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POINTWISE CONVERGENCE OF HERMITE-FEJÉR INTERPOLATION OF HIGHER ORDER FOR FREUD WEIGHTS

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Abstract. This paper is concerned with the approximation by Hermite-Fejér interpolation of higher order based at the zeros of orthogonal polynomials with respect to the typical Freud weight. We will prove a convergence result for even order and a divergence result for odd order.

1. Introduction. The purpose of this paper is to investigate the pointwise convergence of Hermite-Fejér interpolation of higher order based at the zeros of orthogonal polynomials with respect to a Freud weight of the form $\exp(-x^m)$ with an even positive integer m.

Let $Q(x)=x^m/2$ and let $w(x)=\exp(-Q(x))$, where $m=2, 4, 6, \ldots$. The orthonormal polynomials $p_n(w^2; x)=p_n(x)=\gamma_n x^n+\cdots$, where $\gamma_n>0$, are defined by the relation

$$\int_{-\infty}^{\infty} p_l(x)p_n(x)w^2(x)dx=\delta_{ln}.$$

These polynomials were investigated first by Freud, e.g. [Fr1], [Fr2], and recently by many authors in connection with approximation theory. For detailed references and an extensive survey, readers may refer to Nevai [Ne3].

We denote the zeros of $p_n(x)$ by x_{kn}, $k=1, 2, \ldots, n$, where

$$x_{1n}>x_{2n}>\cdots>x_{nn}.$$

Let ν be a positive integer. For a function $f(x)$ defined on the real line $\boldsymbol{R}$, the Hermite-Fejér interpolation polynomial $L_n(\nu; f, x)$ of order ν based at the zeros $x_{1n}, \ldots, x_{nn}$ is defined to be the unique algebraic polynomial of degree at most $\nu n-1$ which satisfies

$$L_n(\nu; f, x_{kn})=f(x_{kn}), \quad k=1, 2, \ldots, n,$$
$$L_n^{(r)}(\nu; f, x_{kn})=0, \quad k=1, 2, \ldots, n; \quad r=1, 2, \ldots, \nu-1,$$

where $L_n^{(r)}(\nu; f, x_{kn})$ is the r-th derivative of $L_n(\nu; f, x)$ at x_{kn}. It is known that, for every $n=1, 2, \ldots$ and $k=1, 2, \ldots, n$, there exists a unique polynomial $h_{kn}(\nu; x)$ of degree $\nu n-1$ satisfying

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$$\begin{aligned}(1.1)\qquad &h_{kn}(\nu; x_{pn})=\delta_{pk}, \quad p=1,2,\ldots,n,\\ &h_{kn}^{(r)}(\nu; x_{pn})=0, \quad p=1,2,\ldots,n, \quad r=1,2,\ldots,\nu-1,\end{aligned}$$

(cf. [Na, Ch. I, §4]). Thus the interpolation polynomial $L_n(\nu; f, x)$ is written in the form

$$(1.2)\qquad L_n(\nu; f, x)=\sum_{k=1}^{n} f(x_{kn})h_{kn}(\nu; x).$$

Since $L_n(\nu; f, x)=1$ for $f(x)=1$, we see that

$$(1.3)\qquad \sum_{k=1}^{n} h_{kn}(\nu; x)=1.$$

We call $h_{kn}(\nu; x)$ the fundamental polynomial of Hermite-Fejér interpolation of order ν. We note that $L_n(1; f, x)$ is the Lagrange interpolation polynomial.

Pointwise convergence of Lagrange interpolation for the Hermite weight $\exp(-x^2)$ was first investigated by Freud [Fr3]. Kis [Ki] studied the pointwise convergence of Lagrange interpolation for the weights $|x|^{\alpha}\exp(-x^2)$, $\alpha>-1$. Nevai [Ne1], [Ne2] discussed in detail the Laguerre weight case. Recently, Knopfmacher [Kn] estimated the order of approximation of pointwise convergence of Lagrange interpolation for the class of regular Freud weights which includes the weights $\exp(-x^m)$, $m=2, 4, 6, \ldots$.

The second author studied in the series of works [Sa1], [Sa2], [Sa3], [Sa4] the approximation by Hermite-Fejér interpolation of order ν for the Hermite weight and the Laguerre weight, and he and Vértesi [SV] studied the convergence of Hermite-Fejér interpolation with base at the zeros of Jacobi polynomials. In this paper, we will consider the approximation and the pointwise convergence problems of Hermite-Fejér interpolation of order ν for the typical Freud weights $\exp(-x^m)$, $m=2, 4, 6, \ldots$ applying the method developed in the previous papers. We will prove a convergence result (Theorem 1) for even order ν, and a divergence result (Theorem 2) for odd order ν.

Let q_n denote the unique positive solution of the equation $q_nQ'(q_n)=n$, that is,

$$q_n=\left(\frac{2n}{m}\right)^{1/m}.$$

Let $C(\boldsymbol{R})$ be the space of continuous functions on $\boldsymbol{R}$. We define the modulus of continuity of $f\in C(\boldsymbol{R})$ on an interval $[a, b]$ by

$$\omega(f; [a,b]; h)=\sup_{\substack{|x-y|\le h\\ x,y\in[a,b]}} |f(x)-f(y)|, \qquad h>0.$$

We may write briefly $\omega(h)$ if there is no possibility of misunderstanding.

Our first theorem is a result on the pointwise convergence of Hermite-Fejér interpolation of even order $\nu=2d$, $d=1, 2, \ldots$ for the weights $\exp(-x^m)$, $m=2, 4, 6, \ldots$.

THEOREM 1. *Let d be a positive integer. Then, there exist positive constants κ, κ', K, μ, C and a number n_0 such that, for $f \in C(\boldsymbol{R})$,*

$$|L_n(2d;f,x)-f(x)| \le C\exp(dQ(x))\Big[k_n^{2d}(x)\Big\{\omega\Big(f;[-\kappa q_n,\kappa q_n];\frac{q_n}{n}\log n\Big)$$

$$+n^{2d}e^{-\mu n}\omega\Big(f;[-Kq_n,Kq_n];\frac{q_n}{n}\log n\Big)\Big\}+\omega\Big(f;[-\kappa' q_n,\kappa' q_n];\frac{q_n}{n}|k_n(x)|\Big)\Big]$$

for $x \in [-\kappa q_n, \kappa q_n]$ and $n \ge n_0$, where $k_n(x)=p_n(x)w(x)q_n^{1/2}$.

It is known that there exists a constant $C'>0$ such that $|k_n(x)| \le C'$ for $x \in [-\kappa q_n, \kappa q_n]$ and $n=1,2,\dots$ (see [Kn, Lemmas 4.1 and 3.7]). If $f \in C(\boldsymbol{R})$ is uniformly continuous, that is, $\omega(f;\boldsymbol{R};h)\to 0$ $(h\to 0)$, then we have by our theorem

$$|L_n(2d;f,x)-f(x)| \le C'' e^{dQ(x)}\Big\{\omega\Big(f;\boldsymbol{R};\frac{q_n}{n}\log n\Big)$$

$$+n^{2d}e^{-\mu n}\omega\Big(f;\boldsymbol{R};\frac{q_n}{n}\log n\Big)+\omega\Big(f;\boldsymbol{R};C'\frac{q_n}{n}\Big)\Big\}$$

for every $x \in [-\kappa q_n, \kappa q_n]$ and $n \ge n_0$ with a constant $C''>0$ not depending on x, n and f. Thus, we have the following corollary:

COROLLARY 1. *Let $f \in C(\boldsymbol{R})$ be a uniformly continuous function on $\boldsymbol{R}$. Then, for every $M>0$, the sequence of Hermite-Fejér interpolation polynomials of even order $2d$ converges uniformly to f in the interval $[-M, M]$, that is,*

$$\lim_{n\to\infty}\max_{-M\le x\le M}|L_n(2d;f,x)-f(x)|=0\,.$$

On the other hand, for the Hermite-Fejér interpolation of odd order, we have a divergence result. Let $\Lambda_n(\nu;[a,b])$ be the Lebesgue constant of Hermite-Fejér interpolation of order $\nu=1,2,\dots$ over an interval $[a,b]$, that is,

$$\Lambda_n(\nu;[a,b])=\max_{a\le x\le b}\sum_{k=1}^{n}|h_{kn}(\nu;x)|\,.$$

Then, we have the following theorem:

THEOREM 2. *Let d be a positive integer. For a and b with $a<b$, there exists a positive constant C such that*

$$\Lambda_n(2d-1;[a,b]) \ge C\log n\,, \qquad n=1,2,\dots\,.$$

Standard argument (cf. [Ri, Theorem 4.3]) leads us to the following:

COROLLARY 2. *For a and b with $a<b$, there exists $f \in C(\boldsymbol{R})$ such that*

$$\limsup_{n\to\infty} \max_{a\le x\le b} |L_n(2d-1;f,x)|=\infty .$$

In §2, for later convenience, we will summarize some known properties of $p_n(x)$. Theorems 1 and 2 will be proved in §3 and §4, respectively.

2. Properties of $p_n(x)$. Let

$$a_n=\gamma_{n-1}/\gamma_n , \qquad n=1,2,\dots ,$$

where γ_n is the leading coefficient of $p_n(x)=\gamma_n x^n+\cdots$.

(A) [BC, Theorem 4]: $a_n=[\beta n^{1/m}]$, where β denotes Freud's constant $\beta=(1/2)\{\pi^{1/2}\Gamma(m/2)/\Gamma(m+1/2)\}^{1/m}$ and, $a_n=[\beta n^{1/m}]$ is the notation of Bonan and Clark [BC, p. 216], that is, $a_n=\beta n^{1/m}+O(n^{-1+1/m})$ as $n\to\infty$.

(B) [BC, Theorem 5 and p. 213]: $p_n'(x)=A_n(x)p_{n-1}(x)-B_n(x)p_n(x)$, where

$$A_n(x)=m\sum_{i=0}^{m/2-1}\binom{2i}{i}[\beta n^{1/m}]^{2i+1}x^{m-2i-2} ,$$

$$B_n(x)=m\sum_{i=0}^{m/2-2}\binom{2i+1}{i}[\beta n^{1/m}]^{2i+2}x^{m-2i-3} \qquad (m>2) ,$$

$$B_n(x)=0 \qquad (m=2) .$$

(C) [BC, p. 220] and [Mh, Proposition 3.5]:

$$A_n(x)p_n''(x)-(mx^{m-1}A_n(x)+A_n'(x))p_n'(x)+\Big\{\frac{a_n}{a_{n-1}}A_n^2(x)A_{n-1}(x)+A_n(x)B_n(x)B_{n-1}(x) - \frac{x}{a_{n-1}}A_n(x)A_{n-1}(x)B_n(x)+A_n(x)B_n'(x)-A_n'(x)B_n(x)\Big\}p_n(x)=0 .$$

(D) [Kn, Lemma 4.1]: (i) There exists a constant $K_1>0$ such that $x_{1n}\le K_1q_n$, $n=1,2,\dots$.

(ii) There exist constants C_1, C_2, $\kappa_1>0$ such that $C_1q_n/n<x_{k-1n}-x_{kn}<C_2q_n/n$, $n=1,2,\dots$ for x_{k-1n}, $x_{kn}\in[-\kappa_1q_n,\kappa_1q_n]$.

(E) [Kn, Lemma 4.5]: There exist constants C_3, C_4, $\kappa_2>0$ such that $C_3q_n^{-1/2}\le|p_{n-1}(x_{kn})|w(x_{kn})\le C_4q_n^{-1/2}$, $n=1,2,\dots$ for x_{kn} with $|x_{kn}|\le\kappa_2q_n$.

