Point degree spectra of represented spaces

Takayuki Kihara
Department of Mathematics
University of California, Berkeley, United States
kihara@math.berkeley.edu

Arno Pauly
Clare College
University of Cambridge, United Kingdom
Arno.Pauly@cl.cam.ac.uk

We introduce the point degree spectrum of a represented space as a substructure of the Medvedev degrees, which integrates the notion of Turing degrees, enumeration degrees, continuous degrees, and so on. The notion of point degree spectrum creates a connection among various areas of mathematics including computability theory, descriptive set theory, infinite dimensional topology and Banach space theory. Through this new connection, for instance, we construct a family of continuum many infinite dimensional Cantor manifolds with property C whose Borel structures at an arbitrary finite rank are mutually non-isomorphic. This provides new examples of Banach algebras of real valued Baire class two functions on metrizable compacta, and strengthen various theorems in infinite dimensional topology such as Pol's solution to Alexandrov's old problem.

1 Introduction

Computability Theory

In computable analysis [60, 75], there has for a long time been an interest in how complicated the set of codes of some element in a suitable spaces may be. Pour-El and Richards [60] observed that any real number, and more generally, any point in a Euclidean space, has a Turing degree. They subsequently raised the question whether the same holds true for any computable metric space. Miller [43] later proved that various infinite dimensional metric spaces such as the Hilbert cube and the space of continuous functions on the unit interval contain points which lack Turing degrees, i.e. have no simplest code w.r.t. Turing reducibility. A similar phenomenon was also observed in algorithmic randomness theory [48]. Day and Miller [14] showed that no neutral measure [37] has Turing degree by understanding each measure as a point in the infinite dimensional space consisting of probability measures on an underlying space.

These previous works convince us of the need for a reasonable theory of degrees of unsolvability of points in an arbitrary represented space. To establish such a theory, we associate a substructure of the Medvedev degrees with a represented space, which we call its *point degree spectrum*. A wide variety of classical degree structures are realized in this way, e.g., Turing degrees [71], enumeration degrees [17], continuous degrees [43], degrees of continuous functionals [27]. What is more noteworthy is that the concept of a point degree spectrum is closely linked to infinite dimensional topology. For instance, all points in a Polish space have Turing degrees if and only if the small transfinite inductive dimension of that space exists.

In a broader context, there are various instances of smallness properties (i.e., σ-ideals) of spaces and sets that start making sense for points in an effective treatment; e.g., arithmetical (Cohen) genericity [48, 50], Martin-Löf randomness [48], and effective Hausdorff dimension [38]. In all these cases, individual points can carry some amount of complexity – e.g. a Martin-Löf random point is in some sense too complicated to be included in a computable G_δ set having effectively measure zero. A recent important example [58, 78] from forcing theory is genericity with respect to the σ-ideal generated by finite-dimensional compact metrizable spaces. Our work provides an effective notion corresponding to topological invariants such as small inductive dimension or metrizability, and e.g. allows us to say that certain points are too complicated to be (computably) a member of a (finite-dimensional) Polish space.

Additionally, the actual importance of point degree spectrum is not merely conceptual, but also applicative. Indeed, unexpectedly, our notion of point degree spectrum turned out to be a powerful tool in descriptive set theory and infinite dimensional topology, in particular, in the study of Banach space theory and involved Borel isomorphism problems, as explained in more depth below.

Descriptive Set Theory

A *Borel isomorphism problem* (see [9, 39, 29, 23]) asks to find a nontrivial isomorphism type in a certain class of Borel spaces (i.e., topological spaces together with their Borel σ-algebras). An *α-th level Borel/Baire isomorphism* between $\mathbf{X}$ and $\mathbf{Y}$ is a bijection f such that $E \subseteq \mathbf{X}$ is of additive Borel/Baire class α if and only if $f[E] \subseteq \mathbf{Y}$ is of additive Borel/Baire class α. These restricted Borel isomorphisms are introduced by Jayne [31], in Banach space theory, to obtain certain variants of the Banach-Stone Theorem and the Gelfand-Kolmogorov Theorem for Banach algebras of the forms $\mathcal{B}^*_\alpha(\mathbf{X})$ for realcompact spaces $\mathbf{X}$. Here, $\mathcal{B}^*_\alpha(\mathbf{X})$ is the Banach algebra of bounded real valued Baire class α functions on a space $\mathbf{X}$ with respect to the supremum norm and the pointwise operation [7, 13, 31]. The first and second level Borel/Baire isomorphic classifications have been studied by several authors (see [32, 33]). However, it is not certain even whether there is an uncountable Polish space whose $G_{\delta\sigma}$-structure is neither isomorphic to the real line nor to the Hilbert cube:

Problem 1 (The Second-Level Borel Isomorphism Problem)**.** Are all uncountable Polish spaces second-level Borel isomorphic either to $\mathbb{R}$ or to $\mathbb{R}^\mathbb{N}$?

Jayne's result [31] shows that this is equivalent to asking the following problem on Banach algebras.

Problem 2 (see also Motto Ros [47])**.** If $\mathbf{X}$ is an uncountable Polish space. Then does there exist $n \in \mathbb{N}$ such that $\mathcal{B}^*_n(\mathbf{X})$ is linearly isometric (or ring isomorphic) either to $\mathcal{B}^*_n([0,1])$ or to $\mathcal{B}^*_n([0,1]^\mathbb{N})$?

The very recent successful attempts to generalize the Jayne-Rogers theorem and the Solecki dichotomy (see [47, 55] and also [35] for a computability theoretic proof) revealed that two Polish spaces are second-level Borel isomorphic if and only if they are σ-homeomorphic. Here, a topological space $\mathbf{X}$ is σ-homeomorphic to $\mathbf{Y}$ (written as $\mathbf{X} \cong^{\mathfrak{T}}_\sigma \mathbf{Y}$) if there are countable covers $\{\mathbf{X}_i\}_{i\in\omega}$ and $\{\mathbf{Y}_i\}_{i\in\omega}$ of $\mathbf{X}$ and $\mathbf{Y}$ such that $\mathbf{X}_i$ is homeomorphic to $\mathbf{Y}_i$ for every $i \in \omega$. Therefore, the second-level Borel isomorphism problem can be reformulated as the following equivalent problem.

Problem 3 (Motto Ros et al. [46])**.** Is any Polish space $\mathbf{X}$ either σ-embedded into $\mathbb{R}$ or σ-homeomorphic to $\mathbb{R}^\mathbb{N}$?

Unlike the classical Borel isomorphism problem, which was able to be reduced to the same problem on zero-dimensional Souslin spaces, the second-level Borel isomorphism problem is inescapably tied to *infinite dimensional topology* [41, 42], since all transfinite dimensional uncountable Polish spaces are mutually second-level Borel isomorphic.

The study of σ-homeomorphic maps in topological dimension theory dates back to a classical work by Hurewicz-Wallman [30] characterizing transfinite dimensionality. Alexandrov [2] asked whether there exists a weakly infinite dimensional compactum which is not σ-homeomorphic to the real line. Roman Pol [59] solved this problem by constructing such a compactum. Roman Pol's compactum is known to satisfy a slightly stronger covering property, called property C [1, 3, 24].

Our notion of *degree spectrum* on Polish spaces serves as an invariant under second-level Borel isomorphism. Indeed, an invariant which we call *degree co-spectrum*, a collection of Turing ideals realized as lower Turing cones of points of a Polish space, plays a key role in solving the second-level Borel isomorphism problem. We show that there is an embedding of an uncountable partial ordering into the σ-embeddability (the second-level Borel embeddability) ordering of metrizable C-compacta.

The key idea is measuring the quantity of all possible Scott ideals realized within the degree co-spectrum of a given space. Our spaces are completely described in the terminology of computability theory (based on Miller's work on the continuous degrees [43]). Nevertheless, the first of our examples turns out to be second-level Borel isomorphic to Roman Pol's compactum. Hence, our solution can also be viewed as a refinement of Roman Pol's solution to Alexandrov's problem.

Summary of Results

This work is part of a general development to study the descriptive theory of represented spaces [52], together with approaches such as synthetic descriptive set theory proposed in [54, 53]. In Section 3, we introduce the notion of point degree spectrum, and clarify the relationship with countable-continuity. In Section 4, we introduce the notion of an ω-left-CEA operator in the Hilbert cube as an infinite dimensional analogue of an ω-CEA operator (in the sense of classical computability theory), and show that the graph of a universal ω-left-CEA operator is an individual counterexample to Problems 1, 2, and 3. In Section 5, we clarify the relationship between a universal ω-left-CEA operator and Roman Pol's compactum. In Section 6, we describe a general procedure to construct uncountably many mutually different compacta under σ-homeomorphism. In Section 7, we characterize represented spaces with effectively-fiber-compact representations (which are relevant for complexity approaches to complexity theory along the lines of Weihrauch 's [77]) as precisely the computable metric spaces. In Section 8, we also look at the degree structures of nonmetrizable spaces. In Section 9, we construct an admissibly represented space whose degree spectrum is strictly larger than that of any second-countable T_0 spaces up to an oracle. The methods used in Sections 7–9 do not depend on those developed in Sections 4–6.

2 Preliminaries

2.1 Represented spaces

We briefly present some fundamental concepts on represented spaces following [51]. A *represented space* is a pair $\mathbf{X} = (X, \delta_X)$ of a set X and a partial surjection $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$. A function between represented spaces is a function between the underlying sets. For $f : \mathbf{X} \to \mathbf{Y}$ and $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, we call F a realizer of f, iff $\delta_Y(F(p)) = f(\delta_X(p))$ for all $p \in \mathrm{dom}(f\delta_X)$, i.e. if the following diagram commutes:

$$\begin{array}{ccc} \mathbb{N}^{\mathbb{N}} & \xrightarrow{F} & \mathbb{N}^{\mathbb{N}} \\ \big\downarrow \delta_{\mathbf{X}} & & \big\downarrow \delta_{\mathbf{Y}} \\ \mathbf{X} & \xrightarrow{f} & \mathbf{Y} \end{array}$$

A map between represented spaces is called *computable* (*continuous*), iff it has a computable (continuous) realizer. Similarly, we call a point $x \in \mathbf{X}$ *computable*, iff there is some computable $p \in \mathbb{N}^{\mathbb{N}}$ with $\delta_{\mathbf{X}}(p) = x$. Thus, a represented space is a kind of space equipped with the notion of computability.

Based on the UTM-theorem, we can introduce the space $\mathcal{C}(\mathbf{X}, \mathbf{Y})$ of continuous functions between $\mathbf{X}$ and $\mathbf{Y}$ such that function evaluation and the other usual notions are computable. In the following, we will want to make use of two special represented spaces, the countable discrete space $\mathbb{N} = (\mathbb{N}, \delta_{\mathbb{N}})$ and the Sierpiński space $\mathbb{S} = (\{\bot, \top\}, \delta_{\mathbb{S}})$. Their representations are given by $\delta_{\mathbb{N}}(0^n 1 0^{\mathbb{N}}) = n$, $\delta_{\mathbb{S}}(0^{\mathbb{N}}) = \bot$ and $\delta_{\mathbb{S}}(p) = \top$ for $p \neq 0^{\mathbb{N}}$. It is straightforward to verify that the computability notion for the represented space $\mathbb{N}$ coincides with classical computability over the natural numbers.

We then have the space $\mathcal{O}(\mathbf{X}) \cong \mathcal{C}(\mathbf{X}, \mathbb{S})$ of open subsets of a represented space $\mathbf{X}$ by identifying a set with its characteristic function, and the usual set-theoretic operations on this space are computable, too. We write $\mathcal{A}(\mathbf{X})$ for the space of closed subsets, where names are names of the open complement. Traditionally in computability theory, a computable element of the hyperspace $\mathcal{O}(\mathbf{X})$ is called a Σ^0_1 *set*, a Σ^0_1 *class* or a *c.e. open set*, and a computable element of the hyperspace $\mathcal{A}(\mathbf{X})$ is called a Π^0_1 *set*, a Π^0_1 *class* or a *co-c.e. closed set*.

The canonic function $\kappa_{\mathbf{X}} : \mathbf{X} \to \mathcal{O}(\mathcal{O}(\mathbf{X}))$ mapping x to $\{U \in \mathcal{O} \mid x \in U\}$ is always computable. If it has a computable inverse, then we call $\mathbf{X}$ *computably admissible*. Admissibility in this sense was introduced by Schröder [66, 65]. Intuitively, the computably admissible represented spaces are those that can be understood fully as topological spaces.

A particularly relevant subclass of represented spaces are the computable Polish spaces, which are derived from complete computable metric spaces by forgetting the details of the metric, and just retaining the representation (or rather, the equivalence class of representations under computable translations). Forgetting the metric is relevant when it comes to compatibility with definitions in effective descriptive set theory as shown in [20].

Example 4. The following are examples of admissible representations.

1. A *computable metric space* is a tuple $\mathbf{M} = (M, d, (a_n)_{n\in\mathbb{N}})$ such that (M, d) is a metric space and $(a_n)_{n\in\mathbb{N}}$ is a dense sequence in (M, d) such that the relation

$$\{(t, u, v, w) \mid \nu_{\mathbb{Q}}(t) < d(a_u, a_v) < \nu_{\mathbb{Q}}(w)\}$$

is recursively enumerable. The *Cauchy representation* $\delta_{\mathbf{M}} : \mathbb{N}^{\mathbb{N}} \rightharpoonup M$ associated with the computable metric space $\mathbf{M} = (M, d, (a_n)_{n\in\mathbb{N}})$ is defined by

$$\delta_{\mathbf{M}}(p) = x \ :\Longleftrightarrow \begin{cases} d(a_{p(i)}, a_{p(k)}) \leq 2^{-i} \text{ for } i < k \\ \text{and } x = \lim_{i\to\infty} a_{p(i)} \end{cases}$$

2. Another, more general subclass are the quasi-Polish spaces introduced by de Brecht [8]. A represented space $\mathbf{X} = (X, \delta_{\mathbf{X}})$ is *quasi-Polish*, if it is countably based, admissible and $\delta_{\mathbf{X}} : \mathbb{N}^{\mathbb{N}} \to \mathbf{X}$ is total. These include the computable Polish spaces as well as ω-continuous domains.

3. Generally, a topological T_0-space $\mathbf{X}$ with a countable base $\mathcal{B} = \langle B_n \rangle_{n\in\mathbb{N}}$ is naturally represented by defining $\delta_{(\mathbf{X},\mathcal{B})}(p) = x$ iff p enumerates the code of a neighborhood basis for x, that is, $\mathrm{range}(p) = \{n \in \mathbb{N} : x \in B_n\}$.

2.2 Degree structures

The Medvedev degrees $\mathfrak{M}$ [40] are a cornerstone of our framework. These are obtained by taking equivalence classes from Medvedev reducibility $\leq_M$, defined on subsets A, B of Baire space $\mathbb{N}^{\mathbb{N}}$ via $A \leq_M B$ iff there is a computable function $F : B \to A$. Important substructures of $\mathfrak{M}$ also relevant to us are the Turing degrees $\mathcal{D}_T$, the continuous degrees $\mathcal{D}_r$ and the enumeration degrees $\mathcal{D}_e$, these satisfy $\mathcal{D}_T \subsetneq \mathcal{D}_r \subsetneq \mathcal{D}_e \subsetneq \mathfrak{M}$.

Turing degrees are obtained from the usual Turing reducibility $\leq_T$ defined on points $p, q \in \mathbb{N}^{\mathbb{N}}$ with $p \leq_T q$ iff there is a computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $F(q) = p$. We thus see $p \leq_T q \Leftrightarrow \{p\} \leq_M \{q\}$, and can indeed understand the Turing degrees to be a subset of the Medvedev degrees. The continuous degrees were introduced by Miller in [43]. Enumeration degrees have received a lot of attention in computability theory, and were originally introduced by Friedberg and Rogers [17]. In both cases, we can provide a simple definition directly as a substructure of the Medvedev degrees later on.

We also use the standard notations from modern computability theory [48, 71]. For instance, Φ_e^x denotes the computation of the e-th Turing machine with oracle x, and $x^{(\alpha)}$ denotes the α-th Turing jump of x.

2.3 Isomorphism and Classification

We are now interested in isomorphisms of a particular kind, this always means a bijection in that function class, such that the inverse is also in that function class. For instance, consider the following morphisms. For a function $f : \mathbf{X} \to \mathbf{Y}$,

1. f is *countably computable* (or *σ-computable*) if there are sets $(X_n)_{n\in\mathbb{N}}$ such that $\mathbf{X} = \bigcup_{n\in\mathbb{N}} X_n$ and each $f|_{X_n}$ is computable.

2. f is *countably continuous* (or *σ-continuous*) if there are sets $(X_n)_{n\in\mathbb{N}}$ such that $\mathbf{X} = \bigcup_{n\in\mathbb{N}} X_n$ and each $f|_{X_n}$ is continuous.

3. f is *$\mathbf{\Gamma}$-piecewise continuous* if there are $\mathbf{\Gamma}$-sets $(X_n)_{n\in\mathbb{N}}$ such that $\mathbf{X} = \bigcup_{n\in\mathbb{N}} X_n$ and each $f|_{X_n}$ is continuous.

4. f is *second-level Borel measurable* is $\mathbf{\Sigma}^0_3$ for every $\mathbf{\Sigma}^0_3$ set $A \subseteq \mathbf{X}$.

Note that if $\mathbf{X}$ and $\mathbf{Y}$ have uniformly proper representations (this includes all computable metric spaces), then the $\mathbf{X}_n$ in the definition of countable continuity may be assumed to be Π^0_2-sets. Moreover, by recent results from descriptive set theory (see [35, 47, 55]), we have the following implication for functions on Polish spaces:

$$\mathbf{\Pi}^0_2\text{-piecewise continuous} \Rightarrow \text{second-level Borel measurable} \Rightarrow \text{countably continuous}$$

Consequently, the second-level Borel isomorphic classification and the countably-continuous isomorphic classification of Polish spaces are exactly the same. More precisely, three classification problems, Problems 1, 2 and 3 in Section 1 are equivalent.

Hereafter, for notation, let $\cong$ be computable isomorphism, $\cong^{\mathfrak{T}}$ continuous isomorphism (i.e., homeomorphism), $\cong_\sigma$ be isomorphism by countably computable functions and $\cong^{\mathfrak{T}}_\sigma$ is countably-continuous isomorphism.

For any of these notions, we write $\mathbf{X} \leq \mathbf{Y}$ with the same decorations on $\leq$ if $\mathbf{X}$ is isomorphic to a subspace of $\mathbf{Y}$ (i.e., $\mathbf{X}$ is embedded into $\mathbf{Y}$) in that way. If $\mathbf{X} \leq \mathbf{Y}$ and $\mathbf{X}$ is not isomorphic to $\mathbf{Y}$ in the designated way, then we also write $\mathbf{X} < \mathbf{Y}$, again with the suitable decorations on $<$. If neither $\mathbf{X} \leq \mathbf{Y}$ nor $\mathbf{Y} \leq \mathbf{X}$, we write $\mathbf{X} \mid \mathbf{Y}$ (again, with the same decorations). The Cantor-Bernstein argument shows the following.

