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CONSTRUCTIVE APPROXIMATION

Pointwise Estimates for Monotone Polynomial Approximation

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Abstract. We prove that if f is increasing on $[-1,1]$, then for each $n = 1, 2, \ldots$, there is an increasing algebraic polynomial P_n of degree n such that $|f(x) - P_n(x)| \le c\omega_2(f, \sqrt{1-x^2}/n)$, where ω_2 is the second-order modulus of smoothness. These results complement the classical pointwise estimates of the same type for unconstrained polynomial approximation. Using these results, we characterize the monotone functions in the generalized Lipschitz spaces through their approximation properties.

1. Introduction

Several results show that in some sense monotone approximation by algebraic polynomials performs as well as unconstrained approximation. For example, Lorentz and Zeller [6] have shown that for each increasing function f in $C(I)$, $I := [-1,1]$, there is an increasing polynomial P_n of degree n that satisfies

$$\|f - P_n\| \le c\omega(f, n^{-1}), \qquad n = 1, 2, \ldots, \tag{1.1a}$$

where ω is the modulus of continuity of f and all norms here and throughout are the uniform norm on I.

More generally, for any $k = 0, 1, \ldots,$ there are increasing P_n that satisfy

$$\|f - P_n\| \le cn^{-k}\omega(f^{(k)}, n^{-1}), \qquad n = 1, 2, \ldots. \tag{1.1b}$$

When $k = 1$, this is a result of Lorentz [5], whereas the general case was proved by DeVore [3]. The cases $k = 0, 1$ are much easier to prove than the general case, since they can be proved using linear methods; in contrast, the proof in [3] uses rather involved nonlinear techniques.

It is well-known that for unconstrained approximation, much improvement can be made in estimates of the form (1.1) when x is near an end point of I. Such

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improvements take the form of replacing n^{-1} in (1.1) with either $\Delta_n(x) := \sqrt{1-x^2}/n + 1/n^2$ or just $\sqrt{1-x^2}/n$. In fact, such improved estimates are needed if we wish to characterize the smoothness of a function by its degree of approximation by algebraic polynomials.

In this article, we are interested in pointwise estimates for monotone approximation. The only result of this type that we know of is by Beatson [1]. He proved that the estimate

$$(1.2)\qquad |f(x) - P_n(x)| \le c\omega(f, \Delta_n(x)), \qquad x \in I, \qquad n = 1, 2, \ldots$$

holds for suitable increasing polynomials P_n whenever f is increasing. Among other things, we shall show that this can be improved to allow second-order smoothness as measured by the second-order modulus of smoothness ω_2.

Theorem 1. *If f is increasing on I, then for each $n = 1, 2, \ldots,$ there is an increasing polynomial P_n of degree n such that*

$$(1.3)\qquad |f(x) - P_n(x)| \le c\omega_2\left(f, \frac{\sqrt{1-x^2}}{n}\right), \qquad x \in I.$$

Notice that we have replaced $\Delta_n(x)$ with the smaller quantity $\sqrt{1-x^2}/n$. For unconstrained approximation, such results were given first by Teljakovskii [7] (for ω), and later by DeVore [2] (for ω_2). From the standard properties of ω_2, we see that (1.3) contains the improved form of (1.2) as well as improved estimates of the form (1.1b) when $k = 1$.

With Theorem 1 and the classical converse theorems for approximation by algebraic polynomials, we have the following result, which characterizes the Lip*α spaces, which are defined as the set of all f such that $\omega_2(f, t) = O(t^\alpha)$, $0 < \alpha < 2$.

Theorem 2. *If $0 < \alpha < 2$, then a function f is increasing and in* Lip*α *if and only if for each $n = 1, 2, \ldots,$ there is an increasing algebraic polynomial P_n of degree n such that*

$$(1.4)\qquad |f(x) - P_n(x)| \le c\left(\frac{\sqrt{1-x^2}}{n}\right)^\alpha, \qquad x \in I.$$

We do not know whether this result holds for $\alpha \ge 2$. This would require at the least a refinement of (1.1b) for $k \ge 2$ in which $1/n$ is replaced by $\Delta_n(x)$.

2. Proofs

Our proof is based on a two-stage approximation. We first approximate f by an increasing piecewise linear function S_n that approximates to the accuracy of the right side of (1.3). We then approximate S_n by an increasing algebraic polynomial.

