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Point Interactions: $\mathcal{PT}$-Hermiticity and Reality of the Spectrum

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Abstract. General point interactions for the second derivative operator in one dimension are studied. In particular, $\mathcal{PT}$-self-adjoint point interactions with the support at the origin and at points $\pm l$ are considered. The spectrum of such non-Hermitian operators is investigated and conditions when the spectrum is pure real are presented. The results are compared with those for standard self-adjoint point interactions.

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1. Introduction

Exactly solvable models are used in quantum mechanics to obtain Hamiltonians describing realistic physical systems but having the important property of being exactly solvable, i.e. that all eigenfunctions, spectrum, and scattering matrix can be calculated in closed form using elementary functions. A large class of such operators can be obtained using the method of point interactions described in detail in [4] in application to the theory of self-adjoint operators. It is not surprising that point interactions can be used to obtain exactly solvable nonself-adjoint operators; see, for example, [24, 25] where the Schrödinger operator on the half-axis with complex boundary condition at the origin has been considered as a model for dissipative operators. Using dissipative operators instead of self-adjoint operator, one obtains Hamiltonians describing irreversible quantum systems which are of great importance especially for evolution problems. It has been discovered recently that Hamiltonians possessing so-called $\mathcal{PT}$-symmetry can be nonself-adjoint and have the real spectrum at the same time. This discovery observed first using numerical computations has been approved by extensive analytical calculations. Many Schrödinger operators which have been studied have complex $\mathcal{PT}$-symmetric potentials but real discrete spectrum. Even if no proof has been discovered that the spectrum is real for the whole class of operators, it was expected that the relations between these two properties are rather close. In the present article we consider $\mathcal{PT}$-symmetric operators with point interactions, which are exactly solvable. This gives us an opportunity to study the relations between the $\mathcal{PT}$-symmetry of the Hamiltonian and the reality of its spectrum in full detail. We show in particular that none of these two properties imply the other one, but that there exist Hamiltonians with point interactions which are $\mathcal{PT}$-symmetric and have real spectrum. Since the operators under investigation are not self-adjoint the classical von Neumann theory cannot be applied without modification. Instead, the method of boundary conditions is used. It is shown how the notion of $\mathcal{PT}$-symmetry has to be modified in application to the extension theory for linear operators. In addition, we introduce the notion of $\mathcal{PT}$-self-adjoint operators. The operators obtained can be used to describe irreversible systems in the framework of the recently appeared $\mathcal{PT}$-symmetric quantum mechanics described below.

Let us start by presenting recent results concerning one-dimensional Schrödinger operator with $\mathcal{PT}$-symmetric potentials determined by the following expression

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x), \tag{1}$$

where the potential $V(x)$ is not a real valued function, but satisfies

$$V(-x) = \overline{V(x)}.$$

Using the space parity operator $\mathcal{P}$: $(\mathcal{P}\psi)(x) = \psi(-x)$ and the complex conjugation operator $\mathcal{T}\psi = \bar{\psi}$ the last property can be written as $\mathcal{PT}V = V$. Then the differential operator L is formally $\mathcal{PT}$-symmetric $\mathcal{PT}\ L = L\ \mathcal{PT}$.

Interest to differential operators with such symmetry became enormous, after it has been discovered that some of these operators have real spectrum (like self-adjoint operators) and therefore can be used in construction of a new, $\mathcal{PT}$-symmetric quantum mechanics.⋆ This fact was first discovered numerically by C. M. Bender, S. Boettcher and collaborators [6–11]. Analytical studies of $\mathcal{PT}$-symmetric Hamiltonians were carried out by M. Znojil and his collaborators [21, 27–34]. Numerous non-self-adjoint operators with real spectrum were studied (see e.g. [15–18]).

The aim of the current paper is to describe exactly solvable $\mathcal{PT}$-Hermitian operators constructing using the method of point interactions [4, 5]. Standard symmetries of (self-adjoint) point interactions in dimension 1 were studied in [1]. We are going now to extend these results to include $\mathcal{PT}$-self-adjoint point interactions. Starting from the second derivative operator in $L_2(\mathbb{R})$ (which is $\mathcal{PT}$-self-adjoint) we construct non trivial point interactions leading to operators having $\mathcal{PT}$-symmetry. Their spectral properties are described in Section 3. Section 4 is devoted to construction of local point interactions concentrated at different points on the real line. One of the main questions investigated is the relations between the $\mathcal{PT}$-symmetry of the operators

⋆It is not true in general that the spectrum of energy $\mathcal{PT}$-symmetric operator is real.

and reality of their spectrum. The family of point interactions with real spectrum is characterized in Section 5.