Observation 5. Let $\mathbf{X}$ and $\mathbf{Y}$ be represented spaces. Then, $\mathbf{X} \cong_\sigma \mathbf{Y}$ if and only if $\mathbf{X} \leq_\sigma \mathbf{Y}$ and $\mathbf{Y} \leq_\sigma \mathbf{X}$

If underlying spaces are admissibly represented spaces, we also use the terminologies such as *σ-homeomorphism* and *σ-embedding* to denote countably-continuous isomorphism and countably-continuous embedding.

2.4 Topological Dimension theory

As general source for topological dimension theory, we point to Engelking [15]. See also van Mill [41, 42] for infinite dimensional topology. A topological space $\mathbf{X}$ is *countable dimensional* if it is the union of countably many finite dimensional subspaces. Recall that a Polish space is countable dimensional if and only if it is *transfinite dimensional*, that is, its transfinite small inductive dimension is less than ω_1 (see [30, pp. 50–51]). One can see that a Polish space $\mathbf{X}$ is countable dimensional if and only if $\mathbf{X} \leq^{\mathfrak{T}}_\sigma \{0,1\}^{\mathbb{N}}$.

To investigate the structure of uncountable dimensional spaces, Alexandrov introduced the notion of weakly/strongly infinite dimensional space. We say that C is a *separator* of a pair (A, B) in a space $\mathbf{X}$ if there are two pairwise disjoint open sets $A' \supseteq A$ and $B' \supseteq B$ such that $A' \sqcup B' = \mathbf{X} \setminus C$. A family $\{(A_i, B_i)\}_{i \in \Lambda}$ of pairwise disjoint closed sets in $\mathbf{X}$ is *essential* if whenever C_i is a separator of (A_i, B_i) in $\mathbf{X}$ for every $i \in \mathbb{N}$, $\bigcap_{i \in \mathbb{N}} C_i$ is nonempty. A space X is said to be *strongly infinite dimensional* if it has an essential family of infinite length. Otherwise, X is said to be *weakly infinite dimensional*.

We also consider the following covering property for topological spaces. Let $\mathcal{O}[\mathbf{X}]$ be the collection of all open covers of a topological space $\mathbf{X}$, and $\mathcal{O}_2[\mathbf{X}] = \{\mathcal{U} \in \mathcal{O}^X : \#\mathcal{U} = 2\}$. Then, $\mathbf{X} \in \mathcal{S}_c(\mathcal{A}, \mathcal{B})$ if for any sequence $(\mathcal{U}_n)_{n \in \mathbb{N}} \in \mathcal{A}[\mathbf{X}]^{\mathbb{N}}$, there is a sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$ of pairwise disjoint open sets such that $\mathcal{V}_n$ refines $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n \in \mathcal{B}[\mathbf{X}]$.

Note that a topological space $\mathbf{X}$ is weakly infinite dimensional if and only if $\mathbf{X} \in \mathcal{S}_c(\mathcal{O}_2, \mathcal{O})$. We say that $\mathbf{X}$ is a *C-space* [1, 24] or *selectively screenable* [6] if $\mathbf{X} \in \mathcal{S}_c(\mathcal{O}, \mathcal{O})$. We have the following implications:

$$\text{countable dimensional} \ \Rightarrow \ C\text{-space} \ \Rightarrow \ \text{weakly infinite dimensional.}$$

Alexandrov's old problem was whether there exists a weakly infinite dimensional compactum $\mathbf{X} >^{\mathfrak{T}}_\sigma \{0,1\}^{\mathbb{N}}$. This problem was solved by R. Pol [59] by constructing a metrizable compactum of the form $R \cup L$ for a strongly infinite dimensional totally disconnected subspace R and a countable dimensional subspace L. Such a compactum is called *Pol-type*. Every Pol-type compactum is a C-space, but is not countable-dimensional. Namely, R. POL's theorem says that there are at least two σ-homeomorphism types of metrizable C-compacta.

There are previous studies on the structure of continuous isomorphism types (Fréchet dimension types) of various kinds of infinite dimensional compacta, e.g., strongly infinite dimensional Cantor manifolds (see [11, 12, 57]). Concerning weakly infinite dimensional Cantor manifolds, ELŻBIETA POL [56] (see also [11]) constructed a C-compactum in which no separator of nonempty subspaces can be hereditarily weakly infinite dimensional. We call such a space a *Pol-type Cantor manifold.*

3 Point Degree Spectra

3.1 Generalized Turing Reducibility

Recall that the notion of a represented space involves the notion of computability. Hence, we can associate analogies of Turing reducibility and Turing degrees with an arbitrary represented space.

Definition 6. Let $\mathbf{X}$ and $\mathbf{Y}$ be represented spaces. We say that $y \in \mathbf{Y}$ is *point-Turing reducible* to $x \in \mathbf{X}$ (written as $y \leq_M^{\mathbf{X},\mathbf{Y}} x$, or simply, $y \leq_M x$) if there is a partial computable function $f :\subseteq \mathbf{X} \to \mathbf{Y}$ such that $f(x) = y$, that is, $\delta_{\mathbf{Y}}^{-1}(y) \leq_M \delta_{\mathbf{X}}^{-1}(x)$.

Based on this idea, we introduce the notion of point degree spectrum of a represented space.

Definition 7. For a represented space $\mathbf{X}$ and a point $x \in \mathbf{X}$, define

$$\begin{aligned} \mathrm{Spec}(x) &= [\delta_{\mathbf{X}}^{-1}(x)]_{\equiv_M} = \text{"the Medvedev degree of } \delta_{\mathbf{X}}^{-1}(x)\text{"} \\ \mathrm{Spec}(\mathbf{X}) &= \{\mathrm{Spec}(x) \mid x \in \mathbf{X}\} \subseteq \mathfrak{M}. \end{aligned}$$

In other words, $\mathrm{Spec}(x)$ is the point-Turing degree of x, and $\mathrm{Spec}(\mathbf{X})$ is the point-Turing degrees of points in the space $\mathbf{X}$. We call $\mathrm{Spec}(\mathbf{X})$ the *point degree spectrum* of $\mathbf{X}$. We also define the relativized point degree spectrum by $\mathrm{Spec}^p(x) = [\{p\} \times \delta_{\mathbf{X}}^{-1}(x)]_{\equiv_M}$ and $\mathrm{Spec}^p(\mathbf{X}) = \{\mathrm{Spec}^p(x) : x \in \mathbf{X}\}$.

Clearly, one can identify the Turing degrees $\mathcal{D}_T$, the continuous degrees $\mathcal{D}_r$ and the enumeration degrees $\mathcal{D}_e$ with degree spectra of some spaces as follows:

- $\mathrm{Spec}(\{0,1\}^{\mathbb{N}}) = \mathrm{Spec}(\mathbb{N}^{\mathbb{N}}) = \mathrm{Spec}(\mathbb{R}) = \mathcal{D}_T$
- (MILLER [43]) $\mathrm{Spec}([0,1]^{\mathbb{N}}) = \mathrm{Spec}(\mathcal{C}([0,1],[0,1])) = \mathcal{D}_r$
- $\mathrm{Spec}(\mathcal{O}(\mathbb{N})) = \mathcal{D}_e$

Observation 8. As any separable metric space embeds into the Hilbert cube $[0,1]^{\mathbb{N}}$, we find in particular that $\mathrm{Spec}(\mathbf{X}) \subseteq \mathcal{D}_r$ for any computable metric space $\mathbf{X}$. As any second-countable T_0 spaces embeds into the Scott domain $\mathcal{O}(\mathbb{N})$, we also have that $\mathrm{Spec}(\mathbf{X}) \subseteq \mathcal{D}_e$ for any second-countable T_0 space $\mathbf{X}$. In the latter case, the point degree of $x \in \mathbf{X}$ corresponds to the enumeration degree of neighborhood basis as in Example 4. The Turing degrees will be characterized in Section 3.2 in the context of topological dimension theory.

The following lemma shows – in MILLER's words – that the continuous degrees are *almost* Turing degrees. To be more precise, any continuous degree is relativized into a Turing degree by all Turing degrees except the smaller ones.

Lemma 9 (Miller)**.** For any non-total continuous degree $q \in \mathcal{D}_r \setminus \mathcal{D}_T$ we find that for all $p \in \mathcal{D}_T$, $(p, q) \in \mathcal{D}_T$ iff $p \not\leq_M q$.

Proof. Let $r = (r(n))_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ be a representative of a non-total continuous degree $q \in \mathcal{D}_r \setminus \mathcal{D}_T$. Let I be the set of all $y \in \{0,1\}^{\mathbb{N}}$ such that $y \leq_M r$, which is a countable set. Choose a real x whose Turing degree is incomparable with I. In particular, x is algebraically transcendent with all reals in I. So, there is an x-computable homeomorphism sending r to a sequence of irrationals. Hence, given any name of (x, r), we first obtain x, and by using x, transform r into irrationals, and then we get the least Turing degree name of (x, r). □

Remark. One can also define the Muchnik degree spectrum of a point as the collection of all Turing degrees of names of the point. Indeed, the degree spectrum of a countable structure S in the sense of computable model theory (see [28, 62]) is defined as the Muchnik degree spectrum of the corresponding point $[S]$ of a quotient space of countable structures by logic action, rather than the Medvedev degree spectrum of the point. The notion of *degree spectrum on a cone* (i.e., degree spectrum up to an oracle) plays an important role in (computable) model theory (see [44, 45]). The detailed investigation on the difference between Medvedev and Muchnik degrees can be found in [25, 26, 72].

However, if an admissibly represented space $\mathbf{X}$ is second countable and T_0, the point degree spectrum is equivalent to the Muchnik degree spectrum. This is because $\mathbf{X}$ is countably based admissible space, then $\mathrm{Spec}(\mathbf{X}) \subseteq \mathcal{D}_e$ by Observation 8, and Medvedev and Muchnik reducibility coincide for $\mathrm{Spec}(\mathcal{O}(\mathbb{N}))$ (see [43, 70]).

3.2 Degree Spectra and Dimension Theory

One of the main tools in our work is the following characterization of the point degree spectra of represented spaces.

Theorem 10. The following are equivalent for admissibly represented spaces $\mathbf{X}$ and $\mathbf{Y}$:

1. $\mathrm{Spec}^r(\mathbf{X}) = \mathrm{Spec}^r(\mathbf{Y})$ for some oracle $r \in \{0,1\}^{\mathbb{N}}$.
2. $\mathbb{N} \times \mathbf{X}$ is countable-continuously isomorphic to $\mathbb{N} \times \mathbf{Y}$, i.e., $\mathbb{N} \times \mathbf{X} \cong^{\mathfrak{T}}_{\sigma} \mathbb{N} \times \mathbf{Y}$.

Moreover, if $\mathbf{X}$ and $\mathbf{Y}$ are Polish, then the following assertions (3) and (4) are also equivalent to the above assertions (1) and (2).

(3) $\mathbb{N} \times \mathbf{X}$ is second-level Borel isomorphic to $\mathbb{N} \times \mathbf{Y}$.

(4) The Banach algebra $\mathcal{B}^*_2(\mathbb{N} \times \mathbf{X})$ is linearly isometric (ring isomorphic and so on) to $\mathcal{B}^*_2(\mathbb{N} \times \mathbf{Y})$.

One can also see that the following assertions are equivalent:

(2′) $\mathbb{N} \times \mathbf{X}$ is G_δ-piecewise homeomorphic to $\mathbb{N} \times \mathbf{Y}$.

(3′) $\mathbb{N} \times \mathbf{X}$ is n-th level Borel isomorphic to $\mathbb{N} \times \mathbf{Y}$ for some $n \geq 2$.

(4′) The Banach algebra $\mathcal{B}^*_n(\mathbb{N} \times \mathbf{X})$ is linearly isometric (ring isomorphic and so on) to $\mathcal{B}^*_n(\mathbb{N} \times \mathbf{Y})$ for some $n \geq 2$.

By our argument in Section 2.3, the assertions ($2'$) is equivalent to (2). Obviously the assertions (3) and (4) imply ($3'$) and ($4'$), respectively. The equivalence between (3) and (4) (and the equivalence between ($3'$) and ($4'$)) has already been shown by Jayne [31] for second-countable (or more generally, realcompact) spaces $\mathbf{X}$ and $\mathbf{Y}$. The implication from the assertion ($3'$) to (2) is, as mentioned in Section 2.3, recently proved by [47, 55], and more recently, an alternative computability-theoretic proof is given by [35] using our framework of point degree spectra of Polish spaces. Consequently, all assertions from (2) to ($4'$) are equivalent.

To see the equivalence between (1) and (2), we characterize the point degree spectra of represented spaces in the context of countably-continuous isomorphism.

Lemma 11. The following are equivalent for represented spaces $\mathbf{X}$ and $\mathbf{Y}$:

1. $\mathrm{Spec}(\mathbf{X}) \subseteq \mathrm{Spec}(\mathbf{Y})$
2. $\mathbf{X} \leq_\sigma \mathbb{N} \times \mathbf{Y}$, i.e., $\mathbf{X}$ is a countable union of subspaces that are computably isomorphic to subspaces of $\mathbf{Y}$.

Proof. We first show that the assertion (1) implies (2). By assumption, for any $x \in \mathbf{X}$ we find $\delta_{\mathbf{X}}^{-1}(x) \equiv_M \delta_{\mathbf{Y}}^{-1}(y_x)$ for some $y_x \in \mathbf{Y}$. Let for $\mathbf{Y}$ any $i, j \in \mathbb{N}$, let $\mathbf{X}_{ij}$ be the set of all points where the reductions are witnessed by Φ_i and Φ_j, and let $\mathbf{Y}_{ij} = \{y_x \mid x \in \mathbf{X}_{ij}\} \subseteq \{0,1\}^{\mathbb{N}}$. Then Φ_i, Φ_j also witness $\mathbf{X}_{ij} \cong \mathbf{Y}_{ij}$, and obviously $\mathbf{X} = \bigcup_{\langle i,j\rangle \in \mathbb{N}} \mathbf{X}_{ij}$.

Conversely, the point spectrum is preserved by computable isomorphism and $\mathrm{Spec}\left(\bigcup_{n\in\mathbb{N}} \mathbf{X}_n\right) = \bigcup_{n\in\mathbb{N}} \mathrm{Spec}(\mathbf{X}_n)$, so the claim follows. □

Proof of Theorem 10 (1) ⇔ (2). It follows from relativizations of Lemma 11 and Observation 5. Here, it is easy to see that the assertion (2) is equivalent to $\mathbb{N} \times \mathbf{X} \leq_\sigma \mathbb{N} \times \mathbf{Y}$. □

This simple argument completely solves a mystery about the occurrence of non-Turing degrees in proper infinite dimensional spaces. Concretely speaking, by combining Lemma 11 and a dimension-theoretic fact (see Section 2.4), we can characterize the Turing degrees by transfinite dimensionality[1].

Corollary 12. The following are equivalent for a Polish space $\mathbf{X}$ endowed with an admissible representation:

1. $\mathrm{Spec}^p(\mathbf{X}) \subseteq \mathcal{D}_T$ for some oracle $p \in \{0,1\}^{\mathbb{N}}$
2. $\mathbf{X}$ is transfinite dimensional.

Now, by Theorem 10, the countable-continuously isomorphic (σ-homeomorphic) classification can be viewed as a kind of degree theory dealing with the *degrees of degree structures* (on a cone). For instance, one may ask whether Post's problem (the Friedberg-Muchnik theorem and so on) is true for degrees of degrees of uncountable Polish spaces.

More details of the structure of degree spectra of Polish space will be investigated in Sections 4 and 6, and those of quasi-Polish space will be in Section 8. We also study the degree spectrum of a non-quasi-Polish space in Section 9.

[1] The same observation was independently made by Hoyrup. Brattka and Miller had conjectured that dimension would be the crucial demarkation line for spaces with only Turing degrees (all personal communication).

4 Intermediate Point Degree Spectra in Computability Theory

4.1 Intermediate Polish Spaces

Let $\mathfrak{P}$ be the set of all uncountable Polish spaces. In this section, we investigate the structure of $\mathfrak{P}/\cong^{\mathfrak{T}}_{\sigma}$, i.e. either of the equivalence classes w.r.t. σ-homeomorphisms, or equivalently, the structure of point degree spectra of uncountable Polish spaces up to relativization.

It is well-known that for every uncountable Polish space X:

$$\{0,1\}^{\mathbb{N}} \leq^{\mathfrak{T}}_{c} X \leq^{\mathfrak{T}}_{c} [0,1]^{\mathbb{N}},$$

where, recall that $\leq^{\mathfrak{T}}_{c}$ is the topological embeddability relation (i.e., the ordering of Fréchet dimension types). The structure of Fréchet types of uncountable dimensional Polish spaces has been developed by several authors (see [11, 12, 56, 57]). In this section, we focus on Problem 3 asking whether there exists a Polish space $\mathbf{X}$ satisfying the following:

$$\{0,1\}^{\mathbb{N}} <^{\mathfrak{T}}_{\sigma} \mathbf{X} <^{\mathfrak{T}}_{\sigma} [0,1]^{\mathbb{N}}.$$

One can see that there is no difference between the structures of σ-homeomorphism types of uncountable Polish spaces and uncountable metrizable compacta.

Fact 13. Every Polish space is σ-homeomorphic to a compact metrizable space.

Proof. All spaces of a given countable cardinality are clearly σ-homeomorphic, and there are compact metrizable spaces of all countable cardinalities.

So let $\mathbf{X}$ be an uncountable Polish space. Lelek [36] showed that every Polish space $\mathbf{X}$ has a compactification $\gamma\mathbf{X}$ such that $\gamma\mathbf{X} \setminus \mathbf{X}$ is countable-dimensional. Clearly $\mathbf{X} \leq_c \gamma\mathbf{X}$. Then, we have $\gamma\mathbf{X} \setminus \mathbf{X} \leq^{\mathfrak{T}}_{\sigma} \{0,1\}^{\mathbb{N}} \leq^{\mathfrak{T}}_{\sigma} \mathbf{X}$, since $\mathbf{X}$ is uncountable Polish and $\gamma\mathbf{X} \setminus \mathbf{X}$ is countable-dimensional. Consequently, $\mathbf{X}, \gamma\mathbf{X} \setminus \mathbf{X} \leq^{\mathfrak{T}}_{\sigma} \mathbf{X}$, and this implies $\gamma\mathbf{X} = \mathbf{X} \cup (\gamma\mathbf{X} \setminus \mathbf{X}) \leq^{\mathfrak{T}}_{\sigma} \mathbf{X}$. □

4.2 The Graph Space of a Universal ω-Left-CEA Operator

Now, we provide a concrete example having an intermediate degree spectrum. We say that a point $(r_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ is *ω-left-CEA in* or an *ω-left-pseudojump of* $x \in \{0,1\}^{\mathbb{N}}$ if r_{n+1} is left-c.e. in $\langle x, r_0, r_1, \dots, r_n\rangle$ uniformly in $n \in \mathbb{N}$. In other words, there is a computable function $\Psi : \{0,1\}^{\mathbb{N}} \times [0,1]^{<\omega} \times \mathbb{N}^2 \to \mathbb{Q}_{\geq 0}$ such that

$$r_n = \limsup_{s\to\infty} \Psi(x, r_0, \dots, r_{n-1}, n, s)$$

for every x, n, s, where $\mathbb{Q}_{\geq 0}$ denotes the set of all nonnegative rationals. Whenever $r_n \in [0,1]$ for all $n \in \mathbb{N}$, such a computable function Ψ generates an operator $J^{\omega}_{\Psi} : \{0,1\}^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ with $J^{\omega}_{\Psi}(x) = (r_0, r_1, \dots)$, which is called an *$\omega$-left-CEA operator*.

