Point process modelling for directed interaction networks

Patrick O. Perry and Patrick J. Wolfe
New York University and University College London

Interaction data

emails
mobile phone calls
transit cards
credit cards
movement in public places
blog entries
online social networks

These transactions leave digital traces that can be compiled into comprehensive pictures of both individual and group behavior

-Lazer et al. (2009)

Raw data + Point process model = Insight

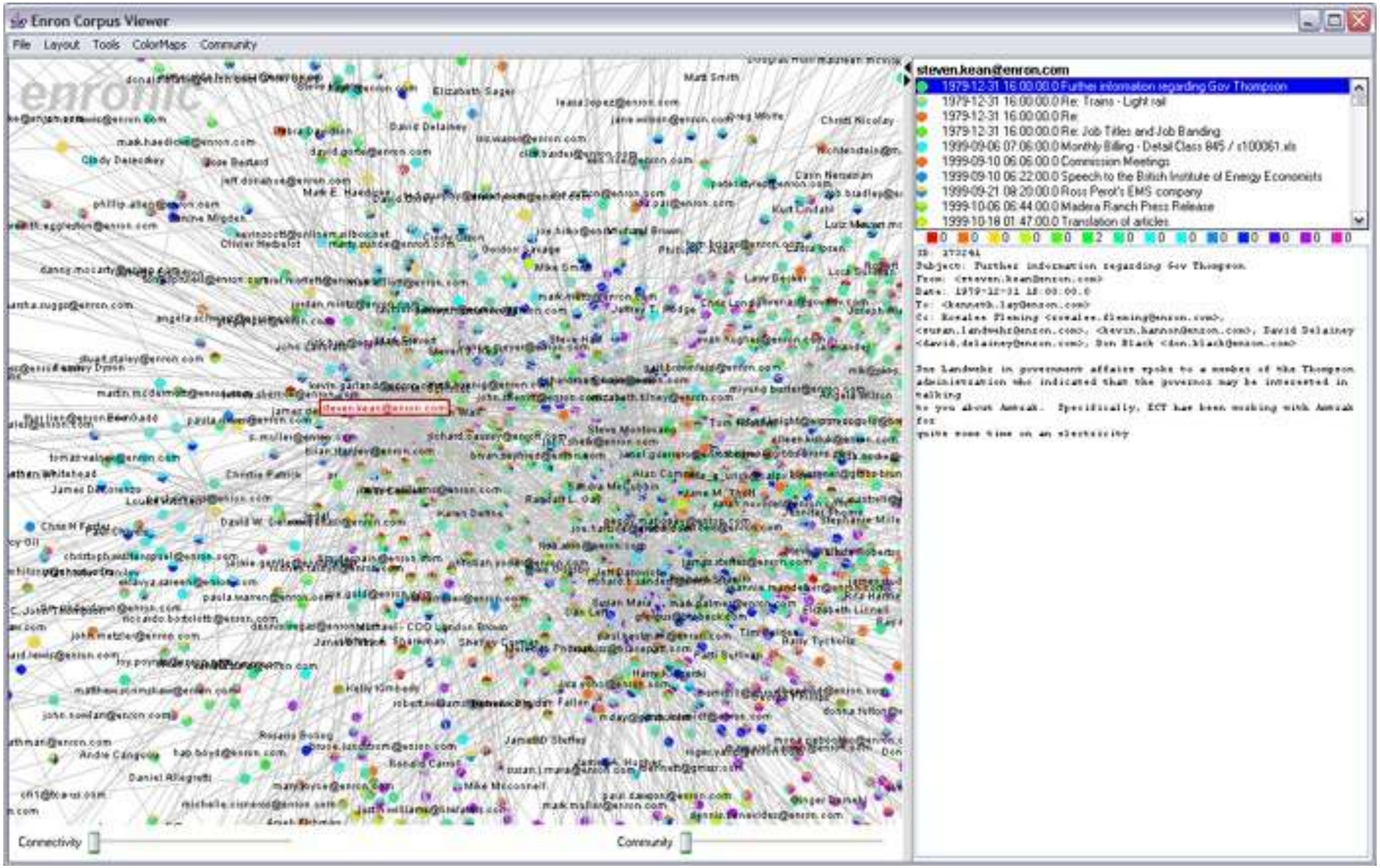
Insight: Which traits and behaviors are predictive of interaction

