, the likelihood ratio statistic is employed by Ramı'rez and Utne [32]. Based on asymptotic normal distribution theory, Toledo *et al.* [33] and Oliveira *et al.* [34] made use of the Weibull generalized renewal processes (WGRP) power transformation from WGRP into homogeneous Poisson process (HPP), and obtained the confidence intervals of the parameters for the WGRP. Their proposed approach involves the computation of the Fisher information matrix and the covariance matrix between the WGRP parameters estimators.

In this paper, extending the study of Wang and Yang [11], a method of the point and interval estimates of reliability indices for repairable systems with imperfect repair is proposed. The point and approximate interval maximum likelihood estimates of model parameters for WGRP Models I and II, as well as various reliability indices of repairable systems, such as reliability at given time, cumulative number of failures, cumulative failure intensity and mean time between failures (MTBF), are derived. Two different cases from real repairable systems are analyzed using the WGRP Models I and II, respectively. One can only calculate the virtual age variance instead of calculating the variance of repair efficiency parameter and its covariances with other model parameters. Therefore, compared with the other exiting method, the method proposed in this paper can also reduce the complexity of calculation.

II. PROPOSED METHODOLOGY

A. INTERVAL ESTIMATION FOR WGRP MODEL PARAMETERS

Let t be the successive failure times, and x be the time between failures (TBF). Now, consider k repairable systems, which are observed from the start time $t_0 = 0$ to the end time T_j $(j = 1, 2, \ldots, k)$. The successive failure times of the jth system are $t_0 < t_{1,j} < t_{2,j} < \ldots < t_{n_j,j} \leq T_j$, and the TBF of the jth system are denoted by $x_0, x_{1,j}, x_{2,j}, \ldots, x_{n_j,j}, T_j - t_{n_j,j}$. So $x_{i,j} = t_{i,j} - t_{i-1,j}(x_0 = 0,\ i = 1, 2, \ldots, n_j)$. Where n_j is the number of failures of the jth system, $t_{i,j}$ is the failure time of the jth system at its ith failure, $x_{i,j}$ is the TBF of the jth system between its (i-1)th and ith failures. Note that in the time truncation case, $t_{n_j,j} < T_j$, and in the failure truncation case, $t_{n_j,j} = T_j$.

For the WGRP model, the conditional probability density function of TBF for k repairable systems is

$$f\left(x_{i,j}\left|v_{i-1,j}\right.\right) = \lambda\beta\left(x_{i,j} + v_{i-1,j}\right)^{\beta-1} \times \exp\left\{-\lambda\left[\left(x_{i,j} + v_{i-1,j}\right)^{\beta} - v_{i-1,j}^{\beta}\right]\right\}, \quad i = 1, 2, \ldots, n_j; \quad j = 1, 2, \ldots, k \tag{1}$$

where $\lambda > 0$ and $\beta > 0$ are model parameters of the WGRP, $v_{i-1,j}$ is virtual age of the jth system after the $(i-1)$th repair.

So, from (1), the conditional reliability R of k repairable systems is

$$R\left(x_{i,j}\left|v_{i-1,j}\right.\right) = \int_{x_{i,j}}^{\infty} f\left(x_{i,j}\left|v_{i-1,j}\right.\right) \mathrm{d}x_{i,j} = \exp\left\{-\lambda\left[\left(x_{i,j} + v_{i-1,j}\right)^{\beta} - v_{i-1,j}^{\beta}\right]\right\}, \quad i = 1, 2, \ldots, n_j; \quad j = 1, 2, \ldots, k \tag{2}$$

The corresponding likelihood function of TBF for k repairable systems with time truncation can be expressed as follows:

$$\begin{aligned} & L\left(x_{i,j}, T_j - t_{n_j,j}\right) \\ & = \prod_{j=1}^{k}\prod_{i=1}^{n_j} f\left(x_{i,j}\right| v_{i-1,j}) R\left(T_j - t_{n_j,j}\right| v_{n_j,j}) \\ & = \prod_{j=1}^{k}\left\{\lambda^{n_j}\beta^{n_j}\exp\left[-\lambda\left(\left(T_j - t_{n_j,j} + v_{n_j,j}\right)^{\beta} - v_{n_j,j}^{\beta}\right)\right]\right. \\ & \quad \times \prod_{i=1}^{n_j}\left[\left(x_{i,j} + v_{i-1,j}\right)^{\beta-1}\exp\left(-\lambda\left(\left(x_{i,j} + v_{i-1,j}\right)^{\beta}\right.\right.\right. \\ & \quad \left.\left.\left.\left. - v_{i-1,j}^{\beta}\right)\right)\right]\right\} \end{aligned} \tag{3}$$

Then, the log-likelihood function is

$$\begin{aligned}\Lambda = \ln L = &\sum_{j=1}^{k} n_j (\ln\lambda + \ln\beta) \\ &-\lambda\sum_{j=1}^{k}\left[\left(T_j - t_{n_j,j} + v_{n_j,j}\right)^{\beta} - v_{n_j,j}^{\beta}\right] \\ &-\lambda\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left[\left(x_{i,j} + v_{i-1,j}\right)^{\beta} - v_{i-1,j}^{\beta}\right] \\ &+(\beta-1)\sum_{j=1}^{k}\sum_{i=1}^{n_j}\ln\left(x_{i,j}+v_{i-1,j}\right)\end{aligned} \tag{4}$$

Note that there are two Kijima-type virtual age models. For Model I, virtual age is defined as

$$v_{i-1,j} = qt_{i-1,j}, \quad v_{n_j,j} = qt_{n_j,j} \ (v_0 = 0) \tag{5}$$

and for Model II, virtual age is

$$v_{i-1,j} = \sum_{j=1}^{k}\sum_{m=1}^{i-1} q^{i-m} x_{m,j}, \quad v_{n_j,j} = \sum_{j=1}^{k}\sum_{m=1}^{n_j} q^{n_j+1-m} x_{m,j} \quad (v_0 = 0, i \geq 2) \tag{6}$$

where q ($0 \leq q \leq 1$) is repair effectiveness parameter, and $v_0 = 0$.