S_n is constructed as follows. We shall select points $-1 = \xi_{-n} < \xi_{-n+1} < \cdots < \xi_n = 1$ (to be defined precisely later) that are more densely distributed near the end points of I. Roughly speaking, these points are uniformly distributed with respect to the Chebyshev measure $dx/\sqrt{1-x^2}$. S_n is then the piecewise linear function that interpolates f at the ξ_k, $k = -n, \ldots, n$. If we let s_j be the slopes

$$s_j := \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j}, \qquad j = -n, \ldots, n-1, \tag{2.1}$$

then S_n can be represented by the truncated power functions $\Phi_j(x) := (x - \xi_j)_+$:

$$S_n(x) = f(-1) + s_{-n}(x+1) + \sum_{j=-n+1}^{n-1} (s_j - s_{j-1})\Phi_j(x). \tag{2.2}$$

It is clear that S_n is increasing if f is. Also, if f is twice continuously differentiable, $\|f''\| \le M$, then (Newton's formula)

$$|f(x) - S_n(x)| \le (M/2)|(x - \xi_j)(x - \xi_{j+1})|, \qquad \xi_j \le x \le \xi_{j+1}. \tag{2.3}$$

We shall now construct a polynomial approximation to S_n. To do this, we first construct polynomials R_j, $j = -n, \ldots, n-1$, that approximate the truncated powers Φ_j. The construction of R_j begins with trigonometric polynomials T_j that are good approximations to the characteristic functions $\chi_0 := 0$ and $\chi_j := \chi_{[-t_j, t_j]}$, $j = 1, \ldots, 2n$, with $t_j := j\pi/2n$, $j = 0, 1, \ldots, 2n$.

Let K_n denote the Jackson kernel

$$K_n(t) := a_n\left(\frac{\sin nt/2}{\sin t/2}\right)^8; \qquad \int_{-\pi}^{\pi} K_n(t)\,dt = 1. \tag{2.4}$$

We recall that the K_n satisfy (see [4, p. 57])

$$\int_{-\pi}^{\pi} |t|^j K_n(t)\,dt \le cn^{-j}, \qquad j = 0, 1, \ldots, 7, \tag{2.5a}$$

$$c_0 n^{-7} \le a_n \le c_1 n^{-7}. \tag{2.5b}$$

Here and throughout c, c_0, and c_1 denote absolute constants; the value of c may vary with each occurrence, even on the same line. We define

$$T_j(t) := \int_{t-t_j}^{t+t_j} K_n(u)\,du, \qquad j = 0, 1, \ldots, 2n. \tag{2.6}$$

In particular, $T_0 = \chi_0$ and $T_{2n} = \chi_{2n}$. We define

$$d_j(t) := \max\left(n \operatorname{dist}\left(t, \{-t_j, t_j\}\right); 1\right).$$

Lemma 3. *For $j = 0, 1, \ldots, 2n$, we have*

$$|\chi_j(t) - T_j(t)| \le c\left(d_j(t)\right)^{-7}, \qquad -\pi \le t \le \pi, \tag{2.7a}$$

$$\int_{-\pi}^{\pi} |\sin t||\chi_j(t) - T_j(t)|\,dt \le c\left(\frac{\sin t_j}{n} + \frac{1}{n^2}\right), \qquad -\pi \le t \le \pi. \tag{2.7b}$$

Proof. When $j=0$ or $j=2n$, these inequalities are obvious. For the other values of j, we let $a := \min(|t-t_j|, |t+t_j|)$. Then

$$|\chi_j(t) - T_j(t)| = \left| \int_{-\pi}^{\pi} \big(\chi_j(t) - \chi_j(t-u)\big) K_n(u)\,du \right|$$

$$\leq \int_{|u|\geq a} K_n(u)\,du \leq \int_{-\pi}^{\pi} \left|\frac{u}{a}\right|^7 K_n(u)\,du \leq c(an)^{-7}$$

because of (2.5). Since we also know that $0 \leq T_j(t) \leq 1$, and hence $|\chi_j(t) - T_j(t)| \leq 1$, we have (2.7a).

