Discrete Comput Geom 6:435–442 (1991)

Discrete & Computational
Geometry

Points and Triangles in the Plane and Halving Planes in Space*

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Abstract. We prove that for any set S of n points in the plane and $n^{3-\alpha}$ triangles spanned by the points in S there exists a point (not necessarily in S) contained in at least $n^{3-3\alpha}/(c \log^5 n)$ of the triangles. This implies that any set of n points in three-dimensional space defines at most $\sqrt[3]{(c/2)}n^{8/3} \log^{5/3} n$ halving planes.

1. Introduction

Let S be a set of n points in the plane and let $\mathscr{T}$ be the set of $\binom{n}{3}$ triangles whose vertices are in S. Boros and Füredi [5] show that there is a point x, not necessarily in S, contained in at least $n^3/27$ triangles and this bound is tight up to an additive

* Work on this paper by Boris Aronov and Rephael Wenger has been supported by DIMACS under NSF Grant STC-88-09648. Work on this paper by Bernard Chazelle has been supported by NSF Grant CCR-87-00917. Work by Herbert Edelsbrunner has been supported by NSF Grant CCR-87-14565. Micha Sharir has been supported by ONR Grant N00014-87-K-0129, by NSF Grant CCR-89-01484, and by grants from the U.S.–Israeli Binational Science Foundation, the Israeli National Council for Research and Development, and the Fund for Basic Research administered by the Israeli Academy of Sciences.

quadratic term. The main result of this paper is a generalization of this result to the case where $\mathcal{T}$ is only a subset of the $\binom{n}{3}$ triangles. Specifically, we show that if $|\mathcal{T}| = n^{3-\alpha}$, then there exists a point x contained in at least $n^{3-3\alpha}/(2^9 \log^5 n)$ of the triangles.[1] An application of this result is that n points in three-dimensional space define at most $2^{8/3}n^{8/3} \log^{5/3} n$ halving planes, where a *halving plane* of a three-dimensional set of n points in general position is a plane passing through three of the points that has an equal number of points on each of its sides (assuming n is odd). This improves the recent $O(n^{3-1/343})$ upper bound of Bárány *et al.* [4].

The organization of this paper is as follows. Section 2 introduces some one-dimensional selection lemmas proved by application of the pigeonhole principle. These form the primary combinatorial tools used to prove the main result in Section 3. Section 4 discusses the application to halving planes of a finite point set in space.

2. One-Dimensional Selection Lemmas

The primary combinatorial tools used to prove the main result of this paper are two variants of a result for points and intervals on a line. They are formulated as parts (i) and (ii) of Lemma 2.1. We refer to (i) as the "unweighted selection lemma" and to (ii) as the "weighted selection lemma." The unweighted selection lemma has also been used in a companion paper [6]; for the sake of completeness we repeat its proof which is not difficult. For two points p and q on the real line we call $I_{pq} = \{x \mid p < x < q \text{ or } q < x < p\}$ the *interval* of the edge $\{p, q\}$.

Lemma 2.1. *Let V be a set of $n \geq 2$ points on the real line and let E be a set or multiset of $m \geq 2n$ edges, that is, unordered point pairs. For a point $x \notin V$, let $E(x)$ denote any subset or submultiset of edges in E whose intervals contain x ($E(x)$ does not necessarily contain all such edges), define $m(x) = |E(x)|$ and let $n(x)$ be the number of points of V incident to edges in $E(x)$.*

(i) *If E is a set, then there is a point x and a set $E(x)$ with $m(x) \geq m^2/(4n^2)$.*
(ii) *If E is a multiset, then there is a point z and a multiset $E(z)$ with $m(z)/n(z) \geq m/(2n \log n)$.*

Proof. In order to show (i) choose $k - 1$ points, none in V, cutting the line into k intervals so that each contains no more than $\lceil n/k \rceil < n/k + 1$ points of V. The number of edges (including those not in E) whose intervals contain none of the $k - 1$ delimiters is therefore at most

$$k\binom{\lceil n/k \rceil}{2} < \frac{n^2 + nk}{2k}.$$

[1] All logarithms in this paper are to the base 2.

The remaining intervals contain at least one of the $k-1$ delimiters each and there are at least $m-(n^2+nk)/2k$ such intervals, which is at least $m/2$ if we choose $k=\lceil n^2/(m-n)\rceil$. By the pigeonhole principle one of the chosen points is contained in at least $m/(2(k-1)) \geq (m^2-mn)/(2n^2) \geq m^2/(4n^2)$ intervals.