2. $\mathcal{PT}$-Self-adjoint Point Interactions

The method of point interactions is well described in several monographs [4, 5, 14]. This method is based on the fact that differential expression which is formally symmetric (or $\mathcal{PT}$-symmetric in our case) does not determine the operator in the Hilbert space uniquely. To determine the operator one has to specify its domain. One can start from a certain standard differential operator (i.e. Laplace operator) and restrict it to a certain densely defined operator. Then extending the restricted operator to another $\mathcal{PT}$-Hermitian operator one can get an operator with nontrivial spectral structure. Thus the main tool to be used is the extension theory for linear operators. The method we are going to present is similar to the method of point interaction developed for self-adjoint operators. Therefore let us introduce two definitions which are similar to the definition of symmetric and self-adjoint operators in conventional quantum mechanics.

DEFINITION 1. An operator L is called $\mathcal{PT}$- *symmetric*★ if it is densely defined and the following inclusion holds

$$L^* \supset \mathcal{P}L\mathcal{P}. \tag{2}$$

A $\mathcal{PT}$-symmetric operator L is called $\mathcal{PT}$-*self-adjoint* if and only if

$$L^* = \mathcal{P}L\mathcal{P}. \tag{3}$$

The second derivative operator $L = -(\mathrm{d}^2/\mathrm{d}x^2)$ with the standard domain $W_2^2(\mathbb{R})$ is both self-adjoint and $\mathcal{PT}$-self-adjoint. In the current section we study point perturbations at the origin of this operator leading to $\mathcal{PT}$-self-adjoint operators. By point interaction we mean any interaction which vanishes on the functions with the support separated from the origin. In other words, a linear operator A is a point perturbation at the origin of the operator L if and only if its restriction to $C_0^\infty(\mathbb{R} \setminus \{0\})$ coincides with the restriction of the operator L:

$$A|_{C_0^\infty(\mathbb{R}\setminus\{0\})} = L|_{C_0^\infty(\mathbb{R}\setminus\{0\})} \equiv L_0.$$

The operator L_0 determined by the last equation is both symmetric and $\mathcal{PT}$-symmetric. Moreover

$$\mathcal{P}L_0\mathcal{P} = L_0, \quad \mathcal{P}L_0^*\mathcal{P} = L_0^*,$$

and thus any $\mathcal{PT}$-symmetric extension of L_0 is a restriction of the maximal operator $L_{\max} = L_0^*$ being the second derivative operator in $L_2(\mathbb{R})$ with the domain

★The operators described by these definitions should be better called $\mathcal{P}$-symmetric and $\mathcal{P}$-self-adjoint, but we prefer to use the notation $\mathcal{PT}$-symmetric and $\mathcal{PT}$-self-adjoint in order to underline the entire relation with $\mathcal{PT}$-symmetric quantum mechanics already described in the literature.

$\mathrm{Dom}(L_{\max}) = W_2^2(\mathbb{R}\setminus\{0\})$. All such $\mathcal{PT}$-self-adjoint extensions and therefore all $\mathcal{PT}$-self-adjoint point interactions at the origin are described by the following

THEOREM 1. *The family of $\mathcal{PT}$-self-adjoint second derivative operators with point interactions at the origin coincides with the set of restrictions of the second derivative operator $L_{\max} = -(\mathrm{d}^2/\mathrm{d}x^2)$, defined originally on $W_2^2(\mathbb{R}\setminus\{0\})$, to the domain of functions satisfying the boundary conditions at the origin of one of the following two types:*

$$\text{(I)} \qquad \begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix} = B \begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix}; \tag{4}$$

with the matrix B equal to

$$B = \mathrm{e}^{i\theta} \begin{pmatrix} \sqrt{1+bc}\,\mathrm{e}^{i\phi} & b \\ c & \sqrt{1+bc}\,\mathrm{e}^{-i\phi} \end{pmatrix}$$

with the real parameters $b \geqslant 0, c \geqslant -1/b$,★ *$\theta, \phi \in [0, 2\pi)$;*

$$\text{(II)} \qquad h_0\psi'(+0) = h_1\mathrm{e}^{i\theta}\psi(+0), \qquad h_0\psi'(+0) = -h_1\mathrm{e}^{-i\theta}\psi(-0) \tag{5}$$

with the real phase parameter $\theta \in [0, 2\pi)$ and with the parameter $\mathbf{h} = (h_0, h_1)$ taken from the (real) projective space $\mathbf{P}^1$.

Proof. We have already proven that any $\mathcal{PT}$-self-adjoint extension of L_0 is a restriction of $L_{\max}$. Consider the vector space $\mathbb{C}^4$ of boundary values of functions from the domain of $L_{\max}$ and the map Γ adjusting to any function $\psi \in W_2^2(\mathbb{R}\setminus\{0\})$ its boundary values

$$\Gamma\colon \psi \mapsto \begin{pmatrix} \psi(+0) \\ \psi'(+0) \\ \psi(-0) \\ \psi'(-0) \end{pmatrix} \in \mathbb{C}^4.$$

The fundamental symmetry operator $\mathcal{P}$ acting in the space of boundary values coincides with the operator of multiplication by the following matrix P

$$P\Gamma = \Gamma\mathcal{P}, \quad \text{where} \quad P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{6}$$

Similarly the operator of complex conjugation $\mathcal{T}$ is mapped by Γ into the operator T of complex conjugation in $\mathbb{C}^4$.