Raw data: Enron e-mail dataset

156 Employees, 21635 Messages, Nov 1998 – June 2002

```
Message-ID: <7303996.1075860726914.JavaMail.evans@thyme>
Date: Wed, 10 Oct 2001 08:51:16 -0700 (PDT)
From: kenneth.lay@enron.com
To: benjamin.r@enron.com
Subject: RE: Power Trading Group

Ben -

I likewise was glad to see you.   Sorry we didn't have a chance to talk.

Good to hear you're doing well.  You're with a great group and, yes, the
company will soon be doing a lot better.

Thanks,

Ken
```

156 nodes, 21635 messages



(Heer, 2004)

The big question

Employee Traits

Gender:
- Female (43)
- Male (113)

Seniority:
- Junior (82)
- Senior (74)

Department:
- Legal (25)
- Trading (60)
- Other (71)

Question: Which traits and behaviors are predictive of interaction?

Raw data

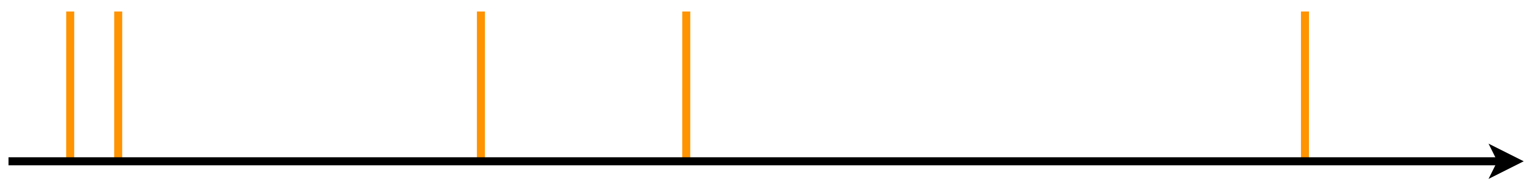
Messages

Time	Sender	Receiver
t_1	i_1	j_1
t_2	i_2	j_2
$\vdots$	$\vdots$	$\vdots$
t_n	i_n	j_n

1. Continuous time
2. Events, not links

Point process model

Messages from i to j :



Time

Model via intensity, $\lambda_t(i,j)$:

$$\lambda_t(i,j)\,dt = \mathrm{Prob}\{i \text{ sends to } j \text{ in } [t, t+dt)\}$$

Employee traits

Variate	Characteristic of actor i	Count
$L(i)$	member of the Legal department	25
$T(i)$	member of the Trading department	60
$J(i)$	seniority is Junior	82
$F(i)$	gender is Female	43

20 edge-specific traits: L(j), L(i)*L(j), T(i)*L(j), J(i) *L(j), ...

Notation: $x(i,j) \in \mathbb{R}^{20}$

First attempt: Cox model

Rate of i–j message exchange

$$\lambda_t(i,j) = \bar{\lambda}_t(i)\exp\{\beta^{\mathrm{T}} x(i,j)\}$$

Baseline send rate

Edge-specific covariate vector

Coefficient vector

$$\lambda : \mathbb{R} \times 156 \times 156 \to \mathbb{R}_+$$
$$\bar{\lambda} : 156 \to \mathbb{R}_+$$
$$x : 156 \times 156 \to \mathbb{R}^{20}$$

Problem: Sparsity

Messages from Tania J.

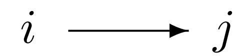
33	0	0	192	0	0	1	0	0	0	0	0	0	0	1	0
0	0	0	0	4	0	0	0	0	275	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	1
405	0	0	0	407	0	0	0	0	5	0	0	1	0	0	0
67	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	126	0	0	1	0
0	3	0	30	0	0	0	0	0	0	0	0	166	0	0	0
1	0	0	0	0	0	0	271	1	0	0	0	0	0	0	0
0	221	0	0	0	0	1	8	0	507	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	26	7	0				