Let $x_{(x,n)}$ denote the zero of $p_n(x)$ closest to x. If x is the midpoint of two zeros, then we define $x_{(x,n)}$ to be the closest zero of $p_n(x)$ to the left.

(F) [Kn, Theorem 3.7]: There exist constants C_5, C_6, $\kappa_3>0$ such that

$$C_5|x-x_{(x,n)}|\frac{n}{q_n}q_n^{-1/2}\le|p_n(x)|w(x)\le C_6|x-x_{(x,n)}|\frac{n}{q_n}q_n^{-1/2} ,$$

$n=1,2,\dots$ for x with $|x|\le\kappa_3q_n$.

Throughout the paper, C_1, C_2, C_3, C_4, C_5, C_6, K_1, κ_1, κ_2, κ_3 with subscript are always the constants in the properties (A)–(F). For the rest of the paper, the letter C denotes a positive constant which may differ at each different occurrence, even in the same chain of inequalities.

3. **Proof of Theorem 1.** We start by a lemma to prove Theorem 1.

LEMMA 1. *Let $\tilde{\kappa}=\min\{\kappa_1, \kappa_2, \kappa_3\}$. Then, there exists $\tilde{\delta}>0$ such that if $1<k<n$, $x_{kn}\in[-\tilde{\kappa}q_{n-1}, \tilde{\kappa}q_{n-1}]$ and $x_{kn}-\tilde{\delta}q_n/n\leq x\leq x_{kn}+\tilde{\delta}q_n/n$, then*

$$C\frac{n}{q_n}q_n^{-1/2}w(x_{kn})^{-1}\leq|p_n'(x)|\leq C\frac{n}{q_n}q_n^{-1/2}w(x_{kn})^{-1},$$

where C is independent of k, n and x.

PROOF. Let $x\in[-\tilde{\kappa}q_n, \tilde{\kappa}q_n]$. By (B) and (F), we have

$$|p_n'(x)|\,w(x)\geq A_n(x)\,|p_{n-1}(x)|\,w(x)-|B_n(x)|\,|p_n(x)|\,w(x)$$

$$\geq C_5A_n(x)\,|x-x_{(x,n-1)}|\frac{n-1}{q_{n-1}}q_{n-1}^{-1/2}-C_6\,|B_n(x)|\,|x-x_{(x,n)}|\frac{n}{q_n}q_n^{-1/2}.$$

Since $A_n(x)\geq Cn/q_n$ and $|B_n(x)|\leq Cn/q_n$ by (B), we have

$$|p_n'(x)|\,w(x)\geq\{D\,|x-x_{(x,n-1)}|-D'\,|x-x_{(x,n)}|\}\left(\frac{n}{q_n}\right)^2q_n^{-1/2}$$

with some positive constants D and D' independent of k, n and x.

We will show that there exist $\delta'>0$ and $\Delta>0$ such that, for every δ with $\delta'>\delta>0$, if $\Delta\delta q_n/n\geq|x-x_{(x,n)}|$, then $|x-x_{(x,n-1)}|\geq\Delta(1-\delta)q_n/n$. Let $x(n)$ be a zero of $p_n(x)$ satisfying $x(n)\in[-\tilde{\kappa}q_{n-1}, \tilde{\kappa}q_{n-1}]$. Let $y(n-1)$ and $y'(n-1)$ be the successive zeros of $p_{n-1}(x)$ such that $y'(n-1)<x(n)<y(n-1)$. Suppose that there exists a subsequence $\{n_i\}$ such that $y(n_i-1)-x(n_i)\leq\varepsilon_iq_{n_i}/n_i$, where $\varepsilon_i\to0$ $(i\to\infty)$. Then, by (F) and (E), we have

$$C_3q_{n_i}^{-1/2}\leq|p_{n_i-1}(x(n_i))|\,w(x(n_i))\leq C_6\,|x(n_i)-y(n_i-1)|\frac{n_i-1}{q_{n_i-1}}q_{n_i-1}^{-1/2}\leq C\varepsilon_iq_{n_i}^{-1/2}$$

with C not depending on i, a contradiction. Thus, we have $y(n-1)-x(n)\geq\lambda q_n/n$ with some constant $\lambda>0$. Similarly, we have $x(n)-y'(n-1)\geq\lambda q_n/n$. Clearly, these imply the existence of δ' and Δ.

Therefore, there exists $\tilde{\delta}>0$ such that $|p_n'(x)|\,w(x)\geq C(n/q_n)q_n^{-1/2}$ for $x_{kn}-\tilde{\delta}q_n/n\leq x\leq x_{kn}+\tilde{\delta}q_n/n$, where C is independent of k, n and x. The opposite inequality is obtained by a similar argument. For $x\in[x_{kn}-\tilde{\delta}q_n/n, x_{kn}+\tilde{\delta}q_n/n]$, we see that $Dw(x_{kn})\leq w(x)\leq D'w(x_{kn})$, where D and D' are some positive constants independent of k, n and x. Thus, we have the desired inequalities. q.e.d.

We begin the proof of Theorem 1. Let $l_{kn}(x)$, $k=1,2,\dots$ be the fundamental polynomials of Lagrange interpolation polynomials $L_n(1;f,x)$, that is, $l_{kn}(x)=h_{kn}(1;x)$. Then,

$$l_{kn}(x)=\frac{W(x)}{(x-x_{kn})W'(x_{kn})}=\frac{p_n(x)}{(x-x_{kn})p_n'(x_{kn})}, \qquad k=1,2,\dots,n, \tag{3.1}$$

where $W(x)=\prod_{k=0}^{n}(x-x_{kn})$. We note that the fundamental polynomial $h_{kn}(v;x)$ of Hermite-Fejér interpolation polynomial $L_n(v;f,x)$ of order v is divisible by $l_{kn}^v(x)$ $(=\{l_{kn}(x)\}^v)$. We define $e_i(v;k,n)$, $i=0,1,\dots,v-1$ to be the coefficients of $h_{kn}(v;x)$ in the expression

$$h_{kn}(v;x)=l_{kn}^v(x)\sum_{i=0}^{v-1}e_i(v;k,n)(x-x_{kn})^i, \qquad k=1,2,\dots,n. \tag{3.2}$$

By (1.1), we have

$$\begin{aligned} &e_i(v;k,n)=-\frac{1}{i!}\sum_{r=0}^{i-1}\frac{i!}{(i-r)!}e_r(v;k,n)(l_{kn}^v)^{(i-r)}(x_{kn}), \qquad i=1,2,\dots,v-1,\\ &e_0(v;k,n)=1, \end{aligned} \tag{3.3}$$

where $(l_{kn}^v)^{(s)}(x_{kn})$ is the s-th derivative of l_{kn}^v at x_{kn}.

We now evaluate $L_n(v;f,x)-f(x)$. By (1.2), (1.3) and (3.2), we have

$$\begin{aligned} L_n(v;f,x)-f(x)&=\sum_{k=1}^{n}(f(x_{kn})-f(x))l_{kn}^v(x)\sum_{i=0}^{v-1}e_i(v;k,n)(x-x_{kn})^i\\ &=\sum_{i=0}^{v-1}\sum_{k=1}^{n}(f(x_{kn})-f(x))l_{kn}^v(x)e_i(v;k,n)(x-x_{kn})^i\\ &=\sum_{i=0}^{v-1}\sum_{k=1}^{n}R_i(v,k,n;x), \quad \text{say}. \end{aligned}$$

Let κ be a constant such that $0<\kappa<\tilde{\kappa}$, where $\tilde{\kappa}$ is the constant in Lemma 1. We choose $\delta>0$ so that $\delta<\min\{C_1/2,\tilde{\delta}\}$, where $\tilde{\delta}$ and C_1 are the constants in Lemma 1 and in (D), (ii), respectively. Let $x\in[-\kappa q_n, \kappa q_n]$. Put

$$\begin{aligned} &J=\{k;0\le|x-x_{kn}|<\delta q_n/n\},\\ &J(j)=\{k;j\delta q_n/n\le|x-x_{kn}|<(j+1)\delta q_n/n, |x_{kn}|\le\kappa q_n\}, \qquad j=1,2,\dots,\\ &I=\{k;\delta q_n/n\le|x-x_{kn}|, \kappa q_n<|x_{kn}|\}. \end{aligned}$$

Although the sets J, $J(j)$ and I may depend on x and n, each of the sets J and $J(j)$, $j=1,2,\dots$ consists of at most two elements and $\{1,2,\dots,n\}=\bigcup_{j=0}^{\lambda(n)}J(j)\cup J\cup I$, where $\lambda(n)$ is the smallest number exceeding $2K_1n/\delta$. Here, K_1 is the constant in (D), (i). Let

$$\sum_1=\sum_{i=0}^{\nu-1}\sum_{k\in J}R_i(\nu,k,n;x)\,;\quad \sum_2=\sum_{i=0}^{\nu-2}\sum_{j=1}^{\lambda(n)}\sum_{k\in J(j)}R_i(\nu,k,n;x)\,,$$

$$\sum_3=\sum_{j=1}^{\lambda(n)}\sum_{k\in J(j)}R_{\nu-1}(\nu,k,n;x)\,;\quad \sum_4=\sum_{i=0}^{\nu-1}\sum_{k\in I}R_i(\nu,k,n;x)\,.$$

Then, $L_n(\nu;f,x)-f(x)=\sum_1+\sum_2+\sum_3+\sum_4$. To estimate $\sum_p$, $p=1,2,3,4$, we need the bounds of $e_i(\nu;k,n)$ given in Lemma 6, which follows from Lemmas 2–5.

LEMMA 2. *Let $\tau>0$. Then, for every x with $|x|\le\tau q_n$ and $r=0,1,\dots,$*

$$|A_n^{(r)}(x)|\le C|x|^{\langle r\rangle}q_n^{m-(r+1+\langle r\rangle)}\,,$$

$$|B_n^{(r)}(x)|\le C|x|^{1-\langle r\rangle}q_n^{m-(r+2-\langle r\rangle)}\,,$$

where

$$\langle r\rangle=\begin{cases}0\ (r:\mathit{even})\,,\\ 1\ (r:\mathit{odd})\,,\end{cases}$$

and C is independent of n and x.

PROOF. Apply the property (B). q.e.d.