★If the parameter b is equal to zero, then the last inequality can be neglected.

The closure of the operator L_0 is defined on the functions having trivial boundary values at the origin and *vice-versa* any function from the domain of $L_{\max}$ with trivial boundary values belongs to $\mathrm{Dom}(\bar{L}_0)$. Thus the dimension of the quotient space $\mathrm{Dom}(L_{\max})/\mathrm{Dom}\,(\bar{L}_0)$ is equal to 4. Any $\mathcal{PT}$-self-adjoint extension of L_0 can be described as the restriction of $L_{\max}$ to the set of functions with boundary values from a certain two-dimensional subspace $\mathcal{L}$ of $\mathbb{C}^4$. Every such subspace can be described by two (linearly independent) boundary conditions using a certain 2×4 dimensional rank 2 matrix $Q=\{q_{ij}\}$ as follows

$$\Gamma\psi \in \mathcal{L} \Leftrightarrow \begin{cases} q_{11}\psi(+0)+q_{12}\psi'(+0)+q_{13}\psi(-0)+q_{14}\psi'(-0)=0, \\ q_{21}\psi(+0)+q_{22}\psi'(+0)+q_{23}\psi(-0)+q_{24}\psi'(-0)=0. \end{cases} \tag{7}$$

The restriction of the maximal operator to a two-dimensional subspace $\mathcal{L}$ possesses the $\mathcal{PT}$-symmetry if and only if

$$\boldsymbol{X}\in\mathcal{L} \Leftrightarrow \boldsymbol{PTX}\in\mathcal{L}.$$

Let us prove now that the $\mathcal{PT}$-self-adjoint point interactions can be parameterized in at least one of the following four ways:

$$(\mathbf{A})\quad \begin{pmatrix}\psi(+0)\\ \psi'(+0)\end{pmatrix} = \mathrm{e}^{i\theta}\begin{pmatrix}\sqrt{1+bc}\,\mathrm{e}^{i\phi} & b\\ c & \sqrt{1+bc}\,\mathrm{e}^{-i\phi}\end{pmatrix}\begin{pmatrix}\psi(-0)\\ \psi'(-0)\end{pmatrix}; \tag{8}$$

$b\geqslant 0, c\geqslant -1/b, \theta,\phi\in[0,2\pi)$;

$$(\mathbf{B})\quad \begin{pmatrix}\psi(+0)\\ \psi'(-0)\end{pmatrix} = \mathrm{e}^{i\theta}\begin{pmatrix}\sqrt{1+bc}\,\mathrm{e}^{i\phi} & b\\ c & \sqrt{1+bc}\,\mathrm{e}^{-i\phi}\end{pmatrix}\begin{pmatrix}\psi(-0)\\ \psi'(+0)\end{pmatrix}; \tag{9}$$

$b\geqslant 0, c\geqslant -1/b, \theta,\phi\in[0,2\pi)$;

$$(\mathbf{C})\quad \begin{pmatrix}\psi(+0)\\ \psi(-0)\end{pmatrix} = \begin{pmatrix}a\mathrm{e}^{i\theta} & b\mathrm{e}^{i\phi}\\ -b\mathrm{e}^{-i\phi} & -a\mathrm{e}^{-i\theta}\end{pmatrix}\begin{pmatrix}\psi'(+0)\\ \psi'(-0)\end{pmatrix}, \tag{10}$$

$a,b\geqslant 0, \theta,\phi\in[0,2\pi)$;

$$(\mathbf{D})\quad \begin{pmatrix}\psi'(+0)\\ \psi'(-0)\end{pmatrix} = \begin{pmatrix}a\mathrm{e}^{i\theta} & b\mathrm{e}^{i\phi}\\ -b\mathrm{e}^{-i\phi} & -a\mathrm{e}^{-i\theta}\end{pmatrix}\begin{pmatrix}\psi(+0)\\ \psi(-0)\end{pmatrix}, \tag{11}$$

$a,b\geqslant 0, \theta,\phi\in[0,2\pi)$.