Thus, based on (4), the second partial derivatives of the log-likelihood function with respect to parameters λ and β of WGRP are given by

$$\Delta_{11} = \frac{\partial^2\Lambda}{\partial\lambda^2} = -\frac{1}{\lambda^2}\sum_{j=1}^{k} n_j \tag{7}$$

$$\begin{aligned}\Delta_{22} = &\frac{\partial^2\Lambda}{\partial\beta^2} \\ = &-\lambda\sum_{j=1}^{k}\left[\left(T_j - t_{n_j,j} + v_{n_j,j}\right)^{\beta}\ln^2\left(T_j - t_{n_j,j} + v_{n_j,j}\right)\right. \\ &\left.-v_{n_j,j}^{\beta}\ln^2 v_{n_j,j}\right] - \lambda\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left[\left(v_{i-1,j}+x_{i,j}\right)^{\beta}\right. \\ &\left.\times\ln^2\left(v_{i-1,j}+x_{i,j}\right) - v_{i-1,j}^{\beta}\ln^2 v_{i-1,j}\right] - \frac{1}{\beta^2}\sum_{j=1}^{k} n_j\end{aligned} \tag{8}$$

$$\begin{aligned}\Delta_{12} = &\frac{\partial^2\Lambda}{\partial\lambda\partial\beta} \\ = &-\sum_{j=1}^{k}\left[\left(T_j - t_{n_j,j} + v_{n_j,j}\right)^{\beta}\ln\left(T_j - t_{n_j,j} + v_{n_j,j}\right)\right. \\ &\left.-v_{n_j,j}^{\beta}\ln v_{n_j,j}\right] - \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left[\left(v_{i-1,j}+x_{i,j}\right)^{\beta}\right. \\ &\left.\ln\left(v_{i-1,j}+x_{i,j}\right) - v_{i-1,j}^{\beta}\ln v_{i-1,j}\right]\end{aligned} \tag{9}$$

Therefore, the variances and covariances of model parameter estimators are estimated by the inverse local Fisher matrix [35] as follows:

$$\Delta = \begin{bmatrix} Var(\hat{\lambda}) & Cov(\hat{\lambda},\hat{\beta}) \\ Cov(\hat{\lambda},\hat{\beta}) & Var(\hat{\beta}) \end{bmatrix} = \begin{bmatrix} -\Delta_{11} & -\Delta_{12} \\ -\Delta_{12} & -\Delta_{22} \end{bmatrix}^{-1}_{\lambda=\hat{\lambda},\beta=\hat{\beta}} \tag{10}$$

where $\hat{\lambda}$, $\hat{\beta}$ are the point estimators of parameters λ and β, respectively. There is no closed-form solution to these estimations. Therefore, a numerical technique has to be employed to obtain the point MLEs of these model parameters. In Ref. [11], to estimate model parameters of WGRP with additional inequality constraints to these parameters, the negative log-likelihood function is minimized directly by a nonlinear programming numerical method. Depending on the normal or lognormal asymptotic distribution of model parameters λ and β, the estimates of model parameter are usually assumed normally or lognormally distributed [36]. The usefulness of lognormal distributions for model parameters has been validated in Guo and Pan [37] by lognormal probability plots. Dauxois and Maalouf [38] also proved that both the estimators of the repair efficiency and the cumulative hazard rate of initial failure time are asymptotically normal distributed. Therefore, we assume that the estimators of model parameters λ and β are either normally or lognormally distributed.

B. APPROXIMATE ASSESSMENT FOR RELIABILITY INDICES OF REPAIRABLE SYSTEMS

Assume that model parameters λ and β are independent with system virtual age v_{i-1}. This assumption can be established because that model parameters λ and β are intrinsic parameters for one system and repair activities do not change them, while virtual age v_{i-1} only has relationship with repair effectiveness and time. Thus, the variance $Var\left(\hat{\boldsymbol{\theta}}\right)$ of a reliability index estimator, $\hat{\boldsymbol{\theta}}$, can be obtained approximately as follows:

$$\begin{aligned}Var\left(\hat{\boldsymbol{\theta}}\,(x;\lambda,\beta,v_{i-1})\right) = &\left(\frac{\partial\hat{\boldsymbol{\theta}}\,(x)}{\partial\lambda}\right)^2 Var\left(\hat{\lambda}\right) \\ &+\left(\frac{\partial\hat{\boldsymbol{\theta}}\,(x)}{\partial\beta}\right)^2 Var\left(\hat{\beta}\right) \\ &+2\left(\frac{\partial\hat{\boldsymbol{\theta}}\,(x)}{\partial\lambda}\right)\left(\frac{\partial\hat{\boldsymbol{\theta}}\,(x)}{\partial\beta}\right)\boldsymbol{Cov}\left(\hat{\lambda},\hat{\beta}\right) \\ &+\left(\frac{\partial\hat{\boldsymbol{\theta}}\,(x)}{\partial v_{i-1}}\right)^2 Var\left(\hat{v}_{i-1}\right)\end{aligned} \tag{11}$$

The first three terms account for the uncertainty of parameter estimation, while the last term considers the uncertainty caused by the virtual age process even when model parameters are fixed. Reducing the computational burden, one only needs to calculate the virtual age variance instead of calculating the variance of parameter q and its covariances with the other two parameters λ and β.

Once the variances $Var\left(\hat{\boldsymbol{\theta}}\right)$ of model parameters or reliability indices $\hat{\boldsymbol{\theta}}$ have been obtained using (10) or (11), the confidence bounds can be found by

$$CB_{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \exp\left(\pm z_{\alpha/2}\sqrt{Var\left(\hat{\boldsymbol{\theta}}\right)}\Big/\hat{\boldsymbol{\theta}}\right) \tag{12}$$

if $\hat{\boldsymbol{\theta}}$ is assumed to be lognormally distributed; or

$$CB_{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \pm z_{\alpha/2}\sqrt{Var\left(\hat{\boldsymbol{\theta}}\right)} \tag{13}$$

if $\hat{\boldsymbol{\theta}}$ is assumed to be normally distributed. Here, α is a given confidence level and $z_{a/2}$ is the percentile of a standard normal distribution.