Proposition 14. There is an effective enumeration $(J^{\omega}_{e})_{e\in\mathbb{N}}$ of all ω-left-CEA operators.

Proof. It is not hard to see that $y \in [0,1]$ is left-c.e. in $x \in \{0,1\}^{\mathbb{N}} \times [0,1]^k$ if and only if there is a c.e. set $W \subseteq \mathbb{N} \times \mathbb{Q}$ such that

$$y = J^k_W(x) := \sup\{\min\{|p|, 1\} : x \in B^k_i \text{ for some } (i,p) \in W\},$$

where B^k_i is the i-th rational open ball in $[0,1]^k$. Thus, we have an effective enumeration of all left-c.e. operators $J : \{0,1\}^{\mathbb{N}} \times [0,1]^k \to [0,1]$ by putting $J^k_e = J^k_{W_e}$, where W_e is the e-th c.e. subset of $\mathbb{N} \times \mathbb{Q}$. Then, we define

$$J^\omega_e(x) = (x, J^0_{\langle e,0\rangle}(x), J^1_{\langle e,1\rangle}(x, J^0_{\langle e,0\rangle}(x)), \dots),$$

that is, J^ω_e is the ω-left-CEA operator generated by the uniform sequence $(J^k_{\langle e,k\rangle})_{k\in\mathbb{N}}$ of left-c.e. operators. Clearly, $(J^\omega_e)_{e\in\mathbb{N}}$ is an effective enumeration of all ω-left-CEA operators. □

Hence, we may define a *universal ω-left-CEA operator* by $J^\omega(e,x) = J^\omega_e(x)$.

Definition 15. The *ω-left-computably-enumerable-in-and-above space* $\omega\mathbf{CEA}$ is a subspace of $\mathbb{N} \times \{0,1\}^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$ defined by

$$\begin{aligned}\omega\mathbf{CEA} &= \{(e,x,r) \in \mathbb{N} \times \{0,1\}^{\mathbb{N}} \times [0,1]^{\mathbb{N}} : r = J^\omega_e(x)\} \\ &= \text{“the graph of a universal } \omega\text{-left-CEA operator.”}\end{aligned}$$

Note that in classical recursion theory, an operator Ψ is called a *CEA-operator* (also known as an *REA-operator* or a *pseudojump*) if there is a c.e. procedure W such that $\Psi(A) = \langle A, W(A)\rangle$ for any $A \subseteq \mathbb{N}$ (see Odifreddi [50, Sections XII and XIII]). An ω-CEA operator is the ω-th iteration of a uniform sequence of CEA-operators. In general, computability theorists have studied α-CEA operators for computable ordinals α in the theory of Π^0_2 singletons.

We say that a continuous degree is *ω-left-CEA* if it contains a point $r \in [0,1]^{\mathbb{N}}$ which is ω-left-CEA in a point $z \in \{0,1\}^{\mathbb{N}}$ such that $z \leq_M r$. The point degree spectrum of the space $\omega\mathbf{CEA}$ (as a subspace of $[0,1]^{\mathbb{N}}$) can be described as follows.

$$\mathrm{Spec}(\omega\mathbf{CEA}) = \{\mathbf{a} \in \mathcal{D}_r : \mathbf{a} \text{ is } \omega\text{-left-CEA}\}.$$

Clearly,

$$\mathrm{Spec}(\{0,1\}^{\mathbb{N}}) \subseteq \mathrm{Spec}(\omega\mathbf{CEA}) \subseteq \mathrm{Spec}([0,1]^{\mathbb{N}}).$$

Lemma 16. The ω-left-CEA space $\omega\mathbf{CEA}$ is Polish.

Proof. It suffices to show that $\omega\mathbf{CEA}$ is Π^0_2. The stage s approximation to J^k_e is denoted by $J^k_{e,s}$, that is, $J^k_{e,s}(z) = \max\{\min\{|p|,1\} : (\exists\langle i,p\rangle \in W_{e,s})\ x \in U_i\}$, where $W_{e,s}$ is the stage s approximation to the e-th computably enumerable set W_e. Note that the function $(e,s,k,z) \mapsto J^k_{e,s}(z)$ is computable. We can easily see that $(e,x,r) \in \omega\mathbf{CEA}$ if and only if

$$(\forall n,k \in \mathbb{N})(\exists s > n)\ d\left(\pi_k(r), J^k_{e,s}(x, \pi_0(r), \pi_1(r), \dots, \pi_{k-1}(r))\right) < 2^{-n},$$

where d is the Euclidean metric on $[0,1]$. □

We devote the rest of this section to a proof of the following theorem.

Theorem 17. The space $\omega\mathbf{CEA}$ has an intermediate σ-homeomorphism type, that is,

$$\{0,1\}^{\mathbb{N}} <^{\mathfrak{T}}_\sigma \omega\mathbf{CEA} <^{\mathfrak{T}}_\sigma [0,1]^{\mathbb{N}}.$$

Consequently, the space $\omega\mathbf{CEA}$ is a concrete counterexample to Problem 3.

4.3 Proof of $\omega\mathrm{CEA} <^{\mathfrak{T}}_{\sigma} [0,1]^{\mathbb{N}}$

The key idea is to measure *how similar the space* $\mathbf{X}$ *is to a zero-dimensional space* by approximating each point in a space $\mathbf{X}$ by a zero-dimensional space. Recall from Section 3.1 that the point degree spectrum coincides with the Muchnik degree spectrum for any second-countable admissibly represented space. Therefore, the spectrum $\mathrm{Spec}(x)$ of a point $x \in \mathbf{X}$ can be identified with its Turing upper cone, that is,

$$\mathrm{Spec}(x) \simeq \{z \in \{0,1\}^{\mathbb{N}} : x \leq_M z\}.$$

We think of the spectrum $\mathrm{Spec}(x)$ as *the upper approximation of* $x \in \mathbf{X}$ *by the zero-dimensional space* $\{0,1\}^{\mathbb{N}}$. Now, we need the notion of *the lower approximation of* $x \in \mathbf{X}$ *by the zero-dimensional space* $\{0,1\}^{\mathbb{N}}$. We introduce the *co-spectrum* of a point $x \in \mathbf{X}$ as its Turing lower cone

$$\mathrm{coSpec}(x) = \{z \in \{0,1\}^{\mathbb{N}} : z \leq_M x\},$$

and moreover, we define the *degree co-spectrum* of a represented space $\mathbf{X}$ as follows:

$$\mathrm{coSpec}(\mathbf{X}) = \{\mathrm{coSpec}(x) : x \in \mathbf{X}\}.$$

Note that the degree spectrum of a represented space fully determines its co-spectrum, while the converse is not true. For every oracle $p \in \{0,1\}^{\mathbb{N}}$, we may also introduce relativized co-spectra $\mathrm{coSpec}^p(x) = \{z \in \{0,1\}^{\mathbb{N}} : z \leq_M x \oplus p\}$, and the relativized degree co-spectra $\mathrm{coSpec}^p(\mathbf{X})$ in the same manner.

Observation 18. Let $\mathbf{X}$ and $\mathbf{Y}$ be admissibly represented spaces. If $\mathrm{Spec}^p(\mathbf{X}) = \mathrm{Spec}^p(\mathbf{Y})$, then we also have $\mathrm{coSpec}^p(\mathbf{X}) = \mathrm{coSpec}^p(\mathbf{Y})$. Therefore, by Theorem 10, the cospectrum of an admissibly represented space up to an oracle is invariant under σ-homeomorphism.

A collection $\mathcal{I}$ of subsets of $\mathbb{N}$ is *realized as the co-spectrum of* x if $\mathrm{coSpec}(x) = \mathcal{I}$. A countable set $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is a *Scott ideal* if it is the standard system of a countable nonstandard model of Peano arithmetic, or equivalently, a countable ω-model of RCA+WKL. Miller [43] showed that every countable Scott ideal is realized as a co-spectrum in $[0,1]^{\mathbb{N}}$.

Example 19. The spectra and co-spectra of Cantor space $\{0,1\}^{\mathbb{N}}$, the space ωCEA, and the Hilbert cube $[0,1]^{\mathbb{N}}$ are illustrated as follows (see also Figure 1):

1. The co-spectrum $\mathrm{coSpec}(x)$ of any point $x \in \{0,1\}^{\mathbb{N}}$ is principal, and *meets with* $\mathrm{Spec}(x)$ exactly at $\deg_T(x)$. The same is true up to some oracle for an arbitrary Polish spaces $\mathbf{X}$ such that $\mathbf{X} \cong^{\mathfrak{T}}_{\sigma} \{0,1\}^{\mathbb{N}}$.
2. For any point $z \in \omega\mathrm{CEA}$, the "*distance*" between $\mathrm{Spec}(z)$ and $\mathrm{coSpec}(z)$ has to be at most the ω-th Turing jump (see Lemma 20).
3. An arbitrary countable Scott ideal is realized as $\mathrm{coSpec}(y)$ of some point $y \in [0,1]^{\mathbb{N}}$. Hence, $\mathrm{Spec}(y)$ and $\mathrm{coSpec}(y)$ can be separated by *an arbitrary distance.*

This upper/lower approximation method clarifies the differences of σ-homeomorphism types of spaces because both relativized point-degree spectra and co-spectra are invariant under σ-homeomorphism by Theorem 10 and Observation 18.

Lemma 20. For any oracle $p \in \{0,1\}^{\mathbb{N}}$, there is a countable Scott ideal which cannot be realized as a p-co-spectrum of an ω-left-CEA continuous degree.

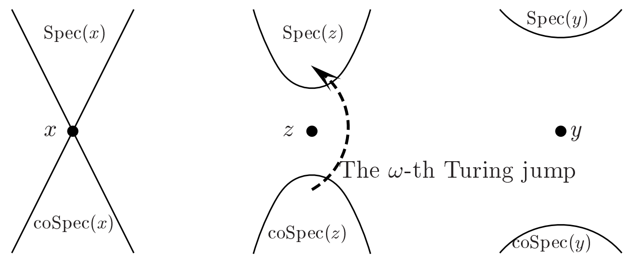

Figure 1: The upper and lower approximations of $\{0,1\}^{\mathbb{N}}$, $\omega\mathbf{CEA}$ and $[0,1]^{\mathbb{N}}$

Proof. Let $y = (e, x, r) \in \omega\mathbf{CEA}$ be an arbitrary point. Then, $x \in \mathrm{coSpec}(y)$, and $x^{(\omega)} \in \mathrm{Spec}(y)$ since r is ω-left-CEA in x. Hence, $\mathrm{coSpec}(y)$ is not closed under the ω-th Turing jump for any $y \in \omega\mathbf{CEA}$. Thus, for any oracle p, the Scott ideal $\mathcal{A}^p = \{x \in \{0,1\}^{\mathbb{N}} : (\exists n \in \mathbb{N})\ x \leq_T p^{(\omega \cdot n)}\}$ cannot be realized as a co-spectrum in $\omega\mathbf{CEA}$. □

Consequently, the ω-left-CEA space is not σ-homeomorphic to the Hilbert cube. Note that Day and Miller [14] showed that every countable Scott set $\mathcal{I}$ is realized by a neutral measure. Hence, we can also conclude that there is a neutral measure whose continuous degree is not ω-left-CEA.

4.4 Proof of $\{0,1\}^{\mathbb{N}} <^{\mathfrak{T}}_{\sigma} \omega\mathbf{CEA}$

Next, we have to show that the ω-left-CEA space is not countable-dimensional. For $\mathbf{X} \subseteq [0,1]^{\mathbb{N}}$, we inductively define $\min \mathbf{X} \in \mathbf{X}$ as follows:

$$\pi_n(\min \mathbf{X}) = \min \pi_n[\{z \in \mathbf{X} : (\forall i < n)\ \pi_i(z) = \pi_i(\min \mathbf{X})\}],$$

where $\pi_n : [0,1]^{\mathbb{N}} \to [0,1]$ is the n-th projection.

Lemma 21. If $\mathbf{X} \subseteq [0,1]^{\mathbb{N}}$ is $\Pi^0_1(p)$ for some $p \in \{0,1\}^{\mathbb{N}}$, then $\min \mathbf{X}$ is ω-left-CEA in p.

Proof. We first note that Hilbert cube $[0,1]^{\mathbb{N}}$ is computably compact in the sense that $A_{[0,1]^{\mathbb{N}}} : \mathcal{O}([0,1]^{\mathbb{N}}) \to \mathbb{S}$ is computable, where $A_{[0,1]^{\mathbb{N}}}(U) = \top$ iff $U = [0,1]^{\mathbb{N}}$. Equivalently, there is a computable enumeration of all finite collections $\mathcal{D}$ of basic open sets which covers the whole space, that is, $\bigcup \mathcal{D} = [0,1]^{\mathbb{N}}$.

It suffices to show that $\pi_{n+1}(\min X)$ is left-c.e. in $\langle \pi_i(\min X)\rangle_{i \leq n}$ uniformly in n relative to p. Given a sequence $\mathbf{a} = (a_0, a_1, \dots, a_n)$ of reals and an real q, we denote by $C(\mathbf{a}, q)$ the set of all points in X of the form $(a_0, a_1, \dots, a_n, r, \dots)$ for some $r \leq q$, that is,

$$C(\mathbf{a}, q) = X \cap \bigcap_{i \leq n} \pi_i^{-1}\{a_i\} \cap \pi_{n+1}^{-1}[0, q].$$

By computable compactness of Hilbert cube, one can see that $C^*(\mathbf{a}) = \{q \in [0,1] : C(\mathbf{a}, q) = \emptyset\}$ is p-c.e. open uniformly in $\mathbf{a}$ since the complement of $C(\mathbf{a}, q)$ is p-c.e. open uniformly in $\mathbf{a}$ and q. Therefore, if $C^*(\mathbf{a})$ is nonempty, then $\sup C^*(\mathbf{a})$ is p-left-c.e. uniformly in $\mathbf{a}$. Finally, we can easily see that $\pi_{n+1}(\min X)$ is exactly $\sup C^*(\langle \pi_i(\min X)\rangle_{i \leq n})$. □

We use the following relativized versions of MILLER's lemmas.

Lemma 22 (MILLER [43, Lemma 6.2])**.** For every $p \in \{0,1\}^{\mathbb{N}}$, there is a multivalued function $\Psi^p : [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ with a $\Pi^0_1(p)$ graph and nonempty, convex images such that, for all $e \in \mathbb{N}$, $\alpha \in [0,1]^{\mathbb{N}}$ and $\beta \in \Psi^p(\alpha)$, if for every representation λ of α, $\varphi_e^{\lambda \oplus p}$ is a representation of $x \in [0,1]$, then $\beta(e) = x$.

Note that KAKUTANI's fixed point theorem ensures the existence of a fixed point of Ψ. If α is a fixed point of Ψ^p, that is, $\alpha \in \Psi^p(\alpha)$, then $\mathrm{coSpec}^p(\alpha) = \{\alpha(n) : n \in \mathbb{N}\}$. Therefore, such an α has no Turing degree relative to p (see [43]).

Lemma 23 (MILLER [43, Lemma 9.2])**.** For every $p \in \{0,1\}^{\mathbb{N}}$, there is an index $e \in \mathbb{N}$ such that for any $x \in [0,1]$, there is a fixed point α of Ψ^p such that $\alpha(e) = x$.

Lemma 24. For any oracle $p \in \{0,1\}^{\mathbb{N}}$, there is an ω-left-CEA continuous degree which is not contained in $\mathrm{Spec}^p(\{0,1\}^{\mathbb{N}})$.

Proof. Let $\mathrm{Fix}(\Psi^p)$ be the set of all fixed points of Ψ^p. Then, $\mathrm{Fix}(\Psi^p)$ is $\Pi^0_1(p)$ since it is the intersection of the graph of a $\Pi^0_1(p)$ set and the diagonal set. Let e be an index in Lemma 23. Clearly, $A = \{\alpha \in \mathrm{Fix}(\Psi^p) : \alpha(e) = p\}$ is again a $\Pi^0_1(p)$ subset of $[0,1]^{\mathbb{N}}$. By Lemma 21, A contains an element α which is ω-left-CEA in p. By the property of A discussed above, α has no Turing degree relative to p. □

Proof of Theorem 17. By Lemma 20, $\mathrm{coSpec}^p(\omega\mathbf{CEA}) \subsetneq \mathrm{coSpec}^p([0,1]^{\mathbb{N}})$ for any oracle p. Moreover, by Lemma 24, $\mathrm{Spec}^p(\{0,1\}^{\mathbb{N}}) \subsetneq \mathrm{Spec}^p(\omega\mathbf{CEA})$ for any oracle p. Therefore, by Theorem 10 and Observation 18, we conclude $\{0,1\}^{\mathbb{N}} <^{\mathfrak{T}}_{\sigma} \omega\mathbf{CEA} <^{\mathfrak{T}}_{\sigma} [0,1]^{\mathbb{N}}$. □

5 Intermediate Point Degree Spectra in Dimension Theory

5.1 Strongly Infinite Dimensional Totally Disconnected Polish Spaces

In this section, we will shed light on dimension-theoretic perspectives of the ω-left-CEA space. Note that $\omega\mathbf{CEA}$ is a totally disconnected infinite dimensional space. We first compare our space $\omega\mathbf{CEA}$ and a totally disconnected infinite dimensional space $\mathbf{RSW}$ which is constructed by RUBIN, SCHORI, and WALSH [64]. A *continuum* is a connected compact metric space, and a continuum is *nondegenerated* if it contains at least two points.

It is known that the hyperspace $\mathsf{CK}(\mathbf{X})$ of continua in a compact metrizable space $\mathbf{X}$ equipped with the Vietoris topology forms a Polish space. Hence, we may think of $\mathsf{CK}(\mathbf{X})$ as a represented space, which corresponds to a positive-and-negative representation of the hyperspace in computable analysis. We consider a closed subspace S of $\mathsf{CK}(\mathbf{X})$ consisting of all continua connecting opposite faces $\pi_0^{-1}\{0\}$ and $\pi_0^{-1}\{1\}$. Then, fix a total Cantor representation of S, i.e., a continuous surjection δ_{CK} from the Cantor set $C \subseteq [0,1]$ onto S. We define the *Rubin-Schori-Walsh space* $\mathbf{RSW}$ [64, 41] as follows:

$$\begin{aligned}\mathbf{RSW} &= \{\min(\delta_{\mathsf{CK}}(p)^{[p]}) : p \in C\},\\ &= \{\min A^{[p]} : A \text{ is the } p\text{-th continuum of } [0,1]^{\mathbb{N}} \text{ with } [0,1] \subseteq \pi_0[A]\},\end{aligned}$$

where $A^{[p]} = A \cap \pi_0^{-1}\{p\} = \{z \in A : \pi_0(z) = p\}$.