To prove (ii) replace the $k-1$ delimiters by $n-1$ points, one between any two adjacent points of V, and build a minimum height binary tree whose nodes are the $n-1$ chosen points, so that the inorder sequence coincides with the left-to-right order of the points. The height of the tree is $\lfloor \log(n-1) \rfloor < \log n$. For a node z define $E(z)$ as the set of edges in E whose intervals contain z but no ancestors of z. In this way each edge is counted at exactly one node which implies $\sum_z m(z) = m$. Because each point of V can be incident to edges associated with at most one node per level we also have $\sum_z n(z) < n(1+\log n)$. Now assume that $m(z)/n(z) < m/(n(1+\log n))$ for each node z. But then

$$\sum_z m(z) < \frac{m}{n(1+\log n)} \sum_z n(z) < m,$$

a contradiction. This implies that there is a point z with

$$\frac{m(z)}{n(z)} \geq \frac{m}{n(1+\log n)} \geq \frac{m}{2n\log n}. \qquad \square$$

Remark. It is easy to see that (i) is tight up to a multiplicative constant in the worst case. In [6] we also show that a set variant of (ii) is tight (up to a multiplicative constant) in the worst case.

3. Selecting a Point in a Set of Triangles

This section studies the problem of finding a point that lies in many members of a given collection of t (open) triangles in the plane. This problem is not interesting in general because all triangles can be disjoint. However, if t is much larger than n, the total number of distinct vertices of the triangles, then we can prove that there must be a point (not necessarily a vertex) contained in many of the triangles. Our result is a generalization of a theorem of Boros and Füredi [5] who prove that the set of all $\binom{n}{3}$ triangles defined by any set of n points in the plane has a subset of size at least $n^3/27$ with nonempty common intersection. Our result is also a generalization of the unweighted selection lemma to two dimensions; other such generalizations can be found in [6]. The theorem and its proof constitute the remainder of this section.

Theorem 3.1. *Let S be a set of n points in the plane and let $\mathcal{T}$ be a set of $t = n^{3-\alpha}$ open triangles spanned by the points of S. Then there exists a point contained in at least $t^3/(2^9 n^6 \log^5 n) = n^{3-3\alpha}/(2^9 \log^5 n)$ of the triangles.*

Remark. In the case $\alpha = 0$ (or, more precisely, when $t = \Omega(n^3)$) our techniques can be slightly modified to prove a lower bound of cn^3 for the number of triangles with nonempty common intersection, thus reproducing (albeit with a smaller constant of proportionality) and extending the result of [5].

All steps of the proof of Theorem 3.1 are elementary and make use of the two selection lemmas presented in the preceding section.

Proof. The first step associates each triangle in $\mathcal{T}$ with one of its edges as follows. Choose a direction in the plane with the property that no line in this direction passes through more than one point of S. Without loss of generality assume this to be the y-direction. For each triangle in $\mathcal{T}$ define its *base* to be the edge with the longest orthogonal x-projection. Each triangle in $\mathcal{T}$ is associated with its base. For each segment ab connecting a pair of points in S define its *multiplicity* μ_{ab} to be the number of triangles that have ab as a base. Clearly, $\sum_{ab} \mu_{ab} = t$.

We next choose an integer $\xi \geq 2$ with the property that

$$\sum_{\xi \leq \mu_{ab} \leq 2\xi} \mu_{ab} \geq \frac{t}{\log n}.$$

The existence of such a ξ can be established if we assume that $t > n^2$ which is no loss of generality as Theorem 3.1 is void otherwise. Cover the interval $[2, n-2]$ of integers with $\lceil \log(n-2) \rceil - 1 \leq \log n$ subintervals of the form $[2^j, 2^{j+1}]$, for $j = 1, \ldots, \lceil \log(n-2) \rceil - 1$, and consider for each subinterval the subset of edges whose multiplicities it contains. By the pigeonhole principle, there must be one subinterval $[2^j, 2^{j+1}]$ so that $\xi = 2^j$ satisfies the above condition.

Let $\mathcal{T}_0$ denote the collection of triangles in $\mathcal{T}$ whose bases have multiplicity in $[\xi, 2\xi]$. The number of such triangles is $t_0 \geq t/(\log n)$. Since t_0 is also bounded from above by $2\xi\binom{n}{2}$, it follows that

$$\xi \geq \frac{t}{n^2 \log n}.$$

From now on we consider only the triangles in $\mathcal{T}_0$.