Since the matrix Q appearing in (7) has rank 2, at least one of its six 2×2 minors is nondegenerate. Depending on which minor is nondegenerate the boundary conditions (7) can be written in different ways using certain 2×2 matrices $B=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$

$$(1)\quad \det\begin{pmatrix}q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}\neq 0 \Rightarrow \begin{pmatrix}\psi(+0)\\ \psi'(+0)\end{pmatrix} = B\begin{pmatrix}\psi(-0)\\ \psi'(-0)\end{pmatrix}.$$

$$(2)\quad \det\begin{pmatrix} q_{11} & q_{13} \\ q_{21} & q_{23} \end{pmatrix} \neq 0 \Rightarrow \begin{pmatrix} \psi(+0) \\ \psi(-0) \end{pmatrix} = B\begin{pmatrix} \psi'(+0) \\ \psi'(-0) \end{pmatrix}.$$

$$(3)\quad \det\begin{pmatrix} q_{11} & q_{14} \\ q_{21} & q_{24} \end{pmatrix} \neq 0 \Rightarrow \begin{pmatrix} \psi(+0) \\ \psi'(-0) \end{pmatrix} = B\begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix}.$$

$$(4)\quad \det\begin{pmatrix} q_{12} & q_{13} \\ q_{22} & q_{23} \end{pmatrix} \neq 0 \Rightarrow \begin{pmatrix} \psi(-0) \\ \psi'(+0) \end{pmatrix} = B\begin{pmatrix} \psi(+0) \\ \psi'(-0) \end{pmatrix}.$$

$$(5)\quad \det\begin{pmatrix} q_{12} & q_{14} \\ q_{22} & q_{24} \end{pmatrix} \neq 0 \Rightarrow \begin{pmatrix} \psi'(+0) \\ \psi'(-0) \end{pmatrix} = B\begin{pmatrix} \psi(+0) \\ \psi(-0) \end{pmatrix}.$$

$$(6)\quad \det\begin{pmatrix} q_{13} & q_{14} \\ q_{23} & q_{24} \end{pmatrix} \neq 0 \Rightarrow \begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix} = B\begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix}.$$

Our aim now is to characterize all matrices B leading to $\mathcal{PT}$-symmetric boundary conditions.

Consider the first case. Suppose that the function ψ satisfies the boundary conditions. The boundary conditions for the function $\mathcal{PT}\psi$ are given by

$$\begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix} = \begin{pmatrix} -\bar{\alpha} - \bar{\beta} \\ -\bar{\gamma} - \bar{\delta} \end{pmatrix}\begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix}.$$

These conditions coincide with the original one if and only if

$$\begin{pmatrix} -\bar{\alpha} & -\bar{\beta} \\ -\bar{\gamma} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1}. \tag{12}$$

Investigating cases (3), (4) and (6) one arrives to exactly the same equation on the matrix B.

Equation (12) implies that $\det B = \alpha\delta - \beta\gamma \neq 0$ and the following four equalities hold

$$\bar{\alpha} = \frac{1}{\det B}\delta, \qquad \bar{\beta} = \frac{1}{\det B}\beta, \qquad \bar{\gamma} = \frac{1}{\det B}\gamma, \qquad \bar{\delta} = \frac{1}{\det B}\alpha.$$

These equations imply in particular that the determinant has absolute value 1, i.e. there exists $\theta \in [0, 2\pi)$ such that $\det B = \mathrm{e}^{2i\theta}$. Consider then the matrix

$$B' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \mathrm{e}^{-i\theta} B$$

with the unit determinant $\alpha'\delta' - \beta'\gamma' = 1$. The entries of B' satisfy the following equations

$$\bar{\alpha}' = \delta', \qquad \bar{\beta}' = \beta', \qquad \bar{\gamma}' = \gamma', \qquad \bar{\delta}' = \alpha'.$$

Hence all such matrices B' can be parameterized by three real parameters:

positive number b;
real number $c, c \geqslant -1/b$;
phase parameter $\phi \in [0, 2\pi)$;

using the following formula

$$B' = \begin{pmatrix} \sqrt{1+bc}\,\mathrm{e}^{i\phi} & b \\ c & \sqrt{1+bc}\,\mathrm{e}^{-i\phi} \end{pmatrix}. \tag{13}$$

This implies that the boundary conditions can be written in the form of (8) in cases (1) and (6) and in the form of (9) in cases (3) and (4).★

It remains to study cases (2) and (5). The boundary conditions determine a $\mathcal{PT}$-symmetric operator if and only if the following equality holds in both cases

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = -\begin{pmatrix} \bar{\delta} & \bar{\gamma} \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{14}$$

This equality is satisfied if and only if $\alpha = -\bar{\delta}, \beta = -\bar{\gamma}$, and such matrices can be parameterized by the following parameters:

two positive numbers $a, b \in \mathbb{R}_+$;
two phases $\theta, \phi \in [0, 2\pi)$;

using the formula

$$B = \begin{pmatrix} a\mathrm{e}^{i\theta} & b\mathrm{e}^{i\phi} \\ -b\mathrm{e}^{-i\phi} & -a\mathrm{e}^{-i\theta} \end{pmatrix}. \quad .$$

We thus get the parameterizations (10) and (11).

It remains to prove that each of the boundary conditions **B**, **C** and **D** can be written as (4) or (5). (The boundary conditions of type **A** coincide with (4).) The boundary conditions of type **B** can be written as (4) if $1 + bc \neq 0$ simply by expressing $\psi'(-0)$ as a function of $\psi(+0)$ and $\psi'(+0)$ from the second boundary condition (9). If $1 + bc = 0$ then the boundary condition (9) is of type (5). The cases **C** and **D** can be studied similarly.