1) RELIABILITY AND WARRANTY TIME

According to (2), the conditional reliability of system after the $(i-1)$th repair and before the ith repair can be calculated by:

$$R\left(x\left|v_{i-1}\right.\right) = \exp\left\{-\lambda\left[\left(x+v_{i-1}\right)^{\beta} - v_{i-1}^{\beta}\right]\right\}, \quad 0 < x < t_i - t_{i-1} \tag{14}$$

So, from (14), the warranty time for a given reliability can be obtained by

$$x\left(R\right|v_{i-1}) = \left(v_{i-1}^{\beta} - \frac{\ln R}{\lambda}\right)^{1/\beta} - v_{i-1}, \quad 0 < x < t_i - t_{i-1} \tag{15}$$

Based on (11), the corresponding variances can be obtained according to (14) and (15), respectively, as follows:

$$\begin{aligned} &Var\left(\hat{R}\left(x\right)\right) \\ &= \exp\left\{-2\lambda\left[\left(x+v_{i-1}\right)^{\beta} - v_{i-1}^{\beta}\right]\right\} \\ &\quad\times\left\{\left[\left(x+v_{i-1}\right)^{\beta} - v_{i-1}^{\beta}\right]^2 Var\left(\hat{\lambda}\right)\right. \\ &\quad+\lambda^2\left[\left(x+v_{i-1}\right)^{\beta}\ln\left(x+v_{i-1}\right) - v_{i-1}^{\beta}\ln v_{i-1}\right]^2 Var\left(\hat{\beta}\right) \\ &\quad+2\lambda\left[\left(x+v_{i-1}\right)^{\beta} - v_{i-1}^{\beta}\right]\left[\left(x+v_{i-1}\right)^{\beta}\ln\left(x+v_{i-1}\right)\right. \\ &\quad\left.-v_{i-1}^{\beta}\ln v_{i-1}\right] Cov\left(\hat{\lambda},\hat{\beta}\right) \\ &\quad\left.+\left(\lambda\beta\right)^2\left[\left(x+v_{i-1}\right)^{\beta-1} - v_{i-1}^{\beta-1}\right]^2 Var\left(\hat{v}_{i-1}\right)\right\} \end{aligned} \tag{16}$$

and

$$\begin{aligned} Var\left(\hat{x}\left(R\right)\right) = &\frac{1}{\beta^2}\left(v_{i-1}^{\beta} - \frac{\ln R}{\lambda}\right)^{2\left(\frac{1}{\beta}-1\right)}\left\{\left(\frac{\ln R}{\lambda^2}\right)^2 Var\left(\hat{\lambda}\right)\right. \\ &+\left[\frac{1}{\beta}v_{i-1}^{\beta}\left(v_{i-1}^{\beta} - \frac{\ln R}{\lambda}\right)\right. \\ &\left.\times\ln v_{i-1}\ln\left(v_{i-1}^{\beta} - \frac{\ln R}{\lambda}\right)\right]^2 Var\left(\hat{\beta}\right) \\ &-2\frac{\ln R}{\lambda^2\beta}v_{i-1}^{\beta}\left(v_{i-1}^{\beta} - \frac{\ln R}{\lambda}\right) \\ &\times\ln v_{i-1}\ln\left(v_{i-1}^{\beta} - \frac{\ln R}{\lambda}\right) Cov\left(\hat{\lambda},\hat{\beta}\right) \\ &\left.-\left(\beta v_{i-1}^{\beta-1} - 1\right)^2 Var\left(\hat{v}_{i-1}\right)\right\} \end{aligned} \tag{17}$$

Thus, the bounds of system reliability can be obtained by

$$CB_R = \hat{R}\exp\left[\pm z_{\alpha/2}\sqrt{Var\left(\hat{R}\right)}\Big/\hat{R}\right] \tag{18}$$

or

$$CB_R = \hat{R} \pm z_{\alpha/2}\sqrt{Var\left(\hat{R}\right)} \tag{19}$$

Same as system reliability, using (12), (13) and (17), the interval estimator of warranty time given reliability can also be obtained.

2) INSTANTANEOUS FAILURE INTENSITY AND INSTANTANEOUS MTBF

Based on (14), the instantaneous failure intensity function of system after the (i-1)th repair can be obtained by

$$h\left(x\right) = -\frac{\mathrm{d}\ln R\left(x\right)}{\mathrm{d}x} = \lambda\beta\left(x+v_{i-1}\right)^{\beta-1}, \quad 0 < x < t_i - t_{i-1} \tag{20}$$

The variance of the instantaneous failure intensity is then calculated by:

$$\begin{aligned} Var\left(\hat{h}\left(x\right)\right) = &\left(x+v_{i-1}\right)^{2(\beta-1)}\left\{\beta^2 Var\left(\hat{\lambda}\right)\right. \\ &+\lambda^2\left[1+\beta\ln\left(x+v_{i-1}\right)\right]^2 Var\left(\hat{\beta}\right) \\ &+2\lambda\beta\left[1+\beta\ln\left(x+v_{i-1}\right)\right] Cov\left(\hat{\lambda},\hat{\beta}\right) \\ &\left.+\left[\frac{\lambda\beta\left(\beta-1\right)}{x+v_{i-1}}\right]^2 Var\left(\hat{v}_{i-1}\right)\right\} \end{aligned} \tag{21}$$

For a given time t, according to reliability engineering definition [25], the expected value of instantaneous MTBF is:

$$\hat{m}\left(x\right) = 1\Big/\hat{h}\left(x\right) \tag{22}$$

So, once the bounds of instantaneous failure intensity $h(x)$ have been given using (12), (13) and (21), the upper and lower bounds of instantaneous MTBF $m(x)$ can be easily obtained respectively from the corresponding bounds as follows:

$$\hat{m}\left(x\right)_U = 1\Big/\hat{h}\left(x\right)_L, \quad \hat{m}\left(x\right)_L = 1\Big/\hat{h}\left(x\right)_U \tag{23}$$

