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Point Sets with Many *k*-Sets*

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Abstract. For any n, k, $n \geq 2k > 0$, we construct a set of n points in the plane with $ne^{\Omega(\sqrt{\log k})}$ k-sets. This improves the bounds of Erdős, Lovász, et al. As a consequence, we also improve the lower bound for the number of halving hyperplanes in higher dimensions.

1. Introduction

For a set P of n points in the d-dimensional space R^d, a k-*set* is subset $P' \subset P$ such that $P' = P \cap H$ for some open half-space H, and $|P'| = k$. The problem is to determine the maximum number of k-sets of an n-point set in R^d. Even in the most studied two-dimensional case, we are very far from the solution, and in higher dimensions even less is known.

The first results in the two-dimensional case are due to Lovász [L] and Erdős et al. [ELSS]. They established an upper bound $O(n\sqrt{k})$, and a lower bound $\Omega(n \log k)$. Despite great interest in this problem [GP1], [W], [E2], [S], [EVW], [AACS], partly due to its importance in the analysis of geometric algorithms [EW2], [CP], [CSY], [E2], there was no progress until the very small improvement due to Pach et al. [PSS]. They improved the upper bound to $O(n\sqrt{k}/\log^* k)$. Recently, Dey [D] obtained an essential improvement of the upper bound; his bound is $O(n\sqrt[3]{k})$. There was no improvement on the lower bound of Erdős et al., besides little improvements on the constant [EW1], [E3], [E1].

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Theorem 1. *For any* $n, k, n \geq 2k > 0$, *there exists a set of n points in the plane with* $ne^{\Omega(\sqrt{\log k})}$ *k-sets*.

In the dual setting, Theorem 1 gives an arrangement of n lines such that the complexity of the k-level (the number of intersection points having exactly k lines above them) is $ne^{\Omega(\sqrt{\log k})}$. A similar bound was obtained by Klawe et al. [KPP] for the complexity of the median level ($k = n/2$) in *pseudoline arrangements* (see also [GP2] and [AW]). However, our construction seems to be essentially different.

Definition 1. Let $n > d \geq 2$, $n - d$ even, and let P be a set of n points in R^d in general position (no $d + 1$ of them lie in the same hyperplane). A hyperplane determined by d points of P is called a *halving hyperplane* (resp. *halving line* for $d = 2$ and *halving plane* for $d = 3$) if it has exactly $(n - d)/2$ points of P on both sides.

In the plane there is a one-to-one correspondence between complementary pairs of $n/2$-sets and halving lines [AG] and, for any fixed d, the number of halving hyperplanes is proportional to the number of $\lfloor n/2 \rfloor$-sets [E2], [DE]. Theorem 1 is based on the following result.

Theorem 2. *For any* $n > 0$ *even, there exists a set of n points in the plane with* $ne^{\Omega(\sqrt{\log n})}$ *halving lines*.

The k-set problem in space seems even harder than in the plane. The most interesting and studied case is $k = n/2$, i.e., finding the maximum number of *halving planes*. The first nontrivial upper bound was given by Bárány et al. [BFL]. It was improved by Aronov et al. [ACE$^+$], Eppstein [E4], and then by Dey and Edelsbrunner [DE] (see also [AACS]). The best known bound, $O(n^{5/2})$, was found very recently by Sharir et al. [SST]. In $d > 3$ dimensions, the trivial upper bound, $O(n^d)$, was only very slightly improved, to $O(n^{d-\varepsilon_d})$, by Živaljević and Vrećica [ZV] (see also [ABFK]). The best known lower bound in $d \geq 3$ dimensions, $\Omega(n^{d-1} \log n)$, follows directly from the lower bound in the plane, as described in [E2]. Using Theorem 1 and the method shown in [E2], we obtain an immediate improvement.

Theorem 3. *For any* $n > 0, d \geq 2$, *there exists a set of n points in* R^d *with* $n^{d-1}e^{\Omega(\sqrt{\log n})}$ *halving hyperplanes*.

2. Idea of the Construction

It is not hard to see and is shown in the next section that it is enough to consider the case $k = n/2$, i.e., the case of *halving lines*. Then the construction for other values of k can be obtained easily.