In the second step of the proof we orthogonally project all points of S onto the x-axis, and define a multiset of intervals delimited by the projected points as follows. For each base ab whose multiplicity is in $[\xi, 2\xi]$ we take every pair of distinct triangles abc, abd in $\mathcal{T}_0$ associated with ab and obtain the corresponding interval $c'd'$, where c' and d' are the projections of the "inner" points c and d. See Fig. 3.1 for an illustration. The number of resulting intervals, counted with multiplicity, is $m_0 \geq t_0(\xi - 1)/2 \geq t_0\xi/4$, since each triangle of $\mathcal{T}_0$ is paired with at least $\xi - 1$ triangles, and each such pair arises twice.

We can thus apply the weighted selection lemma (Lemma 2.1(ii)) to deduce that there exists a point z_0 on the x-axis contained in m_1 intervals (counted with

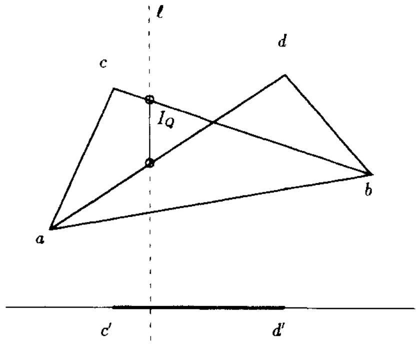

Fig. 3.1. Pairing two triangles with a common base.

multiplicity), which are delimited by at most n_1 distinct endpoints, such that

$$\frac{m_1}{n_1} \geq \frac{m_0}{2n \log n} \geq \frac{t_0 \xi}{8n \log n}.$$

Each of the m_1 resulting intervals $c'd'$ corresponds to a quadruplet $Q = \{a, b, c, d\}$, as in Fig. 3.1. Let l denote the vertical line $x = z_0$. Note that l intersects both segments ab and cd, and that the interval $I_Q = (l \cap ad, l \cap bc)$ is contained in the union of the two corresponding triangles abc, abd of $\mathscr{T}_0$.

In the third step of the proof we consider the system of intervals along l as just defined. We have $m_2 = m_1$ such intevals. By how many endpoints are they delimited? Each such endpoint is the intersection of l with a segment ad so that a is an endpoint of some base while d is one of the at most n_1 "inner" points obtained in the previous step. It follows that the number n_2 of endpoints along l is at most nn_1.

We now apply the unweighted selection lemma (Lemma 2.1(i)) to this system of intervals and points. Note that no two distinct quadruplets can give rise to the same interval (assuming l is chosen so that no two segments connecting points in S meet l at the same point), because the endpoints of the interval uniquely determine the four points a, b, c, d forming the quadruplet. Thus the unweighted selection lemma is indeed applicable, and it yields a point $x_0 \in l$ that is contained in at least

$$\frac{m_2^2}{4n_2^2} \geq \frac{m_1^2}{4n^2 n_1^2} \geq \frac{1}{4n^2}\left(\frac{t_0 \xi}{8n \log n}\right)^2 = \frac{t_0^2 \xi^2}{2^8 n^4 \log^2 n}$$

intervals. Since each interval I_Q is contained in the union of the two triangles forming Q, it follows that x_0 must lie in at least this many triangles of $\mathscr{T}$, where each triangle is now counted with multiplicity, which fortunately is at most 2ξ (recall that a triangle abc can be coupled with at most 2ξ points d to form a quadruple of the type we consider). It follows that x_0 lies in at least

$$\frac{t_0^2 \xi}{2^9 n^4 \log^2 n} \geq \frac{t^3}{2^9 n^6 \log^5 n} = \frac{n^{3-3\alpha}}{2^9 \log^5 n}$$

distinct triangles, where the last expression is obtained by substituting the lower bounds on t_0, ξ that were noted above. □

Remark. Notice that Theorem 3.1 is reasonably strong only as long as α is small; in particular, it is void if $\alpha \geq 1$. We can prove that there is a point contained in at least $t/2n = \frac{1}{2}n^{2-\alpha}$ triangles, which imporves Theorem 3.1 if $\alpha \geq \frac{1}{2}$. For each point $p \in S$ take the sum of the angles at p covered by the incident triangles. Since the sum of angles of any triangle is π, there must be a point p whose sum of angles is at least $t\pi/n$. The full angle at p is 2π which implies that there is a point near p covered by at least $t/2n$ of the angles. This point lies in at least $t/2n = \frac{1}{2}n^{2-\alpha}$ triangles.