Thus we have proven that these boundary conditions describe all maximal $\mathcal{PT}$-symmetric extensions of L_0. One can easily check that all these operators are $\mathcal{PT}$-self-adjoint as well. The theorem is proven. □

Similar results in the theory of self-adjoint point interactions are usually proven using von Neumann extension theory [5, 19, 26]. Instead of developing its counterpart for $\mathcal{PT}$-symmetric operators we preferred to give a constructive proof using boundary conditions.

In what follows the boundary conditions given by (4) will be called ***connected***, since these conditions connect the boundary values on the left and right-hand sides

★One has to take into account that det $B \neq 0$ and therefore the boundary conditions (1) and (6) and (3) and (4) are pairwise equivalent.

of the origin of functions from the domain of the operator. The boundary conditions of the second type given by (5) will be called *separated*. These definitions and the main proposition are quite similar to corresponding description of self-adjoint point interactions given in [19]. The operators appearing in the decomposition corresponding to separated boundary conditions are just the second derivative operators on the half lines with complex boundary condition of the third type at the origin. Such non-self-adjoint operators can easily be studied (see, e.g., [25]).

Thus the set of $\mathcal{PT}$-self-adjoint point interactions can be parameterized by 4 real'parameters. The phase parameter θ is redundant in the sense that the operators corresponding to different values of this parameter are unitary equivalent.★ The parameterization used in (4) and (5) is not optimal in the sense that the correspondence between the parameters and the boundary conditions is not one-to-one (for example, if $b = c = 0$, then changing the phases by π we get the same boundary conditions, similar problem occurs for separated boundary conditions), but we prefer not to dwell on this point. Connected boundary conditions given by the matrix

$$B = e^{i\theta}\begin{pmatrix} a & b \\ c & a \end{pmatrix}, \quad a, b, c \in \mathbb{R}, \theta \in [0, \pi), a^2 - bc = 1 \tag{15}$$

determine operators which are both self-adjoint and $\mathcal{PT}$-self-adjoint. Separated boundary conditions leading to such operators are given by formula (5) with $\theta = 0$.

3. Spectral Problems

In this section we are going to study the spectrum of second derivative operators with $\mathcal{PT}$-self-adjoint point interactions at the origin. Our main result can be formulated as follows

THEOREM 2. *The spectrum of any $\mathcal{PT}$-self-adjoint second derivative operator with point interactions at the origin consists of the branch $[0, \infty)$ of the absolutely continuous spectrum and at most two (counting multiplicity) eigenvalues, which are real negative or are (complex) conjugated to each other.*

Proof. Let us denote by A any $\mathcal{PT}$-self-adjoint second derivative operator with point interaction at the origin (described by Theorem 1). To prove the theorem we are going to calculate its resolvent. Consider the resolvent equation

$$(A - \lambda)U = F, \quad F \in L_2(\mathbb{R}).$$

The unique solution to this equation is the function U from the domain $\mathrm{Dom}(A)$ of the operator A satisfying the differential equation

$$-U'' - \lambda U = F, \tag{16}$$

everywhere outside the origin. We denote by k the square root of the energy parameter $\lambda = k^2$, determined uniquely by $\Im k \geqslant 0$. Let us introduce two functions

★Similar facts for self-adjoint point interactions are described in full details in [2] and [13].

$$e_+(x) = \begin{cases} e^{ikx}, & x > 0 \\ 0, & x < 0 \end{cases}, \quad e_-(x) = \begin{cases} 0, & x > 0 \\ e^{-ikx}, & x < 0 \end{cases}.$$

Then any solution (from $L_2(\mathbb{R})$) to (16) can be written in the form

$$U(x) = \int_{-\infty}^{\infty} \frac{e^{ik|x-y|}}{2ik} F(y)\mathrm{d}y + \rho_+(F)e_+(x) + \rho_-(F)e_-(x), \tag{17}$$

where $\rho_\pm$ are two parameters to be calculated. The boundary values of the function U are

$$U(+0) = \frac{1}{2ik}(f_- + f_+) + \rho_+, \qquad U'(+0) = \tfrac{1}{2}(f_- - f_+) + ik\rho_+,$$
$$U(-0) = \frac{1}{2ik}(f_- + f_+) + \rho_-, \qquad U'(-0) = \tfrac{1}{2}(f_- - f_+) - ik\rho_-$$

where $f_\pm = \int_{\mathbb{R}_\pm} F(y)\mathrm{d}y$.