Messages predicted by model

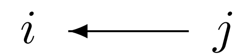
2.3	2.3	1.4	166.7	1.4	1.4	7.1	2.3	7.1	1.6	2.3	0.4	0.4	1.4	166.7	3.2
7.1	2.3	1.4	78.5	3.2	1.4	0.4	4.4	78.5	166.7	3.2	1.4	1.4	2.3	2.3	3.2
2.3	78.5	3.2	0.4	0.4	3.2	1.4	2.3	1.4	1.4	2.3	0.4	78.5	78.2	4.4	4.4
166.7	2.3	1.4	3.2	78.5	2.3	2.3	4.4	4.4	33.7	0.0	4.4	4.4	1.4	4.6	1.4
7.1	4.6	4.4	4.4	0.4	2.3	0.4	7.1	0.4	0.4	2.3	2.3	78.2	2.3	2.3	4.4
4.4	4.4	2.3	2.3	0.4	0.4	0.4	1.6	2.3	2.3	33.7	166.7	1.4	4.6	166.7	0.4
0.4	2.3	1.4	0.4	33.7	0.4	0.4	1.6	0.4	3.2	0.4	1.4	78.2	0.4	0.4	3.2
78.5	1.6	0.4	1.4	166.7	3.2	3.2	166.7	4.4	78.5	2.3	4.4	0.4	0.4	0.4	2.3
3.2	78.2	78.5	0.4	0.4	2.3	2.3	2.3	3.2	78.5	1.4	1.6	0.4	4.6	2.3	4.6
7.1	0.4	4.4	7.1	2.3	0.4	0.4	0.4	4.4	4.4	4.4	2.3				

Solution: Network effects

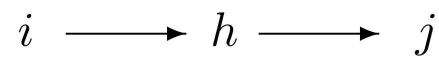
send



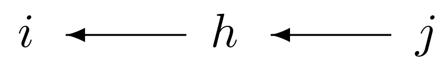
receive



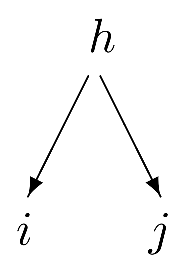
2-send



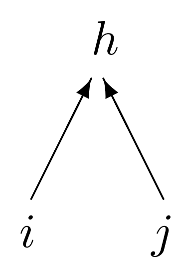
2-receive



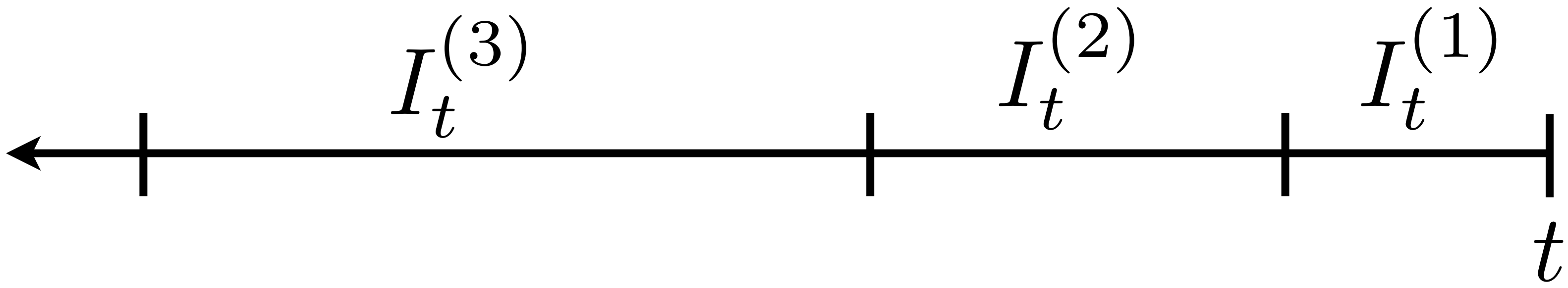
sibling



cosibling



Interval-dependent network effects



$$
\begin{aligned}
\mathbf{send}_t^{(k)}(i,j) &= \#\{i \to j \text{ in } I_t^{(k)}\},\\
\mathbf{receive}_t^{(k)}(i,j) &= \#\{j \to i \text{ in } I_t^{(k)}\};
\end{aligned}
$$

Triadic network effects

$$\textbf{2-send}_t^{(k,l)}(i,j) = \sum_{h \neq i,j} \#\{i \to h \text{ in } I_t^{(k)}\} \cdot \#\{h \to j \text{ in } I_t^{(l)}\},$$

$$\textbf{2-receive}_t^{(k,l)}(i,j) = \sum_{h \neq i,j} \#\{h \to i \text{ in } I_t^{(k)}\} \cdot \#\{j \to h \text{ in } I_t^{(l)}\},$$

$$\textbf{sibling}_t^{(k,l)}(i,j) = \sum_{h \neq i,j} \#\{h \to i \text{ in } I_t^{(k)}\} \cdot \#\{h \to j \text{ in } I_t^{(l)}\},$$

$$\textbf{cosibling}_t^{(k,l)}(i,j) = \sum_{h \neq i,j} \#\{i \to h \text{ in } I_t^{(k)}\} \cdot \#\{j \to h \text{ in } I_t^{(l)}\}.$$