In order to simplify notation, we write the differential equation of (C) as follows: $a(x)p_n''(x)+b(x)p_n'(x)+c(x)p_n(x)=0$, where

$$a(x)=A_n(x)\,,\quad b(x)=-\{mx^{m-1}A_n(x)+A_n'(x)\}\,,$$

$$\begin{aligned}c(x)&=\frac{a_n}{a_{n-1}}A_n^2(x)A_{n-1}(x)+A_n(x)B_n(x)B_{n-1}(x)\\ &\quad-\frac{x}{a_{n-1}}A_n(x)A_{n-1}(x)B_n(x)+A_n(x)B_n'(x)-A_n'(x)B_n(x)\\ &=c_1(x)+c_2(x)+c_3(x)+c_4(x)+c_5(x)\,,\quad\text{say}\,.\end{aligned}\tag{3.4}$$

We differentiate the equation j-times. Then,

$$a(x)p_n''+b(x)p_n'+c(x)p_n=0\,,\qquad j=0\,;\tag{3.5}$$

$$a(x)p_n'''+\{a'(x)+b(x)\}p_n''+\{b'(x)+c(x)\}p_n'+c'(x)p=0\,,\qquad j=1\,;\tag{3.6}$$

$$\begin{aligned}&a(x)p_n^{(j+2)}+\{ja'(x)+b(x)\}p_n^{(j+1)}\\ &\quad+\sum_{s=0}^{j-2}\left\{\binom{j}{s+2}a^{(s+2)}(x)+\binom{j}{s+1}b^{(s+1)}(x)+\binom{j}{s}c^{(s)}(x)\right\}p_n^{(j-s)}\\ &\quad+\{b^{(j)}(x)+jc^{(j-1)}(x)\}p_n'+c^{(j)}(x)p_n=0\,,\qquad j=2,3,\dots\,.\end{aligned}\tag{3.7}$$

For convenience, we denote these equations by

(3.8) $$A_2^{[0]}(x)p_n''+A_1^{[0]}(x)p_n'+A_0^{[0]}(x)p_n=0\,,\qquad j=0\,;$$

(3.9) $$A_3^{[1]}(x)p_n'''+A_2^{[1]}(x)p_n''+A_1^{[1]}(x)p_n'+A_0^{[1]}(x)p_n=0\,,\qquad j=1\,;$$

(3.10) $$A_{j+2}^{[j]}(x)p_n^{(j+2)}+A_{j+1}^{[j]}(x)p_n^{(j+1)}+\sum_{s=0}^{j-2}A_{j-s}^{[j]}(x)p_n^{(j-s)}$$
$$+A_1^{[j]}(x)p_n'+A_0^{[j]}(x)p_n=0\,,\qquad j=2,3,\ldots\,.$$

Our task is to estimate $A_i^{[j]}(x)$, $i=0,1,\ldots,j+2$ for $|x|\le\tau q_n$. To do so, let us consider the bounds of the derivatives $a^{(s)}(x)$, $b^{(s)}(x)$, $c^{(s)}(x)$. Since $a^{(s)}(x)=A_n^{(s)}(x)$, Lemma 2 gives directly an estimate for $a^{(s)}(x)$. We have

$$b^{(s)}(x)=-\left\{A_n^{(s+1)}(x)+\sum_{r=0}^{s}\binom{s}{r}\frac{m!}{(m-r-1)!}x^{m-r-1}A_n^{(s-r)}(x)\right\}.$$

By Lemma 2, we have

$$|b^{(s)}(x)|\le C\left\{|x|^{\langle s+1\rangle}q_n^{m-(s+2+\langle s+1\rangle)}\right.$$
$$\left.+\sum_{r=0}^{s}\binom{s}{r}\frac{m!}{(m-r-1)!}|x|^{m-r-1+\langle s-r\rangle}q_n^{m-(s-r+1+\langle s-r\rangle)}\right\},$$

for $|x|\le\tau q_n$, where C is independent of n and x. The sum $\sum_{r=0}^{s}$ on the right hand-side is bounded by $C|x|^{m-s-1}q_n^{m-1}$ if $s<m-1$. If $s\ge m-1$, then sum $\sum_{r=0}^{s}$ is equal to $\sum_{r=0}^{m-1}$. Thus, it is bounded by $C|x|^{\langle s-(m-1)\rangle}q_n^{m-(s-(m-1)+1+\langle s-(m-1)\rangle)}$. Since m is even, we have

(3.11) $$|b^{(s)}(x)|\le C|x|^{1-\langle s\rangle}q_n^{2m-s-3+\langle s\rangle}\,,\qquad s=0,1,\ldots$$

for $|x|\le\tau q_n$, where C is independent of n and x. To estimate $c^{(s)}(x)$, we first deal with the term $c_1(x)$ in the expression (3.4) for $c(x)$. The s-th derivative $(A_n^2(x)A_{n-1}(x))^{(s)}$ is a linear combination of the terms $A_n^{(t)}A_n^{(u)}A_{n-1}^{(v)}$, $t+u+v=s$. By Lemma 2, we have

$$|c_1^{(s)}(x)|\le C\sum_{\substack{t,u,v\\ t+u+v=s}}|x|^{\langle t\rangle+\langle u\rangle+\langle v\rangle}q_n^{3m-(s+3+\langle t\rangle+\langle u\rangle+\langle v\rangle)}$$

for $|x|\le\tau q_n$, where C is independent of n and x. Here, we use the following fact which follows from (A); $D\le a_n/a_{n-1}\le D'$ with some positive constants D and D' independent of n. For even s, we adopt the estimate $|c_1^{(s)}(x)|\le Cq_n^{3m-s-3}$ which follows immediately from the above inequality. We note that if s is odd, then $\langle t\rangle+\langle u\rangle+\langle v\rangle\ge 1$. Thus, we have $|c_1^{(s)}(x)|\le C|x|q_n^{3m-s-4}$ for odd s. It follows that

$$|c_1^{(s)}(x)|\le C|x|^{\langle s\rangle}q_n^{3m-s-3-\langle s\rangle}\,,\qquad s=0,1,\ldots\,.$$

For the derivative of $c_2(x)$ in $c(x)$, the inequality

$$(3.12)\qquad |c_2^{(s)}(x)|\le C \sum_{\substack{t,u,v\\ t+u+v=s}} |x|^{2+\langle t\rangle-\langle u\rangle-\langle v\rangle} q_n^{3m-(s+5+\langle t\rangle-\langle u\rangle-\langle v\rangle)}$$

follows similarly from Lemma 2. Note that $1+\langle t\rangle-\langle u\rangle-\langle v\rangle\ge 0$ for odd s. Consequently, we see that $c_2^{(s)}(x)$ and $c_1^{(s)}(x)$ have estimates of the same order for $s=0, 1, \dots$. We have

$$(3.13)\qquad c_3^{(s)}(x)=\frac{x}{a_{n-1}}(A_n(x)A_{n-1}(x)B_n(x))^{(s)}+\frac{s}{a_{n-1}}(A_n(x)A_{n-1}(x)B_n(x))^{(s-1)}$$
$$=c_{31}(x)+c_{32}(x),\quad \text{say},$$

for the s-th derivative of $c_3(x)$ in (3.4). For $(A_n(x)A_{n-1}(x)B_n(x))^{(s)}$, we have

$$(3.14)\qquad |(A_n(x)A_{n-1}(x)B_n(x))^{(s)}|\le C \sum_{\substack{t,u,v\\ t+u+v=s}} |x|^{1+\langle t\rangle+\langle u\rangle-\langle v\rangle} q_n^{3m-(s+4+\langle t\rangle+\langle u\rangle-\langle v\rangle)}.$$

From this, it follows that $|(A_n(x)A_{n-1}(x)B_n(x))^{(s)}|\le Cq_n^{3m-s-3}$ for arbitrary s. By this inequality and $D\le q_n/a_{n-1}\le D'$ with some positive constants D and D', we see that $c_{31}(x)$ has the same estimate as $c_1^{(s)}(x)$. For $c_{32}(x)$, we note that if t, u and v satisfy $t+u+v=s-1$, then $\langle t\rangle+\langle u\rangle-\langle v\rangle\ge 0$ for odd s. We see that $c_{32}(x)$ also has the same estimate as $c_1^{(s)}(x)$. Thus, $c_3^{(s)}(x)$ and $c_1^{(s)}(x)$ have the same bound. The s-th derivative of $c_4(x)$ in the expression (3.4) for $c(x)$ is estimated as follows:

$$(3.15)\qquad |c_4^{(s)}(x)|\le C \sum_{\substack{t,u\\ t+u=s}} |A_n^{(t)}(x)|\,|B_n^{(u+1)}(x)|$$
$$\le C \sum_{\substack{t,u\\ t+u=s}} |x|^{1+\langle t\rangle-\langle u+1\rangle} q_n^{2m-(s+4+\langle t\rangle-\langle u+1\rangle)}.$$

We take the estimate $|c_4^{(s)}(x)|\le Cq_n^{2m-s-3}$ if s is even. If s is odd, then $\langle t\rangle=\langle u+1\rangle$. Thus, we have $|c_4^{(s)}(x)|\le C|x|^{\langle s\rangle}q_n^{2m-s-3-\langle s\rangle}$. The s-th derivative $c_5^{(s)}(x)$ of the last term in $c(x)$ satisfies

$$(3.16)\qquad |c_5^{(s)}(x)|\le C \sum_{\substack{t,u\\ t+u=s}} |A_n^{(t+1)}(x)|\,|B_n^{(u)}(x)|$$
$$\le C \sum_{\substack{t,u\\ t+u=s}} |x|^{1+\langle t+1\rangle-\langle u\rangle} q_n^{2m-(s+4+\langle t+1\rangle-\langle u\rangle)}.$$

By the inequality, we see that $c_5^{(s)}$ and $c_4^{(s)}$ have the same bound. Therefore, we have

$$(3.17)\qquad |c^{(s)}(x)|\le C|x|^{\langle s\rangle} q_n^{3m-s-3-\langle s\rangle},\qquad s=0, 1, \dots$$

for $|x|\le \tau q_n$, where C is independent of n and x. Combining (B), Lemma 2, (3.11) and (3.17), we have the following lemma easily.

LEMMA 3. *Let $\tau>0$. Then, for every x with $|x|\le\tau q_n$ and $j=0, 1, \dots$,*

$$Cq_n^{m-1}\le|A_{j+2}^{[j]}(x)|\le Cq_n^{m-1}, \quad |A_{j+1}^{[j]}(x)|\le CM_n(x)q_n^{m-1},$$
$$|A_{j-s}^{[j]}(x)|\le C|x|^{\langle s\rangle}q_n^{3m-s-3-\langle s\rangle}, \qquad s=0, 1, \dots,$$

where

$$M_n(x)=\max\{|x|q_n^{-2}, |x|^{m-1}\},$$

and C is independent of n and x.

The lemma leads us to estimates of the derivatives $p_n^{(r)}(x_{kn})$ at x_{kn}.

LEMMA 4. *Let $r=1, 2, \dots$. Then,*

$$|p_n^{(r)}(x_{kn})|\le CM_n(x_{kn})^{1-\langle r\rangle}q_n^{(r-2+\langle r\rangle)(m-1)}|p_n'(x_{kn})|, \tag{3.18}$$

for $k=1, 2, \dots, n$, where C is independent of k and n.

PROOF. We first note that $|x_{kn}|\le K_1q_n$ by (D), (i). The inequality (3.18) is trivial for $r=1$. By (3.8), (3.9) and Lemma 3, we see that (3.18) holds for $r=2, 3$. We assume that (3.18) holds for $r=1, 2, \dots, j+1$. Let M_n stand for $M_n(x_{kn})$. By (3.10) and Lemma 3, we have

$$|p_n^{(j+2)}(x_{kn})|\le Cq_n^{-(m-1)}\Big\{M_nq_n^{m-1}|p_n^{(j+1)}(x_{kn})| + \sum_{s=0}^{j-1}|x_{kn}|^{\langle s\rangle}q_n^{3m-s-3-\langle s\rangle}|p_n^{(j-s)}(x_{kn})|\Big\}.$$

It follows from the assumption of induction that

$$|p_n^{(j+2)}(x_{kn})|\le C|p_n'(x_{kn})|\Big\{M_n^{1+\langle j\rangle}q_n^{(j-\langle j\rangle)(m-1)} + \sum_{s=0}^{j-1}|x_{kn}|^{\langle s\rangle}q_n^{-\langle s\rangle}\cdot M_n^{1-\langle j-s\rangle}q_n^{(-1+\langle j-s\rangle)(m-1)}\cdot q_n^{(j+1-s)(m-1)-s}\Big\}.$$

The terms $|x_{kn}|^{\langle s\rangle}q_n^{-\langle s\rangle}$ and $M_n^{1-\langle j-s\rangle}q_n^{(-1+\langle j-s\rangle)(m-1)}$ are bounded by some constant, and thus the sum $\sum_{s=0}^{j-1}$ is bounded by $Cq_n^{(j+1)(m-1)}$. Noting $M_n^{1+\langle j\rangle}q_n^{(j-\langle j\rangle)(m-1)}\le Cq_n^{(j+1)(m-1)}$, we have $|p_n^{(j+2)}(x_{kn})|\le Cq_n^{(j+1)(m-1)}|p_n'(x_{kn})|$. This gives (3.18) if $j+2$ is odd. If $j+2$ is even, then we write the sum $\sum_{s=0}^{j-1}$ in the form

$$\sum_{s=0}^{j-1}=\sum_{s=0}^{j-1}|x_{kn}|^{\langle s\rangle}q_n^{-2\langle s\rangle}M_n^{1-\langle j-s\rangle}\cdot q_n^{(j-s+\langle j-s\rangle)(m-1)-s+\langle s\rangle}.$$

Since $|x_{kn}|^{\langle s\rangle}q_n^{-2\langle s\rangle}M_n^{1-\langle j-s\rangle}\le M_n$ for even j, the sum $\sum_{s=0}^{j-1}$ is bounded by $CM_nq_n^{j(m-1)}$. This implies (3.18) with even r. q.e.d.