Consider the case of separated boundary conditions. Then (4) implies

$$\begin{pmatrix} 1 & -e^{i\theta}\left(\sqrt{1+bc}\,e^{i\phi} - ikb\right) \\ ik & -e^{i\theta}\left(c - ik\sqrt{1+bc}\,e^{-i\phi}\right) \end{pmatrix} \begin{pmatrix} \rho_+ \\ \rho_- \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2ik}\left(-1 + e^{i\theta}\sqrt{1+bc}\,e^{i\phi}\right)(f_- + f_+) + e^{i\theta}\frac{b}{2}(f_- - f_+) \\ e^{i\theta}\frac{c}{2ik}(f_- + f_+) + \frac{1}{2}\left(-1 + e^{i\theta}\sqrt{1+bc}\,e^{-i\phi}\right)(f_- - f_+) \end{pmatrix}.$$

This system of equations on $\rho_\pm$ is solvable if and only if the determinant of the matrix at the left hand side is different from zero

$$-e^{i\theta}\left(c - 2ik\cos\phi\sqrt{1+bc} - k^2 b\right) \neq 0. \tag{18}$$

Thus the resolvent equation can be solved for all nonreal k, which are not solutions to the following quadratic equation

$$bk^2 + 2i\cos\phi\sqrt{1+bc}\,k - c = 0. \tag{19}$$

The two solutions to the last equation

$$k_{1,2} = -i\frac{\cos\phi\sqrt{1+bc}}{b} \pm \frac{\sqrt{(1-\cos^2\phi)bc - \cos^2\phi}}{b}$$

are either pure imaginary or symmetric to each other with respect to the imaginary axis. The corresponding energy values are real, or conjugated to each other. Hence the domain $\mathbb{C} \setminus (\mathbb{R}_+ \cup \{k_1^2, k_2^2\})$ belongs to the regularity domain of the operator L.

Formula (17) shows that the difference between the resolvents of the operators A and the unperturbed second derivative operator L has rank two. The spectrum of the (self-adjoint) operator L is pure absolutely continuous and fills in the interval $[0, \infty)$. The complement to the absolutely continuous spectrum is simply connected.

Therefore the perturbed operator A has the branch of absolutely continuous spectrum $[0, \infty)$ as well [22, 23]. Let us study the singularities of the resolvent corresponding to the numbers $k_{1,2}^2$. Let $\Im k_1 > 0$, then the function

$$\psi_1(x) = \begin{cases} e^{i\theta}\left(\sqrt{1+bc}\, e^{i\phi} - ik_1 b\right) e^{ik_1 x}, & x > 0, \\ e^{-ik_1 x}, & x < 0; \end{cases}$$

is a square integrable solution to the equation $-\psi_1'' = k_1^2 \psi_1,\ x \neq 0$ and satisfies the boundary conditions (4). Therefore this function is a (discrete spectrum) eigenfunction for the operator A. Similarly the function $\psi_2 = \mathcal{PT}\psi_1$ is an eigenfunction corresponding to the eigenvalue $\bar{k_1}^2$. If k_1 is pure imaginary (k_1^2 negative real) then the function ψ_1 can be chosen $\mathcal{PT}$-symmetric or -antisymmetric. In the special case $(1 - \cos^2\phi)bc - \cos^2\phi = 0, \cos\phi < 0$ the two solutions to the dispersion equation (19) coincide. Nevertheless the corresponding eigenvalue has multiplicity 1, since the boundary conditions (4) are connected. Therefore we conclude that every solution to the dispersion equation (19) from the upper half plane $\Im k > 0$ determines a simple eigenvalue k^2 of the operator A. Solutions to (19) lying on the nonphysical sheet $\Im k \leqslant 0$ do not determine any eigenvalue of L, since the corresponding solutions to the differential equation do not belong to the Hilbert space. The theorem is proven for connected boundary conditions.

The proof for separated conditions is quite similar. The only difference is that in this case the operator A can be presented as an orthogonal sum of two operators in $L_2(\mathbb{R}_\pm)$. Each of these operators can have one (complex) eigenvalue and these eigenvalues are conjugated to each other. Therefore if the operator A has a negative eigenvalue then this eigenvalue has always multiplicity 2. □

In particular we have proven the following Corollary:

COROLLARY 1. *The eigenvalues corresponding to $\mathcal{PT}$-symmetric eigenfunctions of the operator A are real and negative. Every eigenfunction corresponding to any real eigenvalue of the operator A can be chosen $\mathcal{PT}$-symmetric or -antisymmetric.*

This proposition holds in fact for any $\mathcal{PT}$-self-adjoint operator with nonsingular interactions. Note that the spectrum of the operator A is not always pure real. Let us study the case when the spectrum is real in more detail. The spectrum is pure real if and only if the two solutions to Equation (19) are pure imaginary or are situated on the nonphysical sheet $\Im k \leqslant 0$. The two solutions are pure imaginary only if the discriminant is negative $bc \sin^2\phi \leqslant \cos^2\phi$. If the discriminant is positive ($bc \sin^2\phi > \cos^2\phi$), then the imaginary part of the solutions is given by $-(\cos\phi\sqrt{1+bc})/b$ and is negative only if $\cos\phi > 0$. We have proven the following

PROPOSITION 1. *The spectrum of the $\mathcal{PT}$-self-adjoint second derivative operator with connected point interaction at the origin is pure real if and only if the parameters appearing in (4) satisfy in addition at least one of the following conditions*

(**I**) $bc\sin^2\phi \leqslant \cos^2\phi$;
(**II**) $bc\sin^2\phi \geqslant \cos^2\phi$ and $\cos\phi \geqslant 0$.