Final model

$$\lambda_t(i,j) = \bar{\lambda}_t(i)\exp\{\beta^{\mathrm{T}} x_t(i,j)\}$$

$\lambda_t(i,j)\,dt$	Prob{i sends j a message in time [t,t+dt)}
$\bar{\lambda}_t(i)$	Baseline intensity for sender i
β	Vector of coefficients
$x_t(i,j)$	Vector of time-varying covariates

(cf. Butts 2008 , Vu et al. 2011)

MPLE asymptotics

Theorem (POP & PJW): Under regularity conditions:

1. $$\hat{\beta}_n \xrightarrow{P} \beta$$

2. $$\sqrt{n}(\hat{\beta}_n - \beta) \xrightarrow{d} \text{Normal}\big(0,\ \Sigma(\beta)\big)$$

Cox (1975): heuristic argument ("under mild conditions implying some degree of independence... and that the information values are not too disparate")

Andersen & Gill (1982): survival analysis, fixed time interval

Duplication

```
From: Alice
To: Bob, Carol, Dan
```

=

```
From: Alice
To: Bob
```

```
From: Alice
To: Carol
```

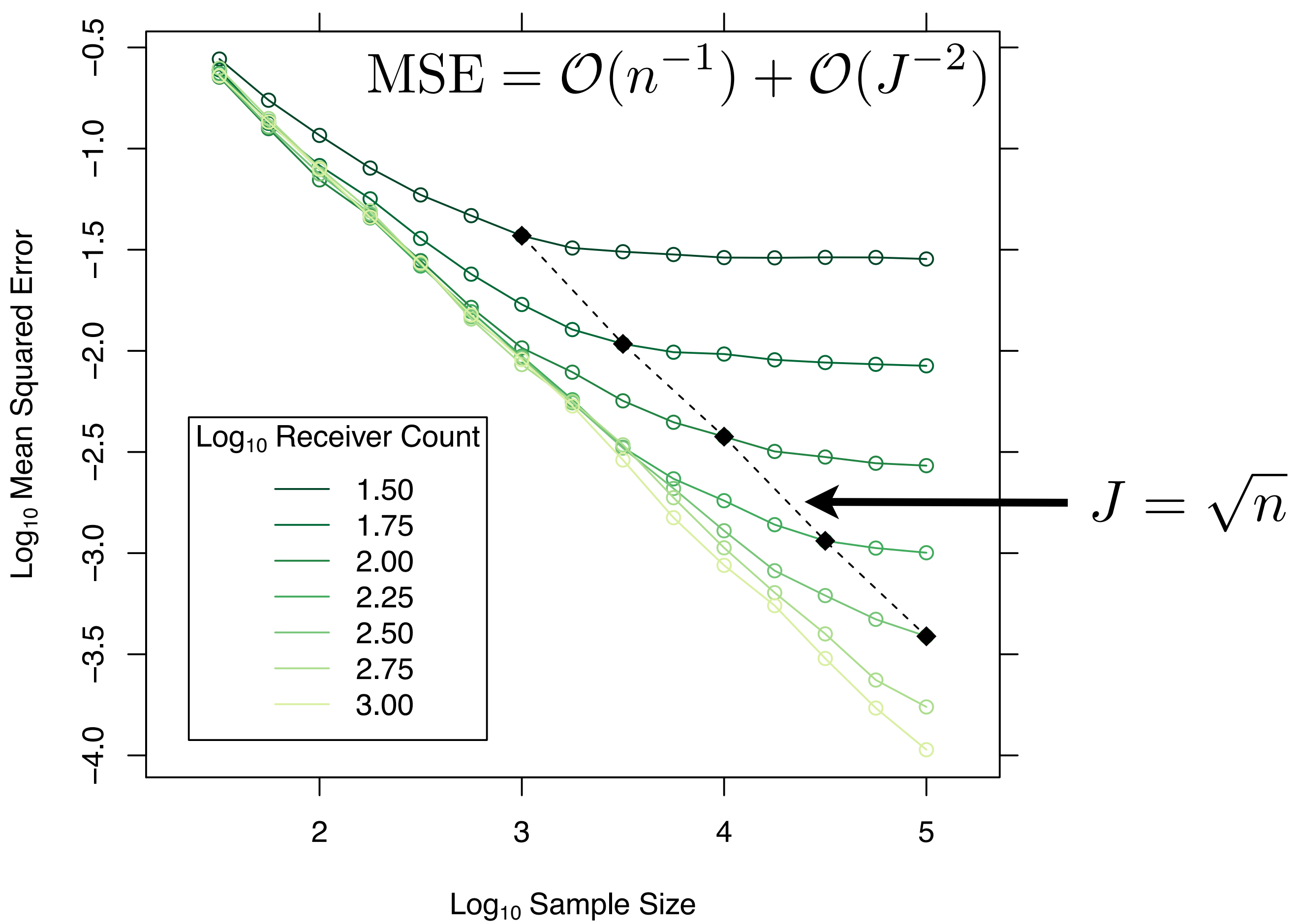
```
From: Alice
To: Dan
```

?

(21635 to 35567)

Approximation error

Theorem (POP & PJW): Under regularity conditions, using message duplication introduces bias of order $(\text{nodes})^{-1}$.



Summary so far

1. Interaction data: (t,i,j) tuples
2. Proportional intensity model; capture group effects and reciprocation through covariates
3. Consistent estimates via MPLE

Next: implementation

Enron results

Data

156 employees
21635 messages

Covariates

20 group-level covariates (static)
216 network effects (dynamic)

Time to fit: 15 minutes

Goodness of fit

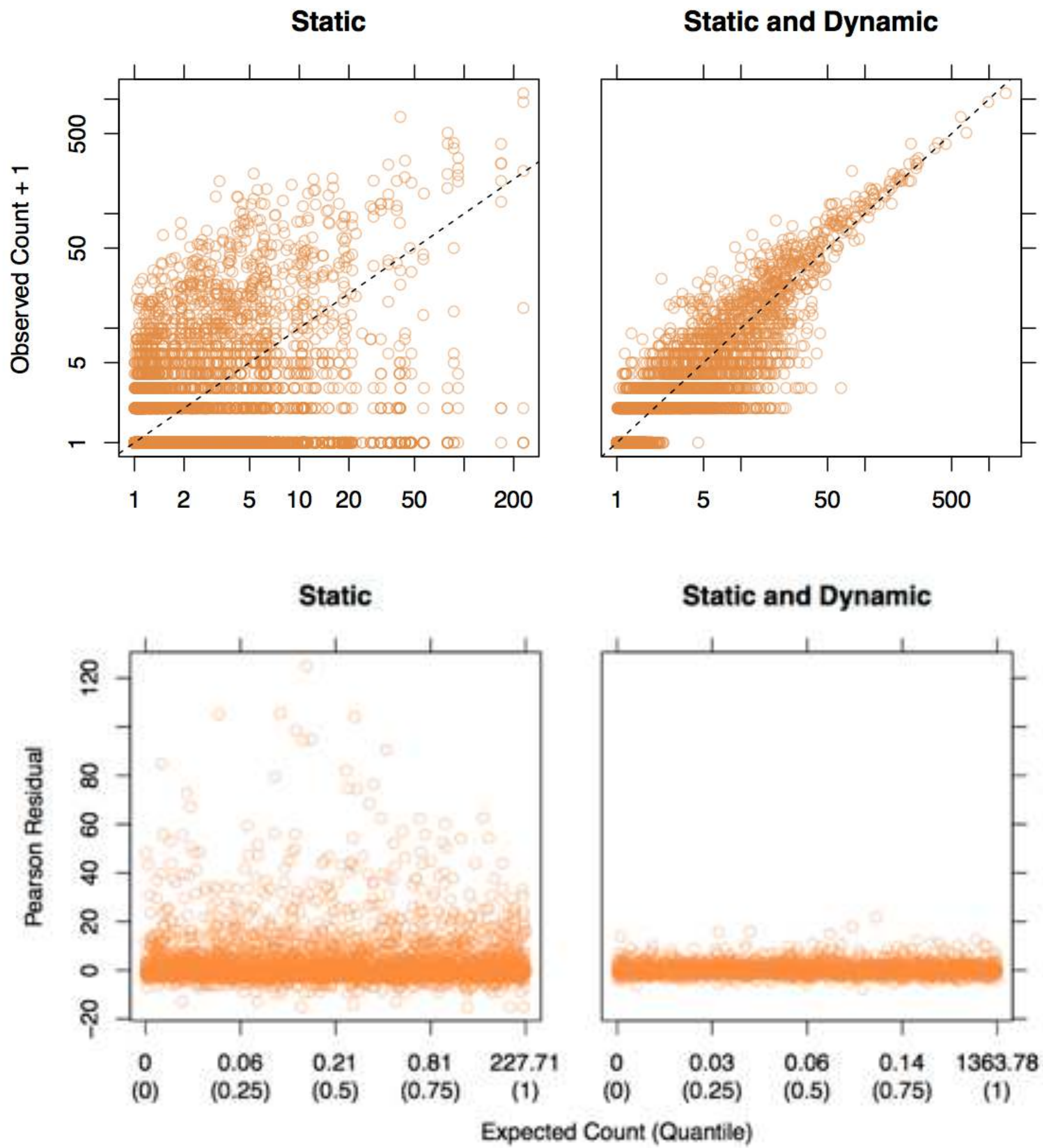
Messages from Tania J.