The next lemma gives an estimate for the r-th derivative $(l_{kn}^{\nu})^{(r)}(x_{kn})$ (=

$(d/dx)^r\{l_{kn}(x)\}^v|_{x=x_{kn}})$, where $l_{kn}(x)$ is the fundamental polynomial of Lagrange interpolation polynomial given by (3.1).

LEMMA 5. *Let v be a positive integer, and let $r=1,2,\ldots$. Then,*

$$|(l_{kn}^v)^{(r)}(x_{kn})| \leq CM_n(x_{kn})^{\langle r\rangle} q_n^{(r-\langle r\rangle)(m-1)}, \qquad k=1,2,\ldots,n, \tag{3.19}$$

where C is independent of k and n.

PROOF. By the equation

$$\begin{aligned} l_{kn}(x) &= \frac{p_n(x)}{(x-x_{kn})p_n'(x_{kn})} \\ &= \frac{1}{p'(x_{kn})}\left\{\frac{p_n'(x_{kn})}{1!} + \frac{p_n^{(2)}(x_{kn})}{2!}(x-x_{kn}) + \cdots + \frac{p_n^{(n)}(x_{kn})}{n!}(x-x_{kn})^{n-1}\right\}, \end{aligned}$$

we have

$$l_{kn}^{(r)}(x_{kn}) = \frac{p_n^{(r+1)}(x_{kn})}{(r+1)p_n'(x_{kn})}, \tag{3.20}$$

which implies (3.19) for $v=1$ by Lemma 4. We have

$$(l_{kn}^{v+1})^{(r)}(x_{kn}) = \sum_{i=0}^{r} \binom{r}{i} (l_{kn}^v)^{(i)}(x_{kn}) l_{kn}^{(r-i)}(x_{kn}).$$

By induction on v, we have

$$|(l_{kn}^{v+1})^{(r)}(x_{kn})| \leq C \sum_{i=0}^{r} M_n(x_{kn})^{\langle i\rangle + \langle r-i\rangle} q_n^{(r-\langle i\rangle - \langle r-i\rangle)(m-1)},$$

where C is independent of k and n. From this, the inequality (3.19) follows immediately for even r. To get (3.19) for odd r, we note that $\langle i\rangle + \langle r-i\rangle = 1$, $i=0,1,\ldots,r$. q.e.d.

We now prove the main lemma. Let $e_i(v;k,n)$ be the coefficients given in (3.2).

LEMMA 6. *Let v be a positive integer, and let $i=0,1,\ldots,v-1$. Then,*

$$|e_i(v;k,n)| \leq CM_n(x_{kn})^{\langle i\rangle} q_n^{(i-\langle i\rangle)(m-1)}, \qquad k=1,2,\ldots,n, \tag{3.21}$$

where C is independent of k and n.

PROOF. We prove this by induction on i. Since $e_0(v;k,n)=1$, (3.21) holds for $i=0$. By (3.3), Lemma 5 and the assumption of induction, we have

$$\begin{aligned} |e_i(v;k,n)| &\leq C \sum_{s=0}^{i-1} |e_s(v;k,n)|\,|(l_k^v)^{(i-s)}(x_{kn})| \\ &\leq C \sum_{s=0}^{i-1} M_n^{\langle s\rangle} q_n^{(s-\langle s\rangle)(m-1)} \cdot M_n^{\langle i-s\rangle} q_n^{(i-s-\langle i-s\rangle)(m-1)} \end{aligned}$$

$$\le C\sum_{s=0}^{i-1} M_n^{\langle s\rangle+\langle i-s\rangle} q_n^{(i-\langle s\rangle-\langle i-s\rangle)(m-1)}.$$

Clearly, we have $|e_i(v;k,n)|\le Cq_n^{i(m-1)}$. This gives (3.21) if i is even. If i is odd, then it is enough to note that $\langle i\rangle=\langle s\rangle+\langle i-s\rangle$, $s=0,1,\dots,i-1$. q.e.d.

We continue the proof of Theorem 1. Let $x\in[-\kappa q_n,\kappa q_n]$. We first estimate $\sum_1$. We may assume that $J\ne\varnothing$. Then, $J=\{k(x,n)\}$, where $k(x,n)$ is the number satisfying $x_{k(x,n)n}=x_{(x,n)}$. We have

$$\begin{aligned}|\textstyle\sum_1|&\le\sum_{i=0}^{v-1}|R_i(v,k(x,n),n;x)|\\&\le\sum_{i=0}^{v-1}|f(x_{k(x,n)n})-f(x)|\,|l_{k(x,n)n}(x)|^v|e_i(v;k(x,n),n)|\,|x-x_{k(x,n)n}|^i.\end{aligned}$$

We have $l_{k(x,n)n}(x)=p_n'(\xi)/p_n'(x_{(x,n)})$ with some ξ between x and $x_{k(x,n)n}$. Let κ' be a constant such that $\kappa<\kappa'<\tilde{\kappa}$. Since $x\in[-\kappa q_n,\kappa q_n]$, we have $x_{k(x,n)n}\in[-\kappa' q_n,\kappa' q_n]\subset[-\tilde{\kappa}q_{n-1},\tilde{\kappa}q_{n-1}]$ for $n\ge n_0$, where n_0 is a number depending only on m, κ' and $\tilde{\kappa}$. Since $|x-x_{k(x,n)n}|<\delta q_n/n$, it follows from Lemma 1 that $|l_{k(x,n)n}(x)|\le C$ for $n\ge n_0$ with C independent of n and x. If follows from (F) that $|x-x_{k(x,n)n}|\le C|k_n(x)|q_n/n$ with C independent of x and n, and thus $|f(x_{k(x,n)n})-f(x)|\le C\omega'(|k_n(x)|q_n/n)$ for $n\ge n_0$, where $\omega'(|k_n(x)|q_n/n)=\omega(f;[-\kappa' q_n,\kappa' q_n];|k_n(x)|q_n/n)$. Thus, Lemma 6 leads us to the inequality

$$\begin{aligned}|\textstyle\sum_1|&\le C\omega'(|k_n(x)|q_n/n)\sum_{i=0}^{v-1}M_n(x_{(x,n)})^{\langle i\rangle}q_n^{(i-\langle i\rangle)(m-1)}\left(\frac{q_n}{n}\right)^i\\&\le C\omega'(|k_n(x)|q_n/n)\sum_{i=0}^{v-1}(q_n^m/n)^i\le C\omega'(|k_n(x)|q_n/n).\end{aligned}$$

We next treat $\sum_2$. Let $0\le i\le v-2$ and $j=1,2,\dots$. Let k be the number satisfying $\delta jq_n/n\le|x-x_{kn}|<\delta(j+1)q_n/n$ and $|x_{kn}|\le\kappa q_n$. We have $|f(x)-f(x_{kn})|\le\omega(|x-x_{kn}|)\le\{1+\delta(j+1)/j\}\omega(jq_n/n)$, where $\omega(jq_n/n)=\omega(f;[-\kappa q_n,\kappa q_n];jq_n/n)$. It follows from (D), (ii), (F) and (E) that $|l_{kn}^v(x)|\le Cj^{-v}\{|k_n(x)|w(x_{kn})w^{-1}(x)\}^v$ with C independent of x, n and k. Combining these inequalities and Lemma 6, we have

$$\begin{aligned}|\textstyle\sum_2|&\le C\sum_{i=0}^{v-2}\sum_{j=1}^{\lambda(n)}\sum_{k\in J(j)}|R_i(v,k,n;x)|\\&\le C\sum_{i=0}^{v-2}\sum_{j=1}^{\lambda(n)}\sum_{k\in J(j)}|f(x_{kn})-f(x)|\,|l_{kn}(x)|^v|e_i(v;k,n)|\,|x-x_{kn}|^i\\&\le C\sum_{i=0}^{v-2}\sum_{j=1}^{\lambda(n)}\sum_{k\in J(j)}\omega\left(\frac{jq_n}{n}\right)\cdot j^{-v}\{|k_n(x)|w(x_{kn})w^{-1}(x)\}^v\end{aligned}$$

$$\cdot M(x_{kn})^{\langle i\rangle} q_n^{(i-\langle i\rangle)(m-1)} \cdot \left(\frac{jq_n}{n}\right)^i.$$

Since $w(x_{kn}) \leq 1$, $M(x_{kn})^{\langle i\rangle} q_n^{(i-\langle i\rangle)(m-1)} \leq C q_n^{i(m-1)}$, $q_n^m/n \leq C$ and $J(j)$ consists of at most two elements, we have

$$\begin{aligned}|\textstyle\sum_2| &\leq C\{|k_n(x)|\, w^{-1}(x)\}^v \sum_{i=0}^{v-2} \sum_{j=1}^{\lambda(n)} \omega\left(\frac{jq_n}{n}\right) \cdot j^{-v+i} \\ &\leq C\{|k_n(x)|\, w^{-1}(x)\}^v \sum_{j=1}^{\lambda(n)} j^{-2} \omega\left(\frac{jq_n}{n}\right) \\ &\leq C\{|k_n(x)|\, w^{-1}(x)\}^v \omega\left(\frac{\log n}{n} q_n\right) \sum_{j=1}^{\lambda(n)} j^{-2}\left(1+\frac{j}{\log n}\right).\end{aligned}$$

Since $\sum_{j=1}^{\lambda(n)} j^{-2}(1+j/\log n) \leq C$, the estimation of $\sum_2$ is complete.

Remark. To estimate $\sum_1$ and $\sum_2$, we do not need the assumption that v is even. We have the following inequality which is essentially proved in the estimation of $\sum_1$ and $\sum_2$: Let $\tau > 0$, and let $v = 2, 3, \ldots$. Then,

$$\max_{|x| \leq \tau} \sum_{i=0}^{v-2} \sum_{k: |x_{kn}| \leq \tau} |l_{kn}(x)|^v |e_i(v; k, n)|\, |x - x_{kn}|^i \leq C,$$

where C is independent of n and x. The inequality will be used in the proof of Theorem 2.