To prove (2.7b) we multiply (2.7a) by $|\sin t|$ and integrate; this leaves us the task of estimating $\Sigma := \int_{-\pi}^{\pi} |\sin t| (d_j(t))^{-7}\,dt$. We write

$$\tfrac{1}{2}\Sigma = \int_0^{\pi} |\sin t| \big(d_j(t)\big)^{-7}\,dt = \int_{I_1} + \int_{I_2},$$

with $I_1 := [t_j - 1/2n, t_j + 1/2n]$ and $I_2 := [0,\pi] - I_1$. Since $|I_1| \leq 1/n$ and $|\sin t| \leq c|\sin t_j|$ for $t \in I_1$, we have $\int_{I_1} \leq c|\sin t_j|/n$. To estimate the integral over I_2, we note that $|\sin t| \leq |\sin t_j| + |t - t_j|$, and therefore

$$\int_{I_2} \leq c \int_{I_2} \big(n|t-t_j|\big)^{-7} \big(|\sin t_j| + |t-t_j|\big)\,dt$$

$$\leq c\left(\frac{|\sin t_j|}{n} + \frac{1}{n^2}\right).$$

■

Now let $r_j(x) := T_{n-j}(t)$, $x = \cos t$, and for $x \in [-1,1]$, we define

$$R_j(x) := \int_{-1}^{x} r_j(u)\,du, \qquad j = -n, \ldots, n.$$

In particular, $R_{-n}(x) = x + 1 = \Phi_{-n}(x)$ and $R_n(x) = 0$. The points ξ_j are defined by the equations

$$1 - \xi_j := R_j(1), \qquad j = -n, \ldots, n.$$

In particular, $\xi_{-n} = -1$ and $\xi_n = 1$. We now develop the relevant properties of the points ξ_j and the polynomials R_j.

From the definition of the polynomials T_j, we have $T_{n-j} - T_{n-(j+1)} \geq 0$. Hence, $r_j - r_{j+1} \geq 0$, and therefore

(2.8) $\qquad R_j - R_{j+1}$ is increasing, $j = -n, \ldots, n-1$.

This gives that $-1 = \xi_{-n} < \xi_{-n+1} < \cdots < \xi_n = 1$. Further properties of the ξ_j are given by

Lemma 4. *With* $\delta_j := n^{-1}(\sin t_{n-j}) + n^{-2}$, *we have*

(2.9a) $\qquad |\xi_j - \cos t_{n-j}| \leq cn^{-1}\sin t_{n-j}, \qquad j = -n, \ldots, n,$

(2.9b) $\qquad |\cos^{-1}\xi_j - t_{n-j}| \leq cn^{-1}, \qquad j = -n, \ldots, n,$

(2.9c) $\qquad c_0\delta_j \leq \xi_{j+1} - \xi_j \leq c_1\delta_j, \qquad j = -n, \ldots, n-1,$

(2.9d) $\qquad c_0\delta_j \leq \delta_{j+1} \leq c_1\delta_j, \qquad j = -n, \ldots, n-1.$

Proof. We shall use the fact that $\sin t_k \geq c \sin t_{k\pm1}$ and $\sin t_k \geq \sin \pi/2n \geq 1/n$, $0 < k < 2n$. In particular, this shows that (2.9d) is valid. Now, (2.9a) and (2.9b) are trivially true when $j = -n$ or $j = n$. Hence, we check these only for other values of j. From (2.7b) and the definition of the ξ_j, we have

$$|\xi_j - \cos t_{n-j}| = \left|\int_0^\pi \left(\chi_{n-j}(t) - T_{n-j}(t)\right) \sin t \, dt\right| \leq c\delta_j \leq cn^{-1} \sin t_{n-j}.$$

This gives (2.9a), and writing

$$\xi_{j+1} - \xi_j = \left(\xi_{j+1} - \cos t_{n-j-1}\right) + \left(\cos t_{n-j-1} - \cos t_{n-j}\right) + \left(\cos t_{n-j} - \xi_j\right),$$

it also gives the right inequality in (2.9c).

For the remaining proofs, we consider only $j \geq 0$. Similar arguments prove the case when $j < 0$. To prove the left inequality in (2.9c), we let $B = [t_{n-j} - \pi/4n, t_{n-j}]$. Then $\sin t \geq c \sin t_{n-j}$ on B, and so

$$\begin{aligned} \xi_{j+1} - \xi_j &= \int_0^\pi \left(T_{n-j}(t) - T_{n-j-1}(t)\right) \sin t \, dt \\ &\geq c|B| \sin t_{n-j} \inf_B \left(T_{n-j} - T_{n-j-1}\right) \\ &\geq cn^{-1} \sin t_{n-j} \inf_B \left(T_{n-j} - T_{n-j-1}\right). \end{aligned} \tag{2.10}$$

Now for $t \in B$, we have $[0, \pi/4n] \subseteq [t - t_{n-j}, t - t_{n-j-1}] =: A$. Hence, from the definition (2.6) of the T_k, we have, for $t \in B$,

$$T_{n-j}(t) - T_{n-j-1}(t) \geq \int_A K_n(u) \, du \geq \int_0^{\pi/4n} K_n(u) \, du \geq c,$$

where the very last inequality uses (2.5b) and the fact that $(\sin nu/2)/(\sin u/2) \geq cn$ for $0 \leq u \leq \pi/4n$. Using this in (2.10) proves the left inequality of (2.9c).