33	0	0	192	0	0	1	0	0	0	0	0	0	0	1	0
0	0	0	0	4	0	0	0	0	275	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	1
405	0	0	0	407	0	0	0	0	5	0	0	1	0	0	0
67	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	126	0	0	1	0
0	3	0	30	0	0	0	0	0	0	0	0	166	0	0	0
1	0	0	0	0	0	0	271	1	0	0	0	0	0	0	0
0	221	0	0	0	0	1	8	0	507	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	26	7	0				

Messages predicted by model

8.9	0.4	0.3	223.6	0.3	0.3	6.0	0.3	0.2	0.4	0.4	0.2	0.2	0.3	19.8	0.3
0.4	0.3	0.3	0.5	5.3	0.3	0.2	0.3	0.5	267.2	0.3	0.3	0.3	0.3	0.3	0.3
0.3	0.9	0.3	0.4	0.2	0.3	0.5	0.3	0.3	0.4	0.3	0.2	29.5	0.5	0.2	3.8
447.3	0.3	0.3	0.3	233.9	0.3	0.3	0.3	0.2	39.9	0.0	0.4	6.6	0.4	0.3	0.3
65.6	0.5	0.3	0.2	0.2	0.3	0.2	0.2	0.2	0.2	0.3	2.7	11.5	0.3	0.4	0.3
0.2	0.3	0.3	0.3	0.3	0.2	0.2	0.3	0.3	0.5	1.2	90.4	0.3	0.3	1.5	0.2
0.2	3.7	0.3	4.8	0.5	0.2	0.2	0.4	0.2	0.3	0.2	0.3	108.0	0.4	0.2	0.3
16.2	0.3	0.2	0.3	0.5	0.3	0.3	226.1	2.5	0.9	0.4	0.3	0.2	0.2	0.2	0.3
0.3	206.6	0.5	0.2	0.2	0.3	7.7	3.9	0.3	655.8	0.3	0.3	0.2	0.3	0.4	0.5
0.2	0.3	0.4	0.3	0.3	0.3	0.2	0.2	0.2	21.6	3.8	0.4				