We will prove the required inequality for $\sum_3$. It is essential in this proof that the order v of Hermite-Fejér interpolation is even: $v = 2d$, $d = 1, 2, \ldots$. By the same argument as in the case of $\sum_2$, we see that

$$\begin{aligned}|\textstyle\sum_3| &\leq C \sum_{j=1}^{\lambda(n)} \sum_{k\in J(j)} \omega\left(\frac{jq_n}{n}\right) \cdot j^{-v}\{|k_n(x)|\, w(x_{kn}) w^{-1}(x)\}^v \cdot M(x_{kn}) q_n^{(v-2)(m-1)} \cdot \left(\frac{jq_n}{n}\right)^{v-1} \\ &\leq C\{|k_n(x)|\, w^{-1}(x)\}^v \sum_{j=1}^{\lambda(n)} \omega\left(\frac{jq_n}{n}\right) \frac{q_n}{jn} \sum_{k\in J(j)} M(x_{kn}) w(x_{kn})^v.\end{aligned}$$

We estimate the sum $U_n = \sum_{j=1}^{\lambda(n)}$. We may assume $x \geq 0$ since the case $x \leq 0$ is similar. Let N be the nonnegative integer such that $N\delta q_n/n \leq x < (N+1)\delta q_n/n$. Since $M(x_{kn}) w(x_{kn})^v \leq \min\{D/|x_{kn}|, D'\}$ with positive constants D and D' independent of k, n and x, we see that

$$\begin{aligned}U_n \leq C\Bigg[&\sum_{j=1}^{\lambda(n)} \omega\left(\frac{jq_n}{n}\right) \frac{q_n}{jn}\left\{(N+j)\frac{\delta q_n}{n}\right\}^{-1} + \sum_{j=1}^{N-1} \omega\left(\frac{jq_n}{n}\right) \frac{q_n}{jn}\left\{(N-j)\frac{\delta q_n}{n}\right\}^{-1} \\ &+ \omega\left(\frac{Nq_n}{n}\right) \frac{q_n}{Nn} + \sum_{j=N+1}^{\lambda(n)} \omega\left(\frac{jq_n}{n}\right) \frac{q_n}{jn}\left\{(j-N)\frac{\delta q_n}{n}\right\}^{-1}\Bigg]\end{aligned}$$

$$\leq C\omega\left(\frac{q_n}{n}\log n\right)\left[\sum_{j=1}^{\lambda(n)}\{j(j+N)\}^{-1}\left(1+\frac{j}{\log n}\right)+\sum_{j=1}^{N-1}\{j(N-j)\}^{-1}\left(1+\frac{j}{\log n}\right)\right.$$
$$\left.+\left(1+\frac{N}{\log n}\right)\frac{q_n}{Nn}+\sum_{j=N+1}^{\lambda(n)}\{j(j-N)\}^{-1}\left(1+\frac{j}{\log n}\right)\right].$$

Since three sums and the term $(1+N/\log n)q_n/(Nn)$ in $[\cdots]$ are uniformly bounded with respect to n and N, we have the desired estimate for $\sum_3$.

Lastly, we will estimate $\sum_4$. We have

$$|\textstyle\sum_4|\leq\displaystyle\sum_{i=0}^{\nu-1}\sum_{k\in I}|f(x_{kn})-f(x)|\,|l_{kn}(x)|^{\nu}\,|e_i(\nu;k,n)|\,|x-x_{kn}|^i.$$

We let $K=K_1$ be the constant in (D), (i). Since $x_{kn}\in[-Kq_n, Kq_n]$ for all k and $x\in[-\kappa q_n, \kappa q_n]\subset[-Kq_n, Kq_n]$, it follows that $|x-x_{kn}|\leq 2Kq_n$ and $|f(x_{kn})-f(x)|\leq(1+2Kn/\log n)\tilde{\omega}(q_n\log n/n)$, where $\tilde{\omega}(q_n\log n/n)$ stands for $\omega(f;[-Kq_n, Kq_n];q_n\log n/n)$. By Lemma 6, we have $|e_i(\nu;k,n)|\leq Cq_n^{i(m-1)}$. Since $|x-x_{kn}|\geq\delta q_n/n$, we have

$$|l_{kn}(x)|=\frac{|k_n(x)|\,w^{-1}(x)q_n^{-1/2}}{|x-x_{kn}|\,|p_n'(x_{kn})|}\leq\delta^{-1}\frac{|k_n(x)|}{w(x)}nq_n^{-3/2}|p_n'(x_{kn})|^{-1}.$$

Thus, we have

$$|\textstyle\sum_4|\leq\displaystyle\sum_{i=0}^{\nu-1}\sum_{k\in I}\left(1+\frac{2Kn}{\log n}\right)\tilde{\omega}\left(\frac{q_n}{n}\log n\right)\cdot\left\{\frac{|k_n(x)|}{w(x)}nq_n^{-3/2}|p_n'(x_{kn})|^{-1}\right\}^{\nu}\cdot q_n^{i(m-1)}\cdot q_n^i,$$
$$\leq C\left\{\frac{|k_n(x)|}{w(x)}\right\}^{\nu}\cdot\tilde{\omega}\left(\frac{q_n}{n}\log n\right)\cdot n^{2\nu}q_n^{-3\nu/2}\sum_{k\in I}|p_n'(x_{kn})|^{-\nu},$$

where C is independent of k, n and x. The sum $\sum_{k\in I}|p_n'(x_{kn})|^{-\nu}$ is treated by the following lemma.

Let λ_{kn}, $k=1,2,\ldots$ be the Cotes numbers which appear in the Gauss-Jacobi quadrature formula

$$\sum_{k=1}^{n}p(x_{kn})\lambda_{kn}=\int_{-\infty}^{\infty}p(x)w^2(x)\,dx$$

valid for all polynomials $p(x)$ of degree at most $2n-1$ (cf. [Ne3]).

LEMMA 7. *Let* $\tau>0$. *Then,*

$$\sum_{k:|x_{kn}|\geq\tau q_n}p_n'(x_{kn})^{-2}\leq Cq_n^{-2m+3}w^2(\tau q_n),$$

where C *is independent of* n.

PROOF. It follows from (D), (i) that $|x_{kn}|\leq K_1q_n$ for all zeros x_{kn}. Then, by

(B), we have $Cn/q_n \le A_n(x_{kn})$, $k=1,2,\dots,n$. By [Ba, (5.2)] and (A), we have $p_n'(x_{kn})^{-2}=a_n\lambda_{kn}A_n(x_{kn})^{-1}\le Ca_n\lambda_{kn}q_n/n\le Cq_n^2\lambda_{kn}/n$. By the formula $\lambda_{kn}\le\int_{x_{k+1n}}^{x_{k-1n}}w^2(x)dx$ (cf. [Ne3, (4.7.27)] and also [Sz, Th. 3.41.1]), we have

$$\sum_{k:|x_{kn}|\ge\tau q_n}\lambda_{kn}\le C\int_{\tau q_n}^{\infty}w^2(x)dx\,.$$

Integration by part (or an asymptotic formula for the incomplete gamma function) shows that the last integral is bounded by $Cq_n^{-m+1}w^2(\tau q_n)$, which gives the inequality.

q.e.d.

By the lemma and the fact that $\nu/2\ge 1$, we have

$$\sum_{k\in I}|p_n'(x_{kn})|^{-\nu}\le\left\{\sum_{k\in I}|p_n'(x_{kn})|^{-2}\right\}^{\nu/2}\le C\{q_n^{-2m+3}w^2(\kappa q_n)\}^{\nu/2}=Cq_n^{(-2m+3)\nu/2}e^{-\mu n}\,,$$

where $\mu=\nu\kappa^m/m$ and C is independent of n and x. Therefore, we have the inequality

$$\left|\sum\nolimits_4\right|\le C\left\{\frac{|k_n(x)|}{w(x)}\right\}^{\nu}\cdot\tilde\omega\left(\frac{q_n}{n}\log n\right)\cdot n^{\nu}e^{-\mu n}\,,$$

which completes the proof of Theorem 1.

4. Proof of Theorem 2. We need to find the lower bound of the coefficient $e_{\nu-1}(\nu;k,n)$ (cf. (3.21)) with odd ν for our purpose (see Lemma 14). To this end, we begin by estimating the following two coefficients $A_{j+2}^{[j]}(x)$ and $A_j^{[j]}(x)$ in (3.8), (3.9) and (3.10).

LEMMA 8. *Let*

$$\alpha=m\left(\frac{m}{2}\right)^{(m-1)/m}\binom{m-2}{m/2-1}\beta^{m-1}\,.$$

Let $j=0,1,\dots$ and let $0<\theta<\Theta$. Suppose $\theta\le|x|\le\Theta$. Let $\varepsilon_n(j;x)$ and $\varepsilon_n'(j;x)$ be functions defined by the equations

$$A_{j+2}^{[j]}(x)=\alpha q_n^{m-1}(1+\varepsilon_n(j;x)q_n^{-2})\,,$$
$$A_j^{[j]}(x)=\alpha^3q_n^{3(m-1)}(1+\varepsilon_n'(j;x)q_n^{-2})\,,$$

respectively. Then, $|\varepsilon_n(j;x)|\le C$ and $|\varepsilon_n'(j;x)|\le C$, where C is independent of n and x, and may depend on j, θ, Θ and m.

PROOF. Clearly, (B) leads to $|\varepsilon_n(j;x)|\le C$. By (3.4)–(3.10), we see that $A_j^{[j]}(x)$ has the following expression

$$A_j^{[j]}(x)=\binom{j}{2}a''(x)+\binom{j}{1}b'(x)+\sum_{r=1}^{5}c_r(x)\,.$$

We first deal with the term $c_1(x)$. We note that $a_n/a_{n-1}=1+(a_n-a_{n-1})/a_{n-1}$, $|(a_n-a_{n-1})/a_{n-1}|\le Cq_n^{-m}$ by (A), and $q_{n-1}/q_n=1+(q_{n-1}-q_n)/q_n$, $|(q_{n-1}-q_n)/q_n|\le Cq_n^{-m}$. From these inequalities and (B), it follows that $c_1(x)=\alpha^3 q_n^{3(m-1)}+\rho_n q_n^{3(m-1)-2}$, $|\rho_n|\le C$. Thus, it is enough to show that the other terms have orders not exceeding q_n^{3m-5}. Since $a''(x)=A_n''(x)$, it follows from Lemma 2 that $|a_n''(x)|\le Cq_n^{m-3}$. By (3.11), we have $|b'(x)|\le Cq_n^{2m-3}$. The inequality $|c_2(x)|\le Cq_n^{3m-5}$ follows from (3.12) with $s=0$. For $c_3(x)$, (3.14) with $s=0$ leads to $|c_3(x)|\le Cq_n^{3m-5}$. It follows from (3.15) with $s=0$ that $|c_4(x)|\le Cq_n^{2m-3}$. By (3.16) with $s=0$, we have $|c_5(x)|\le Cq_n^{2m-5}$. Thus, we have $|\varepsilon_n'(j;x)|\le C$. q.e.d.

By using the lemma, we find the lower bounds for $p_n^{(2s+1)}(x_{kn})$, $s=1,2,\dots$.

LEMMA 9. *Let $s=1,2,\dots$ and let $0<\theta<\Theta$. Suppose $\theta\le|x_{kn}|\le\Theta$. If $\zeta_n(s;x_{kn})$ is defined by*

$$p_n^{(2s+1)}(x_{kn})=(-1)^s\alpha^{2s}q_n^{2s(m-1)}(1+\zeta_n(s;x_{kn})q_n^{-2})p_n'(x_{kn}),$$

then

$$|\zeta_n(s;x_{kn})|\le C, \tag{4.1}$$

where C is independent of n and x_{kn}, but may depend on s, θ, Θ and m.