A compactification of $\mathbf{RSW}$ is well-known in the context of ALEXANDROV's old problem in dimension theory. POL's compactum $\mathbf{RP}$ is given as a compactification in the sense of LELEK of the space **RSW**. Hence, we can see that **RP** and **RSW** have the same point degree spectra (modulo an oracle) as in the proof of Fact 13. Surprisingly, these spaces have the same degree spectra as the space ω**CEA** up to an oracle.

Theorem 25. The following spaces are all σ-homeomorphic to each other.

1. The ω-left-CEA space ω**CEA**.
2. Rubin-Schori-Walsh's totally disconnected strongly infinite dimensional space **RSW**.
3. Roman Pol's counterexample **RP** to Alexandrov's problem.

Lemma 26. Every point of **RSW** is ω-left-CEA.

Proof. By Lemma 21, $\min A^{[p]}$ is ω-left-CEA in p, since $A^{[p]}$ is $\Pi^0_1(p)$. Moreover, clearly, $p \leq_M \min A^{[p]}$. Thus, $\min A^{[p]}$ is ω-left-CEA. □

For notational convenience, without loss of generality, we may assume that the e-th z-computable continuum is equal to the $\langle e, z\rangle$-th continuum.

Lemma 27. Suppose that $x \in [0,1]^{\mathbb{N}}$ is ω-left-CEA in a point $z \in \{0,1\}^{\mathbb{N}}$. Then, there is a nondegenerated z-computable continuum $A \subseteq [0,1]^{\mathbb{N}}$ such that $[0,1] \subseteq \pi_0[A]$ and $\min A^{[p]} = (p, x)$ for a name p of A.

Proof. Given p, we will effectively construct a name $\Psi(p)$ of a continuum A. By Kleene's recursion theorem (see [71]), we may fix p such that the p-th continuum is equal to the $\Psi(p)$-th continuum.

Fix an ω-left-CEA operator J generated by $\langle W_n\rangle_{n\in\mathbb{N}}$ such that $J(z) = x$. Here, as in the proof of Proposition 14, each W_n is a c.e. list of pairs (i, p) indicating $B^n_i \subseteq (J^n_{W_n})^{-1}[p, 1]$. Since $p = \langle e, z\rangle$ for some $e \in \mathbb{N}$, we have a computable function π with $\pi(p) = z$, and then, redefine W_0 to be $W_0 \circ \pi$. In this way, we may assume that $J(p) = x$.

At stage 0, Ψ constructs $A_0 = [0,1] \times [0,1]^{\mathbb{N}}$. At stage $s+1$, if we find some rational open ball $B^n_i \subseteq [0,1]^n$ and a rational $q \in \mathbb{Q}$ such that $W_{n,s}$ declares $B^n_i \subseteq (J^n_{W_n})^{-1}[q, 1]$ by enumerating (i, q), then Ψ removes $\pi_0^{-1}[B(p; 2^{-s})] \cap (B^n_i \times [0, q) \times [0,1]^{\mathbb{N}})$ from the previous continuum A_{s-1}, where $B(p; 2^{-s})$ is the rational open ball with center p and radius 2^{-s}.

Now, we show $\min A^{[p]} = x := (x_0, x_1, \dots)$. Assume that $x_0, \dots, x_{n-1}$ is an initial segment of $\min A^{[p]}$. We will show that $x_n = \pi_n(\min A^{[p]}) = \min \pi_n[\{z \in A^{[p]} : (\forall i < n)\ \pi_i(z) = x_i\}]$. Since $J^n_{W_n}(p, x_0, \dots, x_{n-1}) = x_n$, for any rational $q < x_n$, there is i such that $(i, q) \in W_n$ and $(p, x_0, \dots, x_{n-1}) \in B^n_i$. Therefore, $A \cap (\pi_0^{-1}[B(p; 2^{-s})] \cap (B^n_i \times [0, q) \times [0,1]^{\mathbb{N}})) = \emptyset$. Hence, if $y < x_n$, then no extension of $(p, x_0, \dots, x_{n-1}, y)$ is contained in A. Moreover, if $(p, x_0, \dots, x_{n-1}) \in B^n_i$ and $q < x_n$, then $(i, q) \notin W_n$. Hence, $x_n = \pi_n(\min A^{[p]})$ as desired.

Now, clearly $\min A^{[p]} = (p, x)$. Note that Ψ defines a z-computable continuum A in a uniform manner. The computability is ensured because we only remove a subset of $\pi_0^{-1}[B(p; 2^{-s})]$ after stage s. For the connectivity, if L is any closed subset of $[0,1]^{\mathbb{N}} \setminus A$, then by compactness of L, it is covered by a finite collection of open sets of the form $B^n_i \times [0, q) \times [0,1]^{\mathbb{N}}$. Consider $L^{\complement} = [0,1]^{\mathbb{N}} \setminus L$ If n_0 is a number greater than all such n's, then any $y = (y_n)_{n\in\mathbb{N}} \in L^{\complement}$ is connected to $(y_0, y_1, \dots, y_{n_0-1}, 1, 1, 1, \dots) \in L^{\complement}$ by a line segment inside $L^{\complement}$. Moreover, any $(y_0, \dots, y_k, \vec{1}) \in L^{\complement}$ is connected to $(y_0, \dots, y_{k-1}, 1, \vec{1}) \in L^{\complement}$ by a line segment inside $L^{\complement}$. Therefore, any point $y \in L^{\complement}$ is connected to $\vec{1} \in L^{\complement}$ by a polygonal line inside $L^{\complement}$. Hence, L cannot separate A. Consequently, A is connected. □

Proof of Theorem 25. By Theorem 10, Lemmata 26 and 27. □

The properness of $\mathbf{RSW} <^{\mathfrak{T}}_{\sigma} [0,1]^{\mathbb{N}}$ can also be obtained by some relatively recent work on infinite dimensional topology: the Hilbert cube (indeed, any strongly infinite dimensional compactum) is not σ-hereditary-disconnected (see [61]). However, our alternative proof is naturally extended to a new construction of infinite dimensional spaces, which will be discussed in Section 6.

Now, one can also define the graph $n\mathbf{CEA} \subseteq \mathbb{N} \times \{0,1\}^{\mathbb{N}} \times [0,1]^n$ of a universal n-left-CEA operator (as an analogy of an n-REA operator) in a straightforward manner. Then, the space $n\mathbf{CEA}$ has the following properties.

Theorem 28. The space $n\mathbf{CEA}$ is a totally disconnected n-dimensional Polish space. Moreover, the countable product $n\mathbf{CEA}^{\mathbb{N}}$ is again n-dimensional.

Proof. Clearly, $n\mathbf{CEA}$ is totally disconnected and Polish. To check the n-dimensionality, we think of $n\mathbf{CEA}$ as a subspace of $[0,1]^{n+1}$ by identifying $(e,x) \in \mathbb{N} \times \{0,1\}^{\mathbb{N}}$ with $\iota(0^e 1x) \in [0,1]$, where ι is a computable embedding of $\{0,1\}^{\mathbb{N}}$ into $[0,1]$. We claim that $n\mathbf{CEA}$ intersects with all continua $A \subseteq [0,1]^{n+1}$ such that $[0,1] \subseteq \pi_0[A]$. We have a computable function d such that the $d(e)$-th n-left-CEA procedure $J^n_{d(e)}(x)$ for a given input $x \in \{0,1\}^{\mathbb{N}}$ outputs the value $y \in [0,1]^n$ such that $(\iota(0^e 1x), y) = \min A_{e,x}^{[\iota(0^e 1x)]}$, where $A_{e,x}$ is the e-th x-computable continuum in $[0,1]^{n+1}$ such that $[0,1] \subseteq \pi_0[A_{e,x}]$. By Kleene's recursion theorem (see [71]), there is r such that $J^n_{d(r)} = J^n_r$. Hence, $(\iota(0^r 1x), J^n_r(x)) \in n\mathbf{CEA} \cap A_{e,x}$, which verifies the claim. The claim implies that $n\mathbf{CEA}$ is n-dimensional (see van Mill [41]).

To verify the second assertion, consider the (computably) continuous map g from the square $n\mathbf{CEA}^2$ into $n\mathbf{CEA}$ such that for two points $\mathbf{x} = (e, r, x_0, \dots, x_{n-1})$ and $\mathbf{y} = (d, s, y_0, \dots, y_{n-1})$ in $n\mathbf{CEA}$,

$$g(\mathbf{x}, \mathbf{y}) = (\langle e, d \rangle, r \oplus s, (x_0 + y_0)/2, \dots, (x_{n-1} + y_{n-1})/2).$$

It is not hard to verify that g^{-1} is also (computably) continuous. Hence, $n\mathbf{CEA}^2$ computably embedded into $n\mathbf{CEA}$. In particular, it is n-dimensional. Then, we can conclude that $n\mathbf{CEA}^{\mathbb{N}}$ is also n-dimensional (by the same argument as in van Mill [42, Theorem 3.9.5]). □

5.2 Nondegenerated Continua and ωCEA Degrees

We may extract computability-theoretic contents from the construction of Rubin-Schori-Walsh's strongly infinite-dimensional totally disconnected space $\mathbf{RSW}$. The standard proof of non-countable-dimensionality of $\mathbf{RSW}$ (hence, the existence of a non-Turing degree in $\mathbf{RSW}$) indeed implies the following computability theoretic result.

Theorem 29. There exists a nondegenerated continuum $A \subseteq [0,1]^{\mathbb{N}}$ in which no point has Turing degree.

Proof. Define $\mathbf{H}_{\langle i,j \rangle} \subseteq [0,1]^{\mathbb{N}}$ to be the set of all points which can be identified with an element in $\{0,1\}^{\mathbb{N}}$ via the witnesses Φ_i and Φ_j (as in the proof of Lemma 11). Then, $\bigcup_n \mathbf{H}_n$ is the set of all points in $[0,1]^{\mathbb{N}}$ having Turing degrees. Note that each $\mathbf{H}_n$ is zero-dimensional since it is homeomorphic to a subspace of $\{0,1\}^{\mathbb{N}}$.

Consider the hyperplane $P^i_n = [0,1]^n \times \{i\} \times [0,1]^{\mathbb{N}}$ for each $n \in \mathbb{N}$ and $i \in \{0,1\}$. It is well known that $\{(P^0_n, P^1_n)\}_{n \in \mathbb{N}}$ is essential in $[0,1]^{\mathbb{N}}$. Then, by using the dimension-theoretic fact (see VAN MILL [41, Theorem 4.2.2 (5)]), we can find a separator L_n of (P^0_{n+1}, P^1_{n+1}) in $[0,1]^{\mathbb{N}}$ such that $L_n \cap \mathbf{H}_n = \emptyset$ since $\mathbf{H}_n$ is zero-dimensional.

Put $L = \bigcap_n L_n$. Then, L contains no point having Turing degree, since $L \cap \mathbf{H}_n = \emptyset$ for every $n \in \mathbb{N}$. Moreover, L contains a continuum A from P^0_0 to P^1_0 (see VAN MILL [41, Proposition 4.7.8]). □

Theorem 30. Every nondegenerated continuum $A \subseteq [0,1]^{\mathbb{N}}$ contains a point of an ω-left-CEA continuous degree.

Proof. Note that there is $n \in \omega$ such that $P^{[0,p]}_n$ and $P^{[q,1]}_n$ with some rationals $p < q \in \mathbb{Q}$ intersect with A, since A is nondegenerated, where $P^{[a,b]}_n = [0,1]^n \times [a,b] \times [0,1]^{\mathbb{N}}$. Clearly, there is no separator C of $P^{[0,p]}_n$ and $P^{[q,1]}_n$ with $C \cap A = \emptyset$ (i.e., the pair $(P^{[0,p]}_n, P^{[q,1]}_n)$ is essential in A), since A is not zero-dimensional. Therefore, the pair (P^p_n, P^q_n) is essential in the compact subspace $A^* = A \cap P^{[p,q]}_n$. Hence, $A^* \subseteq P^{[p,q]}_n$ contains a continuum intersecting with P^p_n and P^q_n (see van Mill [41, Proposition 4.7.8]). Consider a computable homeomorphism $h : P^{[p,q]}_n \cong [0,1]^{\mathbb{N}}$ mapping P^p_n and P^q_n to P^0_0 and P^1_0, respectively. Then, $h[A^*]$ is a continuum intersecting with Rubin-Schori-Walsh's space $\mathbf{RSW}$. Hence, it has an ω-left-CEA continuous degree by Theorem 25. □

As a corollary, we can see that every compactum $A \subseteq [0,1]^{\mathbb{N}}$ of positive dimension contains a point of an ω-left-CEA continuous degree. Our proof of Theorem 29 is essentially based on the fact that for any sequence of zero-dimensional spaces $\{X_i\}_{i\in\mathbb{N}}$, there exists a continuum avoiding all X_i's. Contrary to this fact, Theorem 30 says that $\{X_i\}_{i\in\mathbb{N}}$ cannot be replaced with a sequence of totally disconnected spaces.

Corollary 31. There exists a sequence $\{X_i\}_{i\in\mathbb{N}}$ of totally disconnected subspaces of $[0,1]^{\mathbb{N}}$ such that every compact subspace of $Y = [0,1]^{\mathbb{N}} \setminus \bigcup_{i\in\mathbb{N}} X_i$ is zero-dimensional, while Y is infinite dimensional.

Proof. Define $X_{\langle i,j\rangle}$ to be the set of all points which can be identified with an element in $\omega\mathbf{CEA}$ via the witnesses Φ_i and Φ_j. Then, $X_{\langle i,j\rangle}$ is totally disconnected since it is homeomorphic to a subspace of $\omega\mathbf{CEA}$. Clearly, no point $Y = [0,1]^{\mathbb{N}} \setminus \bigcup_{i,j\in\mathbb{N}} X_{\langle i,j\rangle}$ has an ω-left-CEA continuous degree. Assume that Z is a compact subspace of Y of positive dimension. Then Z has a nondegenerated subcontinuum A. However, by Theorem 30, A contains a point of an ω-left-CEA continuous degree. □

6 Structure of σ-Homeomorphism Types

6.1 Almost Arithmetical Degrees

In this section, we generalize our proof idea in Section 4 to construct a compact metrizable space whose points realize a given well-behaved family of *"almost" arithmetical degrees* as cospectra. As a consequence, we obtain the following embeddability result on the structure of σ-homeomorphism types of metrizable compacta.

Theorem 32. There is an embedding of the inclusion ordering $([\omega_1]^{\leq\omega}, \subseteq)$ of countable subsets of the smallest uncountable ordinal ω_1 into the σ-embeddability ordering of Pol-type Cantor manifolds.

As a corollary, there are a continuum chain and a continuum antichain of σ-homeomorphism types of Polish spaces. As seen in the previous sections, the notion of a co-spectrum plays a role of a σ-topological invariant. Roughly speaking, closure properties of co-spectra reflect σ-homeomorphism types of Polish spaces. The following notion estimates the strength of closure properties of functions up to the arithmetical equivalence.

Definition 33. Let g and h be total Borel measurable functions from $\{0,1\}^{\mathbb{N}}$ into $\{0,1\}^{\mathbb{N}}$.

1. We inductively define $g^0(x) = x$ and $g^{n+1}(x) = g^n(x) \oplus g(g^n(x))$.
2. For every oracle $r \in \{0,1\}^{\mathbb{N}}$, consider the following jump ideal defined as

$$\mathcal{J}_a(g,r) = \{z \in \{0,1\}^{\mathbb{N}} : (\exists n \in \mathbb{N})\ x \leq_a g^n(r)\},$$

   where $\leq_a$ denotes the arithmetical reducibility (see [49]), that is, $p \leq_a q$ is defined by $p \leq_T q^{(m)}$ for some $m \in \mathbb{N}$.
3. A function g is *almost arithmetical reducible to* a function h (written as $g \leq_{aa} h$) if for any $r \in \{0,1\}^{\mathbb{N}}$ there is $x \in \{0,1\}^{\mathbb{N}}$ with $x \geq_T r$ such that

$$\mathcal{J}_a(g,x) \subseteq \mathcal{J}_a(h,x).$$

4. Let $\mathcal{G}$ and $\mathcal{H}$ be countable sets of total functions. We say that $\mathcal{G}$ is *aa-included in* $\mathcal{H}$ (written as $\mathcal{G} \subseteq_{aa} \mathcal{H}$) if for all $g \in \mathcal{G}$, there is $h \in \mathcal{H}$ such that $g \equiv_{aa} h$ (i.e., $g \leq_{aa} h$ and $h \leq_{aa} g$).

A function $g : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is said to be *monotone* if $x \leq_T y$ implies $g(x) \leq_T g(y)$. An *oracle* $\mathbf{\Pi}^0_2$*-singleton* is a total function $g : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ whose graph is G_δ. Clearly, every oracle $\mathbf{\Pi}^0_2$-singleton is Borel measurable, whereas there is no upper bound of Borel ranks of oracle $\mathbf{\Pi}^0_2$-singletons. For instance, the α-th Turing jump $j_\alpha(x) = x^{(\alpha)}$ is a monotone oracle $\mathbf{\Pi}^0_2$-singleton for every computable ordinal α (see [49]). The following is the key lemma in our proof.

Lemma 34 (Realization Lemma)**.** There is a map **Rea** transforming each countable set of monotone oracle $\mathbf{\Pi}^0_2$-singletons into a Polish space such that

$$\mathbf{Rea}(\mathcal{G}) \leq^{\mathfrak{T}}_{\sigma} \mathbf{Rea}(\mathcal{H}) \implies \mathcal{G} \subseteq_{aa} \mathcal{H}.$$

6.2 Construction

We construct a Polish space whose co-spectrum codes almost arithmetical degrees contained in a given countable set $\mathcal{G}$ of oracle Π^0_2 singletons. For notational simplicity, given $x \in [0,1]^{\mathbb{N}}$, we write x_n for the n-th coordinate of x, and moreover, $x_{<n}$ and $x_{\leq n}$ for $(x_i)_{i<n}$ and $(x_i)_{i\leq n}$ respectively.

Definition 35. Let $\mathcal{G} = (g_n)_{n\in\mathbb{N}}$ be a countable collection of oracle $\mathbf{\Pi}^0_2$-singletons. The space $\omega\mathbf{CEA}(\mathcal{G})$ consists of $(n,d,e,r,x) \in \mathbb{N}^3 \times \{0,1\}^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$ such that for every i,

1. either $x_i = g^i_n(r)$, or
2. there are $u \leq v \leq i$ such that $x_i \in [0,1]$ is the e-th left-c.e. real in $\langle r, x_{<i}, x_{l(u)}\rangle$ and $x_{l(u)} = g^{l(u)}_n(r)$, where $l(u) = \Phi_d(u, r, x_{<v}) \geq i$.