4. An Application to Halving Planes in Space

Let S be a set of n points in d-dimensional space. A *k-set* of S is a subset $T \subseteq S$ with $|T| = k$ and $T = S \cap h$ for some half-space h. We define $f_{d,k}(n)$ as the maximum number of k-sets realized by any set of n points in d dimensions. Obtaining sharp bounds on $f_{d,k}(n)$ has remained an elusive task for almost 20 years. In the plane the known bounds are $f_{2,k}(n) = \Omega(n \log k)$ [9], [10] and $f_{2,k}(n) = O(n\sqrt{k}/\log^* k)$ [12]. The problem of counting the total number of j-sets for $1 \leq j \leq k$ is much better understood. The maximum number of such j-sets for n points in d dimensions is $\Theta(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$ [1], [7], [11].

The most difficult case in the analysis of $f_{d,k}(n)$ seems to be when k is about half of n. In three dimensions call a triangle abc, a, b, $c \in S$, a *halving triangle* if the plane through these points has at most $(n-3)/2$ points of S on each side; call the plane through a, b, and c a *halving plane.* A recent result of Bárány *et al.* [4] shows that the number of halving planes of n points in three dimensions is $O(n^{3-1/343})$; this is the first nontrivial upper bound on this quantity. The best lower bound is $\Omega(n^2 \log n)$ [8].

Using Theorem 3.1 we can improve the upper bound on the number of halving planes.

Theorem 4.1. *The number of halving planes of a set of n points in three dimensions is less than* $2^{8/3} n^{8/3} \log^{5/3} n$.

Proof. We make use of the fact, established in [4], that no line in space can interesect more than $\binom{n}{2}$ halving triangles of S. Project the points and all halving triangles onto a plane so that no two points have coinciding projections. Theorem

3.1 implies that if S has more than t halving triangles, then there is a point x_0 in this plane contained in the projections of more than $t^3/(2^9 n^6 \log^5 n)$ halving triangles. This implies that the line orthogonal to the projection plane through x_0 intersects that many halving triangles. Hence, by the result of [4], this number must be smaller than $n^2/2$, which implies $t < 2^{8/3} n^{8/3} \log^{5/3} n$. □

Remark. The above bound also serves as an upper bound for the number of k-triangles spanned by a set S of n points, where a *k-triangle* is a triangle whose vettices belong to S and the plane containing it contains exactly k points of S in one of its open half-spaces. In other words, the above bound is also an upper bound on the number of k-sets of a set of n points in three-dimensional space, for any k. Of course, if k is very small, then better bounds are known (see [7]).

The maximum number of halving planes of a set of n points in space is proportional to the maximum number of facets of the so-called median level in an arrangement of n planes in space, see [8]. If we project the facets of a median level vertically onto the xy-plane we get a subdivision known as a higher-order power diagram of a set of n circles in the plane, see [2]. Thus, Theorem 4.1 implies that the number of facets of the median level in space and the number of regions in a higher-order power diagram is $O(n^{8/3} \log^{5/3} n)$.

5. Discussion

This paper proves that for any set of $n^{3-\alpha}$ triangles with a total of n vertices in the plane there exists a point contained in at least $n^{3-3\alpha}/(2^9 \log^5 n)$ of the triangles. A corollary of this result is that any set of n points in three-dimensional space defines at most $2^{8/3} n^{8/3} \log^{5/3} n$ halving triangles. An interesting open problem is the extension of these results to higher dimensions. In view of a result of Bárány [3], who shows that the set of $\binom{n}{d+1}$ simplices defined by n points in d dimensions always contains a constant fraction of simplices with nonempty common intersection, it seems likely that the result on triangles can be generalized to dimensions $d \geq 3$. Such a generalization would then imply new upper bounds on the number of halving hyperplanes in $d + 1$ dimensions.

Another promising line of research is to find other applications of the combinatorial selection lemmas of Section 2. Indeed, these lemmas can be generalized to hyperrectangular boxes and spheres in higher dimensions [6].

Acknowledgments

The authors wish to thank Noga Alon for a helpful discussion that led to an initial significant improvement in the bounds derived in the paper. Parts of the work on this paper were carried out at DEC Systems Research Center in Palo Alto, California, and at the Bellairs Research Institute of McGill University, Barbados, and the authors would like to express their gratitude for the hospitality of both institutions. Finally the authors would like to thank the DIMACS Center for additional support.

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Received January 3, 1990, and in revised form March 2, 1990.