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POINTWISE AND ORDER CONVERGENCE FOR SPACES OF CONTINUOUS FUNCTIONS AND SPACES OF BAIRE FUNCTIONS

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Suppose L is a Riesz space (a lattice ordered vector space). For notation and basic terminology concerning Riesz spaces, the reader is referred to Luxemberg and Zaanen [6]. A *Riesz homomorphism* between two Riesz spaces is a positive linear map that preserves the finite lattice operations. The Riesz space L is *almost σ-complete* if it is Riesz isomorphic to a subspace $L^{\sim}$ of a σ-complete space M with the property that if $m \in M^+$, there is a sequence (u_n) such that $u_n \in L^{\sim}$ $(n = 1, 2, \ldots)$, $0 \leqq u_1 \leqq \leqq u_2 \leqq \ldots$, and $\bigvee_{n=1}^{\infty} u_n = m$. See Aliprantis and Langford [1] or Quinn [10] for some properties of almost σ-complete spaces.

A sequence $f_1, f_2, f_3, \ldots$ is *order Cauchy* if there is a sequence $y_1 \geqq y_2 \geqq y_3 \geqq \ldots \ldots \geqq 0$, $\bigwedge y_n = 0$, such that, for $m \geqq n$, $|f_m - f_n| \leqq y_n$. If every order Cauchy sequence converges then L is *order Cauchy complete.* By Corollary 8.5 of Quinn [12], L is σ-complete if and only if it is almost σ-complete and order Cauchy complete.

Suppose Q is a set and Ω is a Riesz space of real valued functions defined on Q containing the constants. Then $B_1(\Omega)$ (the first Baire class of Ω) is the set of all pointwise limits of sequences of Ω. Also, $USB_1(\Omega)$ is the set of all pointwise limits of non-increasing sequences of $B_1(\Omega)$. For a discussion of Baire spaces see Mauldin [7] or [8].

If X is a topological space, then X is said to be a *P-space,* if $C(X)$, the set of all continuous functions on X, is closed with respect to pointwise convergence. In Gilman and Henriksen [4], P-spaces were introduced. See also Regoli [13] for a discussion of some properties of P-spaces.

It is the purpose of this paper to compare monotone pointwise convergence with monotone order convergence and bounded pointwise convergence with order convergence in $B_1(\Omega)$ and in $C(X)$. All convergence referred to here is sequential convergence.

A first easy observation is that monotone pointwise convergence implies monotone order convergence in both $B_1(\Omega)$ and $C(X)$.

Theorem 1. *Suppose X is either a completely regular Baire space with the countable chain condition or a perfectly normal Baire space. Then bounded pointwise convergence implies order convergence in $C(X)$.*

Proof. Suppose X is a completely regular Baire space with the countable chain condition. Suppose $f_1, f_2, f_3, \ldots$ is a sequence of functions in $C(X)$ pointwise converging to 0 and $0 \leqq f_i \leqq 1$ for each positive integer i. Let $f_{ij} = ((f_i \wedge (1/2)^{j-1}) \vee \vee (1/2)^j) - (1/2)^j$ for each positive integer i and j. Note that $f_i = \sum_{j=1}^{\infty} f_{ij}$ and $0 \leqq f_{ij} \leqq \leqq (1/2)^j$. Also for every point x of X and positive integer j there exists a positive integer k such that if $i > k$, then $f_{ij}(x) = 0$.

For each positive integer i and j let Z_{ij} be the set to which x belongs if $f_{pj}(x) = 0$ for $p \geqq i$ and let U_{ij} be the interior of Z_{ij}. Note that Z_{ij} is closed. For each positive integer j, $Z_{1j} \subset Z_{2j} \subset Z_{3j} \subset \ldots$ and $\bigcup_{i=1}^{\infty} Z_{ij} = X$. Let M_j be the complement of $\bigcup_{i=1}^{\infty} U_{ij}$. Since $\bigcup_{i=1}^{\infty} U_{ij}$ is open, M_j is closed. The interior of M_j is an open set which is the countable union of a collection of sets each of which is the subset of the boundary of Z_{ij}, for some i. As the boundary of each Z_{ij} is nowhere dense and X is a Baire space, M_j does not contain an interior point and M_j is nowhere dense. Therefore $\bigcup_{i=1}^{\infty} U_{ij}$ is an open dense set.