The spectrum does not depend on the phase parameter θ. The operators corresponding to different values of θ are unitary equivalent (see [2], where similar results are proven for self-adjoint point interactions).

4. Local $\mathcal{PT}$-Self-adjoint Point Interactions

We consider now $\mathcal{PT}$-self-adjoint operators with point interactions at positions $x = \pm l$, $l \in \mathbb{R}_+$. Every such operator coincides with a certain restriction of the second derivative operator $-(\mathrm{d}^2/\mathrm{d}x^2)$, defined originally in $L_2(\mathbb{R})$ on the domain $W_2^2(\mathbb{R}\setminus\{\pm l\})$, to the set of functions satisfying some boundary conditions at the points $x = \pm l$. We restrict our consideration to the case of local connected point interaction without aiming to describe all possible restrictions leading to $\mathcal{PT}$-self-adjoint operators. Suppose that the functions from the domain of the restricted operator satisfy the following conditions at $x = l$

$$\begin{pmatrix} \Psi(l^+) \\ \Psi'(l^+) \end{pmatrix} = B \begin{pmatrix} \Psi(l^-) \\ \Psi'(l^-) \end{pmatrix}, \tag{20}$$

where the matrix

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

Then to make the operator $\mathcal{PT}$-symmetric we have to suppose that the following boundary conditions are introduced at $x = -l$

$$\begin{pmatrix} \Psi(-l^-) \\ \Psi'(-l^-) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \bar{B} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \Psi(-l^+) \\ \Psi'(-l^+) \end{pmatrix}, \tag{21}$$

The restriction of the second derivative operator to the set of functions satisfying conditions (20) and (21) will be denoted by A in this section. Let us study the spectrum of this operator. The resolvent of the operator A can be calculated explicitly using methods of the previous section. One concludes that the difference between the resolvents of the operators A and L has rank four and therefore the operator A has a branch of absolutely continuous spectrum $[0, \infty)$. Let us study the discrete spectrum of A. To calculate the eigenfunction one can use the following *Ansatz*:

$$\psi(x) = \begin{cases} c_1 \mathrm{e}^{-ik(x+l)}, & x < -l; \\ c_2 \cos k(x+l) + c_3 \sin k(x+l), & -l < x < l; \\ c_4 \mathrm{e}^{ik(x-l)}, & x > l; \end{cases} \tag{22}$$

where $k^2 = \lambda, \Im k > 0$ and c_1, c_2, c_3, c_4 are arbitrary complex parameters to be determined. Substitution of this function into the boundary conditions (20) and (21) gives the following linear system on c_j

$$\begin{pmatrix} 1 & -\bar{\alpha} & \bar{\beta}k & 0 \\ ik & -\bar{\gamma} & \bar{\delta}k & 0 \\ 0 & \alpha\cos 2kl - \beta k\sin 2kl & \alpha\sin 2kl + \beta k\cos 2kl & -1 \\ 0 & \gamma\cos 2kl - \delta k\sin 2kl & \gamma\sin 2kl + \delta k\cos 2kl & -ik \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0.$$

This system has a nontrivial solution if and only if the determinant of the 4×4 matrix is equal to zero, which gives us the dispersion relation

$$\sin 2kl\{-k^4|\beta|^2 - ik^3(\beta\bar{\delta} + \bar{\beta}\delta) + k^2(|\alpha|^2 - |\delta|^2) + ik(\alpha\bar{\gamma} + \bar{\alpha}\gamma) - |\gamma|^2\}$$
$$+k\cos 2kl\{k^2(\alpha\bar{\beta} + \bar{\alpha}\beta) + ik(\alpha\bar{\delta} + \bar{\alpha}\delta + \beta\bar{\gamma} + \bar{\beta}\gamma) - (\gamma\bar{\delta} + \bar{\gamma}\delta)\} = 0. \quad (23)$$

Each solution to this equation from the upper half plane determines an eigenvalue of the operator A.