Here, $x_{<i}$ is an abbreviation for the sequence $(x_0, \dots, x_{i-1})$. Moreover, for a set $P \subseteq [0,1]^{\mathbb{N}}$, define $\omega\mathbf{CEA}(\mathcal{G}, P)$ to be the set of all points $(d,e,r,x) \in \omega\mathbf{CEA}(\mathcal{G})$ with $(r,x) \in P$.

Lemma 36. Suppose that $\mathcal{G}$ is an oracle $\mathbf{\Pi}^0_2$-singleton, and P is a $\mathbf{\Pi}^0_2$ subset of $[0,1]^{\mathbb{N}}$. Then, $\omega\mathbf{CEA}(\mathcal{G}, P)$ is Polish.

Proof. It suffices to show that $\omega\mathbf{CEA}(\mathcal{G})$ is $\mathbf{\Pi}^0_2$. The condition (1) in Definition 35 is clearly $\mathbf{\Pi}^0_2$. Let $\forall a \exists b > a\ G(a,b,n,l,r,x)$ be a $\mathbf{\Pi}^0_2$ condition representing $x = g^l_n(r)$, and $l(u)[s]$ be the stage s approximation of $l(u)$. The condition (2) is equivalent to the statement that there are $u \leq v \leq i$ such that

$$(\forall t \in \mathbb{N})(\exists s > t)\ l(u)[s] \downarrow \geq i,\ d(x_i, J^{i+1}_{e,s}(r, x_{<i}, x_{l(u)[s]})) < 2^{-t},$$
$$\text{and } G(t,s,n,l(u)[s],r,x_{l(u)[s]}).$$

Clearly, this condition is $\mathbf{\Pi}^0_2$. □

Remark. The space $\omega\mathbf{CEA}(\mathcal{G})$ is totally disconnected for any countable set $\mathcal{G}$ of oracle $\mathbf{\Pi}^0_2$ singletons, since for any fixed $(n,d,e,r) \in \mathbb{N}^3 \times \{0,1\}^{\mathbb{N}}$, its extensions form a finite-branching infinite tree $T \subseteq [0,1]^{<\omega}$.

Recall from Section 4.4 that Miller [43, Lemma 6.2] constructed a Π^0_1 set $\mathrm{Fix}(\Psi)$ such that $\mathrm{coSpec}(x) = \{x_i : i \in \mathbb{N}\}$ for every $x = (x_i)_{i \in \mathbb{N}} \in \mathrm{Fix}(\Psi)$. By Lemma 22, without loss of generality, we may assume that $\mathrm{Fix}(\Psi) \cap \pi_0^{-1}\{r\} \neq \emptyset$ for every $r \in [0,1]$. Now, consider the space $\mathbf{Rea}(\mathcal{G}) = \omega\mathbf{CEA}(\mathcal{G}, \mathrm{Fix}(\Psi))$. To state properties of $\mathbf{Rea}(\mathcal{G})$, for an oracle $\mathbf{\Pi}^0_2$-singleton g and an oracle $r \in \{0,1\}^{\mathbb{N}}$, we use the following Turing ideal:

$$\mathcal{J}_T(g,r) = \{z \in \{0,1\}^{\mathbb{N}} : (\exists n \in \mathbb{N})\ x \leq_T g^n(r)\}.$$

The following is the key lemma, which states that any collection of jump ideals generated by countably many oracle $\mathbf{\Pi}^0_2$-singletons has to be the degree co-spectrum of a Polish space up to the almost arithmetical equivalence!

Lemma 37. Suppose that $\mathcal{G} = (g_n)_{n \in \mathbb{N}}$ is a countable set of oracle $\mathbf{\Pi}^0_2$-singletons.

1. For every $x \in \mathbf{Rea}(\mathcal{G})$, there are $r \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$ such that
$$\mathcal{J}_T(g_n, r) \subseteq \mathrm{coSpec}(x) \subseteq \mathcal{J}_a(g_n, r).$$
2. For every $r \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, there is $x \in \mathbf{Rea}(\mathcal{G})$ such that
$$\mathcal{J}_T(g_n, r) \subseteq \mathrm{coSpec}(x).$$

Proof of Lemma 37 (1). We have $(r,x) \in \mathrm{Fix}(\Psi)$ for every $(n,d,e,r,x) \in \mathbf{Rea}(\mathcal{G})$. For every $i \in \mathbb{N}$, we inductively assume that for every $j < i$, x_j is arithmetical in $g^k_n(r)$ for some $k \in \mathbb{N}$. Now, either $x_i = g^i_n(r)$ or x_i is left-c.e. in $(r, x_{<i}, g^l_n(r))$ for some l. In both cases, x_i is arithmetical in $g^k_n(r)$ for some k. Moreover, $x_i = g^i_n(r)$ for infinitely many $i \in \mathbb{N}$, since either $x_i = g^n_i(r)$ holds or there is $l \geq i$ such that $x_l = g^l_n(r)$ by the condition (2) in Definition 35. Therefore, $g^k_n(r) \leq_T x$ for all $k \in \mathbb{N}$. □

To verify the assertion (2) in Lemma 37, indeed, for any $n \in \mathbb{N}$, we will construct indices d and e such that for every $r \in \{0,1\}^{\mathbb{N}}$, there is x with $(n, d, e, r, x) \in \mathbf{Rea}(\mathcal{G})$, where $x_i = g_n^i(r)$ for infinitely many $i \in \mathbb{N}$. The e-th left-c.e. procedure $J_e^{i+1}(r, x_{<i}, x_{l(u)})$ is a simple procedure extending $r, x_{<i}, x_{l(u)}$ to a fixed point of Ψ. The function Φ_d searches a *safe coding location* $c(n)$ from a given *name* of $x_{\leq c(n-1)}$, where $c(n-1)$ is the previous coding location.

To make sure the search of the next coding location is bounded, as in Definition 35, we have to restrict the set of names of a v-tuple $x_{<v}$ to at most $v+1$ candidates. It is known that a separable metrizable space is at most n-dimensional if and only if it is the union of $n+1$ many zero-dimensional subspaces (see [15]). We say that an admissibly represented Polish space is *computably at most n-dimensional* if it is the union of $n+1$ many subspaces that are computably homeomorphic to subspaces of $\mathbb{N}^{\mathbb{N}}$.

Lemma 38. Suppose that $(\mathbf{X}, \rho_X)$ is a computably at most n-dimensional admissibly represented space. Then, there is a partial computable injection $\nu_X :\subseteq (n+1) \times \mathbf{X} \to \mathbb{N}^{\mathbb{N}}$ such that for every $x \in \mathbf{X}$, there is $k \leq n$ such that $(k, x) \in \mathrm{dom}(\nu_X)$ and $\rho_X \circ \nu_X(k, x) = x$.

Proof. By definition, $\mathbf{X}$ is divided into $n+1$ many subspaces $S_0, \dots, S_n$ such that S_k is homeomorphic to $N_k \subseteq \mathbb{N}^{\mathbb{N}}$ via computable maps τ_k and τ_k^{-1}. Then, the partial computable injection $\tau_k^{-1} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbf{X}$ has a computable realizer τ_k^*, i.e., $\tau_k^{-1} = \rho_X \circ \tau_k^*$. Define $\nu_X(k, x) = \tau_k^* \circ \tau_k(x)$ for $x \in S_k$. Then, we have $\rho_X \circ \nu_X(k, x) = \tau_k^{-1} \circ \tau_k(x) = x$ for $x \in S_k$. $\square$

The Euclidean n-space $\mathbb{R}^n$ is clearly computably n-dimensional, e.g., let S_k be the set of all points $x \in \mathbb{R}^n$ such that exactly k many coordinates are irrational. Furthermore, one can effectively find an index of $\nu_n := \nu_{\mathbb{R}^n}$ in Lemma 38 uniformly in n. Hereafter, let ρ_i be the usual Euclidean admissible representation of $\mathbb{R}^i$. Now, a coding location $c(n)$ will be obtained as a fixed point in the sense of Kleene's recursion theorem (see [71]). Hence, one can effectively find such a location in the following sense.

Lemma 39 (Miller [43, Lemma 9.2])**.** Suppose that $(r, x_{<i})$ can be extended to a fixed point of Ψ, and fix a partial computable function ν which sends $x_{<i}$ to its name, i.e., $\rho_i \circ \nu(x_{<i}) = (x_{<i})$. From an index t of ν and the sequence $x_{<i}$, one can effectively find a location $p = \Gamma(t, r, x_{<i})$ such that for every real y, the sequence $(r, x_{<i})$ can be extended to a fixed point (r, x) of Ψ such that $x_p = y$.

Let $t(n, k)$ be an index of the partial computable function $x \mapsto \nu_n(k, x)$. We define $\Phi_d(u, r, x_{<v})$ to be $\Gamma(t(v, u), r, x_{<v})$ for every $u \leq v$. Note that indices d and e do not depend on g_n.

Proof of Lemma 37 (2). Now, we claim that for every $r \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, there is x with $(n, d, e, r, x) \in \mathbf{Rea}(\mathcal{G})$, where $x_i = g_n^i(r)$ for infinitely many $i \in \mathbb{N}$. We follow the argument by Miller [43, Lemma 9.2]. Suppose that i is a coding location of $g_n^i(r)$, and $(r, x_{\leq i})$ is extendible to a fixed point of Ψ. Then, there is $k \leq i+1$ such that $p = \Phi_d(k, r, x_{\leq i})$ is defined, and then we set $x_p = g_n^p(r)$. By the property of Φ_d, $(r, x_{\leq i}, x_p)$ can be extended to a fixed point of Ψ. Then, the e-th left-c.e. procedure automatically produces $x_{\leq p}$ which is extendible to a fixed point of Ψ. Note that the condition (2) in Definition 35 is ensured via $u = k$, $v = i+1$, and $l(u) = p$. Eventually, we obtain $(r, x) \in \mathrm{Fix}(\Psi)$ such that $z = (n, d, e, r, x) \in \mathbf{Rea}(\mathcal{G})$.

Clearly, $g_n^k(r) \in \mathrm{coSpec}(z)$ for every $k \in \mathbb{N}$, since $\mathrm{coSpec}(z)$ is a Turing ideal, and $g_n^k(r) \leq_T g_n^{k+1}(r)$. Consequently, $\mathcal{J}_T(g_n, r) \subseteq \mathrm{coSpec}(z)$. $\square$

Proof of Lemma 34. Suppose that $\mathbf{Rea}(\mathcal{G}) \leq_\sigma^{\mathfrak{T}} \mathbf{Rea}(\mathcal{H})$. Then, $\mathbb{N} \times \mathbf{Rea}(\mathcal{G}) \leq_\sigma^{\mathfrak{T}} \mathbb{N} \times \mathbf{Rea}(\mathcal{H})$, and by Theorem 11 and Observation 18, the degree cospectrum of $\mathbf{Rea}(\mathcal{G})$ is a sub-cospectrum of that of $\mathbf{Rea}(\mathcal{H})$ up to an oracle p. Fix enumerations $\mathcal{G} = (g_n)_{n\in\mathbb{N}}$ and $\mathcal{H} = (h_n)_{n\in\mathbb{N}}$.

Claim. For any n and u, there are m and v such that $\mathcal{J}_a(g_n, u) = \mathcal{J}_a(h_m, v)$.

By Lemma 37 (2), for any n and $u \geq_T p$, there is $x \in \mathbf{Rea}(\mathcal{G})$ such that $\mathcal{J}_T(g_n, u) \subseteq \mathrm{coSpec}(x) \subseteq \mathcal{J}_a(g_n, u)$. Then, there is $y \in \mathbf{Rea}(\mathcal{H})$ such that $\mathrm{coSpec}^p(x) = \mathrm{coSpec}^p(y)$. We may assume that $p \leq_M y$, otherwise (y, p) has Turing degree by Lemma 9. By Lemma 37 (1), there exist m and v such that $\mathcal{J}_T(h_m, v) \subseteq \mathrm{coSpec}(y) \subseteq \mathcal{J}_a(h_m, v)$. Now, $\mathrm{coSpec}(x) = \mathrm{coSpec}(y)$ holds, and note that $\mathcal{J}_T(h_m, v) \subseteq \mathcal{J}_a(g_n, u)$ implies $\mathcal{J}_a(h_m, v) \subseteq \mathcal{J}_a(g_n, u)$. This verifies the claim.

For a fixed n, $\beta_n(u)$ chooses m fulfilling the above claim for some v. It is not hard to see that there is $m(n)$ such that $\beta_n(u) = m(n)$ for cofinally many u. Then, for cofinally many v, there is u such that $\mathcal{J}_a(g_n, u \oplus v) = \mathcal{J}_a(h_{m(n)}, u \oplus v)$ by monotonicity. Therefore, $g_n \equiv_{aa} h_{m(n)}$. Consequently, $\mathcal{G} \subseteq_{aa} \mathcal{H}$. □

Lemma 40. For any $\mathcal{G}$, there exists a Pol-type Cantor manifold $\mathbf{Z}(\mathcal{G})$ such that $\omega\mathbf{CEA} \oplus \mathbf{Rea}(\mathcal{G}) \equiv_\sigma^{\mathfrak{T}} \mathbf{Z}(\mathcal{G})$.

Proof. Recall from Theorem 25 that $\omega\mathbf{REA}$ is σ-homeomorphic to a strongly infinite dimensional space $\mathbf{RSW}$. Let $\mathbf{R}_0$ and $\mathbf{R}_1$ be homeomorphic copies of $\mathbf{RSW}$, and let $\mathbf{X}$ be a compactification of $R_0 \oplus R_1 \oplus \mathbf{Rea}(\mathcal{G})$ in the sense of Lelek. Then, $\mathbf{X}$ is σ-homeomorphic to $\omega\mathbf{CEA} \oplus \mathbf{Rea}(\mathcal{G})$.

We follow the construction of Elżabieta Pol [56, Example 4.1]. Now, $\mathbf{R}_0$ has a hereditarily strongly infinite dimensional subspace $\mathbf{Y}$ [63]. Choose a point $p \in \mathbf{Y}$ and a closed set $F \subseteq \mathbf{Y}$ containing p such that every separator between p and $\mathrm{cl}_{\mathbf{X}}F$ is strongly infinite dimensional as in [56, Example 4.1 (A)].

Define $\mathbf{K} = \mathbf{X}/\mathrm{cl}_{\mathbf{X}}F$ as in [56, Example 4.1 (A)]. To see that $\mathbf{K}$ is σ-homeomorphic to $\mathbf{X}$, we note that $\mathrm{cl}_{\mathbf{X}}F \cap (\mathbf{R}_1 \cup \mathbf{Rea}(\mathcal{G})) = \emptyset$ since $\mathbf{R}_0$, $\mathbf{R}_1$ and $\mathbf{Rea}(\mathcal{G})$ are separated in $\mathbf{X}$. Therefore, $\mathrm{cl}_{\mathbf{X}}F$ is covered by the union of $\mathbf{R}_0$ (which is homeomorphic to $\mathbf{R}_1$) and a countable dimensional space. Define $\mathbf{Z}$ as a Pol-type Cantor manifold in [56, Example 4.1 (C)]. Then, $\mathbf{Z}(\mathcal{G}) := \mathbf{Z}$ is the union of a finite dimensional space and countably many copies of $\mathbf{K}$. Consequently, $\mathbf{Z}(\mathcal{G})$ is σ-homeomorphic to $\mathbf{Rea}(\mathcal{G})$. □

Proof of Theorem 32. Let S be a countable subset of ω_1. Note that $\sup S$ is countable by regularity of ω_1. Then, there is an oracle p such that $\sup S < \omega_1^{\mathrm{CK},p}$, where $\omega_1^{\mathrm{CK},p}$ is the smallest noncomputable ordinal relative to p. Now, the α-th Turing jump operator j_α^p for $\alpha < \omega_1^{\mathrm{CK},p}$ is defined via a p-computable coding of α. By Spector's uniqueness theorem, the Turing degree of $j_\alpha^p(x)$ for $x \geq_T p$ is independent of the choice of coding of α, so is $\mathcal{J}_a(j_\alpha^p, x)$. Therefore, we simply write j_α for j_α^p.

Define $\mathcal{G}_S = \{j_{\omega^{1+\alpha}} : \alpha \in S\}$. We show that $S \subseteq T$ if and only if $\mathcal{G}_S \subseteq_{aa} \mathcal{G}_T$. Suppose $\alpha \neq \beta$, say $\alpha < \beta$. Clearly, $j_{\omega^\alpha} \leq_{aa} j_{\omega^\beta}$. Suppose for the sake of contradiction that $j_{\omega^\beta} \leq_{aa} j_{\omega^\alpha}$. Then, in particular, for every $x \leq_a \emptyset^{(\omega^\beta \cdot t)}$ with $t \in \mathbb{N}$, we must have $\emptyset^{(\omega^\beta \cdot (t+1))} \leq_a x^{(\omega^\alpha \cdot m)}$ for some $m \in \mathbb{N}$. Thus, there is n such that $\emptyset^{(\omega^\beta \cdot t + \omega^\beta)} \leq_T \emptyset^{(\omega^\beta \cdot t + \omega^\alpha \cdot m + n)} <_T \emptyset^{(\omega^\beta \cdot t + \omega^{\alpha+1})}$. This is a contradiction. □

Corollary 41. There exists a collection $(\mathbf{X}_\alpha)_{\alpha < 2^{\aleph_0}}$ of continuum many Pol-type Cantor manifolds satisfying the following conditions:

1. If $\alpha \neq \beta$, then $\mathbf{X}_\alpha$ is not n-th level Borel isomorphic to $\mathbf{X}_\beta$ for all $n \in \mathbb{N}$.
2. If $\alpha \neq \beta$, then the Banach algebra $\mathcal{B}^*_n(\mathbf{X}_\alpha)$ is not linearly isometric (not ring isomorphic etc.) to $\mathcal{B}^*_n(\mathbf{X}_\beta)$ for all $n \in \mathbb{N}$.

Proof. By Theorems 10 and 32. Here, we note that if $\mathbf{X}$ is n-th level Borel isomorphic to $\mathbf{Y}$, then $\mathbb{N} \times \mathbf{X}$ is again n-th level Borel isomorphic to $\mathbb{N} \times \mathbf{Y}$. $\square$

6.3 An Ordinal Valued σ-Topological Invariant

Although we constructed continuum many mutually different spaces, it is difficult to discern dimension-theoretic differences among these spaces. For instance, all of our spaces have the same transfinite Steinke dimensions [4, 61], game dimensions [16], and so on (see Chatyrko and Hattori [10] for the thorough treatment of the notion of various kinds of transfinite dimensions).

We now focus on an $\aleph_1$ chain of σ-homeomorphism types of Polish spaces:

$$\mathbb{R}^n <^{\mathfrak{T}}_\sigma \mathbf{Rea}(\{j_1\}) <^{\mathfrak{T}}_\sigma \mathbf{Rea}(\{j_\omega\}) <^{\mathfrak{T}}_\sigma \mathbf{Rea}(\{j_{\omega^2}\}) <^{\mathfrak{T}}_\sigma \mathbf{Rea}(\{j_{\omega^3}\}) <^{\mathfrak{T}}_\sigma \dots$$

Our key observation was that closure properties of Scott ideals reflects piecewise homeomorphism types of Polish spaces. The first purpose here is to provide a topological understanding of our method.