We construct a sequence of point sets, $V_0, V_1, V_2, \ldots$, recursively. For $i = 0, 1, 2, \ldots$, point set V_i has n_i points and at least m_i halving lines. Suppose that we already have V_{i-1}

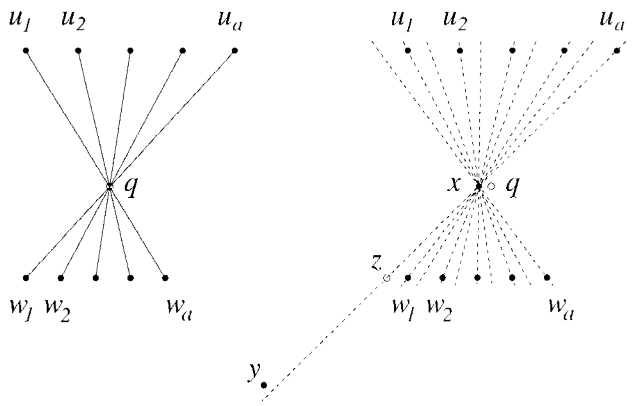

Fig. 1. The extra points x and y, and the new halving lines.

with parameters n_{i-1} and m_{i-1}. We can assume that none of the lines determined by the points is horizontal. Replace each of the points $v \in V_{i-1}$ by $a = a_i$ points, $v_1, v_2, \ldots, v_a$, lying from left to right on a short horizontal segment very close to v. Let the resulting point set be V'_{i-1}. Now we have an_{i-1} points. If the line uw is a halving line of V_{i-1}, then $u_1w_a, u_2w_{a-1}, \ldots, u_aw_1$ are all halving lines of V'_{i-1} (Fig. 1). Therefore, we get am_{i-1} halving lines. Clearly, this recursive construction would give only $m_i = O(n_i)$.

Now suppose that for each $v \in V_{i-1}$, the points $v_1, v_2, \ldots, v_a$ replacing v are placed *equidistantly* on the corresponding very short horizontal segment. Let uw be a fixed halving line of V_{i-1}. Suppose also that u lies higher than w. Then the corresponding a halving lines of V'_{i-1}, $u_1w_a, u_2w_{a-1}, \ldots, u_aw_1$, pass through the same point q (Fig. 1). Add two more points, x and y to V'_{i-1}. Let x be a point on the horizontal line through q, very close to q and to the left of it, and let y be anywhere on the left side of the oriented line $\overline{xu_1}$ and on the right side of $\overline{xw_1}$. Then $u_1w_a, u_2w_{a-1}, \ldots, u_aw_1$ are not halving lines any more, since they have two more points on one of their sides than on the other. Observe, however, that the lines $xu_1, xu_2, \ldots, xu_a$ and $xw_1, xw_2, \ldots, xw_a$ are all halving lines now. Consequently, by adding two extra points, we obtain $2a$ halving lines corresponding to the original halving line uw, instead of a, as in V'_{i-1}. We would like to add those extra points similarly for each pair $u, w \in V_{i-1}$, whenever uw is a halving line of V_{i-1}. The problem is that these extra points x and y work very well *locally* for uw, but they might ruin the other halving lines as they might be on their same side.

Once u and w are replaced by the a equidistant points, q is given, and we have very little freedom in choosing the location of x. On the other hand, we have much more freedom with y. The only way we can essentially relocate q, and hence x, is to change the distance between the consecutive points replacing u and v. In our construction we place the extra points x and y for each halving-pair $u, w \in V_{i-1}$ and introduce some further extra points, in such a way that none of the halving lines is ruined. So, finally every original halving line is replaced by $2a$ halving lines, and the number of points is just slightly more than a times the original number of points. More precisely, $m_i = 2am_{i-1}$ and $n_i \approx an_{i-1}$. With a proper choice of $a = a_i$, this will give the desired bound.

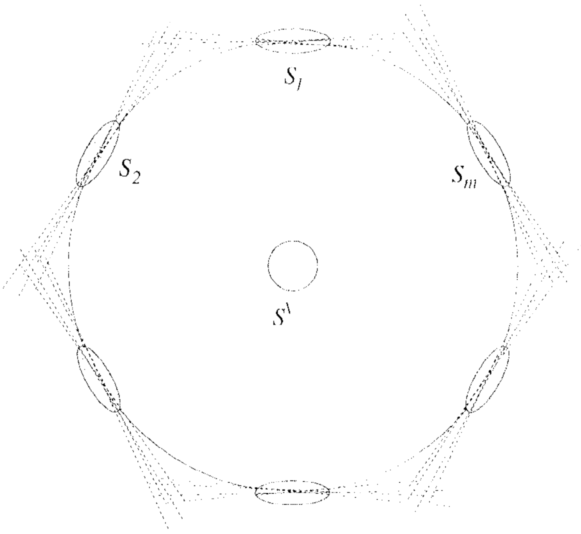

Fig. 2. Constructing a point set with many k-sets from point sets with many halving lines.