3) CUMULATIVE NUMBER OF FAILURES, CUMULATIVE FAILURE INTENSITY AND CUMULATIVE MTBF

For a given time t, according to Guo *et al.* [25], the cumulative number of failures $N(t)$ can be calculated by using the following equation:

$$\begin{aligned} N\left(t\right) = &\sum_{i=1}^{n}\int_0^{x_i}\lambda\beta\left(x+v_{i-1}\right)^{\beta-1}\mathrm{d}x \\ &+\int_0^{t-t_n}\lambda\beta\left(x+v_n\right)^{\beta-1}\mathrm{d}x \end{aligned}$$

$$= \lambda \left\{ \sum_{i=1}^{n} \left[(x_i + v_{i-1})^{\beta} - v_{i-1}^{\beta} \right] + \left[(t - t_n + v_n)^{\beta} - v_n^{\beta} \right] \right\} \quad (24)$$

Then, the corresponding variance is approximated by:

$$\begin{aligned} & Var\left[\hat{N}(t)\right] \\ &= \left\{ \left[(t - t_n + v_n)^{\beta} - v_n^{\beta} \right] + \sum_{i=1}^{n} \left[(x_i + v_{i-1})^{\beta} - v_{i-1}^{\beta} \right] \right\}^2 Var\left(\hat{\lambda}\right) \\ &+ \lambda^2 \left\{ \left[(t - t_n + v_n)^{\beta} \ln(t - t_n + v_n) - v_n^{\beta} \ln v_n \right] + \sum_{i=1}^{n} \left[(x_i + v_{i-1})^{\beta} \ln(x_i + v_{i-1}) - v_{i-1}^{\beta} \ln v_{i-1} \right] \right\}^2 Var\left(\hat{\beta}\right) \\ &+ 2\lambda \left\{ \left[(t - t_n + v_n)^{\beta} - v_n^{\beta} \right] + \sum_{i=1}^{n} \left[(x_i + v_{i-1})^{\beta} - v_{i-1}^{\beta} \right] \right\} \\ &\times \left\{ \left[(t - t_n + v_n)^{\beta} \ln(t - t_n + v_n) - v_n^{\beta} \ln v_n \right] + \sum_{i=1}^{n} \left[(x_i + v_{i-1})^{\beta} \ln(x_i + v_{i-1}) - v_{i-1}^{\beta} \ln v_{i-1} \right] \right\} Cov\left(\hat{\lambda}, \hat{\beta}\right) \\ &+ (\lambda\beta)^2 \left\{ \left[(t - t_n + v_n)^{\beta-1} - v_n^{\beta-1} \right] + \sum_{i=1}^{n} \left[(x_i + v_{i-1})^{\beta-1} - v_{i-1}^{\beta-1} \right] \right\}^2 Var\left(\hat{v}_t\right) \end{aligned} \quad (25)$$

and its bounds can also be obtained by (12), (13) and (25).

At time t, cumulative failure intensity $\hat{h}_c(t)$ and the expected value of cumulative MTBF $\hat{m}_c(t)$ can be calculated using the following equations [24, 25]:

$$\hat{h}_c(t) = \hat{N}(t) \big/ t, \quad \hat{m}_c(t) = t \big/ \hat{N}(t) \quad (26)$$

The bounds can be easily obtained from the corresponding bounds of $\hat{N}(t)$.

$$\hat{h}_c(t)_L = \hat{N}(t)_L \big/ t, \quad \hat{h}_c(t)_U = \hat{N}(t)_U \big/ t \quad (27)$$

$$\hat{m}_c(t)_L = t \big/ \hat{N}(t)_U, \quad \hat{m}_c(t)_U = t \big/ \hat{N}(t)_L \quad (28)$$

III. NUMERICAL EXAMPLE

Case 1: Wang and Yang [11] analyzed 29 field failure data with time truncation for an NC machine tool using the numerical method of nonlinear constrained programming, as shown in Table 1, and considered that the WGRP Model I is the best model for these failure data.

Using the method proposed in this study, one can get the interval estimates of model parameters, as well as the point and interval estimates of reliability indices. The variance-covariance matrix of the model parameters λ and β is

$$\Delta = \begin{bmatrix} 2.832 \times 10^{-3} & -1.339 \times 10^{-2} \\ -1.339 \times 10^{-2} & 6.639 \times 10^{-2} \end{bmatrix}$$

TABLE 1. Failure times of NC machine tool with time truncation.

No.	Failure times	No.	Failure times	No.	Failure times
1	27.51	11	1039.36	21	2867.40
2	367.52	12	1357.80	22	3299.82
3	394.52	13	1680.92	23	3387.57
4	395.64	14	1850.55	24	3468.58
5	406.75	15	1926.98	25	3688.63
6	432.49	16	2398.21	26	3780.33
7	514.17	17	2430.61	27	3862.50
8	855.57	18	2517.04	28	3955.48
9	864.85	19	2600.22	29	4152.00
10	953.02	20	2796.49		

The corresponding variances of reliability indices at the truncated time are calculated as follows:

$$Var\left(\hat{R}(t = 4152.00)\right) = 8.439 \times 10^{-3},$$
$$Var\left(\hat{t}(R = 0.7)\right) = 868.346,$$
$$Var\left(\hat{h}(t = 4152.00)\right) = 2.476 \times 10^{-6},$$
$$Var\left(\hat{N}(t = 4152.00)\right) = 71.153.$$

Table 2 is the point and interval estimates of model parameters and reliability indices with 5% significance level. The warranty time for a given reliability, such as $R = 0.7$, 0.8, 0.9, etc., can be obtained by using the proposed method in this paper. As an example, we give the warranty time with $R = 0.7$ in Table 2.

TABLE 2. Point and interval estimators of model parameters and reliability indices with 5% significance level for case I.