Goodness of fit



Analysis of deviance

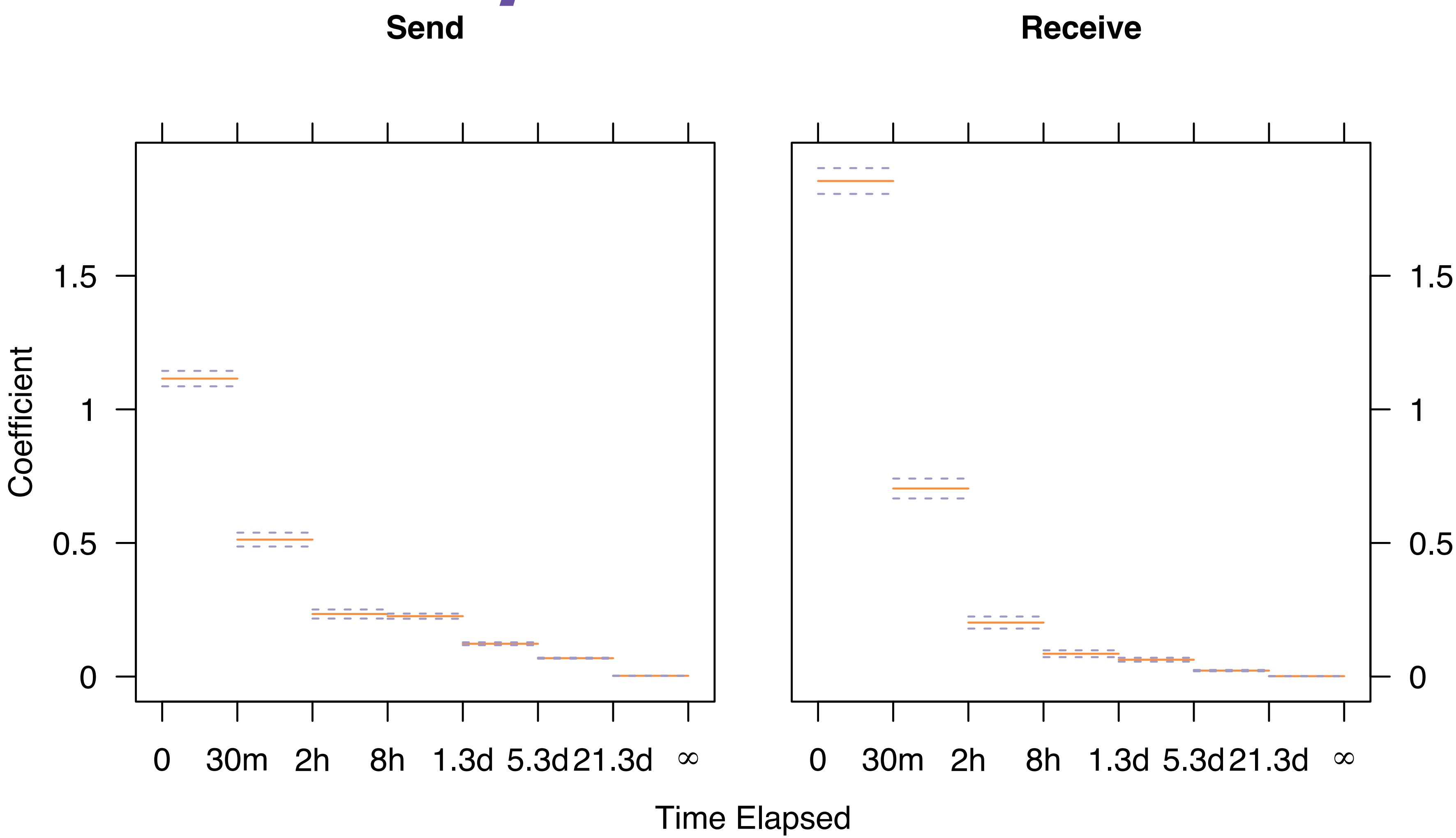
Term	Df	Deviance	Resid. Df	Resid. Dev
Null			32261	325412
Static	20	50365	32241	275047
Send	8	107942	32233	167105
Receive	8	5919	32225	161186
Sibling	50	3601	32175	157585
2-Send	50	516	32125	157069
Cosibling	50	1641	32075	155428
2-Receive	50	158	32025	155270

Group effects

	Receiver			
Sender	L	T	J	F
1	-0.91 (0.04)	-0.36 (0.04)	-0.34 (0.04)	0.04 (0.03)
L	0.63 (0.05)	0.28 (0.05)	0.22 (0.04)	0.15 (0.04)
T	0.32 (0.07)	0.43 (0.05)	0.27 (0.05)	-0.07 (0.05)
J	0.06 (0.05)	0.28 (0.04)	0.37 (0.03)	-0.13 (0.03)
F	0.59 (0.05)	-0.21 (0.05)	-0.09 (0.04)	0.15 (0.03)