3. Proofs of Theorems 1 and 2

First we show how Theorem 1 follows from Theorem 2, and then we prove Theorem 2.

Proof of Theorem 1. Let n, k be fixed, $n \geq 2k > 0$, let $m = \lfloor n/2k \rfloor$, and let $m' = n - 2km$. Let $X_1, X_2, \ldots, X_m$ be the vertices of a regular m-gon, inscribed in a unit circle with center C. Let $\varepsilon > 0$ be very small and let $X_i(\varepsilon)$ be the ε-neighborhood of X_i ($i = 1, 2, \ldots, m$), and let $C(\varepsilon)$ be the ε-neighborhood of C.

By Theorem 2 there exists a $2k$-element point set S, with $2ke^{\Omega(\sqrt{\log k})}$ halving lines. For any $1 \leq i \leq m$ apply a suitable affine transformation A_i to S such that $A_i(S) = S_i \subset X_i(\varepsilon)$ and for any halving line ℓ of S_i, all $X_j(\varepsilon)$, $1 \leq j \leq m$, $j \neq i$, are on the same side of ℓ. Finally, let S' be a set of m' points in $C(\varepsilon)$. Then the set $T = S' \cup_{i=1}^{m} S_i$ has $m2k + m' = n$ points and $m2ke^{\Omega(\sqrt{\log k})} = ne^{\Omega(\sqrt{\log k})}$ k-sets (Fig. 2). □

Definition 2. For a positive integer a and $\varepsilon > 0$, let $P(a, \varepsilon)$ be a set of a equidistant points lying on a horizontal line such that the distance between the first and last points is ε. Then $P(a, \varepsilon)$ is called an (a, ε)-*progression*. We say that a point p is *replaced* by an (a, ε)-progression if p is identical to one of the points in the progression.

Definition 3. A *geometric graph* G is a graph drawn in the plane by (possibly crossing) straight line segments, i.e., it is defined as a pair $G = (V, E)$, where V is a set of points in general position (no three on a line) in the plane and E is a set of closed segments whose endpoints belong to V (see also [PA]).

Proof of Theorem 2. We construct a sequence of geometric graphs $G_0(V_0, E_0)$,

$G_1(V_1, E_1)$, $G_2(V_2, E_2)$, ..., recursively with the property that, for any i, every edge $e \in E_i$ is a halving line of V_i. For $i = 0, 1, 2, \ldots$, graph G_i has $|V_i| = n_i$ vertices and $|E_i| = m_i$ edges. Denote the *maximum degree* of a vertex in G_i by d_i.

Let G_0 have two vertices (points) and an edge connecting them. Suppose that we have already constructed G_{i-1}. Assume without loss of generality that no edge of G_{i-1} is horizontal. Let $\varepsilon = \varepsilon_i > 0$ be very small, and let $v_1, v_2, \ldots, v_{n_{i-1}}$ be the vertices of G_{i-1}. The graph $G_i(V_i, E_i)$ is constructed in three steps:

Step 1. For $j = 1, 2, \ldots, n_{i-1}$, replace v_j by an (a_i, ε^j)-progression. The exact value of $a = a_i$ will be specified later. The resulting point set is V'_{i-1}.

Step 2. Let e be an element of E_{i-1} with endpoints u and w. Then, for some $1 \leq \alpha, \beta \leq n_{i-1}$, we have $u = v_\alpha$, $w = v_\beta$. Suppose without loss of generality that $\alpha < \beta$. Denote the points of the arithmetic progression replacing u (resp. w) by $u_1, u_2, \ldots, u_a$ (resp. $w_1, w_2, \ldots, w_a$). Let q be the intersection of the lines $u_1w_a, u_2w_{a-1}, \ldots, u_aw_1$ (Fig. 1). Add two more points, x and y, to the point set as follows.