Definition 42. Let $\mathbf{X}$ be a topological space. Let $\mathcal{C}({}_\subseteq\mathbf{X},\mathbb{R})$ denote the collection of all continuous functions from subspaces of $\mathbf{X}$ into $\mathbb{R}$. Suppose that a collection $\mathcal{H}({}_\subseteq\mathbb{R},\mathbb{R})$ of functions from subspaces of $\mathbb{R}$ into $\mathbb{R}$ is given. A countable set $F \subseteq \mathcal{C}({}_\subseteq\mathbf{X},\mathbb{R})$ *avoids* $\mathcal{H}$ *on* $\mathbf{X}$ if for any countable set $G \subseteq \mathcal{C}({}_\subseteq\mathbf{X},\mathbb{R})$ and countable set $H \subseteq \mathcal{H}({}_\subseteq\mathbb{R},\mathbb{R})$, there exists a point $x \in \mathbf{X}$ such that

$$(\forall g \in G|_x)(\exists f \in F|_x)(\forall h \in H|_{g(x)})\ f(x) \neq h \circ g(x),$$

where $E|_y := \{f \in E : y \in \mathrm{dom}(f)\}$. We then say that $\mathbf{X}$ is *α-avoiding* if there is a countable set that avoids $\mathcal{B}_\alpha$ on $\mathbf{X}$, where $\mathcal{B}_\alpha$ is the class of all Baire α functions, and $\mathcal{B}_0 = \mathcal{C}$

The *jump dimension* $\mathrm{jdim}(\mathbf{X})$ of $\mathbf{X}$ is the supremum of countable ordinals $\alpha < \omega_1$ such that $\mathbf{X}$ is β-avoiding for all $\beta < \omega^\alpha$, where $\omega^0 = 1$. If such α does not exist, then $\mathrm{jdim}(\mathbf{X}) = -1$. Hence, $\mathrm{jdim}(\mathbf{X}) = -1$ if and only if $\mathbf{X}$ is not 0-avoiding.

This notion provides a new characterization of countable-dimensionality for Polish spaces, and we also see that the jump dimension is invariant under σ-homeomorphism.

Theorem 43. Let $\mathbf{X}$ and $\mathbf{Y}$ be separable metrizable spaces.

1. $\mathbf{X}$ is countable dimensional if and only if $\mathrm{jdim}(\mathbf{X}) = -1$ (i.e., $\mathbf{X}$ is not 0-avoiding).
2. If $\mathbf{X} \leq^{\mathfrak{T}}_\sigma \mathbf{Y}$, then $\mathrm{jdim}(\mathbf{X}) \leq \mathrm{jdim}(\mathbf{Y})$.
3. For every countable ordinal α, there is a Pol-type Cantor manifold $\mathbf{X}$ such that $\mathrm{jdim}(\mathbf{X}) = \alpha$.

To show Theorem 43, we need the following effective interpretation of jump-dimension. We say that $\mathcal{I} \subseteq \{0,1\}^\mathbb{N}$ is *α-principal* if there is $p \in \mathcal{I}$ such that $q \leq_T p^{(\alpha)}$ for all $q \in \mathcal{I}$.

Lemma 44. An admissibly represented separable metrizable space $\mathbf{X}$ is α-avoiding if and only if relative to some oracle r, for all $z \in \{0,1\}^\mathbb{N}$ there is a point $x \in \mathbf{X}$ such that $z \in \mathrm{coSpec}^r(x)$ and $\mathrm{coSpec}^r(x)$ is not α-principal.

Proof. Suppose that $F = \{f_n\}_{n\in\omega}$ is a countable set avoiding $\mathcal{B}_\alpha$ on $\mathbf{X}$. Then, almost all oracles r satisfy that $\mathbf{X}$ is r-computably embedded into Hilbert cube, $\alpha < \omega_1^{\mathrm{CK},r}$ and every $f \in F$ is computable relative to r. Then, clearly $\{f(x) : f \in F,\ x \in \mathrm{dom}(f)\} \subseteq \mathrm{coSpec}^r(x)$ holds for all $x \in \mathbf{X}$. Fix $z \geq_T r$. Let G be the set of all partial z-computable functions from $\mathbf{X}$ into $\mathbb{R}$, and H be the set of all r-effective Baire α functions (i.e., functions of the form $p \mapsto \Phi_e^{p^{(\alpha)}}$). Let $x \in X$ be a point witnessing the avoiding property of F for given G and H. Now, every $p \in \mathrm{coSpec}^r(x)$ is of the form $g(x)$ for some $g \in G$. By the avoiding property of F, there is $f \in F$ such that $f(x) \neq h \circ g(x)$ for all $h \in H$. In other words, for all $p \in \mathrm{coSpec}^r(x)$, we have $f(x) \neq \Phi_e^{p^{(\alpha)}}$ for all $e \in \mathbb{N}$, i.e., $f(x) \not\leq_T p^{(\alpha)}$. Then, put $q := f(x) \in \mathrm{coSpec}^r(x)$. We claim that $z \in \mathrm{coSpec}^r(x)$, i.e., $z \not\leq_M (x, r)$. Otherwise, by Lemma 9, (x, r, z) has a Turing degree since (x, r) has a continuous degree. Therefore, there is $g \in G$ such that $g(x) \equiv_M (x, r, z)$. Thus, $f(x)$ is of the form $h \circ g(x)$ for some r-computable function $h \in H$.

Conversely, suppose that the condition in Lemma 44 holds for r. We show that the set F of all partial r-computable functions from $\mathbf{X}$ into $\mathbb{R}$ avoids $\mathcal{B}_\alpha$ on $\mathbf{X}$. Given G and H, one can find an oracle $z \geq_T r$ such that every $g \in G$ is z-computable and every $h \in H$ is z-effectively Baire α. Then we have a point $x \in \mathbf{X}$ with $z \in \mathrm{coSpec}^r(x)$ such that for all $p \in \mathrm{coSpec}^r(x)$, there exists $q \in \mathrm{coSpec}^r(x)$ such that $q \not\leq_T (z \oplus p)^{(\alpha)}$. Since $\mathrm{coSpec}^r(x) = \mathrm{coSpec}^z(x)$, $\mathrm{coSpec}^r(x)$ contains $g(x)$ for all $g \in G$, and hence, $z \oplus g(x) \in \mathrm{coSpec}^r(x)$. Therefore, $q \neq h \circ g(x)$ since h is of the form $p \mapsto \Phi_e^{(z\oplus p)^{(\alpha)}}$ for some index e. Therefore, F avoids $\mathcal{B}_\alpha$ since $\{f(x) : f \in F,\ x \in \mathrm{dom}(f)\} = \mathrm{coSpec}^r(x)$. $\square$

Proof of Theorem 43. By Miller's result [43], we can deduce that a point $x \in [0,1]^{\mathbb{N}}$ has a Turing degree relative to r if and only if $\mathrm{coSpec}^r(x)$ is principal (i.e., 0-principal). Hence, if $\mathbf{X}$ is countable dimensional, all cospectra are 0-principal up to some oracle. Therefore, $\mathbf{X}$ is not 0-avoiding. Conversely, suppose that $\mathbf{X}$ is not countable dimensional. We claim that for all $z \in \{0,1\}^{\mathbb{N}}$, there is $x \in \mathbf{X}$ such that $z \leq_M x$ and x has no Turing degree. Otherwise, (x, z) has a Turing degree by Lemma 9. In this case, $\mathrm{Spec}^z(\mathbf{X}) \subseteq \mathcal{D}_T$. This implies that $\mathbf{X}$ is countable dimensional. Now, our claim clearly implies the desired condition by Lemma 44.

The second assertion follows from Lemma 44 since the cospectrum is invariant under σ-homeomorphism by Observation 18. Now, we show the third assertion. We first see that the jump-dimension of $\mathbf{Rea}(j_{\omega^\alpha})$ is $\alpha+1$. We have $\mathrm{jdim}(\mathbf{Rea}(j_{\omega^\alpha})) \geq \alpha + 1$ because for any z, there is $x \in \mathbf{Rea}(j_{\omega^\alpha})$ such that $\mathcal{J}_T(j_{\omega^\alpha}, z) \subseteq \mathrm{coSpec}(x) \subseteq \mathcal{J}_a(j_{\omega^\alpha}, z)$ by Lemma 37. If $y \in \mathrm{coSpec}(x)$, then $y \in \mathcal{J}_a(j_{\omega^\alpha}, z)$. Therefore, $y^{(\omega^\alpha+n)} \in \mathcal{J}_T(j_{\omega^\alpha}, z) \subseteq \mathrm{coSpec}(x)$ for all $n \in \mathbb{N}$. Hence, $\mathrm{coSpec}^z(x) = \mathrm{coSpec}(x)$ is closed under the β-th Turing jump for all $\beta < \omega^{\alpha+1}$. To see $\mathrm{jdim}(\mathbf{Rea}(j_{\omega^\alpha})) < \alpha+2$, we note for any $x \in \mathbf{Rea}(j_{\omega^\alpha})$ that $\mathcal{J}_T(j_{\omega^\alpha}, z) \subseteq \mathrm{coSpec}(x) \subseteq \mathcal{J}_a(j_{\omega^\alpha}, z)$ for some z by Lemma 37. Then, $z \in \mathrm{coSpec}(x)$, but $\mathrm{coSpec}(x)$ is covered by the Turing ideal generated by $z^{(\omega^{\alpha+1})}$. If α is a limit ordinal, then consider $\mathbf{X} = \mathbf{Rea}(\{j_{\omega^\beta}\}_{\beta<\alpha})$. $\square$

Example 45.

1. The jump-dimension of Hilbert cube $[0,1]^{\mathbb{N}}$ is ω_1. This is because every countable Scott ideal is realized as a cospectrum in the Hilbert cube [43] and by Lemma 44.
2. The jump-dimension of $\mathbf{Rea}(\mathcal{G})$ cannot be ω_1 for every countable set of $\mathcal{G}$ of oracle $\mathbf{\Pi}^0_2$ singletons. This is because every oracle $\mathbf{\Pi}^0_2$ singleton is Borel measurable. Therefore, there is a countable ordinal α which bounds all Borel ranks of functions contained in $\mathcal{G}$ since $\aleph_1$ is regular. Thus, for any $g \in \mathcal{G}$, we have $g(r) \leq_T (r \oplus z)^{(\alpha)}$ for some oracle z. One can see that the cospectrum of a point in $\mathbf{Rea}(\mathcal{G})$ is $(\alpha \cdot \omega)$-principal.

3. We have $0 \leq \mathrm{jdim}(\omega\mathrm{CEA}) \leq 1$. This is because the cospectrum of a point in ωCEA is ω-principal by the proof of Lemma 20.

7 Internal Characterization of Degree Structures

7.1 Characterizing Continuous Degrees Through a Metrization Theorem

In this section, we will provide a rather strong metrization theorem, namely that any computably admissible space with an effectively fiber-compact representation can be computably embedded in a computable metric space. Our result is a slightly stronger version of a result by SCHRÖDER that an admissible space with a proper representation is metrizable [67]. This also gives us a characterization of the continuous degrees inside the Medvedev degrees that does not refer to represented spaces at all.

For some closed set $A \subseteq \{0,1\}^{\mathbb{N}}$, let $T(A) \subseteq \{0,1\}^{\mathbb{N}}$ be the set of trees for A, where each infinite binary tree is identified with an element of Cantor space. Now let $\delta :\subseteq \{0,1\}^{\mathbb{N}} \to \mathbf{X}$ be an *effectively fiber-compact representation*, i.e. let $x \mapsto \delta^{-1}(\{x\}) : \mathbf{X} \to \mathcal{A}(\{0,1\}^{\mathbb{N}})$ be computable. Then $T(\delta^{-1}(\{x\})) \leq_M \delta^{-1}(\{x\})$. If δ is computably admissible, we also have $\delta^{-1}(\{x\}) \leq_M T(\delta^{-1}(\{x\}))$. Note that being effectively fiber-compact is equivalent to being effectively proper, as the union of compactly many compact sets is compact. It is known that any computable metric space has a computably admissible effectively fiber-compact representation (e.g. [77]). We shall prove that the converse holds, too.

Theorem 46. A represented space $\mathbf{X}$ admits a computably admissible effectively fiber-compact representation iff $\mathbf{X}$ embeds computably into a computable metric space.

Corollary 47. $A \subseteq \{0,1\}^{\mathbb{N}}$ has continuous degree iff there is $B \in \mathcal{A}(\{0,1\}^{\mathbb{N}})$ such that $A \equiv_M B \equiv_M T(B)$.

To prove Theorem 46, we need the following two lemmata and a result by WEIHRAUCH.

Lemma 48. Let $\mathbf{X}$ admit an effectively fiber-compact representation. Then there is a space $\mathbf{Y}$ such that:

1. $\mathbf{X} \hookrightarrow \mathbf{Y}$ (as a closed subspace),
2. $\mathbf{Y}$ has an effectively fiber-compact representation,
3. $\mathbf{Y}$ has a computable dense sequence,
4. if $\mathbf{X}$ is computably admissible, so is $\mathbf{Y}$.

Proof. **Construction of Y:** We start with some preliminary technical notation. Let $\mathrm{Wrap} : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be defined by $\mathrm{Wrap}(p)(2i) = p(i)$ and $\mathrm{Wrap}(p)(2i+1) = 0$. Let $\mathrm{Prefix} :\subseteq \{0,1\}^{\mathbb{N}} \to \{0,1\}^*$ be defined by $\mathrm{Prefix}(p) = w$ iff $p = 0w(1)0w(2)0\ldots011q$ for some $q \in \{0,1\}^{\mathbb{N}}$. Note that $\mathrm{dom}(\mathrm{Prefix}) \cap \mathrm{dom}(\mathrm{Wrap}^{-1}) = \emptyset$ and $\mathrm{dom}(\mathrm{Prefix}) \cup \mathrm{dom}(\mathrm{Wrap}^{-1}) = \{0,1\}^{\mathbb{N}}$.

Let the presumed representation of $\mathbf{X}$ be $\delta_{\mathbf{X}} :\subseteq \{0,1\}^{\mathbb{N}} \to X$. Our construction of $\mathbf{Y}$ will utilize a notation $\nu_{\mathbf{Y}} : \{0,1\}^* \to Y'$ as auxiliary part, this notation (or alternatively, equivalence relation on $\{0,1\}^*$) will be dealt with later. We set $Y = X \cup Y'$ (in particular, we add only countably many elements to $\mathbf{X}$) and then define $\delta_{\mathbf{Y}}$ via $\delta_{\mathbf{Y}}(p) := \delta_{\mathbf{X}}(\mathrm{Wrap}^{-1}(p))$ if $p \in \mathrm{dom}(\delta_{\mathbf{X}} \circ \mathrm{Wrap}^{-1})$ and $\delta_{\mathbf{Y}}(p) = \nu_{\mathbf{Y}}(\mathrm{Prefix}(p))$ if $p \in \mathrm{dom}(\mathrm{Prefix})$.

In order to define $\nu_{\mathbf{Y}}$, we do need to refer to the effective fiber-compactness of $\delta_{\mathbf{X}}$. From the function realizing $x \mapsto \delta_{\mathbf{X}}^{-1}(\{x\}) : \mathbf{X} \to \mathcal{A}(\{0,1\}^{\mathbb{N}})$ we can obtain an indexed family of finite trees $(T_w)_{w \in \{0,1\}^*}$ with the following properties:

1. Each T_w has height $|w|$.
2. If $w \prec u$, then $T_u \cap \{0,1\}^{\leq |w|} = T_w$.
3. $w \in T_w$.
4. For any $p \in \operatorname{dom}(\delta_{\mathbf{X}})$, some $q \in \{0,1\}^{\mathbb{N}}$ is an infinite path through $\bigcup_{n\in\mathbb{N}} T_{p_{\leq n}}$ iff $\delta_{\mathbf{X}}(q) = \delta_{\mathbf{X}}(p)$.

Now we set $\nu_{\mathbf{Y}}(w) = \nu_{\mathbf{Y}}(u)$ iff $T_w = T_u$. Note in particular that $T_w = T_u$ is a decidable property.

Proof of the properties: To see that $\mathbf{X} \hookrightarrow \mathbf{Y}$ it suffices to note that both Wrap and Wrap^{-1} are computable. That $\mathbf{X}$ embeds as a closed subspace follows from $\operatorname{dom}(\mathrm{Wrap}^{-1})$ being closed in $\{0,1\}^{\mathbb{N}}$.

Next we shall see that $\delta_{\mathbf{Y}}$ is effectively fiber-compact by reversing the step from the function $x \mapsto \delta_{\mathbf{X}}^{-1}(\{x\}) : \mathbf{X} \to \mathcal{A}(\{0,1\}^{\mathbb{N}})$ to the family $(T_w)_{w\in\{0,1\}^*}$. First, we define a version of Wrap for finite trees via $\text{T-Wrap}(T) = \{0w(1)0w(2)\ldots w(|w|) \mid w \in T\} \cup \{0w(1)0w(2)\ldots w(|w|)0 \mid w \in T\}$. Given some set $W \subseteq \{0,1\}^*$, let the induced tree of height be defined via $T(W,n) = \{u \exists w \in W \ u \prec w\} \cup \{u \in \{0,1\}^n \mid \exists w \in W \ \wedge w \prec u\}$. Then we define a derived family $(T'_w)_{w\in\{0,1\}^*}$ by $T'_{0w(1)0w(2)\ldots 0w(|w|)} = T'_{0w(1)0w(2)\ldots 0w(|w|)0} = \text{T-Wrap}(T_w)$ and $T'_{0w(1)\ldots w(|w|)1v} = T(\{u \mid T_u = T_w\}, |w|+|v|)$. This construction too satisfies that if $w \prec u$, then $T'_u \cap \{0,1\}^{\leq \mathrm{height}(T'(w))} = T'_w$. Thus, the function that maps p to the set of all infinite pathes through $\bigcup_{n\in\mathbb{N}} T'_{p_{\leq n}}$ does define some function $t : \{0,1\}^{\mathbb{N}} \to \mathcal{A}(\{0,1\}^{\mathbb{N}})$, and one can verify readily that $t(p) = \delta_{\mathbf{Y}}^{-1}(\delta_{\mathbf{Y}}(p))$ whenever $p \in \operatorname{dom}(\delta_{\mathbf{Y}}(p))$.

It is clear that $\mathbf{Y}$ has a computable dense sequence: Fix some standard enumeration $\nu : \mathbb{N} \to \{0,1\}^*$, and consider $(y_n)_{n\in\mathbb{N}}$ with $y_n = \delta_{\mathbf{Y}}(0\nu(n)(1)\ldots 0\nu(n)(|\nu(n)|)1^\omega)$.