Parameters and reliability indices	Point estimation	Interval estimation
λ	3.270×10^{-2}	$[1.193\times10^{-2}, 8.966\times10^{-2}]$
β	0.766	[0.396, 1.481]
q	0.109	[0.024, 0.499]
$R(t = 4152.00)$	0.321	[0.237, 0.419]
$t(R = 0.7)$	60.450	[23.252, 157.160]
$h(t = 4152.00)$	5.548×10^{-3}	$[3.182\times10^{-3}, 9.673\times10^{-3}]$
$m(t = 4152.00)$	180.257	[103.387, 314.291]
$N(t = 4152.00)$	28.000	[15.514, 40.535]
$h_c(t = 4152.00)$	6.744×10^{-3}	$[3.737\times10^{-3}, 1.217\times10^{-2}]$
$m_c(t= 4152.00)$	148.283	[82.160, 267.631]

Figure 1 is the plot of instantaneous failure intensity of this NC machine tool with three different models including minimal repair, perfect repair and imperfect repair.

It can be seen that the failure intensity decreases as the operational time increases in the models of imperfect repair and minimal repair. However, the perfect repair model gives a different description, its failure intensity increases slowly from the initial time to 1000 hours or so, and afterwards it

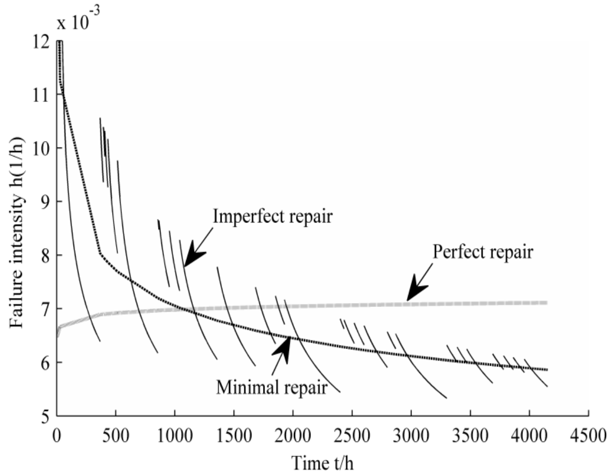

FIGURE 1. Failure intensity of NC machine tool.

tends to a constant value $7 \times 10^{-3}\ h^{-1}$. In this case, the GRP model parameter $\beta < 1$ indicating the machine tool is in an early failure period before it reaches its useful life or steady state condition. So, its failure intensity should decrease as the operational time increases. From this point view, both the hypotheses of imperfect repair and minimal repair describe correctly failure characteristics of the machine tool. On the other hand, compared with minimal repair model, imperfect repair model gives a decreasing failure intensity along with an upward jump at each failure.

The reason for this is that in each instantaneous imperfect repair, the equipment's effective age reduces to a certain value rather than zero, which makes the system much younger than before repair. Since the failure intensity is a function of the effective age and its shape remains unchanged, so the failure intensity value right after each instantaneous repair is not zero, but greater than just before failure in early failure stage, while smaller if it is in a wear-out failure stage. So, in the early failure stage, the failure intensity at each repair time does not decrease, on the contrary, it increases after each instantaneous repair, but the total trend of failure intensity decrease along with the operational time.

As Dijoux and Idée [39] pointed out that the jump discontinuity of the failure intensity after a repair means that the repair action is efficient. However, in an early failure period, repair can be harmful to the overall condition of the system, though necessary for its continued operation. In this situation, a widely accepted procedure is to apply ''burn in'' techniques to screen out defective items and thus improve the performance of the surviving items. This approach reduces the burn-in period and extends the useful life of the system.

Figure 2 is the corresponding reliability plot for this NC machine tool, it shows that at 1000 hours or so, the reliability given by the models of perfect repair and minimal repair is close to zero, but in fact through a repair, this machine tool is still work with a higher reliability. So, if one chooses the model of perfect repair or minimal repair to assess the machine tool's reliability, an incorrect result will be produced.

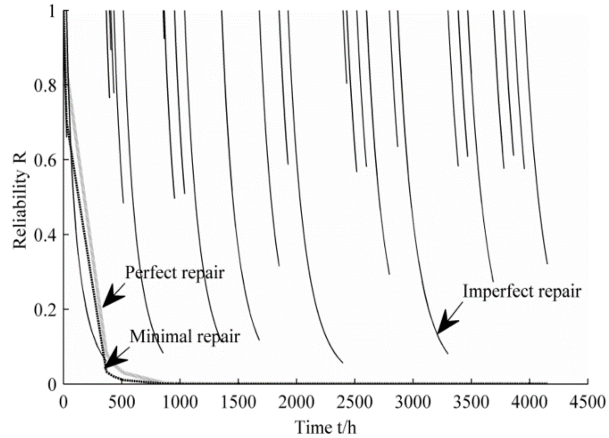

FIGURE 2. Reliability of NC machine tool.

Figure 3 is the plot of the cumulative number of failures versus time for this NC machine tool, it shows that the WGRP model gives a best fitting for failure data than the ORP and NHPP models. The root mean squared errors of fitting for WGRP, ORP and NHPP models are 1.162, 1.871 and 1.184, respectively. The fitting accuracy of WGRP model is also the best, and the ORP model is the worst. This result agrees well with that of the ref. [11].

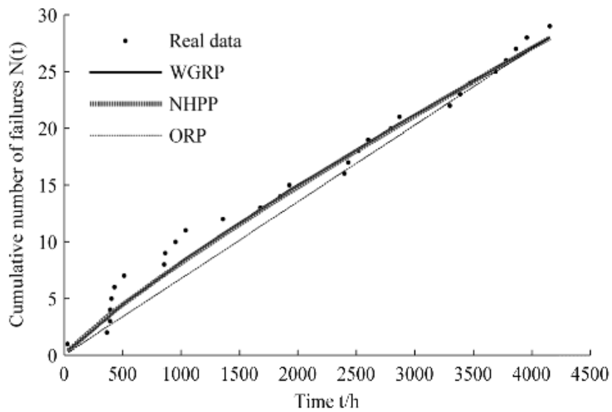

FIGURE 3. Cumulative number of failures versus time for NC machine tool.