Place x so that xq is horizontal, x is to the left of q, and the distance $\overline{xq}$ is so small that, for $1 \leq j < a$, the line xu_j separates $w_1, w_2, \ldots, w_{a-j}$ from $w_{a-j+1}, \ldots, w_a$, and, similarly, the line xw_j separates $u_1, u_2, \ldots, u_{a-j}$ from $u_{a-j+1}, \ldots, u_a$.

Finally, let z be the intersection point of the line xu_a with the line passing through $w_1, w_2, \ldots, w_a$, and place y so that the vectors $\overrightarrow{qz}$ and $\overrightarrow{zy}$ are equal (see Fig. 1).

Add the edges $\{xu_1, xu_2, \ldots, xu_a, xw_1, xw_2, \ldots, xw_a\}$ to E_i.

Since ε is very small and $\alpha < \beta$, we obtain that x and y are in a small neighborhood of w. Moreover, $w_1, w_2, \ldots, w_a$ must be very close to the midpoint of the segment xy. Therefore, any line vw, with $w \in \{w_1, w_2, \ldots, w_a\}$, $v \in V'_{i-1}$, and $v \notin \{u_1, u_2, \ldots, u_a\}$, intersects the segment xy very close to its midpoint, in particular, it separates x and y.

Execute Step 2 for every edge $e \in E_{i-1}$.

Step 3. Let u be an element of V_{i-1}. In Step 1 we replaced u by an (a, ε^j)-progression, say $\{u_1, u_2, \ldots, u_a\}$, from left to right. In Step 2 we possibly placed some pairs of points in a small neighborhood of u. Denote the number of those points by $2D$. For each edge of G_{i-1} adjacent to u, we placed zero or two points in the neighborhood of u, and the number of those edges is at most d_{i-1}. Therefore, we have $D \leq d_{i-1}$.

Place $d_{i-1} - D$ points on the line of $\{u_1, u_2, \ldots, u_a\}$, to the left of u_1, such that their distance from u_1 is between ε and 2ε. Analogously, place $d_{i-1} - D$ points on the line of $\{u_1, u_2, \ldots, u_a\}$, to the right of u_a, such that their distance from u_a is between ε and 2ε (see Fig. 3).

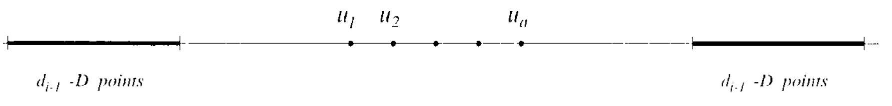

Fig. 3. Place $d_{i-1} - D$ points both to the left and to the right of $u_1, u_2, \ldots, u_a$.

Execute Step 3 for every vertex $u \in V_{i-1}$, and, finally, perturb the points very slightly so that they are in general position. Let $G_i(V_i, E_i)$ be the resulting geometric graph.

Claim 1. *All edges in E_i, introduced in Step 2, are halving lines of V_i.*

Proof of Claim 1. Let $e \in E_{i-1}$ be any edge of G_{i-1} with endpoints $u, w \in V_{i-1}$. Use the notations introduced in Step 2. Let $1 \leq j \leq a$. We know that the line xu_j separates $w_1, w_2, \ldots, w_{a-j}$ from $w_{a-j+1}, \ldots, w_a$. Therefore, it is a halving line of the point set $\{x, y, u_1, u_2, \ldots, u_a, w_1, w_2, \ldots, w_a\}$. All the other points in the neighborhoods of u and w are introduced in pairs, one on each side of the line xu_j. Since uw is a halving line of V_{i-1}, there are exactly $(n_{i-1} - 2)/2$ points of V_{i-1} on both sides of uw, and each of them is replaced by exactly $a + 2d_{i-1}$ points in their small neighborhoods. Therefore, we can conclude that the number of points of V_i lying on different sides of uw are the same. □

Each vertex of G_{i-1} is replaced by $a + 2d_{i-1}$ points. Therefore, $|V_i| = n_i = (a + 2d_{i-1})n_{i-1}$. For each edge $e \in E_{i-1}$, we introduced $2a$ edges in E_i. Consequently, $|E_i| = m_i = 2am_{i-1}$. Let $a = 4d_{i-1}$. Then we have

$$n_i = 6d_{i-1}n_{i-1}, \tag{1}$$
$$m_i = 8d_{i-1}m_{i-1}. \tag{2}$$

Now we calculate d_i. There are three types of points in V_i:

1. Those points which are introduced in Step 1. They have the same degree in G_i as the original point in G_{i-1}. Hence, the maximum degree of those points is d_{i-1}.
2. Those points which are introduced in Step 2. Half of them have degree zero, the other half have degree $2a = 8d_{i-1}$.
3. Those points which are introduced in Step 3. They all have degree zero.