As X is completely regular, for each point x of U_{ij} there is a continuous function h_x such that $0 \leqq h_x \leqq 1$, $h_x(x) = 1$ and $h_x(y) = 0$ for each y not in U_{ij}. Let V_x be the cozero set of h_x. Let ω be a maximal disjoint collection of V_x and B_{ij} be the union of the sets in ω. As X has the countable chain condition, $\omega = \{V_1, V_2, \ldots\}$ is countable. Suppose y belongs to U_{ij} and N is an open set containing y. If N is disjoint from the sets in ω, a V_y containing y and no point of B_{ij} can be constructed as X is completely regular. Then ω is not maximal. Thus y is a limit point of B_{ij} and B_{ij} is dense in U_{ij}. It follows that $\bigcup_{i=1}^{\infty} B_{ij}$ is an open dense set.

Let $\{k_1, k_2, \ldots\}$ be a collection of continuous functions such that for each positive integer p, $0 \leqq k_p \leqq 1$ and V_p is the cozero set of k_p and let $t_{ij} = \sum_{n=1}^{\infty} (1/2)^n k_n$. Then t_{ij} is continuous and its cozero set is B_{ij}. Let $r_{ij} = \bigwedge_{i \geqq p,n} (1 - ((p \cdot t_{nj}) \wedge 1))$. Then $r_{1j} \geqq r_{2j} \geqq r_{3j} \geqq \ldots$ is a sequence of continuous functions converging to zero on an open dense set $W_j = \bigcup_{i=1}^{\infty} B_{ij}$ such that $r_{ij} \geqq f_{ij}$. Let $g_i = \sum_{j=1}^{\infty} r_{ij}$. Thus $g_1 \geqq g_2 \geqq \geqq g_3 \geqq \ldots$ is a sequence of continuous functions converging to zero on $\bigcap_{i=1}^{\infty} W_j$. As X is a Baire space $\bigcap_{j=1}^{\infty} W_j$ is dense. Further $g_i \geqq f_i$. Therefore $f_1, f_2, f_3, \ldots$ order converges to zero.

Suppose X is a perfectly normal Baire space. Define f_{ij} as before. Let $q_{ij} = \bigvee_{p \geqq i} f_{pj}$. Then q_{ij} is lower semi-continuous, q_{ij} converges pointwise to zero as $i \to \infty$, and for every point x in X there exists a positive integer k such that $q_{kj}(x) = 0$. Let m_{ij} be the pointwise limit of $\{(nq_{ij}) \wedge (1/2)^j\}$, $n = 1, 2, 3, \ldots$. Then m_{ij} is lower semi-continuous, is equal to zero wherever q_{ij} is equal to zero, and is equal to $(1/2)^j$ wherever q_{ij} is not equal to zero. Let $r_{ij}(x) = 0$ if x belongs to the interior of $m_{ij}^{-1}(0)$ and equal $(1/2)^j$ otherwise. Since $m_{ij}^{-1}(0)$ is closed, its boundary is nowhere dense. Thus r_{ij} is a non-increasing sequence which converges pointwise to zero as $i \to \infty$ on a set of the second category in X. Also r_{ij} is upper semi-continuous. Since X is perfectly normal there exists a non-increasing sequence h_{ijk}, $k = 1, 2, 3, \ldots$, of continuous functions converging pointwise to r_{ij}. Let $g_{1j} = h_{1j1} \wedge (1/2)^j$ and $g_{pj} = \bigwedge_{i+k=p} h_{ijk} \wedge (1/2)^j$ if $p > 1$. Then g_{pj}, $p = 1, 2, 3, \ldots$, is a non-increasing sequence of continuous functions such that $g_{pj} \geqq r_{pj} \geqq m_{pj} \geqq q_{pj} \geqq f_{pj}$ and g_{pj} converges pointwise to zero on a set of the second category in X.

Let $g_p = \sum_{j=1}^{\infty} g_{pj}$. Then g_p is continuous, $g_p \geqq f_p$, and g_p converges to zero except possibly on a set of the first category in X. In particular g_p converges to zero on a dense subset of X. Thus f_p order converges to zero.