PROOF. We first note by Lemma 3 that, for $j=0,1,\dots$,

$$|A_{j+2}^{[j]}(x_{kn})|\ge Cq_n^{m-1}, \tag{4.2}$$

$$|A_{j+1}^{[j]}(x_{kn})|\le Cq_n^{m-1}, \tag{4.3}$$

$$|A_{j-s}^{[j]}(x_{kn})|\le Cq_n^{3m-3-s-\langle s\rangle}, \qquad s=0,1,\dots,j-1 \tag{4.4}$$

for $\theta\le|x_{kn}|\le\Theta$, where C is independent of k and n, and may depend on θ, Θ, m, and j. By Lemma 4, we have

$$|p_n^{(r)}(x_{kn})|\le Cq_n^{(r-2+\langle r\rangle)(m-1)}|p_n'(x_{kn})|, \qquad r=1,2,\dots \tag{4.5}$$

for $\theta\le|x_{kn}|\le\Theta$, where C is independent of k and n. By Lemma 8, we see that, for $j=0,1,\dots$,

$$-\frac{A_j^{[j]}(x_{kn})}{A_{j+2}^{[j]}(x_{kn})}=(-1)\alpha^2 q_n^{2(m-1)}(1+\rho_{kn}q_n^{-2}), \qquad |\rho_{kn}|\le C. \tag{4.6}$$

Now, we show (4.1) by induction on s. Let $s=1$. It follows from (3.9) that

$$p_n^{(3)}(x_{kn})=-\frac{A_2^{[1]}(x_{kn})}{A_3^{[1]}(x_{kn})}p_n''(x_{kn})-\frac{A_1^{[1]}(x_{kn})}{A_3^{[1]}(x_{kn})}p_n'(x_{kn}).$$

By (4.2), (4.3) and (4.5), the first term on the right side of the above equality is bounded by $C|p_n'(x_{kn})|$. This and (4.6) lead to (4.1) with $s=1$. Suppose that (4.1) with $s-1$ holds.

It follows from the expression (3.10) that

$$(4.7)\qquad p^{(2s+1)}=-\frac{A_{2s}}{A_{2s+1}}p^{(2s)}-\frac{A_{2s-1}}{A_{2s+1}}p^{(2s-1)}-\frac{A_{2s-2}}{A_{2s+1}}p^{(2s-2)}-\cdots-\frac{A_1}{A_{2s+1}}p^{(1)},$$

where A_r and $p^{(r)}$ stand for $A_r^{[2s-1]}(x_{kn})$ and $p_n^{(r)}(x_{kn})$, respectively. By the assumption of induction and (4.6), we see that the second term on the right-hand side of (4.7) has an estimate

$$\begin{aligned}&-\frac{A_{2s-1}}{A_{2s+1}}p^{(2s-1)}\\&=(-1)\alpha^2q_n^{2(m-1)}(1+\rho_nq_n^{-2})\cdot(-1)^{s-1}\alpha^{2(s-1)}q_n^{2(s-1)(m-1)}(1+\zeta_n(s-1;x_{kn})q_n^{-2})p_n'(x_{kn})\\&=(-1)^s\alpha^{2s}q_n^{2s(m-1)}(1+\rho_n'q_n^{-2})p_n'(x_{kn}),\end{aligned}$$

where $\rho_n'=\zeta_n(s-1;x_{kn})+\rho_n+\zeta_n(s-1;x_{kn})\rho_nq_n^{-2}$. Combining (4.2)–(4.5), we see easily that the other terms on the right-hand side of (4.7) are bounded by $Cq_n^{2s(m-1)-2}|p_n'(x_{kn})|$.

q.e.d.

To estimate the lower bound for $e_{\nu-1}(\nu;k,n)$ with odd ν, we need more refined estimates of $(l_{kn}^\nu)^{(2j)}(x_{kn})$. Let $\psi_j(1)=(2j+1)^{-1}$, $j=0,1,2,\ldots$. Let $0<\theta<\Theta$, and suppose $\theta\le|x_{kn}|\le\Theta$. Then, by (3.20) with $r=2j$ and Lemma 9, we have

$$(4.8)\qquad\begin{aligned}&l_{kn}^{(2j)}(x_{kn})=\psi_j(1)(-1)^j\alpha^{2j}q_n^{2j(m-1)}(1+\zeta_n(1;j;x_{kn})q_n^{-2}),\\&|\zeta_n(1;j;x_{kn})|\le C,\qquad j=0,1,\ldots,\end{aligned}$$

where $\zeta_n(1;j;x_{kn})=\zeta_n(j;x_{kn})$ for $j\ge1$, $\zeta_n(1;0;x_{kn})=0$, and C is independent of n and x_{kn}, and may depend on j, θ, Θ and m. By induction on ν, we can estimate $(l_{kn}^\nu)^{(2j)}(x_{kn})$.

LEMMA 10. *For $\nu=1,2,\ldots$, there exists uniquely a sequence $\{\psi_j(\nu)\}_{j=0}^\infty$ of positive numbers and $\zeta_n(\nu;j;x_{kn})$ such that*

$$(4.9)\qquad(l_{kn}^\nu)^{(2j)}(x_{kn})=\psi_j(\nu)(-1)^j\alpha^{2j}q_n^{2j(m-1)}(1+\zeta_n(\nu;j;x_{kn})q_n^{-2}),$$

and

$$|\zeta_n(\nu;j;x_{kn})|\le C,$$

for x_{kn} with $0<\theta\le|x_{kn}|\le\Theta$, where C is independent of n and x_{kn}, and may depend on ν, j, θ, Θ and m.

PROOF. The case $\nu=1$ follows from (4.8). Suppose that the case $\nu-1$ holds. We have

$$(l_{kn}^\nu)^{(2j)}(x_{kn})=\sum_{i=0}^{2j}\binom{2j}{i}(l_{kn}^{\nu-1})^{(i)}(x_{kn})\,l_{kn}^{(2j-i)}(x_{kn})$$

$$= \sum_{r=0}^{j} \binom{2j}{2r} (l_{kn}^{v-1})^{(2r)}(x_{kn}) l_{kn}^{(2j-2r)}(x_{kn})$$
$$+ \sum_{r=1}^{j} \binom{2j}{2r-1} (l_{kn}^{v-1})^{(2r-1)}(x_{kn}) l_{kn}^{(2j-2r+1)}(x_{kn}) .$$

It follows from Lemma 5 that $|(l_{kn}^{v})^{(2t-1)}(x_{kn})| \le C q_n^{(2t-2)(m-1)}$, $t=1,2,\ldots$. Thus, the second sum on the right-hand side of the above equality is bounded by $Cq_n^{(2j-2)(m-1)}$. By (4.8) and the assumption of induction, the first sum $\sum_{r=0}^{j}$ is estimated as

$$\sum_{r=0}^{j} = \sum_{r=0}^{j} \binom{2j}{2r} \psi_r(v-1)(-1)^r \alpha^{2r} q_n^{2r(m-1)}(1+\zeta_n(v-1\,;r\,;x_{kn})q_n^{-2})$$
$$\cdot \psi_{j-r}(1)(-1)^{j-r}\alpha^{2j-2r} q_n^{(2j-2r)(m-1)}(1+\zeta_n(1\,;j-r\,;x_{kn})q_n^{-2})$$
$$= \sum_{r=0}^{j} \frac{1}{2(j-r)+1} \binom{2j}{2r} \psi_r(v-1)(-1)^j \alpha^{2j} q_n^{2j(m-1)}(1+\rho_n q_n^{-2}) , \qquad j=0,1,\ldots,$$

where $\rho_n = \zeta_n(v-1\,;r\,;x_{kn}) + \zeta_n(1\,;j-r\,;x_{kn}) + \zeta_n(v-1\,;r\,;x_{kn})\zeta_n(1\,;j-r\,;x_{kn})q_n^{-2}$. If we put

$$\psi_j(v) = \sum_{r=0}^{j} \frac{1}{2(j-r)+1} \binom{2j}{2r} \psi_r(v-1) , \qquad j=0,1,\ldots, \tag{4.10}$$

then $\{\psi_j(v)\}_{j=0}^{\infty}$ satisfies the required condition. q.e.d.

We rewrite the relation (4.10) in the form

$$\begin{aligned} &\psi_0(v)=1 , \qquad v=1,2,\ldots, \\ &\psi_j(v)-\psi_j(v-1) = \frac{1}{2j+1} \sum_{r=0}^{j-1} \binom{2j+1}{2r} \psi_r(v-1) , \qquad j=1,2,\ldots,\ v=2,3,\ldots . \end{aligned} \tag{4.11}$$

For every j, we will introduce an auxiliary polynomial determined by $\{\psi_j(v)\}_{v=1}^{\infty}$. This polynomial will play an important role in estimating $e_{v-1}(v\,;k,n)$ from below.

LEMMA 11. (i) *For $j=0,1,\ldots$, there exists a unique polynomial $\Psi_j(y)$ of degree j such that $\Psi_j(v)=\psi_j(v)$, $v=1,2,\ldots$.*

(ii) *$\Psi_0(y)=1$, and $\Psi_j(0)=0$, $j=1,2,\ldots$.*

PROOF. Since a polynomial $\Psi_j(y)$ satisfying $\Psi_j(v)=\psi_j(v)$, $v=1,2,\ldots$ is unique, it is enough to show the existence, which we will show by induction on j. In case $j=0$, we can choose $\Psi_0(y)=1$. Suppose that $\Psi_0(y), \Psi_1(y),\ldots,\Psi_{j-1}(y)$ are chosen. By (4.11), we have

$$\psi_j(v) = \psi_j(1) + \sum_{i=2}^{v} \frac{1}{2j+1} \sum_{r=0}^{j-1} \binom{2j+1}{2r} \Psi_r(i-1)$$

$$=\frac{1}{2j+1}\left\{v+\sum_{r=1}^{j-1}\binom{2j+1}{2r}\sum_{i=1}^{v-1}\Psi_r(i)\right\}.$$

If $j=1$, then the last sum is assumed to be zero. Suppose $\Psi_r(y)=\sum_{s=1}^{r}c_s(r)y^s$, $r=1, 2, \ldots, j-1$. Then, it follows that

$$\begin{aligned}\psi_j(v)&=\frac{1}{2j+1}\left\{v+\sum_{r=1}^{j-1}\binom{2j+1}{2r}\sum_{s=1}^{r}c_s(r)\sum_{i=1}^{v-1}i^s\right\},\\&=\frac{1}{2j+1}\left\{v+\sum_{r=1}^{j-1}\binom{2j+1}{2r}\sum_{s=1}^{r}c_s(r)\frac{\varphi_{s+1}(v)}{s+1}\right\},\\&=\frac{1}{2j+1}\left[v+\sum_{s=1}^{j-1}\left\{\sum_{r=s}^{j-1}\binom{2j+1}{2r}c_s(r)\right\}\frac{\varphi_{s+1}(v)}{s+1}\right],\end{aligned}$$

where $\varphi_s(x)$ are the Bernoulli polynomials of degree s defined by the equation $t(e^{xt}-1)(e^t-1)^{-1}=\sum_{s=0}^{\infty}\varphi_s(x)t^s/s!$. Let $\Psi_1(y)=y/3$ and $\Psi_j(y), j=2, 3, \ldots$ be the functions defined by the last formula with y instead of v. Then, since $\varphi_s(0)=0$, $s\geq 2$, we see that $\Psi_j(y)$ has the desired condition. q.e.d.