Finally, we prove (2.9b) for $j = 0, \ldots, n-1$. Let J be the smallest interval that contains $\cos^{-1}\xi_j$ and t_{n-j}. We claim that for n sufficiently large,

$$\sin t \geq \sin t_{n-j}/2 \geq c \sin t_{n-j}, \qquad \text{for half of the } t \in J. \tag{2.11}$$

This is clear if $\cos^{-1}\xi_j \leq t_{n-j}$ ((2.11) holds on $J \cap [t_{n-j}/2, t_{n-j}]$) or if $t_{n-j} \leq \cos^{-1}\xi_j \leq \pi/2$ ((2.11) holds on all of J). On the other hand, if $\cos^{-1}\xi_j \geq \pi/2$, then from (2.9a), $|\xi_j - \cos t_{n-j}| \leq c/n$, and hence $t_{n-j} \geq \pi/4$ and $\cos^{-1}\xi_j \leq 3\pi/4$ provided n is sufficiently large. Therefore (2.11) holds in this case as well. Now integrating (2.11) over J and using (2.9a) gives

$$c \sin t_{n-j} |J| \leq |\xi_j - \cos t_{n-j}| \leq cn^{-1} \sin t_{n-j}.$$

This gives $|J| \leq c/n$ provided n is sufficiently large, and (2.9b) follows. ∎

Lemma 5. *For* $j = -n, \ldots, n-1$, $x = \cos t$ *with* $0 \leq t \leq \pi$, *we have*

$$|\Phi_j(x) - R_j(x)| \leq c\frac{\sin t}{n}\left(d_{n-j}(t)\right)^{-5}, \tag{2.12a}$$

$$|\Phi_j'(x) - R_j'(x)| \leq c\left(d_{n-j}(t)\right)^{-7}. \tag{2.12b}$$

Proof. The case $j = -n$ follows from the fact that $R_{-n} = \Phi_{-n}$. Therefore, we assume $j > -n$. We first prove (2.12b). Since $R_j'(x) = r_j(x) = T_{n-j}(t)$, we have, from (2.7a),

$$\text{(2.13)} \qquad |\Phi_j'(x) - R_j'(x)| \le |\chi_{n-j}(t) - T_{n-j}(t)| + \chi_J(t) \le c\big(d_{n-j}(t)\big)^{-7} + \chi_J(t),$$

where J is the smallest interval that contains t_{n-j} and $\cos^{-1}\xi_j$. It follows from (2.9b) that $|J| \le cn^{-1}$. Hence $d_{n-j}(t) \le c$ for $t \in J$. This means that the second term on the right side of (2.13) is smaller than the first term, and therefore (2.12b) holds.

To prove (2.12a) we shall use the fact that for $k = 0, 1$ and all j,

$$\text{(2.14a)} \qquad \int_0^t |u - t_j|^k \big(d_j(u)\big)^{-7} du \le cn^{-k-1}\big(d_j(t)\big)^{-5}, \qquad 0 \le t \le t_j,$$

$$\text{(2.14b)} \qquad \int_t^\pi |u - t_j|^k \big(d_j(u)\big)^{-7} du \le cn^{-k-1}\big(d_j(t)\big)^{-5}, \qquad t_j \le t \le \pi.$$

For example, the first inequality is proved by writing the integral as a sum of two integrals, the first over $I_1 := [0, t] \cap [t_j - 1/n, t_j]$ and the second over $I_2 := [0, t] \setminus I_1$. Then

$$\int_{I_2} \le n^{-7} \int_{I_2} |u - t_j|^{-7+k} du \le cn^{-7} \min\left(n^{6-k}, |t - t_j|^{-6+k}\right) \le cn^{-k-1}\big(d_j(t)\big)^{-5},$$

where we have used the fact that $d_j(u) \ge 1$ for all u. If the integral over I_1 is not zero, then $|I_1| \le 1/n$ and $d_j(t) = 1$. Hence,

$$\int_{I_1} \le n^{-k-1} \le n^{-k-1}\big(d_j(t)\big)^{-5},$$

as desired. The second inequality in (2.14) is proved in the same way.