In what follows we are going to study the operator with the interaction given by a sum of two delta-functions given by the following formal expression

$$-\frac{d^2}{dx^2} + (u + iv)\delta(x - l) + (u - iv)\delta(x + l), \quad (24)$$

where $u, v \in \mathbb{R}$. This potential is formally $\mathcal{PT}$-symmetric. This operator is determined by the boundary conditions chosen as follows⋆

$$\alpha = \gamma = 1, \quad \beta = 0, \quad \delta = u + iv.$$

In the case $v = 0$ we get a self-adjoint operator. Let us study the case $u = 0$, when the potential is pure imaginary. The dispersion relation takes the following simple form

$$\sin 2kl\{k^2(1 - v^2) + 2ik - 1\} = 0.$$

All solutions to this equation from the open upper half plane are determined by the zeroes of the second order polynomial in the brackets

$$k_{1,2} = -i\frac{1}{1 \pm v}, \quad v \neq \pm 1.$$

These solutions are always pure imaginary. Depending on the absolute value of v one or none solutions are situated on the physical sheet $\Im k > 0$. It is interesting to observe that the spectrum of this operator is always pure real.

5. Point Interactions with Real Spectrum

In this section we are going to discuss connected boundary conditions at the origin leading to second derivative operators with real spectrum. Namely the operator $A = -(d^2/dx^2)$ with the domain

⋆To define the operator correctly one can use for example its quadratic form.

$$\mathrm{Dom}(A) = \left\{ \psi \in W_2^2(\mathbb{R} \setminus \{0\}); \begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix} \right\} \tag{25}$$

will be studied. We suppose that the matrix $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ appearing in the boundary conditions is non degenerate (from $\mathrm{GL}(2, \mathbb{C})$). Again it is easy to prove that the operator has branch of absolutely continuous spectrum $[0, \infty)$ and it remains to study its discrete spectrum only. Let us use the following *Ansatz* for the eigenfunction

$$\psi(x) = \begin{cases} c_1 \mathrm{e}^{-ikx}, & x < 0, \\ c_2 \mathrm{e}^{ikx}, & x > 0, \end{cases} \quad \Im k > 0, \tag{26}$$

corresponding to the energy $\lambda = k^2$. Substituting this function into the boundary conditions we get the 2×2 linear system

$$\begin{pmatrix} \alpha - ik\beta & -1 \\ \gamma - ik\delta & -ik \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0,$$

and the dispersion equation

$$k^2\beta + ik(\alpha + \delta) - \gamma = 0. \tag{27}$$

The last equation has the following two solutions

$$k_{1,2} = \frac{-i(\alpha + \delta) \pm \sqrt{-(\alpha + \delta)^2 + 4\gamma\beta}}{2\beta},$$

if $\beta \neq 0$. Solutions of the equation situated in the closed lower half plane $\Im k \leqslant 0$ do not determine any eigenfunction of the operator A. Hence, the spectrum of the operator A is real if and only if both solutions $k_{1,2}$ to Equation (27) satisfy at least one of the following two conditions:

(1) $\Im k_{1,2} \leqslant 0$;
(2) $\Re k_{1,2} = 0$.

The set of coefficients $\alpha, \beta, \gamma, \delta$ satisfying the first condition can be parameterized by 8 real parameters and leads to operators A with pure absolutely continuous spectrum $[0, \infty)$. Pure imaginary solutions to (27) can lead to nontrivial discrete spectrum. Therefore let us study the set of coefficients satisfying the second condition in more details.

The solutions $k_{1,2}$ are pure imaginary if and only if the following conditions are satisfied:

$$\frac{\alpha + \delta}{\beta} \in \mathbb{R}, \qquad \frac{\gamma}{\beta} \in \mathbb{R}, \qquad \frac{4\gamma}{\beta} \leqslant \frac{(\alpha + \delta)^2}{\beta^2}. \tag{28}$$

The first two conditions imply that the complex numbers $\tau = \alpha + \delta, \beta$, and γ have the same phase. Therefore let us introduce the following parameterization:

$$\tau = t e^{i\theta}, \qquad \beta = b e^{i\theta}, \qquad \gamma = c e^{i\theta},$$

where t, b, c are real numbers. In order to guarantee that solutions $k_{1,2}$ are pure imaginary the real parameters should satisfy the inequality

$$4\frac{c}{b} \leqslant \frac{t^2}{b^2}. \tag{29}$$

We conclude that the second family can be parameterized by 6 real parameters. The intersection between the two families of parameters leading to operators with pure real spectrum is not empty, but the second family is not included in the first one.

The four-parameter family of self-adjoint (connected) boundary conditions is contained in the second (6-parameter) family just described. Parameters leading to $\mathcal{PT}$-self-adjoint boundary satisfy the first two conditions (28). Hence, the family of $\mathcal{PT}$-self-adjoint (connected) boundary conditions leading to operators with real spectrum belongs to the second family of boundary conditions as well.

6. Conclusions

The relations between $\mathcal{PT}$-self-adjoint, self-adjoint and real second derivative operators with point interactions have been studied. One has to investigate whether the operators with real spectrum are similar to certain self-adjoint operators or not. It is easy to generalize these results in order to include point interactions with the support at arbitrary finite (or even infinite) number of points. The results obtained can be applied to construct exactly solvable few-body systems with unusual symmetries (following [3, 20]). This method can be generalized to include point interactions in spaces of higher dimension and higher-order differential operators (like in [12]).

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