It remains to show that if $\delta_{\mathbf{X}}$ is admissible, so is $\delta_{\mathbf{Y}}$. It is this step which requires the identification of some points via $\nu_{\mathbf{Y}}$, and through this, also depends on $\delta_{\mathbf{X}}$ being effectively-fiber-compact. Given some tree encoding some $\delta_{\mathbf{Y}}^{-1}(\{x\})$, we need to be able to compute a path through it. As long as the tree seems to have a path without repeating 1's, we lift the corresponding map for $\delta_{\mathbf{X}}$. If $x = \nu_{\mathbf{Y}}(w)$ for some $w \in \{0,1\}^*$, we notice eventually, and can extend the current path in a computable way by virtue of the identifications. □

The preceding lemma produces spaces with a somewhat peculiar property: The designated dense sequence is an open subset of the space, unlike the usual examples. In [19], Gregoriades has explored a general construction yielding Polish spaces with such properties (cf. [21, Theorem 2.5]), which in particular serves to prevent effective Borel isomorphisms between spaces.

Lemma 49. Let $\mathbf{X}$ admit a computably admissible effectively fiber-compact representation. Then $\mathbf{X}$ is computably regular.

Proof. The properties of the representations mean that we can consider $\mathbf{X}$ as a subspace of $\mathcal{A}(\{0,1\}^{\mathbb{N}})$ containing only pair-wise disjoint sets. Let $A \in \mathcal{A}(\mathcal{A}(\{0,1\}^{\mathbb{N}}))$ be a closed subset in $\mathbf{X}$. Note that we can compute $\bigcup A \in \mathcal{A}(\{0,1\}^{\mathbb{N}})$, as every infinite path computes the relevant tree. Furthermore, given $x \in \mathbf{X} \subseteq \mathcal{A}(\{0,1\}^{\mathbb{N}})$ and A, we can compute $x \cap \bigcup A \in \mathcal{A}(\{0,1\}^{\mathbb{N}})$. As $\mathbf{X}$ only contains pair-wise disjoint points, this set is empty if and only if x and A are disjoint. As $\{0,1\}^{\mathbb{N}}$ is compact, the corresponding tree will have to die out at some finite level, which means that the trees for x and A are disjoint below this level. Let I be the vertices at this level belonging to x. We may now define two open sets $U_I, U_{I^C} \in \mathcal{O}(\mathcal{A}(\{0,1\}^{\mathbb{N}}))$ by letting U_X for $X \in \{I, I^C\}$ accept its input sets A as soon as $A \cap X\{0,1\}^{\mathbb{N}} = \emptyset$ is verified. Then $U_I \cap U_{I^C} = \{\emptyset\}$, thus $\mathbf{X} \cap U_I$ and $\mathbf{X} \cap U_{IC}$ are disjoint open sets. Moreover, we find $A \subseteq U_I$ and $x \in U_{IC}$, so the two open sets are those we needed to construct for computable regularity. □

Proof of Theorem 46. The ⇐-direction is present e.g. in [77]. We can use Lemma 48 to make sure w.l.o.g. that $\mathbf{X}$ has a computable dense sequence. By Lemma 49, the space is computably regular. As shown in [22, 76], a computably regular space with a computable dense sequence admits a compatible metric. □

Miller showed that the Turing degrees below any non-total continuous degree form a Scott ideal [43], heavily drawing on topological arguments. However, based on Corollary 47 we see that the statement itself can be phrased entirely in the language of trees, points and Medvedev reducibility. So far, we do not know of a direct proof involving only these concepts:

Proposition 50. Let $A \subseteq \{0,1\}^{\mathbb{N}}$ be such that $A \equiv_M T(A)$ and that there is no $r \in \{0,1\}^{\mathbb{N}}$ with $A \equiv_M \{r\}$. Then $T(B) \leq_M \{p\} <_M A$ for $p \in \{0,1\}^{\mathbb{N}}$, $B \subseteq \{0,1\}^{\mathbb{N}}$ implies $B \leq_M A$.

7.2 Enumeration Degrees and Overtness

The often overlooked dual notion to compactness is *overtness* (see [73, 74]). Intuitively, overtness makes existential quantification well-behaved: a space $\mathbf{X}$ is overt if $E_{\mathbf{X}} : \mathcal{O}(\mathbf{X}) \to \mathcal{S}$ is continuous, where $E_{\mathbf{X}}(U) = \top$ iff U is nonempty. Therefore, if $\mathbf{X}$ is overt and $P \subseteq \mathbf{X} \times \mathbf{Y}$ is open, then $\{y \in \mathbf{Y} \mid \exists x \in \mathbf{X}\ (x,y) \in P\}$ is open, too. Classically, this is a trivial notion, however, the situation is different from an effective point of view.

One may identify an overt subspace A of $\mathbf{X}$ with E_A, or equivalently, its overtness witness $\{U \in \mathcal{O}(\mathbf{X}) : A \cap U \neq \emptyset\}$ as a point in the represented space $\mathcal{O}(\mathcal{O}(\mathbf{X}))$. Via this identification, we obtain the hyperspace $\mathcal{V}(\mathbf{X})$ of representatives $\overline{A}$ of all overt subspaces A of $\mathbf{X}$ (see also [51]). Note that this corresponds to the lower Vietoris topology on the hyperspace of closed sets. A computable point in $\mathcal{V}(\mathbf{X})$ is also called a *c.e. closed* set in computable analysis.

Now we call a representation $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbf{X}$ *effectively fiber-overt*, iff $\overline{\delta^{-1}} : \mathbf{X} \to \mathcal{V}(\mathbb{N}^{\mathbb{N}})$ is computable. A straightforward argument shows that this is equivalent to δ being effectively open, i.e. $U \mapsto \delta[U] : \mathcal{O}(\mathbb{N}^{\mathbb{N}}) \to \mathcal{O}(\mathbf{X})$ being computable. Now we see that every space with an effectively fiber-overt representation inherits an effective countable basis from $\mathbb{N}^{\mathbb{N}}$, while on the other hand, the standard representations of countably based spaces introduced in Example 4 are all effectively fiber-overt. Thus we see that while effectively fiber-compact representations characterize metrizability, effectively fiber-overt representations characterize second-countability.

8 Point Degree Spectra of Quasi-Polish Spaces

8.1 Lower Reals and Semirecursive Enumeration Degrees

Let us move on to the σ-isomorphic classification of quasi-Polish spaces [8]. We now focus on the following chain of quasi-Polish spaces:

$$\mathcal{D}_T \subsetneq \mathcal{D}_r \subsetneq \mathcal{D}_e, \text{ and } \{0,1\}^{\mathbb{N}} <_\sigma^{\mathfrak{T}} [0,1]^{\mathbb{N}} <_\sigma^{\mathfrak{T}} \mathcal{O}(\mathbb{N}).$$

Here, the proper inclusion $[0,1]^{\mathbb{N}} <_\sigma^{\mathfrak{T}} \mathcal{O}(\mathbb{N})$ follows from relativizing Miller's observation in [43] that no quasi-minimal degree has continuous degree.

In quasi-Polish case, the notion of the specialization order is quite useful. Indeed, Motto Ros has already used the specialization order to give an alternative way to show the properness of $[0,1]^{\mathbb{N}} <_\sigma^{\mathfrak{T}} \mathcal{O}(\mathbb{N})$. Recall that the *specialization order* $\prec$ on a topological space $\mathbf{X}$ is defined via $x \prec y :\Leftrightarrow x \in \overline{\{y\}}$. In particular, the specialization order on $\mathcal{O}(\mathbb{N})$ coincides with subset-inclusion. The T_1 separation property asserts that no two elements are comparable w.r.t. $\prec$, i.e. that the specialization order is a single antichain.

Theorem 51. There is a map $\mathbf{Q}$ transforming each countable set $S \subseteq \omega_1$ into a nonmetrizable quasi-Polish space $\mathbf{Q}(S) \not\leq_\sigma^{\mathfrak{T}} [0,1]^{\mathbb{N}}$ such that

$$S \not\subseteq T \implies \mathbf{Q}(S) \not\leq_\sigma^{\mathfrak{T}} \mathbf{Q}(T).$$

Let $\mathbb{R}_<$ be the real line endowed with the lower topology, that is, its topology is generated by open intervals of the form (p,∞). One can easily see that $\mathbb{R} \mid_\sigma^{\mathfrak{T}} \mathbb{R}_<$ by comparing their specialization orders. From the computability theoretic viewpoint, the property $\mathbb{R} \not\leq_\sigma^{\mathfrak{T}} \mathbb{R}_<$ can be strengthened as follows.

Lemma 52 (Co-spectrum Preservation)**.** Let $\mathbf{X}$ be an admissibly represented Polish space. Then,

$$\mathrm{coSpec}(\mathbf{X} \times \mathbb{R}_<) \subseteq \mathrm{coSpec}(\mathbf{X}) \cup \mathrm{coSpec}(\{0,1\}^{\mathbb{N}}).$$

In particular, if such an $\mathbf{X}$ is uncountable, then there is an oracle $r \in \{0,1\}^{\mathbb{N}}$ such that

$$\mathrm{coSpec}^r(\mathbf{X} \times \mathbb{R}_<) = \mathrm{coSpec}^r(\mathbf{X}).$$

Lemma 53. Let $\mathbf{X}$ admit an effectively fiber-overt representation $\delta_{\mathbf{X}}$ (cf. Subsection 7.2), $x \in \mathbf{X}$, $y \in \mathbb{R}_<$, and $z \in \{0,1\}^{\mathbb{N}}$. If $z \leq_M (x,y)$, then either $z \leq_M x$ or $-y \leq_M x$ holds.

Proof. Let computable $f :\subseteq \mathbf{X} \times \mathbb{R}_< \to \{0,1\}^{\mathbb{N}}$ witness the reduction $z \leq_M (x,y)$. By extending the domain of f if necessary, it can be identified with a c.e. open set $U \subseteq \{0,1\}^{\mathbb{N}} \times \mathbb{Q} \times \mathbb{N} \times \{0,1\}$ satisfying that $f(x,y)(n) = i$ if and only if the following two condition holds:

1. For any $p \in \delta_{\mathbf{X}}^{-1}(x)$ there is some rational $s < y$ such that $(p,s,n,i) \in U$.
2. For any $p \in \delta_{\mathbf{X}}^{-1}(x)$ and any rational $s < y$, $(p,s,n,1-i) \notin U$.

As $\delta_{\mathbf{X}}$ is effectively fiber-overt, the set $U' := \{(x',t,n,i) \mid \exists p \in \delta_{\mathbf{X}}^{-1}(x')\ (p,t,n,i) \in U\}$ is also computable as an open subset of $\mathbf{X} \times \mathbb{Q} \times \mathbb{N} \times \{0,1\}$. Now we can distinguish two cases:

1. For any $\varepsilon > 0$, there exist rationals $t < s < y + \varepsilon$ such that $(x,t,n,i) \in U'$ and $(x,s,n,1-i) \in U'$ for some $n \in \mathbb{N}$ and $i \in \{0,1\}$.
2. Otherwise, there exists $\varepsilon > 0$ such that for all $t < y + \varepsilon$, if $(x,t,n,i) \in U'$ for some $n \in \mathbb{N}$ and $i \in \{0,1\}$, then we must have $i = f(x,y)(n)$.

Note that if $(x,t,n,i) \in U'$ and $(x,s,n,1-i) \in U'$ for $t < s$, then we automatically have $y \leq s$. Therefore, in the first case we can compute $-y \in \mathbb{R}_<$ as the supremum of $-s$ over all witnesses s, thus find $-y \leq_M x$. In the second case, there will be some rational number y_0 with $y \leq y_0 < y + \varepsilon$. Using y_0 in place of y leaves the value f is producing unchanged, thus we have that $z \leq_M x$. $\square$

Proof of Lemma 52. Suppose that $y \in \mathbb{R}_<$ and $x \in \mathbf{X}$. If $-y \not\leq_M x$, then $\mathrm{coSpec}(x,y) = \mathrm{coSpec}(x)$ by Lemma 53. Otherwise, $(x,y,-y) \equiv_M (x,y)$. If $y \leq_M x$, then clearly, $\mathrm{coSpec}(x,y) = \mathrm{coSpec}(x)$. Otherwise, $(y,-y) \not\leq_M x$. Obviously, $(y,-y)$ has Turing degree. By Lemma 9, we have $(x,y,-y) \in \mathcal{D}_T$. Hence, $\mathrm{coSpec}(x,y) \in \mathrm{coSpec}(\{0,1\}^{\mathbb{N}})$. For the latter half of Lemma 52, if $\mathbf{X}$ is uncountable, then there is an r-computable embedding of $\{0,1\}^{\mathbb{N}}$ into $\mathbf{X}$ for some oracle r. □

Proof of Theorem 51. Let $\mathcal{G}_S$ be the countable set of monotone oracle $\mathbf{\Pi}^0_2$ singletons constructed in the proof of Theorem 32. By Lemma 52, the quasi-Polish space $\mathbf{Q}(S) := \mathbf{Rea}(\mathcal{G}_S) \times \overline{\mathbb{R}}_<$ has the same cospectrum as $\mathbf{Rea}(\mathcal{G}_S)$, where $\overline{\mathbb{R}}_< := \mathbb{R}_< \cup \{\infty\}$ is a quasi-completion of $\mathbb{R}_<$. By the proofs of Theorem 32 and Lemma 34, if $S \not\subseteq T$, then the cospectrum of $\mathbf{Rea}(\mathcal{G}_S)$ is not a sub-cospectrum of $\mathbf{Rea}(\mathcal{G}_T)$ relative to all oracles. Therefore, by Observation 18, we have $\mathbf{Q}(S) \not\leq^{\mathfrak{T}}_{\sigma} \mathbf{Q}(T)$. □

As a consequence of Lemma 52, any lower real can compute only a Δ^0_2 real:

$$\mathrm{coSpec}(\mathbb{R}_<) = \{\{x \in \{0,1\}^{\mathbb{N}} : x \leq_T y\} : y \text{ is right-c.e.}\}$$

Indeed, Lemma 53 provides a very simple and natural construction of a quasi-minimal enumeration degree.

Corollary 54 (see also Arslanov, Kalimullin & Cooper [5, Theorem 4])**.** Suppose that $z \in \mathbb{R}$ is neither left-c.e nor right-c.e. Then, the enumeration degree of the cut $\{q \in \mathbb{Q} : q < z\}$ is quasi-minimal.

On the one hand, we deduced the property $[0,1]^{\mathbb{N}} <^{\mathfrak{T}}_{\sigma} \mathcal{O}(\mathbb{N})$ from the topological argument concerning the specialization order on the lower real $\mathbb{R}_<$. On the other hand, Miller's original proof used the existence of a quasi-minimal enumeration degree to show $\mathrm{Spec}([0,1]^{\mathbb{N}}) \subsetneq \mathrm{Spec}(\mathcal{O}(\mathbb{N}))$. Surprisingly, however, the previous argument clarifies that these two seemingly unrelated approaches are essentially equivalent.

Note that the point degree spectrum of the lower real $\mathbb{R}_<$ is indeed strongly connected with the notion of a semirecursive set in the context of the enumeration degrees. Recall from [34] that a set $A \subseteq \mathbb{N}$ is called *semirecursive*, if there is a computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ we find $f(n,m) \in \{n,m\}$, and if $n \in A$ or $m \in A$, then $f(n,m) \in A$. We call an enumeration degree $q \in \mathcal{D}_e$ semirecursive, if it is the degree of a semirecursive point in $\mathcal{O}(\mathbb{N})$.

Jockusch [34] pointed out that every left-cut (i.e., every lower real $x \in \mathbb{R}_<$) is semirecursive, and conversely, Ganchev and Soskova [18] showed that every semirecursive enumeration degree contains a left-cut. Consequently, the point degree spectra of the lower real $\mathbb{R}_<$ can be characterized as follows:

$$\mathrm{Spec}(\mathbb{R}_<) = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \text{ is a semirecursive enumeration degree}\}.$$

8.2 Higher Dimensional Lower Cubes

We can also consider the higher dimensional lower real cubes $\mathbb{R}^n_<$. Surprisingly, the spectra of $\mathbb{R}^n_<$ form a proper hierarchy as follows.

Theorem 55. If $\mathbf{X}$ is a second-countable T_1 space, then $\mathbb{R}^{n+1}_< \,|^{\mathfrak{T}}_{\sigma}\, \mathbf{X} \times \mathbb{R}^n_<$ for every n.

To show the above theorem, we use the following order theoretic lemma. Let $\Lambda^n = (\{0,1\}^n, \leq)$ be a partial order on $\{0,1\}^n$ obtained as the n-th product of the ordering $0 < 1$.

Lemma 56. For every countable partition $(P_i)_{i\in\omega}$ of the n-dimensional hypercube $[0,1]^n$ (endowed with the standard product order), there is $i \in \omega$ such that P_i has a subset which is order isomorphic to the product order Λ^n.

Proof. We use Vaught's "non-meager" quantifier $\exists^* x\varphi(x)$, which states that the set $\{x : \varphi(x)\}$ is not meager in $[0,1]$ (with respect to the standard Euclidean topology). We claim that for every countable partition $(P_i)_{i\in\omega}$ of $[0,1]^n$, there is $i \in \omega$ such that

$$\exists^* x_1 \exists^* x_2 \dots \exists^* x_n \ (x_1, x_2, \dots, x_n) \in P_i$$

Inductively assume that the above claim is true for $n-1$. If the above claim does not hold for n, then by the Baire category theorem, there are comeager many x_1 such that

$$\neg\exists^* x_2 \dots \exists^* x_n \ (x_1, x_2, \dots, x_n) \in \bigcup_i P_i.$$

However, for any such x_1, by the induction hypothesis, the x_1-sections of P_i's do not cover the x_1-section of $[0,1]^n$. In particular, $\bigcup_i P_i$ cannot cover the n-hypercube $[0,1]^n$, which verifies the claim.

Now, let S be a nonmeager set consisting of all x_1's in the above claim. Note that since there are non-meager many $x_1 \in S$, there is a nonempty open set U such that for any nonempty open set $V \subseteq U$, one can find uncountably many such $x_1 \in V \cap S$. Otherwise, S is covered by the closure of the union of the collection $\mathcal{B}$ of all rational open balls B such that $B \cap S$ is countable. Therefore, S is divided into the union of the nowhere dense set $\partial \bigcup \mathcal{B}$ and the countable set $\bigcup_{B\in\mathcal{B}} B \cap S$, which contradicts the fact that S is nonmeager. We fix such a nonempty open set U.