Therefore, for $i > 0$, the maximum degree is $d_i = 8d_{i-1}$. Since $d_0 = 1$, we have $d_i = 8^i$. Using (1) and $n_0 = 2$,

$$n_i = 2 \cdot 6^i \cdot 8^{1+2+\cdots+(i-1)} = 8^{i^2/2+(\log_8 6-1/2)i+1/3}.$$

Analogously, using (2) and $m_0 = 1$,

$$m_i = 8^i \cdot 8^{1+2+\cdots+(i-1)} = 8^{i^2/2+i/2}.$$

Therefore,

$$m_i = n_i 8^{(1-\log_8 6)i-1/3} = n_i e^{\Omega(\sqrt{\log n_i})}.$$

This proves Theorem 2 if n is of the form $2 \cdot 6^i \cdot 8^{1+2+\cdots+(i-1)}$ for some $i \geq 0$. It is not hard to extend the result for every n, using the following easy and well-known results [L], [ELSS], [E2]. Let $f(n)$ be the maximum number of halving lines of a set of n points in the plane.

Claim 2. *For* $a, n > 0$, (i) $f(an) \geq af(n)$, *and* (ii) $f(n+2) \geq f(n)$.

Proof of Claim 2. Let P be a set of n points with $f(n)$ halving lines and suppose that no line determined by the points of P is horizontal. For (i), replace each point of P by an (a, ε)-progression. (See also the previous section and Fig. 1.)

For (ii), add two points to P, one very far from P to the left and one very far to the right. Then all halving lines of P are halving lines of the new point set. □

This concludes the proof of Theorem 2. □

4. Proof of Theorem 3

Let $f_d(n)$ be the maximum number of halving hyperplanes of a set of n points in R^d.

Claim 3. *For* $n > 0$, $f_d(n+2) \geq f_d(n)$.

Proof of Claim 3. The proof is analogous to the proof of Claim 2(ii). □

Suppose for simplicity that d is even. For d odd, the proof is analogous. By Claim 3, we can assume without loss of generality that n is divisible by 6. Let P_1 be a set of $n/3$ points in the intersection of the hyperplanes $x_1 = 0$ and $x_2 = 1$ such that no $d-1$ of them lie in a common $(d-3)$-dimensional affine subspace. Let $P_2 = -P_1$, that is, P_2 is the reflection of P_1 about the origin. Any hyperplane that contains the x_1-axis and avoids P_1, also avoids P_2 and cuts the set $P_1 \cup P_2$ into two equal subsets. Let P_3 be a set of $n/3$ points in the plane spanned by the x_1- and x_d-axes, with $ne^{\Omega(\sqrt{\log n})}$ halving lines, such that the points of P_3 are very close to the origin, and all halving lines have very little angles with the x_1-axis. Now any hyperplane which contains a halving line of P_3 and avoids $P_1 \cup P_2$, is a halving hyperplane of the set $P_1 \cup P_2 \cup P_3$. Since, for any halving line of P_3, there are $\Omega\left(n^{d-2}\right)$ combinatorially different such hyperplanes, Theorem 3 follows. □

Remarks. 1. The proofs of Theorems 1 and 2 imply the lower bound $ne^{0.282\sqrt{\ln k}-2.1}$ for the number of k-sets. If we use a better choice for the value of a_i, a proper ordering of the vertices of G_{i-1} before Step 1, and place the additional points in Step 3 more carefully, we can obtain the lower bound $ne^{0.744\sqrt{\ln k}-2.7} > (n/20)2^{\sqrt{\ln k}}$.

2. Based on Theorem 3 and the proof of Theorem 1, it is not hard to construct an n-element point set in R^d with $nk^{d-2}e^{\Omega(\sqrt{\log k})}$ k-sets.

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