Example 2. Let X be the square of the interval $[0, 1]$ ordered lexicographically. Take as a basis for the topology on X the sets of the form

$$\{(a, x) \mid b > x > c\},$$
$$\{(a, x) \mid 1 \geqq x > b\}, \text{ and}$$
$$\{(a, x) \mid c > x \geqq 0\}.$$

This is a perfectly normal Baire space without the countable chain condition.

The following example is due to H. Cook.

Let M be the subset of the plane to which x belongs if the first coordinate of x is either 0 or 1 and the second coordinate of x is in the closed interval $[0, 1]$. Order M lexicographically. Let a basis for the topology on M be the collection of sets

$$\{(x, y) \mid a < x < b,\ c \leqq y < d\}$$

and

$$\{(x, y) \mid a < x \leqq b,\ c < y < d\}.$$

Let $L = M \times M$. Then L is compact and has the countable chain condition. On the other hand, let R be the subset of L consisting of all pairs of points of M each of whose first coordinate is 1. Then R is the square of an interval of the Sorgenfrey line and is not normal. This implies that L is not completely normal (and thus not perfectly normal).

Theorem 3. *Suppose L is a linear lattice of real valued functions and bounded pointwise convergence implies order convergence in L. Then L is almost σ-complete.*

Proof. Let $\{f_1, f_2, f_3, \ldots\}$ be a bounded disjoint sequnce in L. As $f_1, f_2, f_3, \ldots$, converges pointwise to 0, it must order converge to 0. By Theorem 2.2 of Fremlin [3], L is almost σ-complete.

In [10] Meyer gives a necessary and sufficient condition on a normal topological space X for $C(X)$ to be almost σ-complete. Compare Theorems 1 and 3 here with his Corollary 8. Also see section 9 of [5].

Example 4. There exists a compact Hausdorff space X such that $C(X)$ is not almost σ-complete. Let H denote the set of all continuous functions on the interval $[0, 1]$ and K denote the set of bounded functions in $B_1(H)$. Since $B_1(H)$ is not closed with respect to pointwise convergence it is not σ-complete (Tucker [18], Lemma 4). As it is order Cauchy complete (Tucker [18], Proposition 6), $B_1(H)$ can not be almost σ-complete. If ω denotes the functions it is σ-complete.

Suppose $f_1 \geqq f_2 \geqq f_3 \geqq \ldots$ is a sequence of functions in $C(X)$ converging pointwise to a function f and, further, suppose x is a point of X such that f is not continuous at x. Let $k = f(x) - 1$. Replace each of f_i with $f_i \vee k$ and f with $f \vee k$. Since $\{f_i \vee k\}$ is bounded below and $C(X)$ is σ-complete, $\bigwedge(f_i \vee k)$ is in $C(X)$ and since monotone order convergence implies pointwise convergence $\bigwedge(f_i \vee k) = f \vee k$.

By Theorem 1 of Tucker [14], any function which is the pointwise limit of a sequence of functions in $C(X)$ can be uniformly approximated by the difference of two functions each of which is the pointwise limit of a non-increasing sequence of functions in $C(X)$ and is therefore continuous.

Since $C(X)$ is closed under pointwise convergence, X is a P-space.

(3) implies (2). If X is a P-space, $C(X)$ is closed with respect to pointwise convergence. If $f_1, f_2, f_3, \ldots$ is a bounded sequence converging pointwise to f, let $y_n(x) = \max_{i \geqq n} f_i(x)$. Then $y_1, y_2, y_3, \ldots$ is a nonincreasing sequence of functions converging pointwise to f and $y_n \geqq f_n$. Thus $f_1, f_2, f_3, \ldots$ order converges to f. Also any monotone order convergent sequence is pointwise convergent (Tucker [16], Lemma 3), so that any order convergent sequence is pointwise convergent.

(2) implies (1). By Theorem 3, $C(X)$ is almost σ-complete.