Now, for any $x_1 \in S$, we may inductively assume that the x_1-th section of P_i has a subset $L(x_1)$ which is order isomorphic to Λ^{n-1}. Let $\hat{L}(x_1)$ be the region bounded by $L(x_1)$, which is homeomorphic to $[0,1]^{n-1}$. We may also inductively assume that P_i is dense in $\hat{L}(x_1)$. Therefore, since $\hat{L}(x_1)$ for any $x_1 \in S$ has positive $(n-1)$-dimensional Lebesgue measure, for any nonempty open set $V \subseteq U$ one can find $x_1^0 < x_1^1$ in $V \cap S$ such that the intersection $\pi \circ \hat{L}(x_1^0) \cap \pi \circ \hat{L}(x_1^1)$ also has positive $(n-1)$-dimensional Lebesgue measure, where $\pi : [0,1]^n \to [0,1]^{n-1}$ is the projection defined by $\pi(x_1, x_2, \dots, x_n) = (x_2, \dots, x_n)$. By density of P_i, one can find a smaller $(n-1)$-cubes $L^*(x_1^0), L^*(x_1^1) \subseteq \pi \circ \hat{L}(x_1^0) \cap \pi \circ \hat{L}(x_1^1)$ such that $(\{x_1^0\} \times L^*(x_1^0)) \cup (\{x_1^1\} \times L^*(x_1^1)) \subseteq P_i$ is order isomorphic to Λ^n. □

Proof of Theorem 55. Note that the specialization order on the space $\mathbb{R}^{n+1}_<$ is exactly the same as the standard product order on $\mathbb{R}^{n+1}$. By Lemma 55, for every countable partition $(P_i)_{i\in\omega}$ of $\mathbb{R}^{n+1}_<$, there is $i \in \omega$ such that the specialization order on P_i has a subset which is order isomorphic to the product order Λ^{n+1} whose order dimension is $n+1$. If P_i is embedded into the specialization order on $\mathbf{X} \times \mathbb{R}^n_<$, then the embedded image of an isomorphic copy of Λ^{n+1} has to be contained in a connected component of the order of $\mathbf{X} \times \mathbb{R}^n_<$. However, the specialization order on $\mathbf{X} \times \mathbb{R}^n_<$ is now $\mathrm{card}(\mathbf{X})$ many copies of that on $\mathbb{R}^n_<$ since $\mathbf{X}$ is T_1. Therefore, every connected component of the specialization order on $\mathbf{X} \times \mathbb{R}^n_<$ is isomorphic to the product order on $\mathbb{R}^n$ whose order dimension is n. Hence, $\mathbb{R}^{n+1}_<$ cannot be σ-embedded into $\mathbf{X} \times \mathbb{R}^n_<$.

Conversely, suppose that $\mathbf{X}\times\mathbb{R}^n_<$ is σ-embedded into $\mathbb{R}^{n+1}_<$. By Lemma 55, for every countable partition $(P_i)_{i\in\omega}$ of $\mathbf{X}\times\mathbb{R}^n_<$, there must exist $i\in\omega$ such that P_i contains a uncountable family $(\Lambda^n_\alpha)_{\alpha\in\aleph_1}$ of pairwise incomparable suborders of P_i which are order isomorphic to Λ^n.

Let L_α be the embedded image of Λ^n_α in $\mathbb{R}^{n+1}_<$, and $\hat{L}_\alpha$ be the region bounded by L_α, which is homeomorphic to $[0,1]^n$. As in the proof of Lemma 55, we may also assume that the embedded image $P_i^*\subseteq\mathbb{R}^{n+1}_<$ of P_i is dense in $\hat{L}_\alpha$ for any $\alpha<\aleph_1$. For any $\alpha<\aleph_1$, the projection $\pi_k[\hat{L}_\alpha]=\{(x_0,\dots,x_{k-1},x_{k+1},\dots,x_n):(x_0,\dots,x_n)\in\hat{L}_\alpha\}$ of $\hat{L}_\alpha$ for some $k\leq n$ has positive n-dimensional Lebesgue measure. Fix $k<n+1$ such that $\pi_k[\hat{L}_\alpha]$ has positive n-dimensional Lebesgue measure for uncountably many α. Then, there are $\alpha\neq\beta$ such that $\pi_k[\hat{L}_\alpha]\cap\pi_k[\hat{L}_\beta]$ also has positive $(n+1)$-dimensional Lebesgue measure. It is not hard to see that it contradicts our assumption that L_α and L_β are incomparable. □

Note that Theorem 55 has immediate computability-theoretic corollaries:

Corollary 57. For every $n\in\mathbb{N}$ there are enumeration degrees p_n, q_n such that

- p_n is the product of $n+1$ semirecursive degrees, but not of n semirecursive degrees and a Turing degree.
- q_n is the product of n semirecursive degrees and a Turing degree, but not of $n-1$ semirecursive degrees and a Turing degree, or of $n+1$ semirecursive degrees.

8.3 The co-spectrum of a Universal Quasi-Polish Space

Recall that the co-spectrum of the universal Polish space $[0,1]^{\mathbb{N}}$ consists of all principal countable Turing ideals and all countable Scott ideals. However, there are many non-principal countable Turing ideals that are not Scott ideals, e.g., countable ω-models of $\mathsf{WWKL}+\neg\mathsf{WKL}$, $\mathsf{RT}^2_2+\neg\mathsf{WKL}$ and so on. We now see that every countable Turing ideal is realized as a co-spectrum of the universal quasi-Polish space $\mathcal{O}(\mathbb{N})$ by modifying the standard forcing construction of quasi-minimal enumeration degrees.

Theorem 58. Every countable Turing ideal is realized as a co-spectrum in the universal quasi-Polish space $\mathcal{O}(\mathbb{N})$. In particular, $\mathrm{coSpec}^r([0,1]^{\mathbb{N}})\subsetneq\mathrm{coSpec}^r(\mathcal{O}(\mathbb{N}))$ for every $r\in\mathbb{N}$.

Proof. It suffices to show that, for any sequence $(x_i)_{i\in\mathbb{N}}$ of reals and oracle r, there is $A\in\mathcal{O}(\mathbb{N})$ whose r-co-spectrum $\mathrm{coSpec}^r(A)$ is equal to all $y\in\{0,1\}^{\mathbb{N}}$ such that $y\leq_T r\oplus\bigoplus_{m\leq n}x_m$ for some $n\in\mathbb{N}$. Without loss of generality, we may assume that $x_0=r$. Suppose $\perp\not\in\mathbb{N}$, and let $\mathbb{N}_\perp=\mathbb{N}\cup\{\perp\}$. We say that a sequence $\sigma\in\mathbb{N}_\perp$ strongly extends $\tau\in\mathbb{N}_\perp$ if τ is an initial segment of σ as a $\mathbb{N}_\perp$-valued sequence. A sequence $\sigma\in\mathbb{N}_\perp$ extends $\tau\in\mathbb{N}_\perp$ if σ extends τ as a partial function on $\mathbb{N}$, where the equality $\sigma(n)=\perp$ is interpreted as meaning that $\sigma(n)$ is undefined, that is, $n\not\in\mathrm{dom}(\sigma)$.

Every partial function $\varphi:\subseteq\mathbb{N}\to\mathbb{N}$ generates a tree $T_\varphi\subseteq\mathbb{N}_\perp^{<\omega}$ by

$$T_\varphi=\{\sigma\in\mathbb{N}_\perp^{<\omega}:(\forall n<|\sigma|)\;\varphi(n)\downarrow\to\;\sigma(n)=\varphi(n)\}.$$

Let $\mathbb{P}$ be the collection of pairs (σ,φ) of a string $\sigma\in\mathbb{N}_\perp^{<\omega}$ and a partial function φ such that $\sigma\in T_\varphi$ and $\mathrm{dom}(\varphi)$ is of the form $D(A)=\{(m,n):n\in\mathbb{N}\ \&\ m\in A\}$ for some finite set $A\subseteq\mathbb{N}$. We write $(\tau,\psi)\leq(\sigma,\varphi)$ if τ strongly extends σ, ψ extends φ, and $\psi\upharpoonright|\sigma|=\varphi\upharpoonright|\sigma|$.

By induction, we assume that (σ_0,φ_0) is the pair of an empty string and an empty function, and $(\sigma_s,\varphi_s)\in\mathbb{P}$ has already been defined. Moreover, we inductively assume that the tree T_{φ_s}

is computable in $\bigoplus_{2t<s} x_t$. We now have $\mathrm{dom}(\varphi_s) = D(A_s)$ for some $s \in \mathbb{N}$ by the definition of $\mathbb{P}$. If $s = 2e$ for some $e \in \mathbb{N}$, then choose sufficiently large $m_{s+1} \notin A_s$ with $m_{s+1} > |\sigma_s|$. Then, put $\sigma_{s+1} = \sigma_s$, and define $\varphi_{s+1}(m_{s+1}, n) = x_e(n)$ for every $n \in \mathbb{N}$. Clearly, the tree $T_{\varphi_{s+1}}$ is computable in $\bigoplus_{2t\leq s} x_t$.

If $s = 2e + 1$ for some $e \in \mathbb{N}$, we look for a string $\tau \in T_{\varphi_s}$ strongly extending σ_s which forces the e-th computation Ψ_e to be inconsistent, that is, two different values $\Psi_e(\tau)(n) = i$ and $\Psi_e(\tau)(n) = j$ for some n and $i \neq j$ are enumerated. If there is such a τ, define $\sigma_{s+1} = \tau$ and $\varphi_{s+1} = \varphi_s$.

If there is no such a τ, we look for strings $\eta, \theta \in T_{\varphi_s}$ strongly extending σ_s such that the e-th computations Ψ_e on η and θ split and are consistent, that is, the consistent computations $\Psi_e(\eta)(n) = i$ and $\Psi_e(\theta)(n) = j$ for some n and $i \neq j$ are enumerated. In this case, for a sufficiently large $k > \max |\eta|, |\theta|$, define σ_{s+1} to be the rightmost node of T_{φ_s} strongly extending σ_s, where we declare that $\perp$ is the rightmost element in $\mathbb{N}_\perp$ in the sense that $n < \perp$ for every $n \in \mathbb{N}$. Note that η (resp. θ) (non-strongly) extends $\sigma_{s+1} \upharpoonright |\eta|$ (resp. $\sigma_{s+1} \upharpoonright |\theta|$) since σ_{s+1} chooses as many $\perp$'s as possible. Then, define $\varphi_{s+1} = \varphi_s$.

Otherwise, define $\sigma_{s+1} = \sigma_s$ and $\varphi_{s+1} = \varphi_s$. Finally, we obtain a partial function Φ on $\mathbb{N}$ by combining $\{\varphi_s\}_{s\in\mathbb{N}}$.

As in the usual argument, we will show that Φ is quasi-minimal above the collection $\{\bigoplus_{m\leq n} x_m : n \in \mathbb{N}\}$. Clearly, $\bigoplus_{m\leq e} x_m$ is computable in Φ by our strategy at stage $2e$.

To show quasi-minimality of Φ, consider the e-th computation Ψ_e. If we find an inconsistent computation on some τ at stage $s = 2e + 1$, then clearly, $\Psi_e(\Phi)$ does not define an element of $\{0, 1\}^{\mathbb{N}}$. If we find a consistent e-splitting η and θ on an input n at stage $s = 2e + 1$, $\Psi_e(\Phi)(n)$ is undefined, since otherwise $\Psi_e(\Phi)(n) = k$ implies $\Psi_e(\eta) = \Psi_e(\theta) = k$. Otherwise, for every $n \in \mathbb{N}$, if $\Psi_e(\Phi)(n)$ is defined, then it is consistent, and uniquely determined inside T_{φ_s}. Therefore, $\Psi_e(\Phi)(n) = k$ if and only if there is $\tau \in T_{\varphi_s}$ strongly extending σ_s such that $\Psi_e(\tau)(n) = k$. Consequently, $\Psi_e(\Phi)$ is computable in $\bigoplus_{m\leq e} x_m$, since T_{φ_s} is a pruned $\bigoplus_{m\leq e} x_m$-computable tree by induction. □

Corollary 59. For any separable metrizable space $\mathbf{X}$, we have $\mathbf{X} \times \mathbb{R}_< <^{\mathfrak{T}}_\sigma \mathcal{O}(\mathbb{N})$.

Proof. By Observation 18, Lemma 53 and Theorem 58. □

9 Admissibly Represented Spaces which are not Quasi-Polish

Recall that the class of admissibly represented space coincides with that of T_0 spaces that are quotients of second-countable T_0 spaces [66]. In particular, there is an admissibly represented space which is not second-countable. Schröder and Selivanov have studied hierarchies of such spaces in [69, 68]. In this section, we construct an admissibly represented space which is not σ-embedded into any second-countable T_0 space.

Theorem 60. There is an admissibly represented space $\mathbf{X}$ such that $\mathcal{O}(\mathbb{N}) <^{\mathfrak{T}}_\sigma \mathbf{X}$.

Let $\mathcal{O}_\infty(\mathbb{N})$ be a subspace of $\mathcal{O}(\mathbb{N})$ consisting of infinite subsets of $\mathbb{N}$. The space $\mathbb{Z}_<$ represents the set of integers equipped with lower topology. In this section, we develop the degree spectrum of the function space $C(\mathcal{O}_\infty(\mathbb{N}), \mathbb{Z}_<)$. We represent each continuous functional $H \in C(\mathcal{O}_\infty(\mathbb{N}), \mathbb{Z}_<)$ by an enumeration $\tilde{H}$ of pairs (C, n) of a finite set $C \subseteq \mathbb{N}$ and an integer $n \in \mathbb{Z}$ indicating that $H(Y) \geq n$ for any infinite set $Y \supseteq C$. This representation is automatically given as the usual category-theoretic exponential in the Cartesian closed category of admissibly represented spaces.

Lemma 61. The Scott domain $\mathcal{O}(\mathbb{N})$ is computably embedded into the function space $C(\mathcal{O}_\infty(\mathbb{N}), \mathbb{Z}_<)$. In particular, the degree spectrum of $C(\mathcal{O}_\infty(\mathbb{N}), \mathbb{Z}_<)$ contains all enumeration degrees.

Proof. For any sets $X, A, B \subseteq \mathbb{N}$, define $e_A(X \oplus B) = 1$ if $A \cap B \neq \emptyset$, and $e_A(X \oplus B) = 0$ otherwise. Then, the function $e : \mathcal{O}(\mathbb{N}) \to C(\mathcal{O}_\infty(\mathbb{N}), \mathbb{Z}_<)$ defined by $e(A) = e_A$ is a computable embedding. To see $A \leq_M e_A$, for any $n \in \mathbb{N}$, we have $n \in A$ if and only if $e_A(\mathbb{N} \oplus \{n\}) = 1$. To see $e_A \leq_M A$, for any $Y = Y_0 \oplus Y_1 \subseteq \mathbb{N}$, $e_A(Y) = 1$ if and only if $A \cap Y_1 \neq \emptyset$. Thus, to compute e_A from A, given a finite set $D = D_0 \oplus D_1 \subseteq \mathbb{N}$, we enumerate $(D, 1)$ into $\tilde{e}_A$ when $A \cap D_1 \neq \emptyset$ is witnessed. □

Indeed, $C(\mathcal{O}(\mathbb{N}), \mathbb{Z}_<)$ is computably embedded into $C(\mathcal{O}_\infty(\mathbb{N}), \mathbb{Z}_<)$ by transforming each f into $\hat{f}$ defined by $\hat{f}(X \oplus Y) = f(Y)$ for any $X, Y \subseteq \mathbb{N}$.

Lemma 62. There is a point F in the function space $C(\mathcal{O}_\infty(\mathbb{N}), \mathbb{Z}_<)$ such that F has no enumeration degree.

Proof. We follow the argument by Hinman [27]. We construct a continuous functional $F : \mathcal{O}_\infty(\mathbb{N}) \to \mathbb{Z}_<$ as the limit of an increasing sequence of partial continuous functionals F_n with domain $[\mathcal{A}_n] \subseteq \mathcal{O}_\infty(\mathbb{N})$, where $\mathcal{A}_n$ is a collection of finite sets such that $\bigcup \mathcal{A}_n$ is coinfinite, and $[\mathcal{A}_n]$ denote the set of all infinite sets $X \supseteq L$ for $L \in \mathcal{A}_n$.

At stage n, we look for $G \in C(\mathcal{O}_\infty(\mathbb{N}), \mathbb{Z}_<)$ not exceeding F_{n-1} on $\mathcal{A}_{n-1}$ and $A \in \mathcal{O}(\mathbb{N})$ such that G has the same degree with A via indices $n = \langle a, b\rangle$, i.e., G is computable in A via Φ_b and A is computable in G via Φ_a. If such G and A exist, we choose any $B \subseteq \mathbb{N} \setminus \bigcup \mathcal{A}_{n-1}$.

Since $\Phi_b(A)(B) = G(B)$, we may find a finite set $D \subseteq A$ such that $(E, G(B))$ for some finite set $E \subseteq B$ is enumerated into $\Phi_b(D)$, that is, there are finite sets $D \subseteq A$ and $E \subseteq B$ such that for any $X, Y \subseteq \mathbb{N}$,

$$X \supseteq D \text{ and } Y \supseteq E \longrightarrow \Phi_b(X)(Y) \geq \Phi_b(A)(B) = G(B).$$

Here, we may assume that $G(E) = G(B)$ since the value $G(B)$ is determined by a sufficiently large finite subset of B. Conversely, since $\Phi_a(G) = A \supseteq D$, there is a finite sublist $\tilde{L} \subseteq \tilde{G}$ such that for any $H \in C(\mathcal{O}_\infty(\mathbb{N}), \mathbb{Z}_<)$, if $H(C) \geq n$ for every $(C, n) \in \tilde{L}$, then $\Phi_a(H) \supseteq D$.

By choosing a slow enumeration of G as a name, we may assume that $|C| > |E|$ for any $(C, n) \in \tilde{L}$. Since $\tilde{L}_0 = \{C \subseteq \mathbb{N} : (C, n) \in \tilde{L}\}$ is a finite collection of finite sets, we can find an infinite set $I_n \subseteq \omega \setminus (\bigcup \mathcal{A}_{n-1} \cup \bigcup \tilde{F}_0)$ such that $I_n \cup \bigcup \mathcal{A}_{n-1} \cup \bigcup \tilde{L}_0$ is coinfinite. Define $K_n = I_n \cup E$. Then, $\tilde{F}_n$ is defined as follows:

$$\tilde{F}_n = \tilde{F}_{n-1} \cup \tilde{L} \cup \{(\{t\}, G(E) - 1) : t \in I_n\} \cup J_n,$$

where $J_n = \{(n, z)\}$ for a sufficiently small value $z \in \mathbb{Z}$ if $n \notin \bigcup_{m \leq n} K_m$, and $J_n = \emptyset$ otherwise. Eventually, we get a function $F \in C(\mathcal{O}_\infty(\mathbb{N}), \mathbb{Z}_<)$. Note that $F_n \upharpoonright \mathcal{A}_{n-1} = F_{n-1} \upharpoonright \mathcal{A}_{n-1}$ since G does not exceed F_{n-1} on $\mathcal{A}_{n-1}$.

Now, suppose $F \equiv_M S$ for some $S \subseteq \mathbb{N}$ via an index $n = (a, b)$. So at stage n, the strategy acts via G and A. First note that $\Phi_a(F) = S \supseteq D$ since $\tilde{L} \subset \tilde{F}$. Therefore, by our choice of E,

$$F(E \cup K_n) = \Phi_b(S)(E \cup K_n) \geq \Phi_b(A)(E \cup K_n) = G(E \cup K_n).$$

However, $F(E \cup K_n) \leq G(E) - 1 < G(E \cup K_n)$, a contradiction. □

Proof of Theorem 60. By Lemmata 61 and 62. Here, the relativization of Lemma 62 is obviously true. □

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