Case 2: Mettas and Zhao [24] analyzed 33 time truncated failure data using the GRP Model II, as shown in Table 3.

The variance-covariance matrix of model parameters λ and β can be obtained as follows:

$$\Delta = \begin{bmatrix} 1.476 \times 10^{-8} & -2.771 \times 10^{-5} \\ -2.771 \times 10^{-5} & 5.309 \times 10^{-2} \end{bmatrix}$$

The corresponding variances of reliability indices at the maximum truncated time are

$$Var\left(\hat{R}\,(t = 8760)\right) = 3.858 \times 10^{-3},$$
$$Var\left(\hat{t}\,(R = 0.7)\right) = 10523.268,$$
$$Var\left(\hat{h}\,(t = 8760)\right) = 2.819 \times 10^{-7},$$
$$Var\left(\hat{N}\,(t = 8760)\right) = 0.3048$$

TABLE 3. Failure times of 6 same type repairable systems with time truncation.

No. of failures	Failure times					
	No.1	No.2	No.3	No.4	No.5	No.6
1	2227.080	772.954	900.986	411.407	688.897	105.824
2	2733.229	1034.458	1289.950	1122.740	915.101	500.000
3	3524.214	3011.114	2689.878	1300.000	2650.000	
4	5568.634	3121.458	3928.824			
5	5886.165	3624.158	4328.317			
6	5946.301	3758.296	4704.240			
7	6018.219	5000.000	5052.586			
8	7202.724		5473.171			
9	8760.000		6200.000			

TABLE 4. Point and interval estimators of model parameters and reliability indices with 5% significance level for case 2.

Parameters and reliability indices	Point estimation	Interval estimation
λ	6.800×10^{-5}	$[2.051\times10^{-6}, 2.255\times10^{-3}]$
β	1.358	[0.974, 1.894]
q	0.552	[0.200, 1.522]
$R(t = 8760.00)$	0.140	[0.089, 0.215]
$t(R = 0.7)$	287.360	[142.740, 578.490]
$h(t = 8760.00)$	1.478×10^{-3}	$[7.309\times10^{-4}, 2.989\times10^{-3}]$
$m(t = 8760.00)$	676.490	[381.920, 1198.270]
$N(t = 8760.00)$	8.007	[6.995, 9.165]
$h_c(t = 8760.00)$	9.140×10^{-4}	$[7.985\times10^{-4}, 1.046\times10^{-3}]$
$m_c(t = 8760.00)$	1094.090	[955.772, 1252.410]

Table 4 is the point and interval estimates of model parameters and reliability indices with 5% significance level.

Unlike Case 1 with an early failure stage, the repairable system of Case 2 with $\beta > 1$ indicating it is in a wear-out stage and has an increasing failure intensity. Failure intensity for these 6 same type repairable systems is shown in Figure 4, we can see that all three models possess this characteristic, but the model of imperfect repair gives a more reasonable description for system failure after repair, where its failure intensity jumps down and decreases to a lower value after each instantaneous repair, but the overall failure intensity is increasing with the operational time increases.

The reliability of system in Figure 5 is similar to Case 1, it decreases to a lower value before repair, but just after each instantaneous repair, it jumps up to 1.0 again, and then decays slowly until the next repair, and so on.

Figure 6 is the plot of the cumulative number of failures versus time for 6 same type repairable systems in case 2. In the end of the maximum truncated time 8760h, the cumulative number of failures estimated by the WGRP, ORP and NHPP models are 8.007, 8.795 and 10.242, respectively.

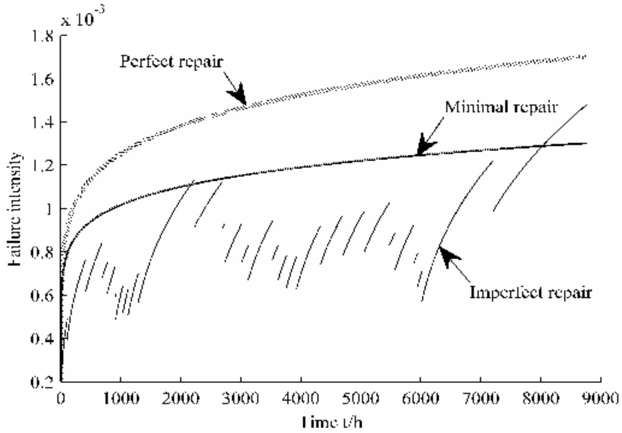

FIGURE 4. Failure intensity for 6 same type repairable systems.

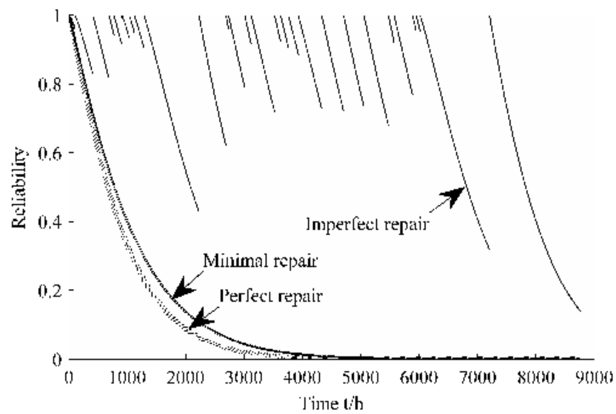

FIGURE 5. Reliability for 6 same type repairable systems.

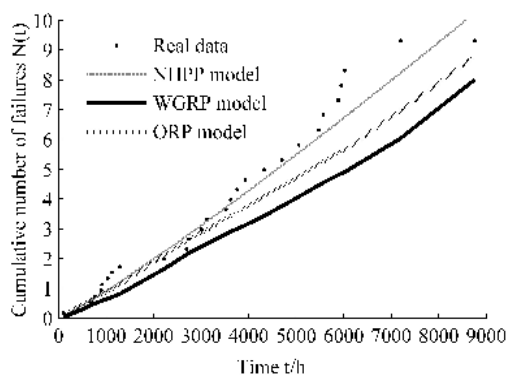

FIGURE 6. Cumulative number of failures versus time for 6 same type repairable systems.