Since $\Psi_j(y)$ is a polynomial of degree j, we can replace $\psi_j(v)$ in (4.10) with $\Psi_j(y)$, that is,

$$\Psi_j(y)=\sum_{r=0}^{j}\frac{1}{2(j-r)+1}\binom{2j}{2r}\Psi_r(y-1) \tag{4.12}$$

for arbitrary real y and $j=0, 1, \ldots$. We use the notation $F_{kn}(x, y)=\{l_{kn}(x)\}^y$ which coincides with l_{kn}^y if y is an integer. Since $l_{kn}(x_{kn})=1$, we have $l_{kn}(x)>0$ for x in a neighborhood of x_{kn}. Thus, $F_{kn}(x, y)$ is well-defined for x in the neighborhood of x_{kn} and arbitrary real y.

We will show that $(\partial/\partial x)^j F_{kn}(x_{kn}, y)$ is a polynomial of degree at most j with respect to y for $j=0, 1, \ldots$, where $(\partial/\partial x)^j F_{kn}(x_{kn}, y)$ is the j-th partial derivative of F_{kn} with respect to x at (x_{kn}, y). We prove these facts by induction on j. The case $j=0$ is trivial. Suppose that the case j holds. To simplify notation, fix y and let $F(x)=F_{kn}(x, y)$ and $l(x)=l_{kn}(x)$. Then, $F'(x)l(x)=yl'(x)F(x)$. By Leibniz's rule, we easily see that

$$F^{(j+1)}(x_{kn})=-\sum_{s=0}^{j-1}\binom{j}{s}F^{(s+1)}(x_{kn})l^{(j-s)}(x_{kn})+y\sum_{s=0}^{j}\binom{j}{s}l^{(s+1)}(x_{kn})F^{(j-s)}(x_{kn}), \tag{4.13}$$

which shows that $F^{(j+1)}(x_{kn})$ is a polynomial of degree at most $j+1$ with respect to y.

Let $P_{kn}^{[j]}(y)$ be defined by

$$\left(\frac{\partial}{\partial x}\right)^{2j}F_{kn}(x_{kn}, y)=(-1)^j\alpha^{2j}q_n^{2j(m-1)}\Psi_j(y)+P_{kn}^{[j]}(y), \qquad j=0, 1, \ldots. \tag{4.14}$$

Then $P_{kn}^{[j]}(y)$ is a polynomial of degree at most $2j$.

LEMMA 12. *Let $j=0, 1, \ldots$. Let $M>0$ and $0<\theta<\Theta$. Suppose that $\theta \le |x_{kn}| \le \Theta$ and $|y| \le M$. Then,*

$$\text{(i)} \qquad \left| \left(\frac{d}{dy} \right)^s P_{kn}^{[j]}(y) \right| \le C q_n^{2j(m-1)-2}, \qquad s=0, 1,$$

and

$$\text{(ii)} \qquad \left| \left(\frac{\partial}{\partial x} \right)^{2j+1} F_{kn}(x_{kn}, y) \right| \le C q_n^{2j(m-1)},$$

where C is independent of n and k, and may depend on j, θ, Θ, M and m.

PROOF. By (4.9), we have

$$P_{kn}^{[j]}(v) = \psi_j(v)(-1)^j \alpha^{2j} q_n^{2j(m-1)} \zeta_n(v; j; x_{kn}) q_n^{-2}, \qquad v=1, 2, \ldots.$$

By Lemma 10, $|P_{kn}^{[j]}(v)| \le D_v q_n^{2j(m-1)-2}$, $v=1, 2, \ldots$, where D_v are positive constants independent of n and k, and may depend on j, θ, Θ, v and m. Since $P_{kn}^{[j]}(y)$ is a polynomial of degree at most $2j$, we see that $P_{kn}^{[j]}(y)$ coincides with the Lagrange interpolation polynomial of degree $2j$ with values $P_{kn}^{[j]}(v)$ at base points v, $v=1, 2, \ldots, 2j+1$. Evaluating the Lagrange interpolation polynomial and its derivative on the interval $[-M, M]$, we see that (i) holds with a constant C determined only by $D_1, \ldots, D_{2j+1}$ and M.

Next, we prove (ii). Since $(\partial/\partial x) F_{kn}(x_{kn}, y) = y l'_{kn}(x_{kn})$, the case $j=0$ follows from Lemma 5. Suppose that (ii) holds for $1, 2, \ldots, j-1$. By (4.13), $(\partial/\partial x)^{2j+1} F_{kn}(x_{kn}, y)$ is written in the form

$$\begin{aligned}
&\left(\frac{\partial}{\partial x} \right)^{2j+1} F_{kn}(x_{kn}, y) = F^{(2j+1)}(x_{kn}) \\
&= -\left\{ \sum_{i=0}^{j-1} \binom{2j}{2i} F^{(2i+1)}(x_{kn}) l^{(2j-2i)}(x_{kn}) + \sum_{i=0}^{j-1} \binom{2j}{2i+1} F^{(2i+2)}(x_{kn}) l^{(2j-2i-1)}(x_{kn}) \right\} \\
&\quad + y \left\{ \sum_{i=0}^{j} \binom{2j}{2i} l^{(2i+1)}(x_{kn}) F^{(2j-2i)}(x_{kn}) + \sum_{i=0}^{j-1} \binom{2j}{2i+1} l^{(2i+2)}(x_{kn}) F^{(2j-2i-1)}(x_{kn}) \right\}.
\end{aligned}$$

From (i), it follows that $|F^{(2s)}(x_{kn})| \le C q_n^{2s(m-1)}$, $s=0, 1, \ldots$. By Lemma 5, we have $|l^{(2s)}(x_{kn})| \le C q_n^{2s(m-1)}$ and $|l^{(2s+1)}(x_{kn})| \le C q_n^{2s(m-1)}$, $s=0, 1, \ldots$. From these inequalities and the assumption of induction, it follows that every sums in the formula is bounded by $C q_n^{2j(m-1)}$. q.e.d.

By (i) of the above lemma, we can prove the following lemma which plays an essential role in estimating the lower bound of $e_{v-1}(v; k, n)$.

LEMMA 13. *If $y<0$, then $\Psi_j(y) \ne 0$ for $j=0, 1, \ldots$.*

PROOF. Since $\Psi_0(y)=1$, we may assume $j\geq 1$. Since $\Psi_j(0)=0$, $\Psi_j(y)$ has an expression

$$\Psi_j(y)=\sum_{i=1}^{j}(-1)^{j-i}a_i(j)y^i\,, \qquad j=1,2,\dots\,.$$

Then, it is enough to show that $a_i(j)>0$, $i=1,2,\dots,j$. For, if $y=-u$, $u>0$, then $\Psi_j(-u)=(-1)^j\sum_{i=1}^{j}a_j(j)u^i\neq 0$.

We will first show that $a_1(j)>0$, $j=1,2,\dots$. It follows from (4.14) and $(-1)^{j-1}a_1(j)=(d/dy)\Psi_j(0)$ that

$$-\alpha^{2j}q_n^{2j(m-1)}a_1(j)=\frac{d}{dy}\left\{\left(\frac{\partial}{\partial x}\right)^{2j}F_{kn}(x_{kn},y)-P_{kn}^{[j]}(y)\right\}\Bigg|_{y=0}.$$

We have

$$\frac{d}{dy}\left\{\left(\frac{\partial}{\partial x}\right)^{2j}F_{kn}(x_{kn},y)\right\}\Bigg|_{y=0}=\left(\frac{d}{dx}\right)^{2j}\left\{\frac{\partial}{\partial y}F_{kn}(x,0)\right\}\Bigg|_{x=x_{kn}}$$

$$=\left(\frac{d}{dx}\right)^{2j}\log l_{kn}(x)\Bigg|_{x=x_{kn}}=-(2j-1)!\sum_{s\neq k}\frac{1}{(x_{kn}-x_{sn})^{2j}}\,.$$

Here, we used the expression (3.1). Thus, we have

$$a_1(j)=\alpha^{-2j}q_n^{-2j(m-1)}\left\{(2j-1)!\sum_{s\neq k}\frac{1}{(x_{kn}-x_{sn})^{2j}}+\frac{d}{dy}P_{kn}^{[j]}(0)\right\}.$$

If n is large enough, then there exists a number $k(n)$ such that $1\leq x_{k(n)n}\leq 2$. From Lemma 12, (i) with $\theta=1$ and $\Theta=2$, it follows that $|(d/dy)P_{k(n)n}^{[j]}(0)|\leq Cq_n^{2j(m-1)-2}$, where C is a positive constant independent of n. From this and (D), (ii), we have $a_1(j)\geq \alpha^{-2j}q_n^{-2j(m-1)}\{C(2j-1)!\,q_n^{2j(m-1)}-Cq_n^{2j(m-1)-2}\}\geq\{C(2j-1)!\,\alpha^{-2j}-Cq_n^{-2}\}$. Letting $n\to\infty$, we see that $a_1(j)>0$.

Next, we treat the other coefficients. We see that

$$(l_{kn}^{2v})^{(2j+2)}(x_{kn})=\sum_{r=0}^{j+1}\binom{2j+2}{2r}(l_{kn}^{v})^{(2r)}(x_{kn})(l_{kn}^{v})^{(2j+2-2r)}(x_{kn})$$

$$+\sum_{r=1}^{j+1}\binom{2j+2}{2r-1}(l_{kn}^{v})^{(2r-1)}(x_{kn})(l_{kn}^{v})^{(2j+2-2r+1)}(x_{kn})\,, \qquad v=1,2,\dots\,.$$

From Lemma 10, it follows that the leading term on the left-hand side of the equation is $\psi_{j+1}(2v)(-1)^{j+1}\alpha^{2(j+1)}q_n^{2(j+1)(m-1)}$. The leading term of the first sum on the right-hand side is

$$\sum_{r=0}^{j+1}\binom{2j+2}{2r}\psi_r(v)\psi_{j+1-r}(v)(-1)^{j+1}\alpha^{2(j+1)}q_n^{2(j+1)(m-1)}\,.$$

Since $|(l_{kn}^{v})^{(2t-1)}(x_{kn})|\le Cq_n^{(2t-2)(m-1)}$, $t=1, 2, \dots$ by Lemma 5, the second sum is bounded by $Cq_n^{2j(m-1)}$. Thus, we have

$$\psi_{j+1}(2v)=\sum_{r=0}^{j+1}\binom{2j+2}{2r}\psi_r(v)\psi_{j+1-r}(v)\,,$$

and thus

$$\psi_{j+1}(2v)-2\psi_{j+1}(v)=\sum_{r=1}^{j}\binom{2j+2}{2r}\psi_r(v)\psi_{j+1-r}(v)\,,\qquad v=1, 2, \dots .$$

This leads to

$$\Psi_{j+1}(2y)-2\Psi_{j+1}(y)=\sum_{r=1}^{j}\binom{2j+2}{2r}\Psi_r(y)\Psi_{j+1-r}(y)\,. \tag{4.15}$$

We replace $\Psi_{j+1}(y)$ in (4.15) by the expression $\Psi_{j+1}(y)=\sum_{i=1}^{j+1}(-1)^{j+1-i}a_i(j+1)y^i$. We have

$$\sum_{i=1}^{j+1}(-1)^{j+1-i}(2^i-2)a_i(j+1)y^i=\sum_{r=1}^{j}\binom{2j+2}{2r}\Psi_r(y)\Psi_{j+1-r}(y)\,.$$

If we assume $a_i(j)>0$, $i=1, 2, \dots, j$, then we see that the right-hand side of the equation is a polynomial of degree $j+1$ whose coefficients are alternating. Thus, we have $(2^i-2)a_i(j+1)>0$, which implies $a_i(j+1)>0$, $i=2, 3, \dots, j+1$. This completes the proof since we have already obtained $a_1(j)>0$, $j=1, 2, \dots$. q.e.d.