A Riesz space L is said to have the *sequential mapping continuity property* (abbreviated s.m.c. property) if every positive linear map from L into an Archimedean, directed, partially ordered vector space is sequentially order continuous. This property was defined in [16] and was shown in K which take on only the values 0 and 1, every function in K is the uniform limit of a sequence of functions each of which is a linear combination of the functions in ω. (Tucker [17], Theorem 7). The functions in ω form a Boolean algebra, so by the Stone representation theorem they are isomorphic to the open and closed sets of a totally disconnected compact Hausdorff space X. The natural mapping between K and $C(X)$ is a Riesz isomorphism. As one can note from the definition of almost σ-complete, it is a property which is preserved by Riesz isomorphisms. Thus $C(X)$ is not almost σ-complete.

Theorem 5. *The following three statements are equivalent:*

(1) *$C(X)$ is almost σ-complete and monotone order convergence implies pointwise convergence,*

(2) *Order convergence in $C(X)$ is equivalent to bounded pointwise convergence, and*

(3) *X is a P-space.*

Proof. (1) implies (3). By Lemma 2.10 of Papangelou [11], $C(X)$ is order Cauchy complete if and only if when each of $x_1 \geqq x_2 \geqq x_3 \geqq \dots$ and $w_1 \leqq w_2 \leqq w_3 \leqq \dots$ is a sequence of functions in $C(X)$ with the property that $x_n \geqq w_n$ for each n and $\{x_n - w_n\}$ order converges to 0 then there is a function u in $C(X)$ such that $\bigwedge x_n = u$ and $\bigvee w_n = u$. Since monotone order convergence is assumed to imply pointwise convergence in $C(X)$, then if x_n and w_n are as described above then $\{x_n - w_n\}$ pointwise converges to 0. Thus the pointwise infimum of x_n is the pointwise supremum of w_n and since it is both upper semi-continuous and lower semi-continuous, it is continuous. So that $C(X)$ is order Cauchy complete and since it is assumed to be almost σ-complete, to apply to a large class of Riesz spaces. Also see [3]. Huijsmans and de Pagter in [5] prove a theorem (Theorem 9.3) for general Riesz spaces which when specialized to $C(X)$ yields that X is a P-space if and only if $C(X)$ is almost σ-complete and has the s.m.c. property. Combining this with Theorem 5 gives that if $C(X)$ is almost σ-complete, then $C(X)$ has the s.m.c. property if and only if order convergence implies pointwise convergence. This is not true for Riesz spaces in general, e.g. let L be the space of bounded sequences. Then L is almost σ-complete and order convergence implies pointwise convergence, but L does not have the s.m.c. property.

Corollary 6. *If $C(X)$ is almost σ-complete then every positive linear functional on $C(X)$ is an integral if and only if X is a P-space.*

Proof. If X is a P-space, then $C(X)$ is closed with respect to pointwise convergence and every positive linear functional on $C(X)$ is sequentially continuous. (Tucker [16], Proposition 4). If every positive linear functional on $C(X)$ is sequentially continuous, then $\varrho_x(f) = f(x)$ is sequentially continuous and order convergence implies pointwise convergence. By Theorem 5, X is a P-space.

Example 7. There exists a topological space X such that $C(X)$ is almost σ-complete but bounded pointwise convergence does not imply order convergence for sequences. Let X be the set of rational numbers in $[0, 1]$ with the topology induced by the ordinary topology on $[0, 1]$. Suppose H is a subset of $C(X)$ and $f = \bigvee_{h \in H} h$. For each x_i in X let $g(x_i) = \text{l.u.b.}_{h \in H} \{h(x_i)\}$. There is a countable subset $\{h_{ij}\}$, $j = 1, 2, \dots$, of H

such that $\text{l.u.b.}_j \{h_{ij}(x_i)\} = g(x_i)$. Thus $\{h_{ij}\}$, $i = 1, 2, \ldots, j = 1, 2, \ldots$, is a countable subset of H such that $\text{l.u.b.}_{ij} \{h_{ij}\} = g$ and $\bigvee_{ij} h_{ij} = f$. So that, $C(X)$ is order separable and thus almost σ-complete. (Aliprantis and Langford [1]). For each positive integer j let N_{ij}, $i = 1, 2, \ldots, j$, be a collection of circles such that N_{ij} has center x_j, the radius of $N_{ij} \to 0$ as $j \to \infty$, and $N_{pj} \cap N_{qj} = \emptyset$, $p < q \leqq j$. Let $f_j(x) = 1$ if x is exterior to each of $N_{1j}, N_{2j}, \ldots, N_{jj}$ and f_j decreases linearly to zero between each circle and its center. If $k_j \geqq f_p$, for all $p \geqq j$, then $k_j \geqq 1$ and thus $\{f_j\}$ does not order converge to 0 even though it converges pointwise.