In this time, the real cumulative number of failures is 8, so we can say that the WGRP model gives a best fitting for failure data than the ORP and NHPP models. The root mean squared errors of fitting for WGRP, ORP and NHPP models are 3.227, 3.748 and 4.370, respectively. The fitting accuracy of WGRP model is still the best, and the NHPP model is the worst. This result agrees well with the result of Mettas and Zhao [24]. Based on the value of log-likelihood for failure data, they considered that the WGRP model is the best fit for this failure data.

IV. CONCLUSION

Based on the asymptotic distribution of the MLE, the point and interval estimators of model parameters for the Kijima's WGRP Models I and II, as well as reliability indices of these repairable systems, including reliability at given time and warranty time given reliability, instantaneous failure intensity and instantaneous MTBF, cumulative failure intensity and cumulative MTBF, are all given. Two different cases are studied to show the validity of our method. The results show that the method proposed in this paper agrees well with the other exiting methods, our method also can reduce the complexity of calculation, it is efficient and powerful.

Imperfect repair makes the instantaneous failure intensity of a repairable system discretely jump either up or down at the time of each failure. In early failure stage, the failure intensity jumps up after repair, and the MTBF will increase. But in wear out stage, the failure intensity jumps down after repair, while still having a decreasing MTBF.

REFERENCES

[1] M. Kijima and U. Sumita, "A useful generalization of renewal theory: Counting processes governed by non-negative Markovian increments," *J. Appl. Probab.*, vol. 23, no. 1, pp. 71–88, Mar. 1986.

[2] M. Kijima, "Some results for repairable systems with general repair," *J. Appl. Probab.*, vol. 26, no. 1, pp. 89–102, Mar. 1989.

[3] L. Doyen and O. Gaudoin, "Classes of imperfect repair models based on reduction of failure intensity or virtual age," *Rel. Eng. Syst. Saf.*, vol. 84, no. 1, pp. 45–56, Apr. 2004.

[4] L. Doyen, "Asymptotic properties of imperfect repair models and estimation of repair efficiency," *Nav. Res. Logistics*, vol. 57, no. 3, pp. 296–307, Apr. 2010.

[5] L. Doyen, R. Drouilhet, and L. Breniere, "A generic framework for generalized virtual age models," *IEEE Trans. Rel.*, vol. 69, no. 2, pp. 816–832, Jun. 2020.

[6] M. Tanwar, R. N. Rai, and N. Bolia, "Imperfect repair modeling using Kijima type generalized renewal process," *Rel. Eng. Syst. Saf.*, vol. 124, pp. 24–31, Apr. 2014.

[7] M. Kaminskiy and V. Krivtsov, "A Monte Carlo approach to repairable system reliability analysis," in *Probabilistic Safety Assessment and Management*. New York, NY, USA: Springer, 1998.

[8] S. Gasmi, C. E. Love, and W. Kahle, "A general repair, proportional-hazards, framework to model complex repairable systems," *IEEE Trans. Rel.*, vol. 52, no. 1, pp. 26–32, Mar. 2003.

[9] C. C. F. de Oliveira, C. T. Cristino, and P. R. A. Firmino, "In the kernel of modelling repairable systems: A goodness of fit test for weibull-based generalized renewal processes," *J. Cleaner Prod.*, vol. 133, pp. 358–367, Oct. 2016.

[10] R. Mullor, J. Mulero, and M. Trottini, "A modelling approach to optimal imperfect maintenance of repairable equipment with multiple failure modes," *Comput. Ind. Eng.*, vol. 128, pp. 24–31, Feb. 2019.

[11] Z. Wang and J. Yang, "Numerical method for Weibull generalized renewal process and its applications in reliability analysis of NC machine tools," *Comput. Ind. Eng.*, vol. 63, no. 4, pp. 1122–1128, 2012.

[12] C. Chauvel, J.-Y. Dauxois, L. Doyen, and O. Gaudoin, "Parametric bootstrap goodness-of-fit tests for imperfect maintenance models," *IEEE Trans. Rel.*, vol. 65, no. 3, pp. 1343–1359, Sep. 2016.

[13] Y. Dijoux, M. Fouladirad, and D. T. Nguyen, "Statistical inference for imperfect maintenance models with missing data," *Rel. Eng. Syst. Saf.*, vol. 154, pp. 84–96, Oct. 2016.

[14] B. Veber, M. Nagode, and M. Fajdiga, "Generalized renewal process for repairable systems based on finite weibull mixture," *Rel. Eng. Syst. Saf.*, vol. 93, no. 10, pp. 1461–1472, Oct. 2008.

[15] W. Si and Q. Yang, "A generalized mixed effect Kijima model and application in optimal maintenance planning," *IEEE Trans. Rel.*, vol. 65, no. 3, pp. 1551–1561, Sep. 2016.

[16] D. T. Nguyen, Y. Dijoux, and M. Fouladirad, "Analytical properties of an imperfect repair model and application in preventive maintenance scheduling," *Eur. J. Oper. Res.*, vol. 256, no. 2, pp. 439–453, Jan. 2017.

[17] S. Gasmi, "Parameter estimation in an alternating repair model," *J. Stat. Planning Inference*, vol. 141, no. 11, pp. 3605–3616, Nov. 2011.

[18] R. Pan and S. E. Rigdon, "Bayes inference for general repairable systems," *J. Qual. Technol.*, vol. 41, no. 1, pp. 82–94, Jan. 2009.

[19] Q. Yang, Y. Hong, N. Zhang, and J. Li, "A copula-based trend-renewal process model for analysis of repairable systems with multitype failures," *IEEE Trans. Rel.*, vol. 66, no. 3, pp. 590–602, Sep. 2017.

[20] W. Si, E. Love, and Q. Yang, "Two-state optimal maintenance planning of repairable systems with covariate effects," *Comput. Oper. Res.*, vol. 92, pp. 17–25, Apr. 2018.