Now to prove (2.12a), we write

$$\text{(2.15)} \qquad \Phi_j(x) - R_j(x) = \int_{-1}^x \big(\Phi_j'(y) - R_j'(y)\big)\, dy = -\int_x^1 \big(\Phi_j'(y) - R_j'(y)\big)\, dy.$$

If $t \ge t_{n-j}$, we use the first representation in (2.15) and (2.12b) to find that

$$\text{(2.16)} \quad |\Phi_j(x) - R_j(x)| \le \int_{-1}^x |\Phi_j'(y) - R_j'(y)|\, dy \le c\int_t^\pi \sin u \big(d_{n-j}(u)\big)^{-7} du.$$

If $\pi/2 \le t$, then $\sin u \le \sin t$ on the interval of integration, and therefore by using (2.14b) with $k = 0$, we get (2.12a).

On the other hand, if $t \le \pi/2$, then we use $\sin u \le \sin t + |u - t| \le \sin t + |u - t_{n-j}|$ on the interval of integration. Putting this in the right integral of (2.16) and using (2.14b) gives

$$(2.17)\qquad |\Phi_j(x) - R_j(x)| \le c\frac{\sin t}{n}\big(d_{n-j}(t)\big)^{-5} + n^{-2}\big(d_{n-j}(t)\big)^{-5}.$$

But $\sin t \ge n^{-1}$, because $t \ge t_1 = \pi/2n$, and therefore (2.12a) follows in this case.

If $t \le t_{n-j}$, we use the second representation in (2.15) together with (2.14a) to arrive at the same conclusion. ∎

If $f \in C(I)$, we define

$$(2.18)\qquad L_n(f) := f(-1) + s_{-n}R_{-n} + \sum_{-n+1}^{n-1}{}^{\cdot}\,(s_j - s_{j-1})R_j,$$

with s_j defined by (2.1). If f is increasing, then $s_j \ge 0$, $j = -n, \dots, n-1$, and since we can also write

$$L_n(f) = f(-1) + \sum_{-n}^{n-1} s_j\big(R_j - R_{j+1}\big),$$

it follows from (2.8) that $L_n(f)$ is increasing. Also, since $R_j(-1) = 0$ and $R_j(1) = 1 - \xi_j = \Phi_j(1)$ for all j, we have $L_n(f, \pm 1) = f(\pm 1)$.

Theorem 6. *If f' is absolutely continuous and $|f''| \le M$ a.e. on I, then for each $n = 1, 2, \dots$ and each $x \in I$, we have*

$$(2.19)\qquad |f(x) - L_n(f,x)| \le cM\left(\frac{\sqrt{1-x^2}}{n}\right)^2.$$

Proof. We will check (2.19) for $x \ge 0$; the case $x < 0$ is proved similarly. If $\xi_j \le x \le \xi_{j+1}$ and $j \le n-2$ (that is, excluding the rightmost interval), then writing $\theta_j := \cos^{-1}\xi_j$, we have $|\theta_{j+1} - t_{n-j-1}| \le c/n$ and hence (2.9) gives

$$\xi_{j+1} - \xi_j \le c\delta_j \le c\delta_{j+1} = c\left(\frac{\sin t_{n-j-1}}{n} + \frac{1}{n^2}\right) \le c\left(\frac{\sin\theta_{j+1}}{n} + \frac{1}{n^2}\right)$$
$$\le c\left(\frac{\sin t}{n} + \frac{1}{n^2}\right) \le c\frac{\sin t}{n} = c\frac{\sqrt{1-x^2}}{n}.$$

Hence,

$$(2.20)\qquad |f(x) - S_n(x)| \le (M/8)\big(\xi_{j+1} - \xi_j\big)^2$$
$$\le cM\left(\frac{\sqrt{1-x^2}}{n}\right)^2, \qquad \xi_j \le x \le \xi_{j+1}.$$

This inequality also holds when $j = n-1$, because $(x - \xi_{n-1})(1-x) \le c\delta_{n-1}(1-x) \le c(1-x)/n^2$ on this interval. Hence, we have

$$(2.21)\qquad |f(x) - S_n(x)| \le cM\left(\frac{\sqrt{1-x^2}}{n}\right)^2, \qquad 0 \le x \le 1.$$