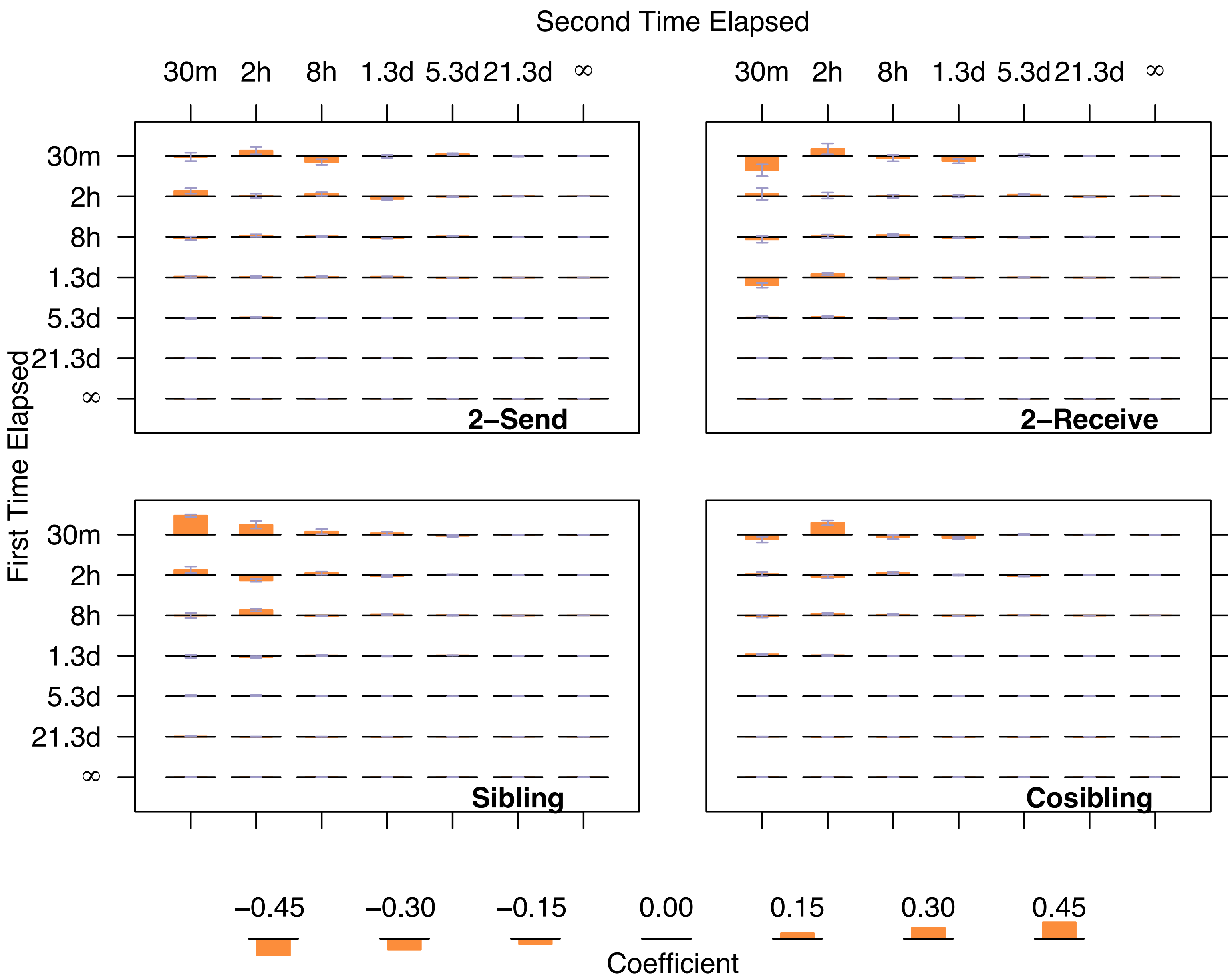
Example: All other factors being equal, **J**unior sends to **J**unior $e^{-0.34 + 0.37} - 1 = 4\%$ more than **J**unior sends to **S**enior; also, **S**enior sends to **S**enior $e^{-(-0.34)} - 1 = 40\%$ more than **S**enior sends to **J**unior.

Dyadic effects



Example: All other factors being equal, every message **j** has sent **i** in the last 30 minutes increases the relative **i-to-j** sending rate by $e^{1.8} = 6$; every message sent between 30 minutes and 2 hours increases the relative rate by $e^{0.7} = 2$.

Triadic effects



What have we learned?

1. Employees exhibit trait-based homophily in their message sending behavior.

2. History-dependent network effects are far more predictive than trait-based effects.

3. The predictive strength of the network effects decays rapidly in time.

References

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