For $v=1, 2, \dots$, define $\eta_{kn}(v; s)$ by the relation

$$e_{2s}(v; k, n)=\frac{1}{(2s)!}(-1)^s\Psi_s(-v)\alpha^{2s}q_n^{2s(m-1)}(1+\eta_{kn}(v; s)q_n^{-2})\,. \tag{4.16}$$

LEMMA 14. *We have*

$$|\eta_{kn}(v; s)|\le C \tag{4.17}$$

for k with $0<\theta\le|x_{kn}|\le\Theta$ *and* $s=0, 1, \dots, \tilde{v}$, *where C is independent of n and k, and may depend on v, s,* θ, Θ *and m, and* $\tilde{v}$ *is the largest integer not exceeding* $(v-1)/2$.

PROOF. We prove (4.17) by induction on s. Since $e_0(v; k, n)=1$ by (3.3) and $\Psi_0(y)=1$, (4.7) holds for $s=0$. Assume it holds until $s-1$. Using (3.3), we write $e_{2s}(v; k, n)$ in the form

$$e_{2s}(v; k, n)=-\frac{1}{(2s)!}\left\{\sum_{r=0}^{s-1}\frac{(2s)!}{(2s-2r)!}e_{2r}(v; k, n)(l_{kn}^{v})^{(2s-2r)}(x_{kn})\right.$$
$$\left.+\sum_{r=1}^{s}\frac{(2s)!}{(2s-2r+1)!}e_{2r-1}(v; k, n)(l_{kn}^{v})^{(2s-2r+1)}(x_{kn})\right\}.$$

We have $|(l_{kn}^{v})^{(2s-2r+1)}| \le Cq_n^{(2s-2r)(m-1)}$ by Lemma 5 and $|e_{2r-1}(v;k,n)| \le Cq_n^{(2r-2)(m-1)}$ by Lemma 6. The second sum $\sum_{r=1}^{s}$ is bounded by $Cq_n^{(2s-2)(m-1)}$. We have by (4.16) and (4.9) that

$$\begin{aligned}\sum_{r=0}^{s-1} &= \sum_{r=0}^{s-1} \frac{(2s)!}{(2s-2r)!} \cdot \frac{1}{(2r)!} (-1)^r \Psi_r(-v) \alpha^{2r} q_n^{2r(m-1)} (1+\eta_{kn}(v;r) q_n^{-2}) \\ &\quad \cdot \psi_{s-r}(v)(-1)^{s-r} \alpha^{2(s-r)} q_n^{2(s-r)(m-1)} (1+\zeta_n(v;s-r;x_{kn}) q_n^{-2}) \\ &= (-1)^s \alpha^{2s} q_n^{2s(m-1)} \sum_{r=0}^{s-1} \binom{2s}{2r} \Psi_r(-v) \psi_{s-r}(v)(1+\rho_n q_n^{-2}),\end{aligned}$$

where $\rho_n = \eta_{kn}(v;r) + \zeta_n(v;s-r;x_{kn}) + \zeta_n(v;s-r;x_{kn})\eta_{kn}(v;r) q_n^{-2}$. Since $|\rho_n| \le C$ by Lemma 10 and the assumption of induction, it is enough to show that $\sum_{r=0}^{s-1} \binom{2s}{2r} \Psi_r(-v)\psi_{s-r}(v) = -\Psi_s(-v)$, $s=1,2,\ldots$, $v=1,2,\ldots$, which is equivalent to $\sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-v)\psi_{s-r}(v) = 0$. Let $C_s(y) = \sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-y)\Psi_{s-r}(y)$. It suffices to show that $C_s(v)=0$, $s=1,2,\ldots$, $v=1,2,\ldots$. We have

$$\begin{aligned}0 &= (l_{kn}^{-1+1})^{(2s)}(x_{kn}) = \sum_{i=0}^{2s} \binom{2s}{i} (l_{kn}^{-1})^{(i)}(x_{kn}) l_{kn}^{(2s-i)}(x_{kn}) \\ &= \sum_{r=0}^{s} \binom{2s}{2r} \left(\frac{\partial}{\partial x}\right)^{2r} F_{kn}(x_{kn}, -1) l_{kn}^{(2s-2r)}(x_{kn}) \\ &\quad + \sum_{r=0}^{s-1} \binom{2s}{2r+1} \left(\frac{\partial}{\partial x}\right)^{2r+1} F_{kn}(x_{kn}, -1) l_{kn}^{(2s-2r-1)}(x_{kn})\end{aligned}$$

for every s. By (4.8), (4.14) and Lemma 12, (i), we see that the first sum $\sum_{r=0}^{s}$ has the form

$$\sum_{r=0}^{s} = (-1)^s \alpha^{2s} q_n^{2s(m-1)} \sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-1)\psi_{s-r}(1) + \xi_n q_n^{2s(m-1)-2},$$

where $|\xi_n| \le C$. By Lemma 5 and Lemma 12, (ii), the second sum $\sum_{r=0}^{s-1}$ is bounded by $Cq_n^{(2s-2)(m-1)}$. Thus, letting $n \to \infty$, we see that

$$0 = \sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-1)\psi_{s-r}(1) = C_s(1)$$

for every s. Suppose $C_s(v)=0$ for every s. We will show that $C_s(v+1)=0$ for every s. Using (4.12) and changing the order of summations, we have

$$C_s(v+1) = \sum_{r=0}^{s} \binom{2s}{2r} \Psi_r(-v-1) \sum_{p=0}^{s-r} \frac{1}{2(s-r-p)+1} \binom{2s-2r}{2p} \Psi_p(v)$$

$$=\sum_{p=0}^{s}\left\{\sum_{r=0}^{s-p}\frac{1}{2(s-r-p)+1}\binom{2s-2r}{2p}\binom{2s}{2r}\Psi_r(-v-1)\right\}\Psi_p(v).$$

By the relation

$$\binom{2s-2r}{2p}\binom{2s}{2r}=\binom{2s}{2p}\binom{2s-2p}{2r}$$

and (4.12), we have

$$\sum_{r=0}^{s-p}\frac{1}{2(s-r-p)+1}\binom{2s-2r}{2p}\binom{2s}{2r}\Psi_r(-v-1)=\binom{2s}{2p}\Psi_{s-p}(-v),$$

which leads to $C_s(v+1)=C_s(-v)$. We are done, since we easily see $C_s(-v)=C_s(v)$.

q.e.d.

Now we are in a position to prove Theorem 2. We find a lower bound for the Lebesgue constants $\Lambda_n(v;[a,b])=\max_{a\le x\le b}\sum_{k=1}^{n}|h_{kn}(v;x)|$ with a positive odd order v and a given interval $[a,b]$, $-\infty<a<b<\infty$. A simple observation allows us to assume $a>0$. By the expression (3.2), we have

$$|h_{kn}(v;x)|\ge|l_{kn}^{v}(x)e_{v-1}(v;k,n)(x-x_{kn})^{v-1}|-|l_{kn}^{v}(x)\sum_{i=0}^{v-2}e_i(v;k,n)(x-x_{kn})^{i}|.$$

Let $c=b+(b-a)$. We have

$$\begin{aligned}\Lambda_n(v;[a,b])&\ge\max_{a\le x\le b}\sum_{k:a\le x_{kn}\le c}|l_{kn}^{v}(x)e_{v-1}(v;k,n)(x-x_{kn})^{v-1}|\\&\quad-\max_{a\le x\le b}\sum_{k:a\le x_{kn}\le c}|l_{kn}^{v}(x)\sum_{i=0}^{v-2}e_i(v;k,n)(x-x_{kn})^{i}|,\\&=\max_{a\le x\le b}F_n(x)-\max_{a\le x\le b}G_n(x),\quad\text{say}.\end{aligned}$$

It follows from Remark in the proof of Theorem 1 that $\max_{a\le x\le b}G_n(x)\le C$ with C independent of n. Thus, it is enough to show that $\max_{a\le x\le b}F_n(x)\ge C\log n$. Let $\Gamma(x;j)$ be the set of numbers defined by

$$\Gamma(x;j)=\left\{k;a\le x_{kn}\le c,j\frac{C_2q_n}{n}\le x_{kn}-x<(j+1)\frac{C_2q_n}{n}\right\},$$

$j=1,2,\dots$, where C_2 is the constant in (D), (ii). We have

$$F_n(x)\ge\sum_{j=1}^{\infty}\sum_{k\in\Gamma(x;j)}|l_{kn}(x)|^{v}|e_{v-1}(v;k,n)|(x_{kn}-x)^{v-1}.$$

Let n_0 be a number such that $c\le q_{n_0}\min\{\kappa_1,\kappa_2,\kappa_3\}$, where κ_1, κ_2 and κ_3 are the constants in (D), (ii), (E) and (F), respectively. By (B), (E) and (F), we have

$$|l_{kn}(x)| = \frac{|p_n(x)|\, w(x)}{(x_{kn}-x)\,|p_n'(x_{kn})|\, w(x_{kn})} \frac{w(x_{kn})}{w(x)}$$

$$\geq C \frac{|x - x_{(x,n)}|}{(x_{kn}-x)} w(c) w^{-1}(a) \geq C \frac{|x - x_{(x,n)}|}{(x_{kn}-x)},$$

for $a \leq x \leq b$, $n \geq n_0$ and $a \leq x_{kn} \leq c$, where C is independent of k, n and x. From this and Lemma 14, it follows that

$$F_n(x) \geq C \sum_{j=1}^{\infty} \sum_{k \in \Gamma(x;j)} \frac{|x - x_{(x,n)}|^{\nu}}{(x_{kn}-x)} |e_{\nu-1}(\nu; k, n)|$$

$$\geq C \sum_{j=1}^{\infty} \sum_{k \in \Gamma(x;j)} \frac{n}{jq_n} |x - x_{(x,n)}|^{\nu}$$

$$\cdot \frac{1}{(\nu-1)!} |\Psi_{(\nu-1)/2}(-\nu)|\, \alpha^{\nu-1} q_n^{(\nu-1)(m-1)} (1 - Dq_n^{-2})$$

for $a \leq x \leq b$ and $n \geq n_0$ with some positive constant D independent of x and n. Let n_1 be a number such that $1 - Dq_{n_1}^{-2} > 1/2$. Let n_2 be a number satisfying $(b-a) > 2C_2 q_{n_2}/n_2$ and $c \leq x_{1n_2}$, where C_2 is the constant in (D), (ii). Then, for every $n \geq n_2$, there exist at least two successive zeros x_{k_0-1n}, x_{k_0n} in the interval (a, b), and so there exists a point $x[n]$ such that $a \leq x[n] \leq b$ and $|x[n] - x_{(x[n],n)}| \geq C_1 q_n/n$ for $n \geq \max\{n_0, n_2\}$, where C_1 is the constant in (D), (ii). Therefore, we have

$$\max_{a \leq x \leq b} F_n(x) \geq C \sum_{j=1}^{\infty} \sum_{k \in \Gamma(x[n];j)} j^{-1}$$

for $n \geq \max\{n_0, n_1, n_2\}$, where C is independent of n. Let $N(n)$ be the number such that $N(n) \leq (c-b)/(C_2 q_n/n) < N(n)+1$. Then, $\Gamma(x;j) \neq \varnothing$ for $x \in [a, b]$, $1 \leq j \leq N(n)$ and $n \geq n_2$. Therefore, $\max_{a \leq x \leq b} F_n(x) \geq C \log N(n) \geq C \log n$ for $n \geq \max\{n_0, n_1, n_2\}$, which completes the proof of Theorem 2.

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