The situation of $B_1(\Omega)$ is much less complicated. It has been shown previously that order convergence implies pointwise convergence. (See Tucker [16], Lemma 3). For the converse we have the following theorem.

Theorem 8. *Bounded pointwise convergence implies order convergence in $B_1(\Omega)$ if and only if $B_1(\Omega)$ is closed with respect to pointwise convergence which implies that it is the set of A measurable functions for some σ-algebra A.*

Proof. If bounded pointwise convergence implies order convergence, by Theorem 3, $B_1(\Omega)$ is almost σ-complete. By Proposition 6 of Tucker [18], $B_1(\Omega)$ is order Cauchy complete. Thus $B_1(\Omega)$ is σ-complete. By Lemma 4 of Tucker [18], it is closed under pointwise convergence. Therefore the conclusion follows. (See Regoli [13] or Bogdan [2].)

If $B_1(\Omega)$ is closed with respect to pointwise convergence and $f_1, f_2, f_3, \ldots$ is a bounded sequence converging pointwise to f, let $y_n(x) = \max_{i \geqq n} f_i(x)$. Then $y_1, y_2, y_3, \ldots$ is a nonincreasing sequence of functions in $B_1(\Omega)$ converging pointwise to f and $y_n \geqq f_n$. Thus $f_1, f_2, f_3, \ldots$ order converges to f.

In [9], Meyer shows that if X is an infinite dispersed compact Hausdorff space then $C(X)$ is not closed with respect to pointwise convergence but $B_1(C(X))$ is.

The following theorems expand on the relationship between P-spaces and spaces of Baire functions. These theorems also supplement the results in Tucker [17].

In the following, $B_2(\Omega) = B_1(B_1(\Omega))$, and in general if α is an ordinal, $\alpha > 0$, $B_\alpha(\Omega)$ is the family of pointwise limits of sequences from $\bigcup_{\alpha > \gamma} B_\gamma(\Omega)$. If ω_1 is the first uncountable ordinal then $B_{\omega_1}(\Omega) = B_{\omega_1 + 1}(\Omega)$ which will be denoted as $B(\Omega)$.

Theorem 9. *If ϱ is a real valued Riesz homomorphism defined on $B_1(\Omega)$, then ϱ can be extended to $B_2(\Omega)$.*

Proof. By Corollary 3 of Tucker [18], ϱ can be extended as a positive linear functional. To show that the extension of ϱ is a Riesz homomorphism consider f and g in $USB_1(\Omega)$ such that $f \wedge g = 0$. There exists a sequence $\{f_i\}$ of $B_1(\Omega)$ such that $f = \bigwedge f_i$ and there exists a sequence $\{g_i\}$ of $B_1(\Omega)$ such that $g = \bigwedge g_i$. Since $\{f_i \wedge g_i\}$ converges pointwise to zero, $\varrho(f_i \wedge g_i) = \varrho(f_i) \wedge \varrho(g_i)$ converges pointwise to zero by Theorem 3 of Tucker [17]. As $\varrho(f) = \bigwedge \varrho(f_i)$ and $\varrho(g) = \bigwedge \varrho(g_i)$ also by Theorem 3, $\varrho(f) \wedge \varrho(g) = 0$. Now suppose h and k are in $B_2(\Omega)$ and $h \wedge k = 0$. There exists a sequence $\{h_i\}$ of points in $(USB_1(\Omega))^+$ and a sequence $\{k_i\}$ of points in $(USB_1(\Omega))^+$ such that $h = \bigvee h_i$ and $k = \bigvee k_i$. Since $h \wedge k = 0$, $h_i \wedge k_i = 0$, which implies $\varrho(h_i) \wedge \varrho(k_i) = 0$, which in turn implies $\varrho(h) \wedge \varrho(k) = 0$. Thus the extension is a Riesz homomorphism.