We now estimate

(2.22) $$E(x) := S_n(x) - L_n(f,x) = \sum (s_j - s_{j-1})(\Phi_j(x) - R_j(x)).$$

Here and in what follows, an unsubscripted "Σ" denotes the sum for $j = -n+1$ to $j = n-1$. Now $|s_j - s_{j-1}| \le cM(\xi_{j+1} - \xi_j) \le cM\delta_j$ for each j [see (2.9c)]. Using this and (2.12a) with $x = \cos t$ gives

$$|E(x)| \le cM\frac{\sin t}{n}\sum \delta_j (d_{n-j}(t))^{-5}.$$

Now for $|j| < n$, $\delta_j \le n^{-1}(\sin t + |t - t_{n-j}|) + n^{-2} \le cn^{-1}(\sin t + |t - t_{n-j}|)$. Hence,

(2.23) $$\begin{aligned}|E(x)| &\le cM\frac{\sin t}{n^2}\sum(\sin t + |t - t_{n-j}|)(d_{n-j}(t))^{-5}\\ &\le cM\frac{\sin t}{n^2}(\sin t + 1/n) \le cM(\Delta_n(x))^2, \qquad 0 \le x \le 1,\end{aligned}$$

where we have used the easily verified inequalities

(2.24) $$\sum_{j=1}^{2n-1} |t - t_j|^k (d_j(t))^{-5} \le cn^{-k}, \qquad k = 0, 1.$$

We can improve (2.23) near the end point 1. Differentiating (2.22) and using (2.12b) gives

$$|E'(x)| \le cM\sum \delta_j (d_{n-j}(t))^{-7} \le cM\Delta_n(x), \qquad x \in I,$$

where the sum has been estimated as in (2.23). Integrating this inequality from x to 1 and using the fact that $E(1) = 0$ gives

(2.25) $$|E(x)| \le cM(1 - x^2)\Delta_n(x).$$

If we superimpose the two inequalities (2.23) and (2.25), we get (2.19). That is, when $1 - x^2 \ge n^{-2}$, (2.23) gives the desired estimate, and when $1 - x^2 \le n^{-2}$, (2.25) gives the desired estimate ($\Delta_n(x) \le 2n^{-2}$ in this case). ■

Proof of Theorem 1. We first prove that the L_n are uniformly bounded on $C(I)$. From (2.12), it follows that

(2.26) $$|\Phi_j(x) - R_j(x)| \le cn^{-1}(\sin t_{n-j} + |t - t_{n-j}|)(d_{n-j}(t))^{-5}.$$

Now from (2.9), $|s_j| \le 2\|f\|/(\xi_{j+1} - \xi_j) \le c\|f\|/\delta_j \le c\|f\|/\delta_{j+1}$. Using this with (2.26) in (2.22) gives

$$\begin{aligned}|L_n(f,x)| &\le |S_n(x)| + \sum_{-n+1}^{n-1} |s_j - s_{j-1}||R_j(x) - \Phi_j(x)|\\ &\le \|f\| + c\|f\| \sum_{-n+1}^{n-1} n^{-1}(\sin t_{n-j} + |t - t_{n-j}|)(d_{n-j}(t))^{-5}\delta_j^{-1}\\ &\le \|f\| + c\|f\| \sum_{-n+1}^{n-1} (1 + n|t - t_{n-j}|)(d_{n-j}(t))^{-5}\\ &\le c\|f\|,\end{aligned}$$

where the last inequality uses (2.24). This shows that the L_n are uniformly bounded.

It is well-known (see, e.g., [2]) that the K-functional

$$K_2(f,u) := \inf_g \left(\|f-g\| + u^2 \|g''\|_\infty \right)$$

is equivalent to $\omega_2(f,u)$ when f is in $C(I)$; in particular,

$$K_2(f,u) \le c\omega_2(f,u), \qquad u > 0. \tag{2.27}$$

Given $x \in I$, we fix x and we take $u := \sqrt{1-x^2}/n$. Then from (2.27), there is a g that satisfies

$$\|f-g\| \le c\omega_2(f,u)$$

and

$$u^2\|g''\|_\infty \le c\omega_2(f,u).$$

From Theorem 6 and the linearity of L_n, we have

$$\begin{aligned} |f(x) - L_n(f,x)| &\le |f(x) - g(x)| + |g(x) - L_n(g,x)| + |L_n(f-g,x)| \\ &\le (1 + \|L_n\|)\|f-g\| + c\|g''\|_\infty u^2 \\ &\le c\omega_2(f,u) = c\omega_2\left(f, \frac{\sqrt{1-x^2}}{n}\right). \end{aligned}$$

Since $L_n(f)$ is increasing and a polynomial of degree $\le 8n$, we have proved Theorem 1. ∎

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