[21] W. Si, Q. Yang, L. Monplaisir, and Y. Chen, "Reliability analysis of repairable systems with incomplete failure time data," *IEEE Trans. Rel.*, vol. 67, no. 3, pp. 1043–1059, Sep. 2018.

[22] I. Shin, T. J. Lim, and C. H. Lie, "Estimating parameters of intensity function and maintenance effect for repairable unit," *Rel. Eng. Syst. Saf.*, vol. 54, no. 1, pp. 1–10, Oct. 1996.

[23] M. Yañez, F. Joglar, and M. Modarres, "Generalized renewal process for analysis of repairable systems with limited failure experience," *Rel. Eng. Syst. Saf.*, vol. 77, no. 2, pp. 167–180, Aug. 2002.

[24] A. Mettas and W. Zhao, "Modeling and analysis of repairable systems with general repair," in *Proc. Annu. Rel. Maintainability Symp. (RAMS)*, Alexandria, VA, USA, Jan. 2005, pp. 176–182.

[25] H. R. Guo, H. Liao, W. Zhao, and A. Mettas, "A new stochastic model for systems under general repairs," *IEEE Trans. Rel.*, vol. 56, no. 1, pp. 40–49, Mar. 2007.

[26] S. Gasmi, "Estimating parameters of a log-linear intensity for a repairable system," *Appl. Math. Model.*, vol. 37, no. 6, pp. 4325–4336, Mar. 2013.

[27] B. H. Lindqvist, "On the statistical modeling and analysis of repairable systems," *Stat. Sci.*, vol. 21, no. 4, pp. 532–551, Nov. 2006.

[28] M. Finkelstein, "On the optimal degree of imperfect repair," *Rel. Eng. Syst. Saf.*, vol. 138, pp. 54–58, Jun. 2015.

[29] F. Zhang and A. K. S. Jardine, "Optimal maintenance models with minimal repair, periodic overhaul and complete renewal," *IIE Trans.*, vol. 30, no. 12, pp. 1109–1119, Dec. 1998.

[30] J. Wang, X. Zhang, J. Zeng, and Y. Zhang, "Optimal dynamic imperfect preventive maintenance of wind turbines based on general renewal processes," *Int. J. Prod. Res.*, vol. 58, no. 22, pp. 6791–6810, Nov. 2019.

[31] Q. Qiu, L. Cui, J. Shen, and L. Yang, "Optimal maintenance policy considering maintenance errors for systems operating under performance-based contracts," *Comput. Ind. Eng.*, vol. 112, pp. 147–155, Oct. 2017.

[32] P. A. P. Ramírez and I. B. Utne, "Decision support for life extension of technical systems through virtual age modelling," *Rel. Eng. Syst. Saf.*, vol. 115, pp. 55–69, Jul. 2013.

[33] M. L. G. D. Toledo, M. A. Freitas, E. A. Colosimo, and G. L. Gilardoni, "ARA and ARI imperfect repair models: Estimation, goodness-of-fit and reliability prediction," *Rel. Eng. Syst. Saf.*, vol. 140, pp. 107–115, Aug. 2015.

[34] C. C. F. de Oliveira, P. R. A. Firmino, and C. T. Cristino, "A tool for evaluating repairable systems based on generalized renewal processes," *Rel. Eng. Syst. Saf.*, vol. 183, pp. 281–297, Mar. 2019.

[35] W. Q. Meeker and L. A. Escobar, *Statistical Methods for Reliability Data*. New York, NY, USA: Wiley, 1998.

[36] W. Nelson and J. Schmee, "Inference for (log) normal life distributions from small singly censored samples and blues," *Technometrics*, vol. 21, no. 1, pp. 43–54, Feb. 1979.

[37] H. Guo and R. Pan, "On determining sample size and testing duration of repairable system test," in *Proc. Annu. Rel. Maintainability Symp.*, Las Vegas, NV, USA, Jan. 2008, p. 6.

[38] J.-Y. Dauxois and E. Maalouf, "Statistical inference in a model of imperfect maintenance with arithmetic reduction of intensity," *IEEE Trans. Rel.*, vol. 67, no. 3, pp. 987–997, Sep. 2018.

[39] Y. Dijoux and E. Idee, "Classes of virtual age models adapted to systems with a burn-in period," *IEEE Trans. Rel.*, vol. 62, no. 4, pp. 754–763, Dec. 2013.

ZHIMING WANG received the M.D. degree from the Lanzhou University of Technology, China, in 2005, and the Ph.D. degree from Shanghai Jiao Tong University, China, in 2011. He is currently an Associate Professor with the School of Mechanical and Electronic Engineering, Lanzhou University of Technology. He has published more than 40 papers in journals and international refereed conferences, such as *Computers and Industrial Engineering*, *Journal of Mechanical Engineering Science*, and *Journal of Risk and Reliability*. His research interests include system reliability modeling, parameter estimation, reliability assessment, mechanical reliability optimization design and evaluation, accuracy reliability, and fatigue reliability.

RONG PAN (Senior Member, IEEE) received the Ph.D. degree in industrial engineering from Penn State University, in 2002. He is currently an Associate Professor of Industrial Engineering with the School of Computing, Informatics, and Decision Systems Engineering, Arizona State University. He published more than 70 journal articles and more than 50 refereed conference papers. His research interests include failure time data analysis, system reliability, design of experiments, multivariate statistical process control, time series analysis, and computational Bayesian methods. His research has been supported by NSF, DOE, Arizona Science Foundation, and Air Force Research Laboratory. He is a Lifetime Member of SRE and a Senior Member of ASQ and IIE. He was a recipient of the Stan Ofsthum Award, in 2008, 2011, and 2018, given by the Society of Reliability Engineers and the William A. Golomski Award, in 2015. His papers won the Best Reliability Paper Award of Quality Engineering, in 2012 and 2013. He is an Associate Editor of IEEE TRANSACTIONS ON RELIABILITY. He also serves on the Editorial Boards of *Journal of Quality Technology* and *Quality Engineering*.