Theorem 10. *If $B_1(\Omega)$ is mapped by a Riesz homomorphism ϱ into $C(X)$ so that for each f in $C^+(X)$ there is a subset ω of $\varrho(B_1(\Omega))$ such that f is the pointwise supremum of the functions in ω, then X is a P-space.*

Proof. By Theorem 2 of Tucker [17], $\varrho = \alpha\beta$ where β is a Riesz homomorphism from $B_1(\Omega)$ to $C(X)$, $\beta(1) = 1$, and α is multiplication by a function h in $C(X)$. Suppose there exists a point x in X such that $h(x) = 0$. Then $f(x) = 0$ for each f in $\varrho(B_1(\Omega))$ and the constant function 1 is not the pointwise supremum of any subset of $\varrho(B_1(\Omega))$. Therefore $h(x) \neq 0$ for each x in X and $1/h$ is a continuous function. Thus $\alpha^{-1}\alpha\beta = \beta$ is a Riesz homomorphism of $B_1(\Omega)$ into $C(X)$ with the property that $\beta(1) = 1$ and for each f in $C^+(X)$ there is a subset ω of $\beta(B_1(\Omega))$ such that f is the pointwise supremum of the functions in ω.

Let Z be a zero set of $C(X)$. There exists a function f in $C(X)$ such that $f(x) = 1$ for each x in Z and $0 \leqq f(x) < 1$ for each x not in Z. There exists a subset ω of $\beta(B_1(\Omega))$ such that f is the pointwise supremum of the functions in ω.

By Theorem 7 of Tucker [17], each bounded function in $B_1(\Omega)$ can be uniformly approximated by a function in $B_1(\Omega)$ with a finite range. Let g be a function in $\beta^{-1}(\omega)$ such that $0 \leqq g \leqq 1$ and g_r be a function in $B_1(\Omega)$ with a finite range that uniformly approximates $g - 1/2^{r+1}$ within $1/2^{r+1}$. Note that $g_r \leqq g$ and $\{g_r\}$ converges pointwise to g. Thus $\beta(g_r) \leqq \beta(g)$ and by Theorem 3 of Tucker [17], $\{\beta(g_r)\}$ converges pointwise to $\beta(g)$. The function g_r is the sum of a finite number of disjoint functions in $B_1(\Omega)$ each of which takes on only one non-zero value. Denote the set of all such functions for all r and all g as Γ. Then f is the pointwise supremum of the functions in $\beta(\Gamma)$. Now if h is a function in $B_1(\Omega)$ that takes on only the values 0 and 1, $h \leqq 1$, $(1 - h) \leqq 1$, and $h \wedge (1 - h) = 0$. This implies that $\beta(h)$ takes on only the values 0 and 1. Thus each function in $\beta(\Gamma)$ takes on only one non-zero value.

Pick x in Z. There exists a subsequence $f_1, f_2, f_3, \ldots$ of $\beta(\Gamma)$ such that $0 < f_1(x) \leqq f_2(x) \leqq f_3(x) \ldots$ and $\{f_i(x)\}$ converges to 1. For each positive integer i let c_i be a positive number and t_i be a function that takes on only the values 0 and 1 such that $\beta(c_i t_i) = f_i$. Let $s_i = \beta(c_i(t_1 \wedge t_2 \wedge \ldots \wedge t_i))$. Then $s_i(x) = f_i(x)$ so that $\{s_i(x)\}$ converges to 1. The sequence $\{c_i(t_1 \wedge t_2 \wedge \ldots \wedge t_i)\}$ converges pointwise to a function k in $B_2(\Omega)$. By Theorem 9, β can be extended to $B_2(\Omega)$. Let $\beta(k) = v$. By Theorem 4 of Tucker [17], v is continuous. By Theorem 3 of Tucker [17], v is the pointwise limit of $\{s_i\}$. Therefore $v(x) = 1$, $v^{-1}(1)$ is open, and Z is open. This implies that x is a P-space by Theorem 5.3 of Gillman and Henrikson [4].

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